# Limit curve theorems for incomplete metric spaces and the null distance on Lorentzian manifolds

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**Abstract**

We prove a limit curve theorem for incomplete metric spaces. Our main application is to Sormani and Vegas' null distance, where our results give strong control on the Lorentzian lengths of limit curves. We also show that regular cosmological time functions and the surface function of a Cauchy surface in a globally hyperbolic manifold define such a null distance.

## Contents



## 1 Introduction

This paper proves limit curve theorems for incomplete distances on Lorentzian manifolds and establishes new sufficient conditions for the upper semi-continuity of the Lorentzian length functional on curves.

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There are natural situations in which one wants the Lorentzian length functional to be upper semi-continuous over sequences of curves with common non-compact domain. We give necessary conditions for this in Lemma 2.14. Existing limit curve theorems, however, assume a complete metric which may have little relation to the Lorentzian distance. The compatibility of the null distance with the Lorentzian distance makes it an attractive alternative, but there are no guarantees of completeness for the null distance. This was a strong motivator for our limit curve theorem, which allows for incomplete distances, Theorem 3.3.

Theorem 3.3 only requires the metric to induce the manifold topology and need not be metrically complete. In Section 4 we specialise our limit curve theorem to the null distance of Sormani and Vega [22] so that we can make use of Lemma 2.14. We require additional conditions given in Proposition 4.3 and Theorem 4.4. We show that the null distance induced by a regular cosmological time or a suitable surface function [20] satisfy our conditions.

The strangest feature of our definitions and results is that limit curves can have strictly smaller domains than the curves in the defining sequence. This is a direct consequence of using incomplete metrics, because portions of the "obvious" limit curve may not exist, or be obstructed by an incompleteness. See Sections 2.5 and 2.6 for details and examples.

In Section 2 we recall the required background results, including the null distance, surface functions and known limit curve theorems. We also study the relationship of the Lorentzian length to limit curve theorems, how incompleteness damages these relations, and provide our definition of limit curve in incomplete spaces.

Section 3 proves our limit curve theorems for incomplete distances, and in Section 4 we exploit null distances associated to (suitable) time functions to control the Lorentzian length of limit curves. We show that regular cosmological time functions satisfy our requirements. In Section 5, we show that surface functions of $C^1$ Cauchy surfaces give null distances with strong control on the Lorentzian lengths of limit curves induce null distances which can be used with Theorem 3.3, see Corollaries 5.12 and 5.13

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# 2 Background

The material below recalls ideas relied on in this paper. General references for the needed Lorentzian geometry are [2] and [19]. A review of material relating to generalised time functions can be found in [14, 15]. A detailed introduction to the limit curve theorem can be found in [12].

## 2.1 Some Lorentzian geometry

Our manifolds, denoted by $M$, are Hausdorff, paracompact and smooth. We allow an arbitrary number $\geq 1$ of spacelike dimensions. The metric is $g$ with signature $(-, +, \ldots, +)$ and assumed to be smooth. Since $M$ is separable, any compact exhaustion of $M$ is independent of the choice of any metric or distance.

We work with continuous curves in $M$. A curve is a continuous function $\gamma : I \to M$ where $I \subset \mathbb{R}$ is a connected interval. In particular, $\gamma(I)$ is connected. A change of parameter is a continuous, bijective, strictly increasing or decreasing function $s : J \to I$, where $J \subset \mathbb{R}$ is a connected interval.

We extend Penrose' definition of causal curves, [19, Definition 2.25], to curves that may not have compact image.

**Definition 2.1** (Causal, timelike and null curves)**.** A continuous function $\gamma : I \to M$ from a connected, not compact, interval $I$ of $\mathbb{R}$ is a future directed causal / timelike / null geodesic if for all $K \subset I$ a connected compact subinterval, the subcurve $\gamma|_K$ is a future directed causal / timelike / null curve in the sense of [19, Definition 2.25]. ▲

A past directed, continuous causal curve is defined by time duality. By a causal curve we will always mean a continuous causal curve.

Every causal curve $\gamma : I \to M$ has a re-parametrisation $s : J \to I$ so that for any chart $\phi : U \to \mathbb{R}^n$, the curve $\phi \circ \gamma \circ s$ is locally Lipschitz with respect to the Euclidean distance on $\mathbb{R}^n$, [2, Page 75ff, Equation 3.14]. See also [14, Theorem 2.12]. If $h$ is any Riemannian metric on $M$ then as $(\gamma \circ s)'$ exists for almost all $t \in J$, the (extended) number

$$a = \int_J \sqrt{h((\gamma \circ s)'(t), (\gamma \circ s)'(t))}\, \mathrm{d}t \quad \in \quad [0, \infty],$$

is well-defined. If the interval $I$ is closed then let $A = [0, a]$, otherwise let $A$ be one of $(0, a)$, $[0, a)$, $(0, a]$ depending on the closedness of $I$ to the left and right. Then there is a change of parameter $r : A \to J$ so that

$$h((\gamma \circ s \circ r)'(t), (\gamma \circ s \circ r)'(t)) = 1,$$

for almost all $t \in A$. We call $r$ the arc-length parametrisation induced by $h$.

A function is Cauchy if all of its level surfaces are Cauchy hypersurfaces. We allow Cauchy functions to not be surjective onto $\mathbb{R}$. Thus, we differ from [3].

Penrose has shown that continuous causal curves with compact image have well defined Lorentzian length [19, Definition 7.4ff]. We can extend this definition to our continuous curves. If $\gamma : I \to M$ is a continuous causal curve with an arbitrary connected interval as a domain, we define

$$L(\gamma) = \sup\{L(\gamma|_{[a,b]} : a, b \in I,\ a < b\}, \tag{1}$$

where $L(\gamma|_{[a,b]})$ is the Lorentzian length as defined by Penrose [19, Definition 7.4ff]. Thus, by definition if $\gamma : [0, a) \to M$ is a continuous causal curve then $L(\gamma) = \lim_{t \to a} L(\gamma|_{[0,t]})$.

A function $f : M \to \mathbb{R}$ is locally Lispschitz if on any compact subset $C$ of a chart, $f$ is Lipschitz with respect to any metric inducing the manifold topology on the chart.

When $\gamma : I \to M$ is locally Lipschitz we have

$$L(\gamma) = \int_I \sqrt{-g(\gamma', \gamma')}\mathrm{d}t, \tag{2}$$

by [13, Theorem 2.37]. Minguzzi presents a similar discussion of causal curves with an emphasis on absolute continuity [13, Sections 2.3, 2.4, and 2.5].

## 2.2 Cosmological time, surface functions and the null distance

The Lorentzian distance will be denoted $d_L$. If $h$ is an auxiliary Riemannian metric we write $d(\cdot,\cdot;h)$ for the (actual) distance induced by $h$. The subscript $L$ on the Lorentzian distance is intended to remind the reader that $d_L$ is not a distance.

**Definition 2.2** (Cosmological time, [1])**.** The cosmological times $\tau : M \to \mathbb{R}$ is defined by

$$\tau(x) = \sup\left\{d_L(y,x) : y \in I^-(x)\right\}$$

If $\tau(x) < \infty$ for all $x \in M$ (so that the earliest time is a finite time ago) and $\lim_t \tau(\gamma(t)) = 0$ for all past-directed inextendible causal curves then we say that $\tau$ is regular. ▲

Given $S \subset M$ we define

$$d_L(S,x) = \sup\{d_L(s,x) : s \in S\} \qquad \text{and} \qquad d_L(x,S) = \sup\{d_L(x,s) : s \in S\}.$$

**Definition 2.3** (Surface function)**.** If $S \subset M$ is an achronal set such that $M = I^+(S) \cup S \cup I^-(S)$ and for all $x \in M$, $d_L(S,x) < \infty$ and $d_L(x,S) < \infty$ then the function $\tau_S : M \to \mathbb{R}$ given by

$$\tau_S(x) = \begin{cases} d_L(S,x), & x \in I^+(S) \\ 0, & x \in S, \\ -d_L(x,S), & x \in I^-(S), \end{cases}$$

is well-defined. We call $\tau_S$ the surface function associated to $S$. ▲

Surface functions, and in particular suitable surfaces, can be constructed on any Lorentzian manifold $(M,g)$ with finite Lorentzian distance, meaning that for all $x,y \in M$, $d_L(x,y) < \infty$, see [20]. Surface functions are increasing on timelike curves and non-decreasing on null curves. Surface functions are continuous almost everywhere on $M$, [20, Corollary A.2], and are differentiable almost everywhere on $M$, [13, Theorem 1.19].

We now introduce the null distance of a generalised time function [22]. A generalised time function is any function that is increasing on causal curves. An alternating causal curve is a $C^0$ piecewise $C^\infty$ function $\gamma : [a,b] \to M$, $[a,b] \subset \mathbb{R}$, with a finite partition $\{t_1,\dots,t_k\}$, $[t_1,t_k] = [a,b]$ so that for all $i = 1,\dots,k-1$, $\gamma|_{[t_i,t_{i+1}]}$ is a smooth *future or past* directed causal curve. Note that we use the phrase "alternating causal curve" where [22] uses the phrase "piecewise causal curve", which means something different for us.

Any generalised time function $\tau$ induces a functional $L(\cdot;\tau)$ on alternating causal curves,

**Definition 2.4** ([22, Definition 3.2])**.** Let $\tau$ be a generalised time function. If $\gamma : [a,b] \to M$, $[a,b] \subset \mathbb{R}$, with a finite partition $\{t_1,\dots,t_k\}$, $[t_1,t_k] = [a,b]$, is an alternating causal curve then we define $L(\gamma;\tau) = \sum_{i=1}^{k-1} |\tau(\gamma(t_{i+1})) - \tau(\gamma(t_i))|$. The function $d(\cdot,\cdot;\tau) : M \times M \to \mathbb{R}$ called the null distance of $\tau$, [22, Definition 3.2], is defined by

$$d(x,y;\tau) = \inf\{L(\gamma;\tau) : \gamma \text{ is an alternating causal curve from } x \text{ to } y\}.$$ ▲

Note that null distances may only be pseudo-distances and may not induce the topology of the manifold. If $\gamma : [0,1] \to M$ is a future directed, piecewise smooth, causal curve then $L(\gamma;\tau) = \tau(\gamma(1)) - \tau(\gamma(0))$, [22, Lemma 3.6(2)]. If $y \in J^+(x)$ then $d(x,y;\tau) = \tau(y) - \tau(x)$, [22, Lemma 3.11].

To describe when a null distance is a distance that induces the manifold topology we need to introduce the concept of "anti-Lipschitz". Anti-Lipschitz functions were first introduced in [4, Lemma 4.1ff]. Anti-Lipschitz functions are also defined in [13, Section 2.2, Item (e), Page 21] and [22, Definition 4.4]. Equivalence of these various definitions is proven in [22, Prop 4.9].

**Definition 2.5.** Let $(M, g)$ be a Lorentzian manifold and $f : M \to \mathbb{R}$ a function. The function $f$ is anti-Lipschitz on an open set $U \subset M$ if there exists a Riemannian metric $h : TU \times TU \to M$ so that for all $\gamma : [0, 1] \to U$ a future directed causal curve

$$f \circ \gamma(1) - f \circ \gamma(0) \geq L(\gamma; h).$$

A function $f$ is locally anti-Lipschitz if it is anti-Lipschitz on a neighbourhood of each $p \in M$. ▲

An anti-Lipschitz function $f$ is increasing on all future directed causal curves and therefore is a generalised time function, [22, Definition 4.4]. We recall the following characterisation of when the null distance of a generalised time function is a metric compatible with the manifold topology.

**Proposition 2.6.** *[22, Propositions 3.15, 4.5] Let $f : M \to \mathbb{R}$ be a generalised time function. If $f$ is locally anti-Lipschitz then $d(\cdot, \cdot; f)$ is a distance. If, in addition, $f$ is continuous then $d(\cdot, \cdot; f)$ induces the manifold topology.*

## 2.3 Smooth approximation of Lipschitz functions

We shall use Czarnecki and Rifford's approximation theorem, [6, Theorem 2.2, page 4475], to smoothly approximate a locally Lipschitz function. This result relies on uses Clarke's generalised gradient [5]. We briefly review the required results and definitions here.

**Definition 2.7** (Clarke's generalised directional derivative, [5, Page 25])**.** Let $Y \subset \mathbb{R}^n$, $v, x \in \mathbb{R}^n$. If $f : Y \to \mathbb{R}$ is Lipschitz on a neighbourhood of $x$ then the generalised directional derivative of $f$ at $x$ in the direction $v$ is denoted $f^\circ(x; v)$ and is defined by

$$f^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}. \qquad ▲$$

The generalised directional derivative is similar to a partial strong derivative but uses a lim sup rather than lim. See [5, Proposition 2.1.1ff] for proofs of some properties.

**Definition 2.8** (Clarke's generalised gradient, [5, Page 27])**.** Let $Y \subset \mathbb{R}^n$, $v, x \in \mathbb{R}^n$. If $f : Y \to \mathbb{R}$ is Lipschitz on a neighbourhood of $x$ then the generalised gradient of $f$ at $x$ is denoted $\partial^\circ f(x)$ or $\partial^\circ f|_x$ and is defined by

$$\partial^\circ f(x) = \{w \in \mathbb{R}^n : \forall v \in \mathbb{R}^n,\ f^\circ(x; v) \geq w \cdot v\},$$

where $w \cdot v$ is the dot product, the standard Euclidean norm on $\mathbb{R}^n$. ▲

Strictly, Clarke's generalised gradient is a subset of the dual space. Throughout this paper we identify differential forms over $\mathbb{R}^n$ with vectors over $\mathbb{R}^n$ via the isomorphism given by the standard Euclidean metric.

Properties and applications of the generalised gradient can be found in [5]. In Theorem 2.9 for any $A \subset \mathbb{R}^n$ we denote the closure of the convex hull of $A$ by $\overline{\text{co}}\{A\}$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ function. The gradient of $f$, in the standard Euclidean metric, will be denoted $Df$.

**Theorem 2.9** ([5, Thm 2.5.1]). *Let $Y \subset \mathbb{R}^n$ $f : Y \to \mathbb{R}$ be a locally Lipschitz function, and let $S$ be any subset of Lebesgue measure zero. Then*

$$\partial^\circ f(x) = \overline{\text{co}}\{\lim Df(x_i) : \, x_i \to x, \; x_i \notin S, \; f \textit{ differentiable at } x_i\}.$$

**Theorem 2.10** ([6, Theorem 2.2, page 4475]). *Let $B \subset \mathbb{R}^n$ be the open unit ball with respect to the standard Euclidean metric.*

*Let $Y \subset \mathbb{R}^n$ be an open subset and let $f : Y \to \mathbb{R}$ be a locally Lipschitz function. Then, for every continuous function $\epsilon : Y \to \mathbb{R}^+$, there exists a smooth function $f_\epsilon : Y \to \mathbb{R}$ such that for all $x \in Y$,*

$$|f_\epsilon(x) - f(x)| \leq \epsilon(x)$$

*and*

$$Df_\epsilon(x) \in \partial^\circ f|_{(x+\epsilon(x)B)\cap Y} + \epsilon(x)B, \qquad \partial^\circ f(x) \subset Df|_{(x+\epsilon(x)B)\cap Y} + \epsilon(x)B.$$

### 2.4 Limit curve theorems for complete spaces

We will make use of standard limit curve theorems [9, Page 369] and the upper semi-continuity of the Lorentzian length function $L$ with respect to the topology of uniform convergence on compact sets [9, Proposition, page 369]. To the best of our knowledge these results were first presented in [9], though Galloway does indicate that the first edition of [2] contains similar results.

The following is a typical version of "the" limit curve theorem. For a modern and more flexible statement of the result see [12].

**Theorem 2.11** (A paraphrase of [2, Lemma 14.2]). *Let $\gamma_i : \mathbb{R}^+ \to M$ be a sequence of future directed causal curves parametrised with respect to the arc-length induced by a complete Riemannian metric. If $x$ is an accumulation point of $(\gamma_i(0))_i$ then there exists an inextendible future directed causal curve $\gamma : \mathbb{R}^+ \to M$ so that $\gamma(0) = x$ and a subsequence $(\gamma_{i_k})$ which converges to $\gamma$ uniformly on compact subsets of $\mathbb{R}^+$ with respect to the distance induced by the Riemannian metric.*

The upper semi-continuity of the Lorentzian length functional $L$ has been expressed in a handful of different ways, for example [9, Proposition, page 369], [19, Theorem 7.5], [2, Proposition 14.3], and [12, Theorem 2.4]. We rephrase Geroch's formulation [9, Proposition, page 369].

**Theorem 2.12.** *Let $I \subset \mathbb{R}$ be a compact interval and let $d$ be a metric on $M$ which is compatible with the manifold topology. For each $i \in \mathbb{N}$, let $\gamma_i : I \to M$ be a past directed inextendible causal curve. If the sequence $(\gamma_i)$ converges uniformly, with respect to a metric $d$, to a past directed inextendible causal curve $\gamma : I \to M$, then*

$$L(\gamma) \geq \limsup_i L(\gamma_i).$$

A proof can be found in [2, Proposition 14.3]. Theorem 2.12 does not necessarily hold if the curves are defined over a non-compact interval. Examples 2.16 and 2.17 below demonstrate this.

In limit curve theorems for Lorentzian manifolds, the limit curve is causal. This has been proven elsewhere and is well-known, see for instance [12, Lemma 2.7] or [2, Second paragraph page 77]. For completeness we include the following result, using a proof that we have not seen elsewhere which is inspired by [2, Lemma 3.29].

**Lemma 2.13.** *Let $(M, g)$ be a Lorentzian manifold. For each $i \in \mathbb{N}$, let $\gamma_i : (a, b) \to M$ be a future directed causal curve. If there exists a continuous curve $\gamma : (a, b) \to M$ so that for all $t \in (a, b)$, $\gamma_i(t) \to \gamma(a)$ then $\gamma$ is a continuous causal curve.*

*Proof.* Let $t \in (a, b)$ and choose $U$ an open convex normal neighbourhood containing $\gamma(t)$. Since $U$ is open there exists $\epsilon > 0$ so that $\gamma(t - \epsilon, t + \epsilon) \subset U$. Let $t_1, t_2 \in (t - \epsilon, t + \epsilon)$, $t_1 < t_2$. By restricting to a subsequence, if necessary we can assume that for all $i \in \mathbb{N}$, $\gamma_i(t_1), \gamma_i(t_2) \in U$.

Since each $\gamma_i$ is future directed causal and as $U$ is convex normal, for each $i \in \mathbb{N}$ there exists a future directed causal geodesic in $U$ from $\gamma_i(t_1)$ to $\gamma_i(t_2)$. Let $v_i = \exp^{-1}_{\gamma_i(t_1)}(\gamma_i(t_2))$. Then by the assumption of pointwise convergence of $\gamma_i$ and joint continuity of $\exp^{-1}$, the sequence of tangent vectors $(v_i)_{i\in\mathbb{N}}$ converges to $v = \exp^{-1}_{\gamma(t_1)}(\gamma(t_2))$.

Let the vector field $T$ define our time orientation. Since each $\gamma_i$ is future directed and timelike we have $g(T, v_i) \leq 0$ and $g(v_i, v_i) \leq 0$. Taking the limit with respect to $i$ shows that $g(T, v) \leq 0$ and $g(v, v) \leq 0$. Hence $v \in T_{\gamma(t_1)}M$ is future directed and causal. By construction $\exp_{\gamma(t_1)}(v) = \gamma(t_2)$. Thus as $U$ is convex normal, the unique geodesic between $\gamma(t_1)$ and $\gamma(t_2)$ is the curve $t \mapsto \exp_{\gamma(t_1)}(tv)$, which is future directed. Thus $\gamma(t_2) \geq \gamma(t_1)$ and so $\gamma$ is a continuous causal curve as required. □

## 2.5 Lorentzian length control

In this section we prove one result, Lemma 2.14, which gives sufficient conditions to know when control of Lorentzian length over compact subsets of the domain of the curves implies control of Lorentzian length over the entire domain of the curves. Application of Lemma 2.14 depends, in this paper, on a careful analysis of parametrisations of causal curves in Lorentzian manifolds and other additional strong hypotheses: see Section 4. After proving Lemma 2.14, we give our definition of convergence of curves in incomplete metric spaces, Definition 2.15, and then present two examples illustrating what can go wrong.

**Lemma 2.14.** *Let $(M, g)$ be a Lorentzian manifold, and fix $a \in \mathbb{R}^+$, finite. Let $d$ be a metric on $M$ which is compatible with the manifold topology. Let $(\gamma_i : [0, a) \to M)_i$ be a sequence of causal curves which converge uniformly on compact subsets of $[0, a)$, with respect to $d$, to a causal curve $\gamma : [0, a) \to M$. If*

1. *the Lorentzian arc-lengths $L(\gamma_i)$ are all finite, the sequence $(L(\gamma_i))_i$ is bounded above, and*
2. *there exists $b > 0$ so that for all $i \in \mathbb{N}$ and all $t_1, t_2 \in [0, a)$*

$$L(\gamma_i|_{[t_1,t_2]}) \leq b\,|t_2 - t_1| \quad \textit{and} \quad L(\gamma_i|_{[t_1,a)}) \leq b\,|a - t_1|\,,$$

*then there exists a subsequence $(\gamma_{i_k})_k$ so that $L(\gamma) \geq \limsup_k L(\gamma_{i_k})$. Moreover, if for all $t \in [0, a)$,*

$$L(\gamma|_{[0,t]}) = \lim_k L(\gamma_{i_k}|_{[0,t]})$$

*then* $L(\gamma) = \lim_k L(\gamma_{i_k})$.

*Proof.* We begin by noting that $(L(\gamma_i))_i$ is a bounded sequence in $\mathbb{R}^+$. We can therefore choose a subsequence $(L(\gamma_{i_k}))_k$ so that $\lim_k L(\gamma_{i_k}) = \limsup_i L(\gamma_i)$.

We can define a function $f : [0, a) \times \mathbb{N} \to \mathbb{R}^+$ by $f(t, k) = L(\gamma_{i_k}|_{[0,t]})$. By definition of the Lorentzian length, Equation (1), $L(\gamma_{i_k}|_{[0,t]}) \to L(\gamma_{i_k})$ as $t \to a$. Therefore we can extend $f$ to the domain $[0, a] \times \mathbb{N}$ by defining $f(a, k) = L(\gamma_{i_k})$ and ensure that, for fixed $k$, $f(\cdot, k)$ is continuous.

Choose $\epsilon > 0$. We need to check three cases. For all $t, s \in [0, a)$, and all $k \in \mathbb{N}$ if $0 < t - s < \epsilon/b$ then

$$f(t, k) - f(s, k) = L(\gamma_{i_k}|_{[0,t]}) - L(\gamma_{i_k}|_{[0,s]}) = L(\gamma_{i_k}|_{[s,t]}) \leq b(t - s) < \epsilon.$$

A similar calculation can be performed if $s > t$ to show that if $|t - s| < \epsilon/b$ then $|f(t, k) - f(s, k)| \leq \epsilon$. Further, for all $t \in [0, a]$ and all $k \in \mathbb{N}$ if $0 < a - t < \epsilon/b$ then

$$f(a, k) - f(t, k) = L(\gamma_{i_k}|_{[0,a)}) - L(\gamma_{i_k}|_{[0,t]}) = L(\gamma_{i_k}|_{[t,a)}) \leq b(a - t) < \epsilon.$$

That is, for all $t, s \in [0, a]$ and all $k \in \mathbb{N}$, $|t - s| < \epsilon/b$ implies that $|f(t, k) - f(s, k)| < \epsilon$.

Thus the family of functions $f(\cdot, k)$ is uniformly bounded and equicontinuous. Arzelá-Ascoli, [7, Theorem XII.6.4], therefore implies that there exists a function $f : [0, a] \to \mathbb{R}^+$ and a subsequence $(f(\cdot, k_j))_{j\in\mathbb{N}}$ which converges to $f$ uniformly on compact subsets of $[0, a]$, and hence on $[0, a]$ (this was the point of assuming $a < \infty$). The upper semicontinuity of the Lorentzian length function $L$ and Theorem 2.12 tell us that

$$L(\gamma) = \lim_{t\to a} L(\gamma|_{[0,t]}) \geq \lim_{t\to a} \limsup_j L(\gamma_{i_{k_j}}|_{[0,t]}) = \lim_{t\to a} \limsup_j f(t, k_j) = \lim_{t\to a} \lim_j f(t, k_j).$$

Since $f(\cdot, k) \to f(\cdot)$ uniformly on $[0, a]$ we can apply the Moore-Osgood theorem, [21, Theorem 7.11] or [10, Theorem VII.2, page 100]. That is, we can interchange the limits and compute that

$$\begin{aligned} L(\gamma) = \lim_{t\to a} L(\gamma|_{[0,t]}) &\geq \lim_{t\to a} \limsup_j L(\gamma_{i_{k_j}}|_{[0,t]}) \\ &= \lim_{t\to a} \lim_j f(t, k_j) \\ &= \lim_j \lim_{t\to a} f(t, k_j) \\ &= \lim_j L(\gamma_{i_{k_j}}) = \lim_k L(\gamma_{i_k}) = \limsup_i L(\gamma_i). \end{aligned} \tag{3}$$

If $L(\gamma|_{[0,t]}) = \lim_k L(\gamma_{i_k}|_{[0,t]})$ then it is clear that the inequality in Equation (3) is an equality. Thus the result holds. □

Lemma 2.14 gives conditions under which the Lorentzian length function is upper semi-continuous on non-compact inextendible causal curves. To use a result like Lemma 2.14, one would typically select a suitable sequence of causal curves and then control the length of the limit curve given by Theorem 2.11.

Theorem 2.11 requires the curves in the sequence to be 1-Lipschitz parametrised with respect to the chosen complete distance on $M$. Hence to achieve the conditions of Lemma 2.14, the chosen distance must be tied to the Lorentzian geometry: the null distance [22] of a suitable time function provides such a distance. That said, the null distance of a suitable time function has no guarantee to be complete.

In the remainder of this paper we prove a generalisation of Theorem 2.11 for distances that are not complete and then show that the null distance associated to various time functions allows us to use Lemma 2.14.

### 2.6 What goes wrong in incomplete spaces

In this section we present two examples that demonstrate how, for a sequence $(\gamma_i)_i$, of causal curves, that uniformly converge to a causal curve $\gamma : \mathbb{R}^+ \to M$ it is possible for

$$L(\gamma) < \limsup_i L(\gamma_i)$$

despite Theorem 2.12 which shows that on compact subsets $K \subset \mathbb{R}^+$ we have

$$L(\gamma|_K) \geq \limsup_i L(\gamma_i|_K).$$

Ultimately the problem stems from needing to choose an auxiliary metric on the manifold to define convergence, as the Lorentzian distance is not suitable for this purpose. Having chosen a distance in order to define uniform convergence, the parametrisation induced by the distance is typically not at all related to the Lorentzian length functional. For example, the usual method is to choose a complete Riemannian metric. In order to ensure an appropriate relation between the distance and the Lorentzian length, we typically lose completeness, and so it is necessary to allow the metric space $(M, d)$ to be incomplete. This causes a secondary problem: what does it mean for curves to converge in an incomplete metric space?

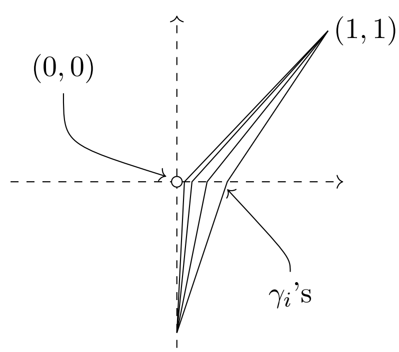


Figure 1: An illustration of Example 2.16. The circle represents the removed origin in $\mathbb{R}^2$. The dashed lines represent the $x$ and $y$ axis. The remaining curves present the images of three of the $\gamma_i$'s. The intersection of these curves with the $x$ axis tends to the removed origin as $i \to \infty$.

Figure 1 illustrates the problems we face when defining what we mean by a limit curve in an incomplete space. The figure illustrates a sequence of past directed causal curves $\gamma_i : [0, 2] \to \mathbb{R}^2 \setminus \{(0, 0)\}$ whose images are two straight lines between the points $(1, 1)$, $(1/i, 0)$ and $(0, 1)$. Whilst it is intuitively clear that the sequence of curves converges pointwise to a "disconnected curve",

$$t \mapsto \begin{cases} (1,1) - t(1,1), & t \in [0,1), \\ (0,0) - (t-1)(0,1), & t \in (1,2], \end{cases}$$

this will not do as a basis for the definition of limit curve.

Our definition of curve requires a connected domain (and so image). As well as satisfying an intuitive notion of curve, this requirement is essential for discussion of causality: *causal curves must be traversable.*

Thus if we take the curves in Figure 1 to start at the top right, the point $(1,1)$, and be past directed, the best we can hope for is that the "top half" of the curves converge to the "top half" of the intuitive limit curve. The second halves of the limit curves will not be said to be converging to a limit curve. They "fall off" the limit. For us to talk about a limit curve requires specification of a point which is the start of the limit curve. In this case the point is $(1,1)$. We present the details in Example 2.16.

**Definition 2.15.** Let $M$ be a manifold, and $d : M \times M \to \mathbb{R}^+$ a metric on $M$. Let $\gamma_i : [0, a_i) \to M$, $a_i \in \mathbb{R}^+ \cup \{\infty\}$, be a sequence of $C^0$ curves in $M$ with $\gamma_i(0) \to x \in M$. We say that $(\gamma_i)$ $d$-converges uniformly on compacta to the continuous curve $\gamma : [0, a) \to M$ if $\gamma(0) = x$ and for all $\delta < a$, there exists $N_\delta \in \mathbb{N}$ so that $i \geq N_\delta$ implies that $a_i > \delta$ and $(\gamma_i|_{[0,\delta]})_{i \geq N_\delta}$ converges uniformly to $\gamma|_{[0,\delta]} : [0, \delta] \to M$ with respect to the distance $d$. ▲

If $\gamma : [0, a) \to M$ is a limit curve then for all $t \in [0, a)$ we have that $\gamma|_{[0,t)} : [0, t) \to M$ is also a limit curve. That is, sub-curves of a limit curve are also limit curves. In this sense limit curves are non-unique. Uniqueness can be achieved by taking the maximal limit curve, whose graph is the union of the graphs of all possible limit curves. In practice, we will work with the maximal curve.

**Example 2.16.** Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$ have the metric $dx^2 - dy^2$, where $x, y$ are the standard coordinates on $M$ induced by inclusion into $\mathbb{R}^2$. For $i \geq 2$, let $\gamma_i : [0, 2] \to M$ be defined by

$$\gamma_i(t) = \begin{Bmatrix} (1,1) - t(1 - 1/i, 1), & t \in [0, 1) \\ (1/i, 0) - (t-1)(1/i, 1), & t \in [1, 2]. \end{Bmatrix}$$

See Figure 1 for a diagram representing $M$ and the $\gamma_i$'s. Each $\gamma_i$ is a $C^0$ piecewise smooth causal curve, and direct calculation shows that the length of $\gamma_i$ defined by the Lorentzian metric is $L(\gamma_i) = \sqrt{1 - 1/i^2} + \sqrt{1 - (1 - 1/i)^2}$. Thus $L(\gamma_i) \to 1$. According to Definition 2.15, the maximal limit curve from the point $(1,1)$ is $\gamma : [0, 1) \to M$ defined by $\gamma(t) = (1,1) - t(1,1)$. Direct calculation gives $L(\gamma) = 0$.

Thus, in particular, $L(\gamma) < \limsup_i L(\gamma_i)$. Since the Lorentzian arc length is invariant to changes in parametrisation no amount of fiddling with parametrisation or the distance used to to define the limit curve will result in $L(\gamma) = \limsup_i L(\gamma_i)$. The situation described in this example can be excluded by assuming that $M$ is globally hyperbolic, since the problems stem from the set $J^+((0,-1)) \cap J^-((1,1))$ not being compact. ▲

**Example 2.17.** Continuing Example 2.16, we show what we need to do to the $\gamma_i$'s to apply Theorem 2.11 and how this relates to the conditions of Lemma 2.14. Let $h = \frac{1}{(x^2+y^2)^2}(dx^2 + dy^2)$. This is a complete Riemannian metric on $M$.

Classical limit curve theorems use a distance induced by a complete Riemannian metric such as $h$, and assume that the curves they apply to are inextendible. Doing so ensures that the inextendible curves being considered will have domains $\mathbb{R}^+$ when re-parametrised to be 1-Lipschitz with respect to $d(\cdot, \cdot; h)$. To compare with classical limit theorem statements, we now construct inextendible extensions of the $\gamma_i$. We construct the extensions by adding a null curve, showing that the problem with the limits of Lorentzian lengths does not arise from the extension process.

Let $\lambda : [2, \infty) \to M$ be given by $\lambda(t) = (0, -1) - (t-2)(1, 1)$. We now extend each $\gamma_i$ to be a past directed, inextendible curve by appending $\lambda$. That is, our extension is the curve given by

$$t \mapsto \begin{cases} \gamma_i(t), & t \in [0, 2), \\ \lambda(t), & t \in [2, \infty). \end{cases}$$

We shall denoted these extensions by $\gamma_i : \mathbb{R}^+ \to M$. The limit curve $\gamma$ described in Example 2.16 is already inextendible.

We now compute the $h$ arc-length parametrisations of our inextendible $\gamma_i$'s and $\gamma$. Let $s : [0, 1) \to \mathbb{R}^+$ be defined by

$$s(t) = L(\gamma|_{[0,t]}; h) = \int_0^t \frac{1}{\sqrt{2}(1-\tau)^2} \mathrm{d}\,\tau = \frac{1}{\sqrt{2}} \left( \frac{t}{1-t} \right).$$

Note that $s(t) \to \infty$ as $t \to 1$. We can define $t : \mathbb{R}^+ \to [0, 1)$ by, $s(t(u)) = u$. That is, $t(s) = \frac{\sqrt{2}s}{1+\sqrt{2}s}$ so that $\gamma \circ t : \mathbb{R}^+ \to M$ is an $h$ arc-length parametrised curve.

The curves $\gamma_i : \mathbb{R}^+ \to M$ are inextendible and therefore $L(\gamma_i; h) = \infty$. Thus, for each $i \in \mathbb{N}$ define a function $s_i : \mathbb{R}^+ \to \mathbb{R}^+$, by $s_i(t) = L(\gamma_i|_{[0,t)}; h)$. That is,

$$s_i(t) = \int_0^t \frac{\sqrt{\left(1 - \frac{1}{i}\right)^2 + 1}}{\left(1 - \tau\left(1 - \frac{1}{i}\right)\right)^2 + (1-\tau)^2} \mathrm{d}\,\tau,$$

for $t \in [0, 1)$, and

$$s_i(t) = \lim_{t \to 1} s_i(t) + \int_1^t \frac{\sqrt{\left(\frac{1}{i}\right)^2 + 1}}{\left(\frac{2-\tau}{i}\right)^2 + (\tau - 1)^2} \mathrm{d}\,\tau$$

for $t \in [1, 2)$ and

$$s_i(t) = \lim_{t \to 2} s_i(t) + \int_2^t \frac{\sqrt{2}}{\left(\tau^2\right)^2 + (\tau - 2)^2} \mathrm{d}\,\tau$$

for $t \in [2, \infty)$. We can now define $t_i : \mathbb{R}^+ \to \mathbb{R}^+$ by $s_i(t_i(u)) = u$.

All curves $\gamma \circ t$ and $\gamma_i \circ t_i$ now have the same domain $\mathbb{R}^+$ and are such that for all $u_1, u_2 \in \mathbb{R}^+$ $d(\gamma \circ t(u_1), \gamma \circ t(u_2); h) \leq |u_1 - u_2|$ and $d(\gamma_i \circ t_i(u_1), \gamma_i \circ t_i(u_2); h) \leq |u_1 - u_2|$.

Since

$$L(\gamma \circ t) = 0 < 1 = \limsup_i L(\gamma_i \circ t_i)$$

we know that Lemma 2.14 does not hold for the sequence $(\gamma_i \circ t_i)$ and the limit curve $\gamma \circ t$. We now investigate the conditions of Lemma 2.14 to determine which fails. Since all curves, $\gamma$ and $\gamma_i$, have domain $\mathbb{R}^+$ and Lemma 2.14 requires our curves to have a domain with compact closure in $\mathbb{R}^+$ we shall first need to re-parametrise. For each $i \in \mathbb{N}$ we consider $\gamma_i \circ t_i \circ \tan : [0, \pi/2) \to M$ and $\gamma \circ t \circ \tan : [0, \pi/2) \to M$. We immediately see that the sequence $(\gamma_i \circ t_i \circ \tan)_i$ converges uniformly to $\gamma \circ t \circ \tan$. It is also clear that $L(\gamma_i \circ t_i \circ \tan)$ is finite for all $i \in \mathbb{N}$ and the sequence $(L(\gamma_i \circ t_i \circ \tan))_i$ is bounded above.

The sequence $(s_i(1))_i$ is increasing with limit $\infty$. In particular, $(\arctan \circ s_i(1))_i$ is increasing with limit $\pi/2$. This implies that

$$1 = L(\gamma_i \circ t_i \circ \tan|_{[\arctan(s_i(1)),\pi/2)}) \leq b\left(\frac{\pi}{2} - \arctan(s_i(1))\right).$$

While the left hand side is constant the right hand side tends to 0 as $i \to \infty$. This violates the second listed assumption of Lemma 2.14. ▲

## 3 The limit curve theorem for incomplete metric spaces

In this section we generalise Theorem 2.11 to incomplete metric spaces. We begin by considering the limit curve theorem in a compact subset of $M$. First we prove a lemma that will help us re-parametrise our curves.

**Lemma 3.1.** *For each $i \in \mathbb{N}$ let $a_i \in \mathbb{R}^+ \cup \{\infty\}$ and define*

$$Y_i = \begin{cases} [0, a_i], & a_i < \infty, \\ [0, a_i), & a_i = \infty. \end{cases}$$

*Let $a = \limsup_i a_i$ and define*

$$X = \begin{cases} [0, a], & a < \infty, \\ [0, a), & a = \infty. \end{cases}$$

*For each $i \in \mathbb{N}$ define $f_i : X \to Y_i$ a bijective, continuous, increasing function by*

$$f_i(x) = \begin{cases} x, & a_i = a = \infty, \\ \dfrac{2a_i}{\pi} \arctan\left(\dfrac{\pi}{2a_i} x\right), & a = \infty,\ a_i < \infty, \\ \dfrac{a_i}{a} x, & a_i,\ a < \infty. \end{cases}$$

*For all $\epsilon > 0$ there exists a subsequence $(a_{i_k})_k$ of $(a_i)_i$ such that for all $k \in \mathbb{N}$, $a_{i_k} \leq a + \epsilon$ and so $\sup\{f'_{i_k}(t) : t \in X\} \leq 1 + \epsilon$.*

*Proof.* Let $\epsilon > 0$. Since $a = \limsup_i a_i$ there exists a subsequence $(a_{i_k})_k$ so that $a_{i_k} \to a$. Consequently, there exists $N$ big enough so that the subsequence $(a_{i_{k+N}})_{k+N}$ is such that for all $k \in \mathbb{N}$, $a_{i_{k+N}} \leq a + \epsilon$.

Passing to such a subsequence if necessary, for sufficiently large $i$ we have

$$0 < f'_i(x) = \begin{cases} 1, & a_i = a = \infty, \\ \dfrac{1}{1 + \left(\frac{\pi}{2a_i} x\right)^2}, & a = \infty,\ a_i < \infty, \\ \dfrac{a_i}{a}, & a_i,\ a < \infty. \end{cases} \leq \begin{cases} 1, & a_i = a = \infty, \\ 1, & a = \infty,\ a_i < \infty, \\ 1 + \epsilon, & a_i,\ a < \infty. \end{cases}$$

Thus the result holds. □

The following lemma is a “local” version of Theorem 2.11. In Theorem 3.3, below, we show how to use Proposition 3.2 to prove a global version of the limit curve theorem for incomplete distances. In this local version, we manage incompleteness by working in a compact set.

**Proposition 3.2.** *Let $M$ be a manifold, let $d : M \times M \to \mathbb{R}$ be a distance on $M$ compatible with the manifold topology, and let $B$ be a compact subset of $M$. Let $\gamma_i : Y_i = [0, a_i) \to M$, $a_i \in \mathbb{R}^+ \cup \{\infty\}$, be a sequence of $C^0$ curves in $M$ so that for some $N \in \mathbb{N}$, $n > N$ implies that $\gamma_n \subset B$. Let the functions $f_i : X \to Y_i$ be constructed as in Lemma 3.1. If for all $i \in \mathbb{N}$ and for all $t_1, t_2 \in I_i$,*

$$d(\gamma_i(t_1), \gamma_i(t_2)) \leq |t_1 - t_2|.$$

*then there is a subsequence $(\gamma_{i_k} \circ f_{i_k})_k$ of $(\gamma_i \circ f_i)_i$ which*

1. *converges uniformly on compact subsets of $X$ to a $C^0$ curve $\gamma : X \to B$ with respect to $d$, and*
2. *is such that $\lim_k a_{i_k} = \limsup_i a_i$.*

*Proof.* Let $\epsilon > 0$. Since $a = \limsup\{a_i : i \in \mathbb{N}\}$, for all sufficiently large $i \in \mathbb{N}$ we have $a_i < a + \epsilon$. Thus there exists a subsequence $(a_{i_k})_k$ so that $\lim_k a_{i_k} = a$ and for all $k \in \mathbb{N}$, $a_{i_k} < a + \epsilon$. Without loss of generality we assume that $i_k = k$. That is, we assume that, $\lim_i a_i = a$ and for all $i \in \mathbb{N}$ $a_i < a + \epsilon$. Lemma 3.1 shows that $\sup\{f_i'(t) : t \in X\} \leq 1 + \epsilon$.

By assumption for all $i \in \mathbb{N}$ and all $t_1, t_2 \in I_i$ we know that $d(\gamma_i(t_1), \gamma_i(t_2)) \leq |t_1 - t_2|$. Thus for all $t_1, t_2 \in X$ we know that

$$d(\gamma_i \circ f_i(t_1), \gamma_i \circ f_i(t_2)) \leq |f_i(t_1) - f_i(t_2)| \leq (1 + \epsilon) |t_1 - t_2| .$$

Thus $\{\gamma_i \circ f_i : i \in \mathbb{N}\}$ is uniformly equicontinuous. By assumption, for all sufficiently large $i \in \mathbb{N}$ we have $\gamma_i \subset B$. Thus $\{\gamma_i \circ f_i(t) : i \in \mathbb{N}\}$ has compact closure for each $t \in X$.

Arzelà and Ascoli’s theorem [7, Theorem XII.7.6.4 and Theorem XII.7.7.2], implies that there exists some $C^0$ curve $\gamma : X \to M$ such that there is a subsequence of $(\gamma_i \circ f_i)_{i\in\mathbb{N}}$ which converges uniformly to $\gamma$ on compact subsets of $X$ with respect to $d$. Since, for $i \in \mathbb{N}$ large enough we know that $\gamma_i \subset B$, and as $\gamma(t)$ is the pointwise limit of some subset of $(\gamma_i \circ f_i(t))_i$, we see that $\gamma \subset B$. Thus the result holds. ☐

The first part of Proposition 3.2 is not a new result: similar ideas are used in [11, Theorem 3.7]. For Lorentzian length control we need to understand the relationship between $\limsup_i a_i$, $\lim_k a_{i_k}$ and $a$. This part of Proposition 3.2 is new.

We can patch the limit curves on intersecting compact subsets together. When we do so, the “second half” of the limit curve may “fall off” as in Example 2.16. This occurs as the patching mechanism restricts us to working on the traversable portion of the limit curve from its assumed starting point.

**Theorem 3.3** (The limit curve theorem)**.** *Let $M$ be a manifold and let $d : M \times M \to \mathbb{R}$ be a distance on $M$ whose topology agrees with that of $M$. Let $(\gamma_i : [0, a_i) \to M)_{i\in\mathbb{N}}$, $a_i \in \mathbb{R}^+ \cup \{\infty\}$, be a sequence of $C^0$ curves in $M$. If there exists $x \in M$ so that*

$$\gamma_i(0) \to x \in M$$

*and if, for all $i \in \mathbb{N}$ and for all $t_1, t_2 \in [0, a_i)$,*

$$d(\gamma_i(t_1), \gamma_i(t_2)) \leq |t_1 - t_2| ,$$

*then there exists a $C^0$ curve $\gamma : [0, a) \to M$, $a \in (0, \limsup_i a_i]$, with $\gamma(0) = x$, and a subsequence $(\gamma_{i_k})_k$ of $(\gamma_i)_i$ so that*

1. *$a \leq \limsup_k a_{i_k}$, and*
2. *for all compact $C \subset [0, a)$ there exists $J \in \mathbb{N}$ so that $k \geq J$ implies that $C \subset [0, a_{i_k})$ and $(\gamma_{i_k}|_C)_{k \geq J}$ converges uniformly to $\gamma|_C$ with respect to $d$. That is, $(\gamma_{i_k})_k$ $d$-converges uniformly to $\gamma$ on compacta, as in Definition 2.15.*

*Proof.* Let $\{K_i : i \in \mathbb{N}\}$ be a compact exhaustion of $M$ with $x \in K_1$. If there exists $j \in \mathbb{N}$ and $N \in \mathbb{N}$ so that $i \geq N$ implies that $\gamma_i \subset K_j$ then the result follows from Proposition 3.2.

Otherwise for all $j \in \mathbb{N}$ and $N \in \mathbb{N}$ there exists $i \geq N$ so that $\gamma_i \not\subset K_j$. We can, therefore, by taking a subsequence, assume that for all $i, j \in \mathbb{N}$, $\gamma_i \cap K_j \neq \varnothing$ and that $d(K_j, M \setminus K_{j+1}) > 0$ for each $j$. Thus, if $a_i^j \in [0, a_i)$ is such that $\gamma_i|_{[0,a_i^j]}$ is the connected component of $\gamma_i \cap K_j$ containing $\gamma_i(0)$, then we know that each $a_i^j$ is well-defined and for all $j \in \mathbb{N}$,

$$\limsup_i a_i^j < \limsup_i a_i^j + d(K_j, M \setminus K_{j+1}) \leq \limsup_i a_i^{j+1} \leq \limsup_i a_i.$$

In particular, as $K_j$ is compact, $\limsup_i a_i^j$ is finite, though $\limsup_i a_i$ may be infinite.

We shall construct $\gamma$ inductively over the compact exhaustion using Proposition 3.2. Without loss of generality we assume that for all $i \in \mathbb{N}$, $\gamma_i(0) \in K_1$.

Let $a^1 = \limsup\{a_i^1 : i \in \mathbb{N}\}$. By compactness of $K_1$, $a^1 < \infty$. Apply Proposition 3.2 with $\epsilon_1 > 0$ to construct $\gamma^1 : [0, a^1] \to K_1$ a limit curve of a subsequence of $(\gamma_i \circ f_i^1)_i$ which converges uniformly to $\gamma^1$, where, for each $i \in \mathbb{N}$, $f_i^1 : [0, a^1] \to [0, a_i^1]$ is defined by $f_i^1(t) = a_i^1 t / a^1$. Let $h_1 : \mathbb{N} \to \mathbb{N}$ be the function that selects the uniformly convergent subsequence. That is, the sequence $(\gamma_{h_1(i)} \circ f_{h_1(i)}^1)_i$ converges uniformly to $\gamma^1$. Note that Proposition 3.2 implies that $\lim_i a_{h_1(i)}^1 = a^1$.

We can repeat this construction, starting with the sequence of curves $(\gamma_{h_1(i)})_i$ and the compact set $K_2$. Let $a^2 = \limsup\{a_{h_1(i)}^2 : i \in \mathbb{N}\}$. Apply Proposition 3.2 with $\epsilon_2 > 0$ to construct $\gamma^2 : [0, a^2] \to K_2$ a limit curve of a subsequence of $(\gamma_{h_1(i)} \circ f_{h_1(i)}^2)_i$ which converges uniformly to $\gamma^2$, where, for each $i \in \mathbb{N}$, $f_i^2 : [0, a^2] \to [0, a_i^2]$ is defined by $f_i^2(t) = a_i^2 t / a^2$. Let $h_2 : \mathbb{N} \to \mathbb{N}$ be the function that selects the uniformly convergent subsequence from $(\gamma_{h_1(i)})_i$. That is, the sequence $(\gamma_{h_2(i)} \circ f_{h_2(i)}^2)_i$ converges uniformly to $\gamma^2$ and is a subsequence of $(\gamma_{h_1(i)} \circ f_{h_1(i)}^1)_i$. Note again that Proposition 3.2 implies that $\lim_i a_{h_2(i)}^2 = a^2$. We can repeat the construction for all $j \in \mathbb{N}$.

With this construction completed we now give the definition of the limit curve $\gamma$ and the required subsequence. We show that for all $j \in \mathbb{N}$, the curve $\gamma^j$ is a subcurve of $\gamma^{j+1}$. By construction $(\gamma_{h_j(i)} \circ f_{h_j(i)}^j)_i$ converges uniformly to $\gamma^j$ and $(\gamma_{h_{j+1}(i)} \circ f_{h_{j+1}(i)}^{j+1})_i$ converges uniformly to $\gamma^{j+1}$. By construction $(\gamma_{h_{j+1}(i)})_i$ is a subsequence of $(\gamma_{h_j(i)})_i$. Therefore $(\gamma_{h_{j+1}(i)} \circ f_{h_{j+1}(i)}^j)_i$ converges to $\gamma^j$ uniformly while $(\gamma_{h_{j+1}(i)} \circ f_{h_{j+1}(i)}^{j+1}(i))_i$ converges to $\gamma^{j+1}$ uniformly. By construction

$$\lim_{i \to \infty} \frac{a_{h_j(i)}^j}{a^j} t = t = \lim_{i \to \infty} \frac{a_{h_{j+1}(i)}^{j+1}}{a^{j+1}} t.$$

Thus

$$\lim_{i \to \infty} \left( f_{h_{j+1}(i)}^{j+1} \right)^{-1} \circ f_{h_{j+1}(i)}^j(t) = t$$

and hence

$$\left(\gamma_{h_{j+1}(i)} \circ f^{j+1}_{h_{j+1}(i)} \circ \left(f^{j+1}_{h_{j+1}(i)}\right)^{-1} \circ f^{j}_{h_{j+1}(i)}(t)\right)_i$$

converges to both $\gamma^j(t)$ and $\gamma^{j+1}(t)$. Uniform convergence and the Hausdorffness of $M$ imply that $\gamma^j(t) = \gamma^{j+1}(t)$. Since this is true for all $t \in [0, a^j]$ we see that $\gamma^j$ is a subcurve of $\gamma^{j+1}$.

We choose the subsequence which has the given uniform convergence property. Let $k \in \mathbb{N}$ and let $i_k = h_k(k)$. Let $j \in \mathbb{N}$ then $(\gamma_{i_k} \circ f^j_{i_k})_{k\geq j}$ is a subsequence of $(\gamma_{h_j(i)} \circ f^j_{h_j(i)})_i$ and therefore converges uniformly to $\gamma^j$.

We define the curve $\gamma$. As $K_i$ is a compact exhaustion we know that for all $j \in \mathbb{N}$, $a^j < a^{j+1} < \limsup_i a_i$. Thus the sequence $(a^j)$ is increasing and $\lim_j a^j$ exists (possibly infinite). We let $a = \lim_j a^j$. Thus, we can define a curve $\gamma : [0, a) \to M$ by $\gamma(t) = \gamma^j(t)$ for any $j \in \mathbb{N}$ with $t < a^j$. Since $a^j = \lim_k a^j_{i_k}$ we know that $a^j < \limsup_k a_{i_k}$ and therefore that $a \leq \limsup_k a_{i_k}$ (again, possibly infinite).

We now show that if $C \subset [0, a)$ is compact then there exists $J \in \mathbb{N}$ so that $k \geq J$ implies that $\sup\{c : c \in C\} \leq a_{i_k}$ and $(\gamma_{i_k}|_C)_{k\geq J}$ converges uniformly to $\gamma|_C$.

Let $t = \sup\{c : c \in C\}$. Then the image $\gamma([0, t])$ is compact and hence lies in some $K_j$, $j \in \mathbb{N}$. This implies that $\gamma([0, t]) \subset \gamma^j$ and therefore that $t < a^j$. Since $\lim_k a^j_{i_k} = a^j$ we know that there exists the required $J$ and that the claimed uniform convergence occurs. □

Unlike similar results, Theorem 3.3 does not require the distance to be complete, e.g. [2, Theorem 14.2, see the last sentence on page 510], [12, Theorem 3.1] [11, Theorem 3.14].

## 4 Length control over limit curves via null distances

We left open the choice of distance in Theorem 3.3. In this section we show that the null distance induced by a suitable time function is sufficient to show that Lemma 2.14 holds for the limit curve constructed in Theorem 3.3.

By a "suitable time function" we mean any function whose null distance is a metric compatible with the manifold topology as in Proposition 2.6 and which satisfies a further condition given in Proposition 4.3. To prepare, the next two results make weaker assumptions, and so the null distances my not satisfy Proposition 2.6.

**Proposition 4.1.** *Let $(M, g)$ be a Lorentzian manifold. Let $f : M \to \mathbb{R}$ be a locally Lipschitz function which is increasing on all timelike curves, and let $d(\cdot, \cdot; f)$ be the associated null distance. Let $\gamma : [0, a) \to M$, $a \in \mathbb{R}^+ \cup \{\infty\}$ be a past directed causal curve. If there exists a constant $b > 0$ such that for all points where $\nabla f$ exists we have $g(\nabla f, \nabla f) \leq -b^2$, then for all $t_1, t_2 \in [0, a)$*

$$d(\gamma(t_1), \gamma(t_2); f) = |f(\gamma(t_2)) - f(\gamma(t_1))| \geq bL(\gamma|_{[t_1,t_2]}),$$

*where $d(\cdot, \cdot; f)$ is the null distance defined by $f$ and $L$ is the Lorentzian length.*

*Proof.* The equality

$$d(\gamma(t_1), \gamma(t_2); f) = |f(\gamma(t_2)) - f(\gamma(t_1))|$$

is proven in [22, Lemma 3.11]. We will show that

$$|f(\gamma(t_2)) - f(\gamma(t_1))| \geq bL(\gamma|_{[t_1,t_2]}).$$

We may assume that $\gamma([t_1, t_2]) \subset U$ where $\phi : U \to \mathbb{R}^n$ is a chart. Let $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ be the Euclidean norm on $\mathbb{R}^n$. Throughout the proof below we freely identify differential forms in $T^*\mathbb{R}^n$ with tangent vectors $T\mathbb{R}^n$ with points in $\mathbb{R}^n$ using the isomorphisms induced by the standard frame and metric on $\mathbb{R}^n$. The open Euclidean unit ball in $\mathbb{R}^n$ will be denoted $B$. In an abuse of notation, let $g$ and $g^{-1}$ denote the push forward of $g$ and $g^{-1}$ by $\phi$ onto $\phi(U)$. Since $g, g^{-1}$ are coordinate dependent metrics we must take some care with them when working in $\mathbb{R}^n$ with the identifications above. Note, in particular, that by taking the points of evaluation, e.g. $x, y \in \phi(U)$, arbitrarily close we can ensure that $g^{-1}|_x$ is arbitrarily close to $g^{-1}|_y$. We have $f \circ \gamma = f \circ \phi^{-1} \circ \phi \circ \gamma$, so we can consider the function $f \circ \phi^{-1}$ on $\phi(U)$, which contains the curve $\phi \circ \gamma$.

Since $\gamma([t_1, t_2])$ is compact and $U$ is open there exists $\epsilon^* > 0$ so that the compact set

$$V := \{x \in \mathbb{R}^n : \exists t \in [t_1, t_2],\ \|x - \gamma(t)\| \leq \epsilon^*\} \subset \phi(U).$$

The set-valued Clarke generalised gradient of $f \circ \phi^{-1}$ at $y \in \phi(U)$ is denoted $\partial^\circ(f \circ \phi^{-1})(y)$, see Definition 2.8. Minguzzi [14, Theorem 1.19] shows that $f$ has a gradient defined almost everywhere on the manifold. Theorem 2.9 implies that, for each $y \in \phi(V)$, vectors $v \in \partial^\circ(f \circ \phi^{-1})(y)$ satisfy $g^{-1}(v, v) \leq -b^2$, because the past-pointing vectors of Lorentzian length $\leq -b^2$ form a convex set.

The function $f$ is locally Lipschitz and so there is a constant $D > 0$ so that at any $y \in \phi(V)$ where $df$ exists we have $\|df\|_2 \leq D$. Hence, again by Theorem 2.9, any element $v \in \partial^\circ(f \circ \phi^{-1})(y)$ also satisfies $\|v\|_2 \leq D$. Thus we know that for each $y \in \phi(U)$ the set $K_y = \overline{\partial^\circ(f \circ \phi^{-1})(y + \epsilon^* B) + \epsilon^* B}$ is compact. The set $K = \bigcup_{y \in V} K_y$ is a compact subset of $T\mathbb{R}^n$. Hence as $g$ is continuous there exists $C > 0$ so that for all $y \in V$ and all $u, v \in K_y$, $g(u, v) \leq C$.

Fix $\epsilon$ so that $0 < \epsilon < \epsilon^*$ and so that $-b^2 + (\epsilon^2 + 2\epsilon)C + \epsilon < 0$.

Since $\gamma$ is a continuous past directed causal curve we can, by [19, Definition 7.4], approximate $\phi \circ \gamma|_{[t_1,t_2]}$ by a past directed causal curve which is a piecewise smooth $g$-geodesic $\tilde{\gamma} : [t_1, t_2] \to \phi(V)$ so that $\left|L(\phi \circ \gamma|_{[t_1,t_2]}) - L(\tilde{\gamma})\right| < \epsilon$, $\tilde{\gamma}(t_1) = \phi \circ \gamma(t_1)$, and $\tilde{\gamma}(t_2) = \phi \circ \gamma(t_2)$. Note that by construction, wherever $\tilde{\gamma}'$ exists it is a past directed $g$-causal vector.

Since $g$ is smooth on $\phi(U)$, there exists a function $\delta : \phi(U) \to \mathbb{R}^+$ so that for all $x \in \phi(U)$,

1. $\delta(x) < \epsilon$,
2. $\{y \in \mathbb{R}^n : \|x - y\|_2 \leq \delta(x)\} \subset \phi(U)$, and
3. for all $x, y \in \phi(U)$, if $\|x - y\|_2 < \delta(x)$ then $\left|g^{-1}|_x(d\tilde{f}(x), d\tilde{f}(x)) - g^{-1}|_y(d\tilde{f}(x), d\tilde{f}(x))\right| < \delta(x)$.

Since $f \circ \phi^{-1}$ is Lipschitz, Theorem 2.10 implies that there exists a smooth function $\tilde{f} : \phi(U) \to \mathbb{R}$ such that for all $y \in \phi(U)$ we have $\left|f \circ \phi^{-1}(y) - \tilde{f}(y)\right| \leq \delta(y) < \epsilon$, and

$$d\tilde{f}(y) \in \partial^\circ(f \circ \phi^{-1})((y + \delta(y)B) \cap \phi(U)) + \delta(y)B.$$

Note that while $d\tilde{f}$ is evaluated at $y$, the membership relation is for the union of Clarke's generalised gradients of $f \circ \phi^{-1}$ over the set of points $(y + \delta(y)B) \cap \phi(U)$. We want to use knowledge of

$\partial^\circ(f\circ\phi^{-1})(y)$ to construct a bound on $g(\nabla\tilde f,\nabla\tilde f)$. In doing so we will need to be careful about where the generalised gradient and the differential $d\tilde f$ are evaluated.

With these approximations we have

$$\begin{aligned}\left|\tilde f(\tilde\gamma(t_2))-\tilde f(\tilde\gamma(t_1))\right| &= \left|\tilde f(\phi\circ\gamma(t_2))-\tilde f(\phi\circ\gamma(t_1))\right|\\ &= \left|f(\gamma(t_2))+\tilde f(\phi\circ\gamma(t_2))-f(\gamma(t_2))-f(\gamma(t_1))-(\tilde f(\phi\circ\gamma(t_1))-f(\gamma(t_1)))\right|\\ &< |f(\gamma(t_2))-f(\gamma(t_1))|+2\epsilon.\end{aligned}$$

We now show that $x\in\phi(U)$ implies that $g^{-1}|_x(d\tilde f(x),d\tilde f(x))<0$, thus showing that $\tilde f$ has $g$-causal gradient. By construction of $\tilde f$ there exists $y\in(x+\delta(x)B)\cap\phi(U)$ so that

$$d\tilde f(x)\in\partial^\circ(f\circ\phi^{-1})(y)+\delta(x)B.$$

By definition there exists $w\in\partial^\circ f\circ\phi^{-1}(y)$ and $u\in B$ so that $d\tilde f(x)=w+\delta(x)u$. We can compute that

$$g|_y(d\tilde f(x),d\tilde f(x)=g|_y(w,w)+\delta(x)^2g|_y(u,u)+2\delta(\gamma(t))g|_y(w,u)\le -b^2+(\epsilon^2+2\epsilon)C.$$

Since $y\in(x+\delta(x)B)\cap\phi(U)$, by definition of $B$, we see that $\|x-y\|_2<\delta(x)$ and therefore know that $\left|g^{-1}|_x(d\tilde f(x),d\tilde f(x))-g^{-1}|_y(d\tilde f(x),d\tilde f(x))\right|<\delta(x)$. Thus we have that

$$g^{-1}|_x(d\tilde f(x),d\tilde f(x))\le g^{-1}|_y(d\tilde f(x),d\tilde f(x))+\delta(x)<-b^2+(\epsilon^2+2\epsilon)C+\epsilon<0,$$

by construction of $\epsilon$. Hence the $g$-gradient of $\tilde f$ is $g$-causal.

By [18, Prop 5.30], if $\tilde\gamma'$ and $\nabla\tilde f$ are both time-like the reverse Cauchy inequality holds,

$$\left|g(\nabla\tilde f,\tilde\gamma')\right|\ge\sqrt{-g(\nabla\tilde f,\nabla\tilde f)}\sqrt{-g(\tilde\gamma',\tilde\gamma')},$$

where $\nabla\tilde f$ is the gradient of $\tilde f$ with respect to the push forward of $g$, the $g$-gradient. If $\nabla\tilde f$ or $\tilde\gamma'$ is null then it is clear that this inequality continues to hold. Using this inequality we have

$$\begin{aligned}\left|\tilde f(\tilde\gamma(t_2))-\tilde f(\tilde\gamma(t_1))\right| &= -\int_{t_1}^{t_2}\frac{d}{dt}\tilde f(\tilde\gamma(t))\,\mathrm{d}t=-\int_{t_1}^{t_2}d\tilde f(\tilde\gamma')\,\mathrm{d}t=\int_{t_1}^{t_2}\left|g(\nabla\tilde f,\tilde\gamma')\right|\,\mathrm{d}t\\ &\ge\int_{t_1}^{t_2}\sqrt{-g(\nabla\tilde f,\nabla\tilde f)}\,\sqrt{-g(\tilde\gamma',\tilde\gamma')}\,\mathrm{d}t\\ &>\int_{t_1}^{t_2}\sqrt{b^2-(\epsilon^2+2\epsilon)C-\epsilon}\,\sqrt{-g(\tilde\gamma',\tilde\gamma')}\,\mathrm{d}t\\ &=\sqrt{b^2-(\epsilon^2+2\epsilon)C-\epsilon}\;L(\tilde\gamma|_{[t_1,t_2]}).\end{aligned}$$

The result now follows by letting $\epsilon\to 0$. □

**Lemma 4.2.** *Let $(M,g)$ be a Lorentzian manifold and let $f:M\to\mathbb{R}$ be a time function and $d(\cdot,\cdot;f)$ the associated null distance. If $\gamma:[0,b)\to M$ is a past directed causal curve then there exists a change of parameter $s:[0,a)\to[0,b)$, where $a=\lim_{t\to b}(f\circ\gamma(0)-f\circ\gamma(t))$, so that $f\circ\gamma\circ s(t)=f\circ\gamma(0)-t$ and, for all $t_1,t_2\in[0,a)$,*

$$d(\gamma\circ s(t_1),\gamma\circ s(t_2);f)=|t_1-t_2|\,.$$

*Proof.* Let $a = \lim_{t\to b} f(\gamma(0)) - f(\gamma(t))$. Since $\gamma_i$ is past directed and causal $f\circ\gamma_i$ is a decreasing, continuous, and bijective function. It therefore has an inverse. Thus we can define $s : [0, a) \to [0, b)$ by $s(u) = (f\circ\gamma)^{-1}(f(\gamma(0)) - u)$. Note that $u = f(\gamma(0)) - f(\gamma\circ s(u))$. By construction $s$ is continuous, bijective and increasing.

We can compute, by [22, Lemma 3.11], that for all $u_1, u_2 \in [0, a]$,

$$d(\gamma\circ s(u_1), \gamma\circ s(u_2); f) = |f\circ\gamma\circ s(u_1) - f\circ\gamma\circ s(u_2)| = |u_1 - u_2|\,.$$

Thus the result holds. □

The next result says that if the range of $f$ and the parametrisation of the curves in a sequence by Lemma 4.2 are compatible then the domains of the curves converge.

**Proposition 4.3.** *Let $(M, g)$ be a Lorentzian manifold, $f : M \to \mathbb{R}$ be a locally Lipschitz, locally anti-Lipschitz time function and let $d(\cdot,\cdot; f)$ be the null distance associated to $f$, Definition 2.4. Let $(\gamma_i : [0, b_i) \to M)_{i\in\mathbb{N}}$ be a sequence of past directed inextendible continuous causal curves in $M$, with $b_i \in \mathbb{R}^+\cup\{\infty\}$, such that there exists $x \in M$ with $\gamma_i(0) \to x$. For each $i \in \mathbb{N}$ let $s_i : [0, a_i) \to [0, b_i)$ be the change in parameter constructed in Lemma 4.2. Let $\gamma : [0, a) \to M$ be the limit curve constructed in Theorem 3.3 from the sequence $(\gamma_i\circ s_i)_i$, and let $(\gamma_{i_k}\circ s_{i_k})_k$ be the subsequence of $(\gamma_i)$ that $d(\cdot,\cdot; f)$-converges uniformly on compacta to $\gamma$,*

*If $f$ is such that for all $r \in (-\infty, f(\gamma(0))) \cap \operatorname{ran}(f)$ we have $\gamma\cap f^{-1}(r) \neq \varnothing$ and there exists $N \in \mathbb{N}$ so that for all $k \geq N$ we have $\gamma_{i_k}\cap f^{-1}(r) \neq \varnothing$ then there exists a subsequence $(\gamma_{i_{k_j}})_j$ of $(\gamma_{i_k})$ so that*

$$a = \limsup_j a_{i_{k_j}}\,.$$

*Proof.* By Theorem 3.3 we know that $a \leq \limsup_k a_{i_k}$.

We first show that $f\circ\gamma(u) = f\circ\gamma(0) - u$. Let $u \in [0, a)$. Since $\gamma$ is past directed $r = f\circ\gamma(u) \in (-\infty, f(\gamma(0)) \cap \operatorname{ran}(f)$. Thus by assumption there exists $N \in \mathbb{N}$ so that $k \geq N$ implies that there exists $u_{i_k} \in [0, a_{i_k})$ so that $f\circ\gamma_{i_j}\circ s_{i_j}(u_{i_j}) = f\circ\gamma(u)$. The set $f^{-1}(r)$ is achronal and as each $\gamma_{i_k}$ and $\gamma$ are causal we know that the sets $\gamma_{i_k}\cap f^{-1}(r)$ and $\gamma\cap f^{-1}(r)$ are singletons. Since $f$ is continuous $f^{-1}(r)$ is closed and therefore, by the uniform convergence of $(\gamma_{i_k}\circ s_{i_k})$ to $\gamma$, we know that $u_{i_k} \to u$. Thus

$$f\circ\gamma(u) = f\circ\gamma_{i_k}\circ s_{i_k}(u_{i_k}) = \lim_{k\to\infty} f\circ\gamma_{i_k}\circ s_{i_k}(u_{i_k}) = \lim_{k\to\infty} f\circ\gamma_{i_k}(0) - u_{i_k} = f\circ\gamma(0) - u,$$

as claimed.

We now prove that $a = \limsup_j a_{i_j}$. Let $r \in (-\infty, f(x))\cap\operatorname{ran}(f)$ by assumption there exists $u \in [0, a)$ so that $f\circ\gamma(u) = r$. As $\gamma$ is past directed we now know that

$$\inf\operatorname{ran}(f) = \lim_{u\to a} f\circ\gamma(u) = f\circ\gamma(0) - \lim_{u\to a} u = f\circ\gamma(0) - a.$$

By construction of each $s_i$, Lemma 4.2, we know that $f\circ\gamma_{i_k}\circ s_{i_k}(t) = f\circ\gamma_{i_k}(0) - t$. By assumption there exists $N \in \mathbb{N}$ so that $k \geq K$ implies that there exists $u_{i_k} \in [0, a_{i_k})$ so that $r = f\circ\gamma_{i_k}\circ s_{i_k}(u_{i_k})$. Thus as for $\gamma$,

$$\inf\operatorname{ran}(f) = f\circ\gamma_{i_j}(0) - a_{i_j}\,.$$

By taking a subsequence, if necessary, we can assume that $\lim_k a_{i_k} = \limsup_k a_{i_k}$. Since $a \leq \limsup_k a_{i_k}$,

$$\inf \operatorname{ran}(f) = f \circ \gamma(0) - a \geq f \circ \gamma(0) - \limsup_k a_{i_k} = \lim_k f \circ \gamma_{i_k}(0) - a_{i_k} = \inf \operatorname{ran}(f).$$

Thus $a = \limsup_k a_{i_k}$ as required. □

We now give the "null distance limit curve theorem" which demonstrates that for the null distance induced by a locally Lipschitz, locally anti-Lipschitz time function it is possible to have global length control over the limit curve.

**Theorem 4.4** (Null distance limit curve theorem)**.** *Let $(M, g)$ be a Lorentzian manifold, $f : M \to \mathbb{R}$ be a locally Lipschitz, locally anti-Lipschitz time function and let $d(\cdot, \cdot; f)$ be the null distance associated to $f$, Definition 2.4. Assume that there is a constant $b > 0$ such that for all points where $\nabla f$ exists we have $g(\nabla f, \nabla f) \leq -b^2$.*

*Let $(\gamma_i : [0, b_i) \to M)_{i \in \mathbb{N}}$ be a sequence of past directed inextendible continuous causal curves in $M$, with $b_i \in \mathbb{R}^+ \cup \{\infty\}$, such that there exists $x \in M$ with $\gamma_i(0) \to x$. For each $i \in \mathbb{N}$ let $s_i : [0, a_i) \to [0, b_i)$ be the change in parameter constructed in Lemma 4.2.*

*Let $\gamma : [0, a) \to M$ be the limit curve constructed in Theorem 3.3 from the sequence $(\gamma_i \circ s_i)_i$, and let $(\gamma_{i_k} \circ s_{i_k})_k$ be the subsequence of $(\gamma_i)$ that $d(\cdot, \cdot; f)$-converges uniformly on compacta to $\gamma$. If*

1. *both sequences $(L(\gamma_{i_k} \circ s_{i_k}))_i$ and $(a_{i_k})_k$ are bounded above, and*
2. *$\limsup_k a_{i_k} = a$,*

*then there exists a subsequence $(\gamma_{i_{k_j}})_j$ of $(\gamma_{i_k})$ so that*

$$L(\gamma) \geq \limsup_j L(\gamma_{i_{k_j}}).$$

*Moreover if for all $t \in [0, a)$,*

$$L(\gamma|_{[0,t]}) = \lim_k L(\gamma_{i_k} \circ s_{i_k}|_{[0,t]})$$

*then $L(\gamma) = \lim_k L(\gamma_{i_k} \circ s_{i_k})$.*

*Proof.* Since $f$ is locally anti-Lipschitz $d(\cdot, \cdot; f)$ is a definite metric, and as $f$ is continuous $d(\cdot, \cdot; f)$ is compatible with the manifold topology as in Proposition 2.6. Hence, Lemmas 4.1 and 4.2 allow us to apply Theorem 3.3 using the null distance of $f$ to the sequence $(\gamma_i \circ s_i)$.

To prove the theorem we need to use Lemma 2.14. Let $B \in \mathbb{R}^+$ be such that for all $i \in \mathbb{N}$ we have $a_i \leq B$. Thus $a \leq \limsup_i a_i \leq B$. In particular, $a$, and $\limsup_i a_i$ are finite. By taking a subsequence we can assume that $\lim_k a_{i_k} = \limsup_k a_{i_k}$. For each $i \in \mathbb{N}$ define $f_i : [0, a) \to [0, a_i)$ by $f_i(x) = a_i x / a$.

For each $k \in \mathbb{N}$ we have $\gamma_{i_k} \circ s_{i_k} \circ f_{i_k} : [0, c) \to M$. By assumption the sequence $(L(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k}))_i$ is bounded above, which implies that each $L(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k})$ is finite.

Choose $\epsilon > 0$. Lemma 3.1 implies that by taking a further subsequence we can arrange that $\sup_k \sup\{f'_{i_k}(t) : t \in [0, a]\} \leq 1 + \epsilon$. By assumption and Lemmas 4.1, and 4.2 we therefore know that, for each $k \in \mathbb{N}$ and all $t_1, t_2 \in [0, a)$,

$$\begin{aligned} L\left(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k}|_{[t_1,t_2]}\right) &= L\left(\gamma_{i_k} \circ s_{i_k}|_{[f_{i_k}(t_1), f_{i_k}(t_2)]}\right) \\ &\leq \frac{1}{b}\left|f_{i_k}(t_1) - f_{i_k}(t_2)\right| \leq \frac{1+\epsilon}{b}\left|t_1 - t_2\right|. \end{aligned}$$

Since, for all $k \in \mathbb{N}$ and all $t_1, t_2 \in [0, a)$, we have

$$L\left(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k}|_{[t_1,t_2]}\right) \leq \frac{1+\epsilon}{b}\left|t_1 - t_2\right|$$

we know that

$$L\left(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k}|_{[t_1,a)}\right) = \lim_{t_2 \to a} L\left(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k}|_{[t_1,t_2]}\right) \leq \lim_{t_2 \to a} \frac{1+\epsilon}{b}\left|t_1 - t_2\right| = \frac{1+\epsilon}{b}\left|t_1 - a\right|$$

as $a$ is finite.

Therefore to use Lemma 2.14 it remains to show that $(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k})_k$ converges uniformly to $\gamma$ with respect to $d(\cdot, \cdot; f)$ on compact subsets of $[0, a)$. This is a consequence of $\lim_k a_{i_k} = a$.

Let $C \subset [0, a)$ be compact and define $K = \sup C$. Choose $\tilde{\epsilon} > 0$. Let $J_1 \in \mathbb{N}$ be such that $k \geq J_1$ implies that

$$K\left|\frac{a_{i_k}}{a} - 1\right| < \tilde{\epsilon}.$$

Since $(\gamma_{i_k} \circ s_{i_k}|_C)_k$ uniformly converges to $\gamma|_C$ there exists $J_2$ so that $k \geq J_2$ implies that, for all $c \in C$,

$$d(\gamma_{i_k} \circ s_{i_k}(c), \gamma(c); f) < \tilde{\epsilon}.$$

Hence if $k \geq \max\{J_1, J_2\}$ we can compute, for all $c \in C$, that

$$\begin{aligned} d\gamma_{i_k} \circ s_{i_k} \circ f_{i_k}(c), \gamma(c); f) &\leq d(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k}(c), \gamma_{i_k} \circ s_{i_k}(c); f) \\ &\quad + d(\gamma_{i_k} \circ s_{i_k}(c), \gamma(c); f) \\ &\leq \left|f_{i_k}(c) - c\right| + \tilde{\epsilon} \\ &\leq K\left|\frac{a_{i_k}}{a} - 1\right| + \tilde{\epsilon} \\ &< 2\tilde{\epsilon}. \end{aligned}$$

That is, the conditions of Lemma 2.14 hold and hence there exists a subsequence $(\gamma_{i_{k_j}} \circ s_{i_{k_j}} \circ f_{i_{k_j}})_j$ of $(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k})_k$ so that

$$L(\gamma) \geq \limsup_j L(\gamma_{i_{k_j}} \circ s_{i_{k_j}} \circ f_{i_{k_j}}) = \limsup_j L(\gamma_{i_{k_j}} \circ s_{i_{k_j}}),$$

as required.

Suppose further that for all $t \in [0, a)$,

$$L(\gamma|_{[0,t]}) = \lim_k L(\gamma_{i_k} \circ s_{i_k}|_{[0,t]}).$$

Supposing that for all $k \in \mathbb{N}$ we have $f_{i_k}(t) > t$,

$$0 \leq \lim_k L(\gamma_{i_k} \circ s_{i_k}|_{[t, f_{i_k}(t)]}) \leq \lim_k \frac{1}{b}\left|t - \frac{a_{i_k}}{a}t\right| = 0,$$

hence

$$L(\gamma|_{[0,t]}) = \lim_k L(\gamma_{i_k} \circ s_{i_k}|_{[0,t]}) = \lim_k L(\gamma_{i_k} \circ s_{i_k} \circ f_{i_k}|_{[0,t]})$$

and hence by Lemma 2.14,

$$L(\gamma) = \lim_j L(\gamma_{i_{k_j}} \circ s_{i_{k_j}} \circ f_{i_{k_j}}) = \lim_k L(\gamma_{i_{k_j}} \circ s_{i_{k_j}}).$$

A similar argument holds if for all $k \in \mathbb{N}$ we have $f_{i_k}(t) \leq t$. Since we can restrict to a subsequence so that one of these two conditions hold the theorem is true. ☐

The global bounds on lengths in Theorem 4.4 can be achieved for space-times whose past is bounded.

**Theorem 4.5.** *If $(M, g)$ is a globally hyperbolic manifold with regular cosmological time $\tau$, then for any sequence $(\gamma_i : [0, b_i) \to M)$, $b_i \in \mathbb{R}^+ \cup \{\infty\}$, of past directed inextendible continuous causal curves so that $\gamma_i(0) \to x \in M$ conditions 1. and 2. of Theorem 4.4 hold.*

*Proof.* By [1, Theorem 1.2(2 and 5)] the regular cosmological time is continuous and locally Lipschitz. By [22, Theorem 5.4] the regular cosmological time is anti-Lipschitz.

Since the cosmological time $\tau$ is regular on any past directed inextendible curve, $\gamma : [0, b_i) \to M$, we know that $\lim_{t \to b_i} \tau \circ \gamma(t) = 0$. This implies that the cosmological time satisfies the conditions of Proposition 4.3. As the cosmological time is regular it is also finite, hence the result holds. ☐

# 5 Null distances of surface functions

To apply Theorem 4.4 we need a function $f : M \to \mathbb{R}$ that is

1. locally Lipschitz,
2. locally anti-Lipschitz,
3. a time function, and such that
4. there exists some $b \in \mathbb{R}^+$ so that $g(\nabla f, \nabla f) \leq -b^2$ wherever $\nabla f$ exists,

and from Proposition 4.3, the sequence of curves needs to satisfy: for all $r \in (-\infty, f(\gamma(0))) \cap \mathrm{ran}\,(f)$ we have $\gamma \cap f^{-1}(r) \neq \varnothing$ and there exists $N \in \mathbb{N}$ so that for all $k \geq N$ we have $\gamma_{i_k} \cap f^{-1}(r) \neq \varnothing$.

We shall call the first four of these conditions the regularity conditions and we shall call the last condition the geometric condition. Theorem 4.5 proves that regular cosmological time satisfies both the regularity and the geometric conditions.

The geometric condition depends on the given sequence of inextendible curves, and so the function $f$ could in principle be tailored for application to a specific sequence of curves. The geometric condition is always true if the level sets of $f$ are Cauchy. In the remainder of this section we shall show that the surface function associated to a $C^1$ Cauchy surface satisfies the regularity conditions, Corollary 5.12.

Before we prove our claim it is worth noting that Müller and Sánchez, [17, Theorem 1.2], have shown the existence of a smooth surjective Cauchy time function $f : M \to \mathbb{R}$ on any globally hyperbolic manifold so that $g(\nabla f, \nabla f) \leq -1$. Since $f$ is Cauchy it satisfies the geometric condition. Since the function is smooth it is locally Lipschitz. Sormani and Vega have shown, [22, Corollary 4.16], that such a function is also anti-Lipschitz. Thus Müller and Sánchez' time function also satisfies the regularity conditions except for the bound on gradient length.

If one could prove that a lower bound for $g(\nabla f, \nabla f)$ existed then Müller and Sánchez's time function would satisfy both the regularity and the geometric conditions. The existence of such a lower bound is tied to the geometry of the manifold. In truncated Minkowski space, i.e. $\mathbb{R} \times (0, \infty)$, no such bound exists since $f$ surjects onto $\mathbb{R}$, though, of course, the standard time function given by projection does satisfy the regularity and geometric conditions.

We remind the reader that our Cauchy surfaces are, by definition, acausal [2, page 65].

We shall rely on the following technical result.

**Lemma 5.1.** *Let $(M, g)$ be globally hyperbolic, let $S \subset M$ be a Cauchy surface, let $\tau_S : M \to \mathbb{R}$ the surface function of $S$, let $x \in I^+(S)$ and let $h$ be an auxiliary Riemannian metric. Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of points so that $(x_i) \subset I^+(x)$ and $x_{i+1} \in I^-(x_i)$, $x_i \to x$. If $(\gamma_i : [0, b_i] \to M)_{i \in \mathbb{N}}$, $b_i \in \mathbb{R}^+$, is a sequence of $h$-arc length parametrised, past directed, causal curves so that for all $i \in \mathbb{N}$, $\gamma_i(0) = x_i$, $\gamma_i(b_i) \in S$, and $L(\gamma_i) \geq \tau_S(x_i) - 1/i$, then there exists an $h$-arc length parametrised, past directed, timelike curve $\gamma : [0, a] \to M$, $a \in \mathbb{R}^+$, so that $\gamma(0) = x$ and $\gamma(a) \in S$ and is such that*

1. *there exists a subsequence $(\gamma_{i_k})_k$ of $(\gamma_i)_i$ which $d(\cdot, \cdot; h)$-converges to $\gamma$ uniformly on compacta,*
2. *$\tau_S(x) = L(\gamma) = d_L(\gamma(a), \gamma(0)) = \lim_k \tau_S(x_{i_k})$, and*
3. *$\gamma$ can be re-parametrised as a smooth timelike geodesic.*

*Proof.* Choose $S'$ a second Cauchy surface so that $S' \subset I^-(S)$. Since $S'$ is Cauchy we can extend each $\gamma_i$ to $\tilde{\gamma}_i \in \Omega_{S', x_i}$. We will parametrise each extended $\tilde{\gamma}_i$ by $h$ induced arc-length. Thus we have $\tilde{\gamma}_i : [0, a_i) \to M$, $a_i \in (0, \infty]$, so that $L(\tilde{\gamma}_i \cap I^+(S)) \geq \tau_S(x_i) - 1/i$ and for all $i \in \mathbb{N}$ and $t_1, t_2 \in [0, a_i)$ we have $d(\tilde{\gamma}_i(t_1), \tilde{\gamma}_i(t_2); h) \leq |t_1 - t_2|$.

Choose $y \in I^+(x)$. Thus $y \in I^+(x) \subset I^+(S) \subset I^+(S') = \text{int}(D^+(S'))$ as $S'$ is achronal, by definition, [19, Proposition 5.20] implies that $J^-(y) \cap J^+(S')$ is compact. As $J^-(y) \cap J^+(S')$ is compact and as each $\tilde{\gamma}_i$ is extendible, we see that in fact the $a_i$ are finite and indeed there exists $b \in \mathbb{R}^+$ so that $a_i \leq b$, [14, Theorem 1.35].

Since $J^-(y) \cap J^+(S')$ is compact, Proposition 3.2 implies that there exists

1. a continuous curve $\gamma : [0, a) \to M$, $a = \limsup_i a_i < \infty$,
2. a sequence of re-parametrisations $f_i : [0, a) \to [0, a_i)$,
3. a subsequence $(\gamma_{i_k})_k$ of $(\gamma_i)$ so that $(\gamma_{i_k} \circ f_{i_k})_k$ will $d(\cdot, \cdot; h)$-converge uniformly on compact subsets of $[0, a)$ to $\gamma$ and so that $\lim_k a_{i_k} = a$.

Since for all $i \in \mathbb{N}$, $a_i < \infty$ and $a < \infty$ we know that $f_i(x) = a_i x / a$, Lemma 3.1. As $a = \lim_k a_{i_k}$ we know that there exists $K \in \mathbb{R}^+$ so that $\sup\{f_i'(t) : t \in [0, a]\} = a_i / a \leq K$. This implies that $(\gamma_{i_k} \circ f_{i_k})_k$ will $d(\cdot, \cdot; h)$-converge uniformly to $\gamma$ on compacta, in the sense of Definition 2.15.

Since for all $i \in \mathbb{N}$, $\gamma_i \cap S \neq \varnothing$ we see that $\gamma \cap S \neq \varnothing$. Thus there exists $t \in [0, a)$ so that $\gamma(t) \in S$. Similarly for each $k$ there exists $t_k \in [0, a)$ so that $\gamma_{i_k} \circ f_{i_k}(t_k) \in S$. Similarly to the proof that

$J^{-}(y) \cap J^{+}(S')$ is compact, [19, Proposition 5.20] implies that the set $J^{-}(y) \cap J^{+}(S)$ is compact. Hence $t_k \to t$ by the uniform convergence of the $(\gamma_{i_k})$. By Proposition 2.12

$$L(\gamma|_{[0,t]}) \geq \limsup_k L(\gamma_{i_k} \circ f_{i_k}|_{[0,t]}) = \limsup_k L(\gamma_{i_k} \circ f_{i_k}|_{[0,t_k]}).$$

Since $x_{i_k} \in I^{+}(x)$ we have $\tau_S(x_{i_k}) \geq \tau_S(x)$. Hence we also have $\limsup_k \tau_S(x_{i_k}) \geq \tau_S(x)$. Therefore

$$\begin{aligned}
\limsup_k \tau_S(x_{i_k}) \geq \tau_S(x) &\geq L(\gamma|_{[0,t]}) \\
&\geq \limsup_k L(\gamma_{i_k} \circ f_{i_k}|_{[0,t]}) = \limsup_k L(\gamma_{i_k} \circ f_{i_k}|_{[0,t_k]}) \\
&\geq \limsup_k \tau_S(x_{i_k}) - 1/i_k \\
&= \limsup_k \tau_S(x_{i_k}). \qquad (4)
\end{aligned}$$

That is, we have shown that $\tau_S(x) = L(\gamma|_{[0,t]}) = \limsup_k \tau_S(x_{i_k})$. Since $x_{i+1} \in I^{-}(x_i)$ the sequence $(\tau_S(x_{i_k}))_k$ is decreasing. Thus $\limsup_k \tau_S(x_{i_k}) = \lim_k \tau_S(x_{i_k})$. Since $d_L(\gamma(t), \gamma(0)) \geq L(\gamma|_{[0,t]}) = \tau_S(\gamma(0)) \geq d_L(\gamma(t), \gamma(0))$, Theorem 4.13 of [2] implies that $\gamma|_{[0,t]}$ can be reparametrised as a smooth timelike geodesic. This implies that $\gamma|_{[0,t]}$ is a timelike curve and so gives the result. □

As a corollary we get the following result.

**Corollary 5.2.** *If $(M, g)$ is globally hyperbolic and $S \subset M$ is a Cauchy surface then $M$ is the disjoint union $M = I^{+}(S) \cup S \cup I^{-}(S)$. Moreover for all $x \in I^{+}(S)$, $d_L(S, x) < \infty$ and for all $x \in I^{-}(S)$, $d_L(x, S) < \infty$.*

*Proof.* By assumption and as $S$ is Cauchy the interior of $D^{+}(S)$ is $I^{+}(S)$. Since $S$ is acausal we have that $M$ is the disjoint union $M = I^{+}(S) \cup S \cup I^{-}(S)$. Lemma 5.1 implies that there exists a curve with compact image which attains the distance to the surface. This implies that the distance to the surface is finite. The result follows by time duality. □

**Lemma 5.3.** *If $(M, g)$ is globally hyperbolic and $S \subset M$ is a Cauchy surface, then $\tau_S$ is continuous.*

*Proof.* Suppose that $\tau_S$ is discontinuous at $x \in \overline{I^{+}(S)}$. That is, there exists a sequence $(y_i)_{i \in \mathbb{N}} \subset M$ with $y_i \to x$ and $\lim_{i\to\infty} \tau_S(y_i) > \tau_S(x)$. Choose $(x_i)_i \in \mathbb{N} \subset I^{+}(x)$ so that $x_{i+1} \subset I^{-}(x_i)$ and $x_i \to x$. Since $x \in I^{-}(x_i)$ we see that $\lim_{i\to\infty} \tau_S(x_i) \geq \lim_{i\to\infty} \tau_S(y_i) > \tau_S(x)$. We can now find curves $\gamma_i \in \Omega_{S,x_i}$ with $\tau_S(x_i) = L(\gamma_i) - 1/i$, by definition of $\tau_S$. Consequently the existence of such a sequence of points $(y_i)$ contradicts Lemma 5.1, hence we have the result. □

**Proposition 5.4.** *Let $(M, g)$ be globally hyperbolic, let $S \subset M$ be a Cauchy surface, and $\tau_S : M \to \mathbb{R}$ the surface function of $S$. For all $x \in I^{+}(S)$ there exists $\gamma : [0, a] \to M$, $a \in \mathbb{R}^{+}$, a future directed, $g$-arc length parametrised, timelike smooth geodesic from $S$ to $x$ so that*

1. *for all $u, v \in [0, a]$, $u < v$, $L(\gamma|_{[u,v]}) = d_L(\gamma(u), \gamma(v)) = \tau_S(\gamma(v)) - \tau_S(\gamma(u))$;*
2. *for all $u \in [0, a]$, if $\nabla\tau_S$ exists at $\gamma(u)$ then $\gamma'(u) = -\nabla\tau_S|_{\gamma(u)}$ and, in particular,*

$$g(\nabla\tau_S|_{\gamma(u)}, \nabla\tau_S|_{\gamma(u)}) = -1.$$

*Proof.* Choose a sequence $(x_i)_i \subset I^+(x)$ so that $x_{i+1} \subset I^-(x_i)$. For each $i \in \mathbb{N}$ choose $\gamma_i : [0, b_i] \to M$, $b_i \in \mathbb{R}^+$, be a past directed causal curve from $x_i$ to $S$ so that $L(\gamma_i) \geq \tau_S(x_i) - 1/i$. Let $\gamma : [0, \tilde{a}] \to M$ be the curve constructed by Lemma 5.1 using the sequence of curves $(\gamma_i)_i$. We re-parametrise $\gamma$ to be arc-length parametrised with respect to $g$. In an abuse of notation we denote this re-parametrised curve by $\gamma : [0, a] \to M$, $a \in \mathbb{R}^+$. Thus, by definition $g(\gamma', \gamma') = -1$.

Remark 4.11 of [2] shows that for all $u, v \in [0, a]$, $u < v$, $L(\gamma|_{[u,v]}) = d_L(\gamma(u), \gamma(v))$. Suppose that for some $u \in [0, a]$ we have $\tau_S(\gamma(u)) > L(\gamma|_{[0,u]})$. Then

$$L(\gamma) = \tau_S(x) \geq d_L(\gamma(u), x) + \tau_S(\gamma(u)) > L(\gamma|_{[u,a]}) + L(\gamma|_{[0,u]}) = L(\gamma),$$

which is a contradiction. We find that for all $u \in [0, a]$ we have $\tau_S(\gamma(u)) = L(\gamma|_{[0,u]})$. Hence, for all $u, v \in [0, a]$, $u < v$

$$L(\gamma|_{[0,v]}) = \tau_S(\gamma(v)) \geq d_L(\gamma(u), \gamma(v)) + \tau_S(\gamma(u)) = L(\gamma|_{[u,v]}) + L(\gamma|_{[0,u]}) = L(\gamma|_{[0,v]}),$$

and therefore

$$L(\gamma|_{[u,v]}) = d_L(\gamma(u), \gamma(v)) = \tau_S(\gamma(v)) - \tau_S(\gamma(u)).$$

Since $\gamma$ is arc length parametrised, for all $u \in [0, a]$ we have $\tau_S(\gamma(u)) - \tau_s(\gamma(0)) = L(\gamma|_{[0,u]}) = u$. Thus, by the reverse Cauchy inequality, [18, Proposition 5.30], and the direct calculation in [20, Propostion 2.16], wherever $\nabla\tau_S$ exists we have

$$1 = g(\nabla\tau_S, \gamma') \geq \sqrt{-g(\nabla\tau_S, \nabla\tau_S)}\sqrt{-g(\gamma', \gamma')} \geq 1$$

and so $\nabla\tau_s = -\gamma'$ and $g(\nabla\tau_S, \nabla\tau_S) = -1$. □

**Lemma 5.5.** *If $(M, g)$ is globally hyperbolic and $S \subset M$ is a Cauchy surface, then for all $t \in \operatorname{ran}(\tau_S)$ the set $\tau_S^{-1}(t)$ is acausal.*

*Proof.* Suppose that $t > 0$. Let $x, y \in \tau_S^{-1}(t)$ be such that $y \in J^+(x) \setminus \{x\}$. As $\tau_S$ is increasing on timelike curves we know that $y \notin I^+(x)$. Therefore there exists $\gamma$ a null curve from $x$ to $y$ on which $\tau_S$ is constant.

Proposition 5.4 implies that there exists $\lambda \in \Omega_{S,x}$ a timelike geodesic so that $L(\lambda) = \tau_S(x)$.

The concatenation of $\gamma$ and $\lambda$, denoted $\sigma$, is a causal curve so that $L(\sigma) = \tau_S(y)$. Theorem 4.13 of [2] implies that $\sigma$ can be reparametrised as a smooth timelike geodesic. This is a contradiction as, at $x$, $g(\gamma', \gamma') = 0$ and $g(\lambda', \lambda') < 0$.

The result now follows by time duality and as our Cauchy surfaces are necessarily acausal [2, Page 65]. □

**Lemma 5.6.** *Surface functions of Cauchy surfaces in globally hyperbolic manifolds are generalised time functions, i.e. increasing on all future directed causal curves.*

*Proof.* Let $\tau_S$ be the surface function of the Cauchy surface $S$. Let $\gamma$ be a future directed causal curve. We know that $\tau_S$ is non-decreasing on $\gamma$. If $\tau_S$ is constant over some subcurve of $\gamma$ then there exists a level surface of $\tau_S$ that contains a causal curve. This contradicts Lemma 5.5 and the assumption that $S$ is acausal. Thus $\tau_S$ is a non-constant non-decreasing function on $\gamma$ and therefore is increasing. □

**Lemma 5.7.** *If $(M, g)$ is globally hyperbolic and $S \subset M$ is a $C^1$ Cauchy surface, then there exists an open neighbourhood $U \subset M$ of $S$ and a continuous vector field $X : U \to TM$ so that wherever $\nabla \tau_S$ exists $X = \nabla \tau_S$.*

*Proof.* As $S$ is $C^1$ we know that the unit normal vector field $n$ to $S$ is continuous and timelike everywhere. Let $NS$ denote the normal bundle to $S$, that is $N_s S$ is the span of $n(s) \in T_s M$ for each $s \in S$. Let $V \subset NS$ be an open neighbourhood of the zero section of $NS$ over $S$ so that $\exp(V)$ is a normal neighbourhood of $S$.

Let $s \in S$ and let $I_s = \{t \in \mathbb{R} : tn(s) \in V\}$. Define $\gamma_s(t) : I_s \to M$ by $\gamma_s = \exp_s(tn(s))$. By definition $\gamma_s$ is focal point free and therefore for all $t \in I_s \cap \mathbb{R}^+$, $t = L(\gamma_s|_{[0,t]}) = d_L(S, \gamma_s(t)) = \tau_S(\gamma_s(t))$, see [2, Propositions 12.25, 12.29]. Similarly, for all $t \in I_s \cap \mathbb{R}^-$, $-t = L(\gamma_s|_{[t,0]}) = d_L(\gamma_s(t), S) = -\tau_S(\gamma_s(t))$. Thus $\tau_S(\gamma_s(t)) = t$ for all $t \in I_s$. Therefore, if $\nabla \tau_S|_{\gamma_s(t)}$ exists then $\nabla \tau_S|_{\gamma_s(t)} = -\gamma_s'(t)$. Let $\partial_t \in TI_s$ be the standard unit length tangent vector. Thus $\gamma_s'(t) = (d\exp_s(\partial_t))|_{tn(s)}$. Let $X(\exp(tn(s))) = -\gamma_s'(t) = -(d\exp_s(\partial_t))|_{tn(s)}$. As $n$ is continuous and $d\exp$ is smooth $X$ is a continuous vector field so that if $\nabla \tau_S$ exists at $x$ then $X(x) = \nabla \tau_S$. □

**Lemma 5.8.** *Let $(M, g)$ be globally hyperbolic, $\tau_S$ the surface function associated to a $C^1$ Cauchy surface $S \subset M$ and $h$ an auxiliary Riemannian metric. For all $C \subset M$, a compact subset, there exists $K \in \mathbb{R}^+$ so that for all $c \in C$ where $\nabla \tau_S|_c = g(d\tau_S|_c, \cdot)$ exists we have $h(\nabla \tau_S|_c, \nabla \tau_S|_c) < K$.*

*Proof.* Lemma 5.7 implies that there exists a neighbourhood $U \subset M$ of $S$ and a continuous vector field $X : U \to TM$ so that $X = \nabla \tau_S$ wherever $\nabla \tau_S|_U$ exists. If we suppose that $C \subset U$, then since $h(X, X)$ is continuous and $C$ is compact the result holds.

More generally, we start by considering $C \cap U$. Let $\phi_+, \phi_-, \phi_0$ be a partition of unity whose supports are in $I^+(S)$, $I^-(S)$ and $U$ respectively. Let $C_0 = \{x \in C \cap U : \phi_0 \geq 1/2\}$. Then $C_0$ is a closed subset of the compact set $C$, and so is compact, and $C_0 \subset U$. Now define $C_+$ to be the closure of $(C \cap I^+(S)) \setminus C_0$, and similarly set $C_-$ to be the closure of $(C \cap I^-(S)) \setminus C_0$.

We have already shown that the result is true for $C_0$. Therefore, by time duality, if the result holds for any compact subset $C \subset I^+(S)$ then the result will hold for any compact subset $C \subset M$.

With this in mind suppose that there exists a compact set $C \subset I^+(S)$ and $(x_i) \subset C$ a sequence of points in $C$ so that $\nabla \tau_S|_{x_i}$ exists and $\lim_{i\to\infty} h(\nabla \tau_S|_{x_i}, \nabla \tau_S|_{x_i}) = \infty$. By taking a subsequence we can further assume that $x_i \to x \in C$.

Since $M$ is globally hyperbolic $A = J^-(C) \cap J^+(S)$ is compact. Thus, by Proposition 5.4, for each $i \in \mathbb{N}$ there exists a smooth past directed timelike geodesic $\gamma_i : [0, a_i] \to A$ so that $\gamma_i(0) = x_i$, $\gamma_i(a_i) \in S$ and $L(\gamma_i) = \tau_S(x_i)$. Without loss of generality we can assume that each $\gamma_i$ is $h$ arc length parametrised. That is, we can assume that $h(\gamma_i', \gamma_i') = 1$.

Let $\gamma : [0, a) \to A$ be the smooth timelike geodesic from $x$ given by applying Lemma 5.1 to the sequence $(\gamma_i)$. Lemma 5.1 tells us that $\tau_S(x) = L(\gamma) = \lim_k \tau_S(x_{i_k})$. In particular, $\gamma$ is timelike.

Due to our parametrisation, Proposition 5.4 implies that

$$\gamma_i'(0) = \frac{1}{\sqrt{h(\nabla \tau_S|_{x_i}, \nabla \tau_S|_{x_i})}} \nabla \tau_S|_{x_i}.$$

By taking a convex normal neighbourhood about $x$ we see that there exists $\tau \in \mathbb{R}^+$ so that for $i$ large enough $\tau$ is in the domain of $\gamma_i$. Since $\gamma_i$ is a geodesic we know that $\gamma_i(t) = \exp_{\gamma_i(0)}(t\gamma_i'(0))$, at least for $t \in [0, \tau]$. The world function, [19, Definition 2.13], on $U$, $\Phi : U \times U \to \mathbb{R}$ is defined by

$$\Phi(p, q) = g(\exp_p^{-1}(q), \exp_p^{-1}(q)).$$

We can compute that

$$\Phi(\gamma_i(0), \gamma_i(t)) = t^2 g(\gamma_i'(0), \gamma_i'(0)) = -\frac{t^2}{h(\nabla\tau_S|_{x_i}, \nabla\tau_S|_{x_i})}.$$

Since $\Phi$ is continuous, [19, Definition 2.13] we see that

$$\Phi(\gamma(0), \gamma(t)) = \lim_{i\to\infty} \Phi(\gamma_i(0), \gamma_i(t)) = \lim_{i\to\infty} t^2 g(\gamma_i'(0), \gamma_i'(0)) = \lim_{i\to\infty} -\frac{t^2}{h(\nabla\tau_S|_{x_i}, \nabla\tau_S|_{x_i})} = 0. \qquad (5)$$

As $\gamma$ is a timelike geodesic we know that $\Phi(\gamma(0), \gamma(t)) < 0$, [19, Lemma 2.15], so Equation (5) gives us a contradiction. Hence the result holds for $C \subset I^+(S)$ and thus for any compact subset of $M$. $\square$

**Corollary 5.9.** *Let $(M, g)$ be globally hyperbolic, $\tau_S$ the surface function associated to a $C^1$ Cauchy surface $S \subset M$ and $h$ an auxiliary Riemannian metric. If $C \subset M$ is compact then*

$$\{\nabla\tau_S|_c : c \in C,\ \nabla\tau_S \text{ exists at } c\} \subset \{v \in T_cC : g(v, v) = -1,\ h(v, v) \le K\},$$

*which is a compact subset of the tangent bundle $TC$.*

*Proof.* This is an immediate consequence of Lemma 5.8 and Proposition 5.4. $\square$

**Corollary 5.10.** *Let $(M, g)$ be globally hyperbolic. If $\tau_S$ is the surface function associated to a $C^1$ Cauchy surface, then $\tau_S$ is locally anti-Lipschitz with respect to any auxiliary Riemannian metric $h$.*

*Proof.* Let $p \in M$, $C \subset M$ be a compact neighbourhood of $p$ and let $U \subset C$ be an open neighbourhood of $p$. Lemma 5.8 implies that there exists $K \in \mathbb{R}^+$ so that wherever $\nabla\tau_S|_C$ exists $h(\nabla\tau_S, \nabla\tau_S) < K$. We also know that $g(\nabla\tau_S, \nabla\tau_S) = -1$, by Proposition 5.4. Therefore

$$\sqrt{|g(\nabla\tau_S, \nabla\tau_S))|} = 1 \ge \frac{1}{\max\{1, K\}} \max\{1, \sqrt{h(\nabla\tau_S, \nabla\tau_S)}\},$$

and so $\nabla\tau_S$ is bounded away from light cones. The local anti-Lipschitz property now follows from [22, Definition 4.13 and Theorem 4.18]. $\square$

**Lemma 5.11.** *Let $M$ be globally hyperbolic. If $\tau_S$ is the surface function associated to a $C^1$ Cauchy surface, then $\tau_S$ is locally Lipschitz with respect to any auxiliary Riemannian metric $h$.*

*Proof.* Since the statement is local, and all Riemannian metrics are locally equivalent, it suffices to check local Lipschitzness with respect to a single metric.

Let $x \in M$ and let $\phi : U \to \mathbb{R}^n$ be a chart about $x$. Let $\partial_1, \ldots, \partial_n$ be the coordinate frame on $U$. Choose $e_1, \ldots, e_n$ a pseudo-orthonormal frame over $U$, with $g(e_1, e_1) = -1$, $g(e_i, e_j) = \delta_{ij}$ for $i = 1, \ldots, n$ and $j = 2, \ldots, n$. Define $h : TU \times TU \to \mathbb{R}^+$ a Riemannian metric

$$h(u, v) = g(u, v) - 2\frac{g(u, e_1)g(v, e_1)}{g(e_1, e_1)},$$

so that $h_{ij} = |g_{ij}| = \delta_{ij}$.

For any open set $V \subset \mathbb{R}^n$ let $L^\infty(V)$ be the space of bounded Lebesgue measurable functions $f : V \to \mathbb{R}$ so that $\|f\|_\infty := \operatorname{ess\,sup}_V |f| < \infty$ and let $W^{1,\infty}_{\text{loc}}(V)$ to be the Sobolev space of all functions $f : V \to \mathbb{R}$ so that if $C \subset V$ is compact then $f|_C \in L^\infty(C)$, the weak partial derivatives $\partial_i f|_C$, $i = 1, \ldots, n$, exist and are such that $\partial_i f|_C \in L^\infty(C)$, see [8, Notation on pages 26 and 36, Definition 4.2]. Since the coordinate map $\phi$ is smooth, $\tau_S$ is locally Lipschitz on $M$ with respect to $h$ if and only if $\tau_S \circ \phi^{-1}$ is locally Lipschitz with respect to the Euclidean metric. By [8, Theorem 4.5], $\tau_S \circ \phi^{-1}$ is locally Lipschitz if and only if $\tau_S \circ \phi^{-1} \in W^{1,\infty}_{\text{loc}}(\phi(U))$.

Let $C \subset \phi(U)$ be compact. Since $\tau_S \circ \phi^{-1}$ is continuous, Lemma 5.3, we see that $\tau_S \circ \phi^{-1}$ is Lebesgue measurable and $\left\|\tau_S \circ \phi^{-1}|_C\right\|_\infty < \infty$. That is $\tau_S \circ \phi^{-1}|_C \in L^\infty(C)$.

Lemma 5.8 implies that there exists $K_1 \in \mathbb{R}^+$ so that wherever the $g$-gradient $\nabla\tau_S|_{\phi^{-1}(C)}$ exists we have $h(\nabla\tau_S|_{\phi^{-1}(C)}, \nabla\tau_S|_{\phi^{-1}(C)}) < {K_1}^2$.

With $(f_i)_{i=1}^n$ the standard orthonormal basis of $\mathbb{R}^n$, the partial derivatives of $\tau_S \circ \phi^{-1}$ are given by

$$\partial_i(\tau_S \circ \phi^{-1}) = D(\tau_S \circ \phi^{-1}) \cdot f_i.$$

As $\tau \circ \phi^{-1}$ is continuous, the main result of [16] implies that the sets where the partial derivatives exist are measurable, and the functions $\partial_i(\tau_S \circ \phi^{-1})$ are measurable on these sets. As $D(\tau_S \circ \phi^{-1})$ exists a.e., these sets are of full measure. As $D(\tau_S \circ \phi^{-1})$ is bounded by Lemma 5.8, so too are the partial derivatives and $\tau_S \circ \phi^{-1} \in W^{1,\infty}$. Hence $\tau_S \circ \phi^{-1}$ and so $\tau_S$ are locally Lipschitz. □

Summarising the results of this section, we have the following.

**Corollary 5.12.** *Let $(M, g)$ be globally hyperbolic. If $S \subset M$ is a $C^1$ Cauchy surface, then the surface function $\tau_S$ associated to $S$ is*

1. *a locally anti-Lipschitz, locally Lipschitz, time function,*
2. *such that $\nabla\tau_S$ exists almost everywhere and $g(\nabla\tau_S, \nabla\tau_S) = -1$ wherever $\nabla\tau_S$ exists, and*
3. *the null distance defined by $\tau_S$ is an actual metric which induces the manifold topology.*

*Proof.* Lemma 5.2 proves that any Cauchy surface has an associated surface function. That is, the lemma proves that $\tau_S$ is well defined by Definition 2.3 for any Cauchy surface. Minguzzi, [14, Theorem 1.19], shows that $\tau_S$ has an almost everywhere defined gradient. Proposition 5.4 shows that wherever $\nabla\tau_S$ exists $g(\nabla\tau_S, \nabla\tau_s) = -1$. Lemma 5.3 shows that $\tau_S$ is continuous and so Lemma 5.6 completes the proof that $\tau_S$ is a time function. Corollary 5.10 proves that $\tau_S$ is locally anti-Lipschitz. We now know [22, Theorem 4.6] that the null distance induced by $\tau_S$ is a distance and induces the manifold topology. Lemma 5.11 proves that $\tau_S$ is locally Lipschitz. □

**Corollary 5.13.** *Let $(M, g)$ be globally hyperbolic. If $S \subset M$ is a $C^1$ Cauchy surface, then the restriction of the surface function $\tau_S$ associated to $S$ to $I^+(S)$ is a regular cosmological time function for $I^+(S)$. Hence by Theorem 4.5, $\tau_S$ restricted to $I^+(S)$ satisfies the assumptions of Theorems 4.3 and 4.4.*

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