Curvature and Weitzenböck formula for the Podleś quantum sphere

Bram Mesland§*, Adam Rennie†*

§Mathematisch Instituut, Universiteit Leiden, Netherlands

†School of Mathematics and Applied Statistics, University of Wollongong
Wollongong, Australia

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Abstract

We prove that there is a unique Levi-Civita connection on the one-forms of the Dabrowski-Sitarz spectral triple for the Podleś sphere S^2_q. We compute the full curvature tensor, as well as the Ricci and scalar curvature of the Podleś sphere using the framework of [MRLC]. The scalar curvature is a constant, and as the parameter $q \to 1$, the scalar curvature converges to the classical value 2. We prove a generalised Weitzenböck formula for the spinor bundle, which differs from the classical Lichnerowicz formula for $q \neq 1$, yet recovers it for $q \to 1$.

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*email: b.mesland@math.leidenuniv.nl, renniea@uow.edu.au

1 Introduction

As an example of the constructions presented in [MRLC], we compute the Levi-Civita connection and curvature of the Podleś quantum sphere S^2_q. The geometry of the Podleś sphere has been investigated by numerous authors [BM2, DS, KW, M, NT, RS, S, SW, W]. One important feature is that the classical ($q = 1$) version has positive curvature, and there is no zero curvature metric, so this represents an example outside the conformally flat setting. Our results are broadly in line with those of the existing algebraic works [BM2, M], with the principal difference being the starting point and the techniques.

We work with the differential one-forms built from the spectral triple [DS] for the Podleś sphere. As is known [SW, W], this differential calculus agrees with the usual abstract differential calculus of [P]. The bimodule of one-forms in this case has trivial centre, putting it outside of the scope of techniques developed in [BGJ2] to construct a Levi-Civita connection for it. The general theory of [MRLC, MRC] can be applied in this context and the first main result of the paper reads

Theorem 1. *The module of differential one-forms on the Podleś sphere S^2_q equipped with the quantum metric defined in Section 3.2 admits a unique Levi-Civita connection whose scalar curvature equals $r = [2]_q(1 + (q^{-2} - q^2)^2)$.*

The relevance of this example goes beyond being able to compute junk, exterior derivative and Levi-Civita connection. Unlike classical and θ-deformed manifolds, both the uniqueness of the Levi-Civita connection and the Weitzenböck formula for the Podleś sphere hold with a non-trivial generalised braiding. Indeed to view the Podles sphere as two-dimensional requires a non-trivial twisting of its Hochschild homology and we utilise this fact in our constructions [HK, K, RS, S]. Thus our approach starting from connections on modules and spectral triples, is entirely compatible with the known algebraic theory, where braidings appear naturally from the quantum group structure and to deal with dimension drop.

To state a Weitzenböck formula requires a special setting. The Dirac spectral triple of the Podleś sphere involves the Dirac operator $\mathcal{D}$ acting on a spinor module $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. The generalised Weitzenböck formula relates $\mathcal{D}^2$ to the spinor Laplacian $\Delta^{\mathcal{S}}$. Our second main result establishes a close relationship between these operators and the Clifford representation of the curvature of the spinor bundle.

Theorem 2. *The Dirac operator* $\mathcal{D}$ *and spin Laplacian* $\Delta^{\mathcal{S}}$ *for the Podleś sphere* S^2_q *are related by the Weitzenböck formula*

$$\mathcal{D}^2 = \Delta^{\mathcal{S}} + \frac{1}{q^{-2}+q^2}\begin{pmatrix} q^2 & 0 \\ 0 & q^{-2}\end{pmatrix},$$

and the spin Laplacian $\Delta^{\mathcal{S}}$ *is a positive operator.*

The matrix operator appearing in the Weitzenböck formula is to be interpreted as acting on the graded module $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. Thus, the close relationship between the curvature of the spinor module and the curvature of the module of one-forms breaks down for $q \neq 1$. Whilst a Weitzenböck formula holds, the specific Lichnerowicz relationship that $\mathcal{D}^2 = \Delta + \frac{1}{4}r$ where r is the scalar curvature does not hold for $q \neq 1$. That said, as $q \to 1$, we do recover the classical formula.

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2 Background

We present a summary of the theoretical framework from [MRLC, MRC] which we use to define and compute the Levi-Civita connection and curvature.

2.1 Differentials and junk

Throughout this article we are looking at the differential structure provided by a spectral triple. See [MRLC, Examples 2.4–2.6].

Definition 2.1. Let B be a C^*-algebra. A spectral triple for the unital $*$-algebra B is a triple $(\mathcal{B}, \mathcal{H}, \mathcal{D})$ where $\mathcal{B} \subset B$ is a dense $*$-subalgebra, $\mathcal{H}$ is a Hilbert space equipped with a unital $*$-representation $B \to \mathbb{B}(\mathcal{H})$, and $\mathcal{D}$ an unbounded self-adjoint operator $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset \mathcal{H} \to \mathcal{H}$ such that for all $a \in \mathcal{B}$

$$a \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D}) \quad \text{and} \quad [\mathcal{D}, a] \text{ is bounded},$$

and $(\mathcal{D} \pm i)^{-1}$ is compact.

It is worthwhile to point out that the constructions of this paper do not require the compact resolvent condition.

Given a spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D})$, the module of one-forms is the $\mathcal{B}$-bimodule

$$\Omega^1_{\mathcal{D}}(\mathcal{B}) := \mathrm{span}\,\{a[\mathcal{D}, b] : a, b \in \mathcal{B}\} \subset \mathbb{B}(\mathcal{H}).$$

We obtain a first order differential calculus $\mathrm{d} : \mathcal{B} \to \Omega^1_{\mathcal{D}}(\mathcal{B})$ by setting $\mathrm{d}(b) := [\mathcal{D}, b]$. This calculus carries an involution $(a[\mathcal{D}, b])^\dagger := [\mathcal{D}, b]^* a^*$ induced by the operator adjoint. Thus $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger)$ is a first order differential structure in the sense of [MRLC].

We recollect some of the constructions of [MRLC] for $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger)$. Write $T^k_{\mathcal{D}}(\mathcal{B}) := \Omega^1_{\mathcal{D}}(\mathcal{B})^{\otimes_{\mathcal{B}} k}$ and $\Omega^1_{\mathcal{D}}(\mathcal{B})^k = \mathrm{span}\{b_0[\mathcal{D}, b_1] \cdots [\mathcal{D}, b_k] : b_j \in \mathcal{B}\}$. The universal differential forms $\Omega^*_u(\mathcal{B})$ [L96] admit representations (with $\delta(b) = 1 \otimes b - b \otimes 1$ for $b \in \mathcal{B}$)

$$\pi_{\mathcal{D}} : \Omega^k_u(\mathcal{B}) \to T^k_{\mathcal{D}}(\mathcal{B}) \quad \pi_{\mathcal{D}}(a_0\delta(a_1) \cdots \delta(a_k)) = a_0[\mathcal{D}, a_1] \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} [\mathcal{D}, a_k], \tag{2.1}$$

$$m \circ \pi_{\mathcal{D}} : \Omega^k_u(\mathcal{B}) \to \Omega^k_{\mathcal{D}}(\mathcal{B}) \quad m \circ \pi_{\mathcal{D}}(a_0\delta(a_1) \cdots \delta(a_k)) = a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_k], \tag{2.2}$$

where $m : T^k_{\mathcal{D}}(\mathcal{B}) \to \Omega^k_{\mathcal{D}}(\mathcal{B})$ is the multiplication map. Neither $\pi_{\mathcal{D}}$ nor $m \circ \pi_{\mathcal{D}}$ are maps of differential algebras, but are $\mathcal{B}$-bilinear maps of associative $*$-$\mathcal{B}$-algebras, [Lan, MRLC]. The $*$-structure on $\Omega^*_{\mathcal{D}}(\mathcal{B})$ is determined by the adjoint of linear maps on $\mathcal{H}$, while the $*$-structure on $\oplus_k T^k_{\mathcal{D}}(\mathcal{B})$ is given by the operator adjoint and

$$(\omega_1 \otimes_{\mathcal{B}} \omega_2 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \omega_k)^\dagger := \omega_k^* \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \omega_2^* \otimes_{\mathcal{B}} \omega_1^*.$$

We will write $\omega^\dagger := \omega^*$ for one forms ω as well.

The maps $\pi_{\mathcal{D}} : \Omega^*_u(\mathcal{B}) \to T^*_{\mathcal{D}}$ and $\delta : \Omega^k_u(\mathcal{B}) \to \Omega^{k+1}_u(\mathcal{B})$ are typically not compatible in the sense that δ need not map $\ker \pi_{\mathcal{D}}$ to itself. Thus in general, $T^*_{\mathcal{D}}(\mathcal{B})$ can not be made into a differential algebra. The issue to address is that there are universal forms $\omega \in \Omega^n_u(\mathcal{B})$ for which $\pi_{\mathcal{D}}(\omega) = 0$ but $\pi_{\mathcal{D}}(\delta(\omega)) \neq 0$. The latter are known as *junk forms*, [C, Chapter VI]. We denote the $\mathcal{B}$-bimodules of junk forms and junk 2-tensors by

$$J^k_{\mathcal{D}}(\mathcal{B}) = \{m \circ \pi_{\mathcal{D}}(\delta(\omega)) : m \circ \pi_{\mathcal{D}}(\omega) = 0\} \quad \text{and} \quad JT^k_{\mathcal{D}}(\mathcal{B}) = \{\pi_{\mathcal{D}}(\delta(\omega)) : \pi_{\mathcal{D}}(\omega) = 0\}.$$

Observe that the junk submodules depend only on the representation of the universal forms. In particular, the bimodule $JT^k_{\mathcal{D}}(\mathcal{B}) \subset T^2_{\mathcal{D}}(\mathcal{B})$ is exactly the module that one needs to quotient out in the construction of the so-called "maximal prolongation" of a given first order calculus, see [BM2, Lemma 1.32].

Definition 2.2. A second order differential structure $(\Omega^1_{\mathcal{D}}, \dagger, \Psi)$ is a first order differential structure $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger)$ together with an idempotent $\Psi : T^2_{\mathcal{D}} \to T^2_{\mathcal{D}}$ satisfying $\Psi \circ \dagger = \dagger \circ \Psi$ and $JT^2_{\mathcal{D}}(\mathcal{B}) \subset \mathrm{Im}(\Psi) \subset m^{-1}(J^2_{\mathcal{D}}(\mathcal{B}))$. A second order differential structure is Hermitian if $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is a finitely generated projective right $\mathcal{B}$-module with right inner product $\langle \cdot | \cdot \rangle_{\mathcal{B}}$, such that $\Psi = \Psi^2 = \Psi^*$ is a projection, and the left action of $\mathcal{B}$ is adjointable.

A second order differential structure admits an exterior derivative $\mathrm{d}_\Psi : \Omega^1_{\mathcal{D}}(\mathcal{B}) \to T^2_{\mathcal{D}}(\mathcal{B})$ via

$$\mathrm{d}_\Psi(\rho) = (1 - \Psi) \circ \pi_{\mathcal{D}} \circ \delta \circ \pi_{\mathcal{D}}^{-1}(\rho). \tag{2.3}$$

The differential satisfies $\mathrm{d}_\Psi([\mathcal{D}, b]) = 0$ for all $b \in \mathcal{B}$. A differential on one-forms allows us to define curvature for modules, and formulate torsion for connections on one-forms.

For an Hermitian differential structure $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger, \Psi, \langle \cdot \mid \cdot \rangle)$, the module of one-forms $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is also a finite projective left module [MRLC, Lemma 2.12] with inner product ${}_{\mathcal{B}}\langle \omega \mid \rho \rangle = \langle \omega^\dagger \mid \rho^\dagger \rangle_{\mathcal{B}}$. Thus all tensor powers $T^k_{\mathcal{D}}(\mathcal{B})$ carry right and left inner products. In particular, the inner product on $T^2_{\mathcal{D}}(\mathcal{B})$ is given by

$$\langle \omega \otimes \rho \mid \eta \otimes \tau \rangle_{\mathcal{B}} = \langle \rho \mid \langle \omega \mid \eta \rangle_{\mathcal{B}} \tau \rangle_{\mathcal{B}}.$$

Inner products on $\Omega^k_{\mathcal{D}}(\mathcal{B})$ do not arise automatically, but see [MRLC, Lemma 3.5].

The two inner products on $\Omega^1_{\mathcal{D}}(\mathcal{B})$ give rise to equivalent norms on $\Omega^1_{\mathcal{D}}(\mathcal{B})$, and using results of [KPW], $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is a bi-Hilbertian bimodule of finite index. To explain what this means for us, recall [FL02] that a (right) frame for $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is a (finite) collection of elements (ω_j) that satisfy

$$\rho = \sum_j \omega_j \langle \omega_j \mid \rho \rangle_{\mathcal{B}}$$

for all $\rho \in \Omega^1_{\mathcal{D}}(\mathcal{B})$. A finite projective bi-Hilbertian module has a "line element" or "quantum metric" [BM2] given by

$$G = \sum_j \omega_j \otimes \omega_j^\dagger. \tag{2.4}$$

The line element G is independent of the choice of frame, is central, meaning that $bG = Gb$ for all $b \in \mathcal{B}$, and

$$\mathrm{span}_{\mathcal{B}}\Big\{ \sum_j \omega_j \otimes \omega_j^\dagger : \text{for any frame } (\omega_j) \Big\}$$

is a complemented submodule of $T^2_{\mathcal{D}}$. The inner product is computed via

$$-\, g(\omega \otimes \rho) := \langle G \mid \omega \otimes \rho \rangle_{\mathcal{B}} = \langle \omega^\dagger \mid \rho \rangle_{\mathcal{B}}. \tag{2.5}$$

Such bilinear inner products appear in [BM2, BGJ2, BGJ1]. The element

$$e^\beta := \sum_j {}_{\mathcal{B}}\langle \omega_j \mid \omega_j \rangle = -g(G) \in \mathcal{B} \tag{2.6}$$

is independent of the choice of right frame, and is central, positive and invertible (provided the left action of $\mathcal{B}$ on $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is faithful [KPW, Corollary 2.28]), and we define the normalised version of G by $Z = e^{-\beta/2} \sum_j \omega_j \otimes \omega_j^\dagger$.

2.2 Hermitian torsion-free connections

Our standard references for connections on modules are [BM2, Lan]. A right connection on a right $\mathcal{B}$-module $\mathcal{X}$ is a $\mathbb{C}$-linear map

$$\overrightarrow{\nabla} : \mathcal{X} \to \mathcal{X} \otimes_{\mathcal{B}} \Omega^1_{\mathcal{D}}, \quad \text{such that} \quad \overrightarrow{\nabla}(xa) = \overrightarrow{\nabla}(x)a + x \otimes [\mathcal{D}, a].$$

There is a similar definition for left connections on left modules. Connections always exist on finite projective modules, [Lan].

Given a right inner product $\mathcal{B}$-module $\mathcal{X}$ we have sesquilinear $\Omega^1_{\mathcal{D}}$-valued pairings

$$\begin{aligned} \mathcal{X}\otimes_{\mathcal{B}}\Omega^1_{\mathcal{D}}\times\mathcal{X}\to\Omega^1_{\mathcal{D}}, &\quad \langle x\otimes \mathrm{d}(b)\mid y\rangle_{\mathcal{B}}=\mathrm{d}(b)^*\langle x\mid y\rangle_{\mathcal{B}},\\ \mathcal{X}\times\mathcal{X}\otimes_{\mathcal{B}}\Omega^1_{\mathcal{D}}\to\Omega^1_{\mathcal{D}}, &\quad \langle x\mid y\otimes \mathrm{d}(b)\rangle_{\mathcal{B}}=\langle x\mid y\rangle_{\mathcal{B}}\,\mathrm{d}(b). \end{aligned} \tag{2.7}$$

A connection $\overset{\rightarrow}{\nabla}$ on a right inner product $\mathcal{B}$-module $\mathcal{X}$ is Hermitian [Lan, Equation (7.42)], [MRLC, Definition 2.23] if for all $x, y\in\mathcal{X}$ we have

$$-\langle\overset{\rightarrow}{\nabla}x\mid y\rangle_{\mathcal{B}}+\langle x\mid\overset{\rightarrow}{\nabla}y\rangle_{\mathcal{B}}=[\mathcal{D},\langle x\mid y\rangle_{\mathcal{B}}].$$

For left connections we instead require (with the pairings (2.7) appropriately modified)

$${}_{\mathcal{B}}\langle\overset{\leftarrow}{\nabla}x\mid y\rangle-{}_{\mathcal{B}}\langle x\mid\overset{\leftarrow}{\nabla}y\rangle=[\mathcal{D},{}_{\mathcal{B}}\langle x\mid y\rangle].$$

If furthermore $\mathcal{X}$ is a $\dagger$-bimodule [MRLC, Definition 2.8] such as $\mathcal{X}=T^k_{\mathcal{D}}$, then for each right connection $\overset{\rightarrow}{\nabla}$ on $\mathcal{X}$ there is a conjugate left connection $\overset{\leftarrow}{\nabla}$ given by $\overset{\leftarrow}{\nabla}=-\dagger\circ\overset{\rightarrow}{\nabla}\circ\dagger$ which is Hermitian if and only if $\overset{\rightarrow}{\nabla}$ is Hermitian.

Example 2.3. Given a (right) frame $v=(x_j)\subset\mathcal{X}$ for a $\dagger$-bimodule $\mathcal{X}$ we obtain left- and right Grassmann connections via

$$\overset{\leftarrow}{\nabla}{}^v(x):=[\mathcal{D},{}_{\mathcal{B}}\langle x\mid x_j^\dagger\rangle]\otimes x_j^\dagger,\quad \overset{\rightarrow}{\nabla}{}^v(x):=x_j\otimes[\mathcal{D},\langle x_j\mid x\rangle_{\mathcal{B}}],\qquad x\in\mathcal{X}.$$

The Grassmann connections are Hermitian and conjugate, that is $\overset{\leftarrow}{\nabla}{}^v=-\dagger\circ\overset{\rightarrow}{\nabla}{}^v\circ\dagger$. A pair of conjugate connections on $\mathcal{X}$ are both Hermitian if and only if for any right frame (x_j) [MRLC, Proposition 2.31]

$$\overset{\rightarrow}{\nabla}(x_j)\otimes x_j^\dagger+x_j\otimes\overset{\leftarrow}{\nabla}(x_j^\dagger)=0. \tag{2.8}$$

The differential (2.3) allows us to ask whether a connection on $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is torsion-free, meaning [MRLC, Section 4.1] that for any frame

$$1\otimes(1-\Psi)\big(\overset{\rightarrow}{\nabla}(\omega_j)\otimes\omega_j^\dagger+\omega_j\otimes\mathrm{d}_\Psi(\omega_j^\dagger)\big)=0.$$

For a Hermitian right connection, being torsion-free is equivalent to $(1-\Psi)\circ\overset{\rightarrow}{\nabla}=-\mathrm{d}_\Psi$. For the conjugate left connection this becomes $(1-\Psi)\circ\overset{\leftarrow}{\nabla}=\mathrm{d}_\Psi$, [MRLC, Definition 4.3].

Definition 2.4. A frame (ω_j) for $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is *exact* if there exist $b_j\in\mathcal{B}$ such that $\omega_j=[\mathcal{D},b_j]$.

The existence result we will employ for Hermitian torsion-free connections is tied to the existence of an exact, or even closed, frame for the one-forms.

Theorem 2.5. *[MRLC, Corollary A.3, Corollary A.4] Let $(\Omega^1_{\mathcal{D}},\dagger,\Psi,\langle\cdot|\cdot\rangle_{\mathcal{B}})$ be an Hermitian differential structure. Suppose there is an exact frame $v=(\omega_j)$ for $\Omega^1_{\mathcal{D}}$. Then for the Grassmann connection $\overset{\rightarrow}{\nabla}{}^v:\Omega^1_{\mathcal{D}}\to\Omega^1_{\mathcal{D}}\otimes_{\mathcal{B}}\Omega^1_{\mathcal{D}}$ of the frame (ω_j) there is an equality of $\mathcal{B}$-bimodules*

$$JT^2_{\mathcal{D}}=\mathcal{B}\cdot\overset{\rightarrow}{\nabla}{}^v(\mathrm{d}(\mathcal{B}))\cdot\mathcal{B}. \tag{2.9}$$

Moreover the Grassmann connection $\overset{\rightarrow}{\nabla}{}^v$ is Hermitian and torsion-free.

For uniqueness, which we discuss in the Appendix, we need the definition of a special kind of bimodule connection.

Definition 2.6. Suppose that $\sigma : T^2_{\mathcal{D}}(\mathcal{B}) \to T^2_{\mathcal{D}}(\mathcal{B})$ is an invertible bimodule map such that $\dagger \circ \sigma = \sigma^{-1} \circ \dagger$ and such that the conjugate connections $\overrightarrow{\nabla}, \overleftarrow{\nabla}$ satisfy

$$\sigma \circ \overrightarrow{\nabla} = \overleftarrow{\nabla}.$$

Then we say that σ is a braiding and that $(\overrightarrow{\nabla}, \sigma)$ is a $\dagger$-bimodule connection.

Under some further technical assumptions on the Hermitian differential structure [MRLC, Sections 4,5], an Hermitian torsion-free σ-bimodule connection is unique if it exists. See Theorem A.7 in the Appendix for details on uniqueness.

2.3 Curvature and Weitzenböck formula

Given a second order differential structure, the curvature of connections can be defined in the well-known algebraic manner. For Hermitian differential structures, can also define Ricci- and scalar curvature. Curvature and the square of the Dirac operator are related via a Weitzenböck formula on a class of Dirac spectral triples which encode the key features of Dirac bundles over manifolds.

Definition 2.7. Given a second order differential structure $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger, \Psi)$ we set

$$\Lambda^2_{\mathcal{D}}(\mathcal{B}) := (1 - \Psi) T^2_{\mathcal{D}}(\mathcal{B}).$$

To define curvature, we require the second order differential $\mathrm{d}_\Psi : \Omega^1_{\mathcal{D}} \to \Lambda^2_{\mathcal{D}}$, but not any higher degree forms. Then the classical definition of curvature is available.

Definition 2.8. If $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger, \Psi)$ is a second order differential structure, define the curvature of any right connection $\overrightarrow{\nabla}$ on a finite projective right module $\mathcal{X}_{\mathcal{B}}$ to be

$$R^{\overrightarrow{\nabla}}(x) = (1 \otimes (1 - \Psi)) \circ (\overrightarrow{\nabla} \otimes 1 + 1 \otimes \mathrm{d}_\Psi) \circ \overrightarrow{\nabla}(x) \in \mathcal{X} \otimes_{\mathcal{B}} \Lambda^2_{\mathcal{D}}(\mathcal{B}), \qquad x \in \mathcal{X}.$$

For a connection $\overleftarrow{\nabla}$ on a left module ${}_{\mathcal{B}}\mathcal{X}$ we define the curvature to be

$$R^{\overleftarrow{\nabla}}(x) = ((1 - \Psi) \otimes 1) \circ (1 \otimes \overleftarrow{\nabla} - \mathrm{d}_\Psi \otimes 1) \circ \overleftarrow{\nabla}(x) \in \Lambda^2_{\mathrm{d}}(\mathcal{B}) \otimes_{\mathcal{B}} \mathcal{X}, \qquad x \in \mathcal{X}.$$

Lemma 2.9. *If $v = (x_j)$ is a right frame for $\mathcal{X}$ and $\overrightarrow{\nabla}^{\mathcal{X}} = \overrightarrow{\nabla}^{v}$ the associated Grassmann connection then*

$$R^{\overrightarrow{\nabla}^{\mathcal{X}}}(x) = 1 \otimes (1 - \Psi)\Big(x_k \otimes [\mathcal{D}, \langle x_k \mid x_j \rangle_{\mathcal{B}}] \otimes [\mathcal{D}, \langle x_j \mid x_p \rangle_{\mathcal{B}}] \langle x_p \mid x \rangle_{\mathcal{B}}\Big)$$

Similarly if $v = (x_j)$ is a left frame for a left module $\mathcal{X}$ and $\overleftarrow{\nabla}^{\mathcal{X}} = \overleftarrow{\nabla}^{v}$ then

$$R^{\overleftarrow{\nabla}^{\mathcal{X}}}(x) = (1 - \Psi) \otimes 1\Big({}_{\mathcal{B}}\langle x \mid x_p \rangle [\mathcal{D}, {}_{\mathcal{B}}\langle x_p \mid x_j \rangle] \otimes [\mathcal{D}, {}_{\mathcal{B}}\langle x_j \mid x_k \rangle] \otimes x_k\Big).$$

If $\mathcal{X}$ is a †-bimodule, the corresponding elements in $\mathcal{X}\otimes_{\mathcal{B}}\Lambda^2_{\mathcal{D}}\otimes_{\mathcal{B}}\mathcal{X}$ are

$$R^{\overrightarrow{\nabla}^{\mathcal{X}}} = x_k\otimes(1-\Psi)\left([\mathcal{D},\langle x_k\mid x_j\rangle_{\mathcal{B}}]\otimes[\mathcal{D},\langle x_j\mid x_p\rangle_{\mathcal{B}}]\right)\otimes x_p^\dagger,$$

and

$$R^{\overleftarrow{\nabla}^{\mathcal{X}}} = x_p^\dagger\otimes(1-\Psi)\left([\mathcal{D},{}_{\mathcal{B}}\langle x_p\mid x_j\rangle]\otimes[\mathcal{D},{}_{\mathcal{B}}\langle x_j\mid x_k\rangle]\right)\otimes x_k.$$

Proof. This follows directly from [MRC, Proposition 3.3]. □

Definition 2.10. Let $(\Omega^1_{\mathcal{D}}(\mathcal{B}),\dagger,\Psi,\langle\cdot|\cdot\rangle)$ be a Hermitian differential structure and $\overrightarrow{\nabla}$ a right connection on $\Omega^1_{\mathcal{D}}$ with curvature $R^{\overrightarrow{\nabla}}\in\Omega^1_{\mathcal{D}}(\mathcal{B})\otimes\Lambda^2_{\mathcal{D}}(\mathcal{B})\otimes\Omega^1_{\mathcal{D}}(\mathcal{B})$. The Ricci curvature of $\overrightarrow{\nabla}$ is

$$\mathrm{Ric}^{\overrightarrow{\nabla}} = {}_{\mathcal{B}}\langle R^{\overrightarrow{\nabla}}\mid G\rangle\in T^2_{\mathcal{D}}(\mathcal{B})$$

and the scalar curvature is

$$r^{\overrightarrow{\nabla}} = \langle G\mid \mathrm{Ric}^{\overrightarrow{\nabla}}\rangle_{\mathcal{B}}.$$

These definitions mirror those of [BM2] and references therein, and agree when both apply.

We now recall from [MRC] a class of spectral triples for which the Weitzenböck formula holds. Given a left inner product module $\mathcal{X}$ and a positive functional $\phi:\mathcal{B}\to\mathbb{C}$, the Hilbert space $L^2(\mathcal{X},\phi)$ is the completion of $\mathcal{X}$ in the scalar product $\langle x,y\rangle:=\phi({}_{\mathcal{B}}\langle x\mid y\rangle)$.

Definition 2.11. Let $(\mathcal{B},\mathcal{H},\mathcal{D})$ be a spectral triple equipped with a braided Hermitian differential structure $(\Omega^1_{\mathcal{D}}(\mathcal{B}),\dagger,\Psi,\langle\cdot|\cdot\rangle,\sigma)$. Then $(\mathcal{B},\mathcal{H},\mathcal{D})$ is a *Dirac spectral triple* relative to $(\Omega^1_{\mathcal{D}}(\mathcal{B}),\dagger,\Psi,\langle\cdot\mid\cdot\rangle,\sigma)$ if

1. for $\omega,\eta\in\Omega^1_{\mathcal{D}}(\mathcal{B})$ we have
$$(m\circ\Psi)(\rho\otimes\eta) = e^{-\beta}m(G)\langle\rho^\dagger\mid\eta\rangle_{\mathcal{B}} = -e^{-\beta}m(G)g(\rho\otimes\eta); \tag{2.10}$$
2. there is a left inner product module $\mathcal{X}$ over $\mathcal{B}$ and a positive functional $\phi:\mathcal{B}\to\mathbb{C}$ such that $\mathcal{H}=L^2(\mathcal{X},\phi)$ and the natural map $c:\Omega^1_{\mathcal{D}}(\mathcal{B})\otimes_{\mathcal{B}}L^2(\mathcal{X},\phi)\to L^2(\mathcal{X},\phi)$ restricts to a map $c:\Omega^1_{\mathcal{D}}(\mathcal{B})\otimes_{\mathcal{B}}\mathcal{X}\to\mathcal{X}$;
3. there is a left connection $\overleftarrow{\nabla}^{\mathcal{X}}:\mathcal{X}\to\Omega^1_{\mathcal{D}}(\mathcal{B})\otimes_{\mathcal{B}}\mathcal{X}$ such that $\mathcal{D}=c\circ\overleftarrow{\nabla}^{\mathcal{X}}:\mathcal{X}\to L^2(\mathcal{X},\phi)$;
4. there is an Hermitian torsion free †-bimodule connection $(\overrightarrow{\nabla}^G,\sigma)$ on $\Omega^1_{\mathcal{D}}$ such that
$$\mathcal{D}(\omega x) = c\circ\overleftarrow{\nabla}^{\mathcal{X}}(c(\omega\otimes x)) = c\circ(m\circ\sigma\otimes 1)(\overrightarrow{\nabla}^G(\omega)\otimes x+\omega\otimes\overleftarrow{\nabla}^{\mathcal{X}}(x)) \tag{2.11}$$

We note that for spectral triples of Dirac-type operators on Riemannian manifolds we have $e^{-\beta}m(G)=\mathrm{Id}$, [MRC, Lemma 4.4]. A spectral triple satisfying the conditions has a natural connection Laplacian $\Delta^{\mathcal{X}}$, [MRC, Definition 4.3], defined on $x\in\mathcal{X}$ by

$$\Delta^{\mathcal{X}}(x) = e^{-\beta}m(G)\langle G\mid(\overrightarrow{\nabla}^G\otimes 1+1\otimes\overleftarrow{\nabla}^{\mathcal{X}})\circ\overleftarrow{\nabla}^{\mathcal{X}}(x)\rangle_{\mathcal{B}}, \tag{2.12}$$

and there is a Weitzenböck formula relating $\mathcal{D}^2$ and $\Delta^{\mathcal{X}}$.

Theorem 2.12. *Let $(\mathcal{B}, L^2(\mathcal{X}, \phi), \mathcal{D})$ be a Dirac spectral triple relative to the braided Hermitian differential structure $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger, \Psi, \langle\cdot\mid\cdot\rangle, \sigma)$, and let $\Delta^{\mathcal{X}}$ be the connection Laplacian of the left connection $\overleftarrow{\nabla}^{\mathcal{X}}$. If $m\circ\sigma\circ\Psi = m\circ\Psi$ and $\Psi(G) = G$ then*

$$\mathcal{D}^2(x) = \Delta^{\mathcal{X}}(x) + c\circ(m\circ\sigma\otimes 1)\big(R^{\overleftarrow{\nabla}^{\mathcal{X}}}(x)\big). \tag{2.13}$$

In Section 5 we will check that these conditions hold for the Podleś sphere spectral triple, and then derive the Weitzenböck formula in that case. We also use [MRC, Corollary 4.9] to show that $\Delta^{\mathcal{X}} \geq 0$.

3 The Levi-Civita connection for the Podleś sphere

For the remainder of this paper we will study connections on the one-forms $\Omega^1_{\mathcal{D}}(\mathcal{B}) = \{\sum a^i[\mathcal{D}, b^i] : a^i, b^i \in \mathcal{B}\}$ of a specific spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D})$. Here $\mathcal{B}$ is the (coordinate algebra of) the Podleś sphere, which we describe along with the spectral triple below. In the Appendix we will require a completion of $\mathcal{B}$ to use the framework of [MRLC] to address the uniqueness of the algebraic Hermitian torsion-free connection we construct below.

3.1 Coordinates, spectral triple and inner product for the Podleś sphere

We start with the polynomial algebra $\mathcal{A} := \mathcal{O}(SU_q(2))$ on quantum $SU(2)$ spanned by the matrix elements t^l_{ij}, with $l \in \frac{1}{2}\mathbb{N}$ and $i, j \in \{-l, -l+1, \ldots, l-1, l\}$. While the coproduct

$$\Delta(t^l_{ij}) = \sum_k t^l_{ik}\otimes t^l_{kj},$$

is easy to describe in this picture, the product involves (quantum) Clebsch-Gordan coefficients. We summarise the basic algebraic relations we require using the conventions of [RS]. The generators and relations of $\mathcal{O}(SU_q(2))$ are

$$\begin{gathered} ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb \\ ad = 1 + qbc, \quad da = 1 + q^{-1}bc, \end{gathered} \tag{3.1}$$

with adjoints

$$a^* = d, \quad b^* = -qc, \quad c^* = -q^{-1}b, \quad d^* = a. \tag{3.2}$$

The relation to the matrix elements of the defining corepresentation is

$$a = t^{\frac12}_{-\frac12,-\frac12}, \quad b = t^{\frac12}_{-\frac12,\frac12}, \quad c = t^{\frac12}_{\frac12,-\frac12}, \qquad d = t^{\frac12}_{\frac12,\frac12}. \tag{3.3}$$

The C^*-algebra $C(SU_q(2))$ and polynomial algebra $\mathcal{O}(SU_q(2))$ carry a one parameter group of automorphisms which for $s \in \mathbb{R}$ is given by $U_s(t^l_{ij}) = q^{\sqrt{-1}sj}t^l_{ij}$. The one parameter group is periodic and thus gives an action of the circle $\mathbb{T}$. The fixed point C^*-algebra $C(SU_q(2))^{\mathbb{T}}$ for this circle action is the C^*-algebra $C(S^2_q)$ of the Podleś sphere,

$$C(S^2_q) := C(SU_q(2))^{\mathbb{T}} \subset C(SU_q(2)).$$

For future reference we recall the definition of the q-numbers

$$[x]_q = \frac{q^{-x} - q^x}{q^{-1} - q}.$$

The Podleś sphere $\mathcal{B} := \mathcal{O}(S_q^2) \subset C(S_q^2)$ is the polynomial dense $*$-subalgebra of $C(S_q^2)$ spanned by the matrix elements t^l_{i0}. The generators of the Podleś sphere are

$$\begin{aligned} A &= -q^{-1}bc = c^*c = t^{1/2*}_{1/2,-1/2}t^{1/2}_{1/2,-1/2} = q^{-2}t^{1/2}_{-1/2,1/2}t^{1/2*}_{-1/2,1/2} = -q^{-1}[2]_q^{-1}t^1_{00} \\ B &= ac^* = -q^{-1}ab = t^{1/2}_{-1/2,-1/2}t^{1/2*}_{1/2,-1/2} = -q^{-1/2}[2]_q^{-1/2}t^1_{-10} \\ B^* &= cd = t^{1/2}_{1/2,-1/2}t^{1/2}_{1/2,1/2} = q^{-1/2}t^{1/2}_{1/2,-1/2}t^{1/2*}_{-1/2,-1/2} = [2]_q^{-1/2}q^{1/2}t^1_{10}. \end{aligned}$$

The equalities with matrix elements t^1_{k0} come from Clebsch-Gordan relations summarised in [S, Appendix A]. The generators A, B, B^* obey the relations

$$BA = q^2AB, \quad AB^* = q^2B^*A, \quad B^*B = A - A^2, \quad BB^* = q^2A - q^4A^2.$$

The spinor module $S = S^+ \oplus S^-$ is realised as the direct sum of the finitely generated projective modules $S^\pm := P_\pm C(S_q^2)^{\oplus 2}$. In our conventions [S, Section 4.3] the projections $P_\pm$ are given by

$$P_+ = \begin{pmatrix} t^{1/2}_{1/2,1/2}t^{1/2*}_{1/2,1/2} & t^{1/2}_{1/2,1/2}t^{1/2*}_{-1/2,1/2} \\ t^{1/2}_{-1/2,1/2}t^{1/2*}_{1/2,1/2} & t^{1/2}_{-1/2,1/2}t^{1/2*}_{-1/2,1/2} \end{pmatrix} = \begin{pmatrix} 1-A & -B^* \\ -B & q^2A \end{pmatrix},$$

and

$$P_- = 1 - P_+ = \begin{pmatrix} t^{1/2}_{1/2,-1/2}t^{1/2*}_{1/2,-1/2} & t^{1/2}_{1/2,-1/2}t^{1/2*}_{-1/2,-1/2} \\ t^{1/2}_{-1/2,-1/2}t^{1/2*}_{1/2,-1/2} & t^{1/2}_{-1/2,-1/2}t^{1/2*}_{-1/2,-1/2} \end{pmatrix} = \begin{pmatrix} A & B^* \\ B & 1-q^2A \end{pmatrix}.$$

Observe that $P_+ + P_- = \mathrm{Id}_2$ and $P_+P_- = 0$. Hence $\mathrm{End}_{C(S_q^2)}(S^+ \oplus S^-) \cong M_2(C(S_q^2))$.

The (smooth sections of the) spinor bundle over the Podleś sphere is the module

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^- = \mathrm{span}\{(t^l_{i,1/2}b_+, t^l_{i,-1/2}b_-)^T : b_\pm \in \mathcal{B}\},$$

with (right) $\mathcal{B}$-valued inner product

$$\langle (w_+, w_-)^T \mid (z_+, z_-)^T \rangle_{\mathcal{B}} = w_+^*z_+ + w_-^*z_-.$$

The formula for the multiplication in terms of Clebsch-Gordan coefficients shows that $\mathcal{S}$ is also a left $\mathcal{B}$-module. Together with the Haar state $h : C(SU_q(2)) \to \mathbb{C}$ we can then build a Hilbert space $\mathcal{H} = L^2(S, h)$ which carries a left representation of $\mathcal{B}$ (see [S] for formulae in our conventions).

The natural Dirac operator yielding a spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D})$ is [DS, NT]

$$\mathcal{D} = \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix}$$

where $\partial_e t^l_{i,j} = \sqrt{[l+1/2]_q^2 - [j+1/2]_q^2}\; t^l_{i,j+1}$ and $\partial_f t^l_{i,j} = \sqrt{[l+1/2]_q^2 - [j-1/2]_q^2}\; t^l_{i,j-1}$.

We will abbreviate the coefficients in ∂_e, ∂_f as $\kappa^l_k = \sqrt{[l+1/2]_q^2 - [k-1/2]_q^2}$.

With $\partial_k(t^l_{ij}) := q^j t^l_{ij}$, and $a, b \in \mathcal{O}(SU_q(2))$ we have

$$\partial_e(ab) = \partial_e(a)\partial_k(b) + \partial_k^{-1}(a)\partial_e(b) \qquad \partial_f(ab) = \partial_f(a)\partial_k(b) + \partial_k^{-1}(a)\partial_f(b). \tag{3.4}$$

For $b \in \mathcal{B}$ we have $\partial_e\partial_f(b) = \partial_f\partial_e(b)$. For $b \in \mathcal{B}$, the commutator of $\mathcal{D}$ with the left multiplication by b is

$$\mathrm{d}(b) := [\mathcal{D}, b] = \begin{pmatrix} 0 & q^{-1/2}\partial_e(b) \\ q^{1/2}\partial_f(b) & 0 \end{pmatrix}. \tag{3.5}$$

We denote by $\mathcal{C}_\mathcal{D}(\mathcal{B}) \subset \mathbb{B}(\mathcal{H})$ the "Clifford algebra" generated by $\mathcal{B}$ and commutators $[\mathcal{D}, \mathcal{B}]$. We define an operator-valued weight $\Phi : \mathcal{C}_\mathcal{D}(\mathcal{B}) \to \mathcal{B}$ by

$$\Phi(\rho) = \mathrm{Tr}\left(\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}\rho\right).$$

The functional Φ gives a right inner product $\langle \rho \,|\, \eta\rangle_\mathcal{B} := \Phi(\rho^*\eta)$ on the Clifford algebra which restricts to one-forms as

$$\begin{aligned} \langle [\mathcal{D}, b_1]a_1 \,|\, [\mathcal{D}, b_2]a_2\rangle_\mathcal{B} &= a_1^* \,\mathrm{Tr}\left(\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}[\mathcal{D}, b_1]^\dagger[\mathcal{D}, b_2]\right) a_2 \\ &= a_1^*(q^2\partial_f(b_1)^*\partial_f(b_2) + q^{-2}\partial_e(b_1)^*\partial_e(b_2))a_2. \end{aligned} \tag{3.6}$$

Remark 3.1. This metric reduces to the round metric on S^2 for $q = 1$. Our main reason for this choice of metric is that, like the classical round metric, the metric (3.6) enjoys an exact frame, see Lemma 3.4. The inner product is a right $SU_q(2)$-comodule map, see remark 3.6 below.

Remark 3.2. For $q = 1$ we should consider $\Phi(T) = \frac{1}{2}\mathrm{Tr}(T)$ for $T \in \mathrm{End}^*_\mathcal{B}(S)$ so that $\Phi(b) = b$ for $b \in \mathcal{B}$. This is to access the actual (inverse) metric via $\Phi((\mathrm{d}x^\mu)^*\mathrm{d}x^\nu) = g^{\mu\nu}$. This means that our current definition scales the inverse metric by 2, and so the metric $g_{\mu\nu}$ by $\frac{1}{2}$. This is compatible with using the Dolbeault Laplacian $(\sqrt{2}(\partial + \bar\partial))^2$, [H05, Section 3.1]. The combined effect of not normalising Φ and using the Dolbealt Dirac cancel out, and we will see that the inner product on one-forms corresponds to the metric of radius 1 when $q = 1$.

We recall the following facts as we will use them extensively in the computations to come:

$$(t^l_{ij})^* = (-q)^{j-i}t^l_{-i,-j} \quad \text{and} \quad t^l_{ij} = (-q)^{j-i}(t^l_{-i,-j})^*. \tag{3.7}$$

$$\delta_{ij} = \sum_{p=-l}^{l}(t^l_{pi})^*t^l_{pj} = \sum_{p=-l}^{l} t^l_{ip}(t^l_{jp})^* \quad \text{Orthogonality relations.} \tag{3.8}$$

$$\kappa^1_1 = \kappa^1_0 = [2]_q^{1/2} = \sqrt{q+q^{-1}} \qquad \kappa^l_1 = \kappa^l_0. \tag{3.9}$$

Lemma 3.3. *Using the spectral triple* $(\mathcal{B}, \mathcal{H}, \mathcal{D})$, *we define* $\mathrm{d} : \mathcal{B} \to \Omega^1_\mathcal{D}(\mathcal{B})$ *as* $\mathrm{d}(b) = [\mathcal{D}, b]$. *Then* $\Omega^1_\mathcal{D}(\mathcal{B})$ *is a first order differential structure and a* $\dagger$*-bimodule.*

3.2 Frame and quantum metric for the Podleś sphere

We now define a finite exact frame for $\Omega^1_{\mathcal{D}}(\mathcal{B})$. We will use this frame in all our computations.

Lemma 3.4. *A frame for the right $\mathcal{B}$-module $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is given by the three module elements*

$$\omega_j = q^{-2+j}(\kappa^1_1)^{-1}[\mathcal{D}, t^1_{-2+j,0}] = q^{-2+j}\begin{pmatrix} 0 & q^{-1/2}t^1_{-2+j,1} \\ q^{1/2}t^1_{-2+j,-1} & 0 \end{pmatrix} \tag{3.10}$$

$$= (-1)^{1-j}\begin{pmatrix} 0 & q^{1/2}(t^1_{2-j,-1})^* \\ q^{-1/2}(t^1_{2-j,1})^* & 0 \end{pmatrix}, \qquad j = 1, 2, 3. \tag{3.11}$$

Proof. We first compute $\omega_j^\dagger$ using the formula for adjoints (3.7) and the orthogonality relations (3.8), finding that

$$\omega_j^\dagger = (-1)^{1-j}\begin{pmatrix} 0 & q^{-1/2}t^1_{2-j,1} \\ q^{1/2}t^1_{2-j,-1} & 0 \end{pmatrix}. \tag{3.12}$$

Then for $\rho = \begin{pmatrix} 0 & \rho_+ \\ \rho_- & 0 \end{pmatrix} \in \Omega^1_{\mathcal{D}}(\mathcal{B})$ the orthogonality relations (3.8) and definition of the adjoints (3.7) yield

$$\sum_j \omega_j \langle \omega_j \mid \rho\rangle_{\mathcal{B}} = \sum_j \omega_j q^{-2+j}(q^{3/2}(t^1_{-2+j,-1})^*\rho_- + q^{-3/2}(t^1_{-2+j,1})^*\rho_+)$$

$$= \sum_j q^{-4+2j}\begin{pmatrix} 0 & qt^1_{-2+j,1}(t^1_{-2+j,-1})^*\rho_- + q^{-2}t^1_{-2+j,1}(t^1_{-2+j,1})^*\rho_+ \\ q^2t^1_{-2+j,-1}(t^1_{-2+j,-1})^*\rho_- + q^{-1}t^1_{-2+j,-1}(t^1_{-2+j,1})^*\rho_+ & 0 \end{pmatrix}$$

$$= \sum_j q^{-4+2j}\begin{pmatrix} 0 & q(-q)^{4-2j}(t^1_{2-j,-1})^*t^1_{2-j,1}\rho_- + q^{-2}(-q)^{6-2j}(t^1_{2-j,-1})^*t^1_{2-j,-1}\rho_+ \\ q^2(-q)^{2-2j}(t^1_{2-j,1})^*t^1_{2-j,1}\rho_- + q^{-1}(-q)^{4-2j}(t^1_{2-j,1})^*t^1_{2-j,-1}\rho_+ & 0 \end{pmatrix}$$

$$= \sum_j \begin{pmatrix} 0 & q(t^1_{2-j,-1})^*t^1_{2-j,1}\rho_- + (t^1_{2-j,-1})^*t^1_{2-j,-1}\rho_+ \\ (t^1_{2-j,1})^*t^1_{2-j,1}\rho_- + q^{-1}(t^1_{2-j,1})^*t^1_{2-j,-1}\rho_+ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \rho_+ \\ \rho_- & 0 \end{pmatrix} = \rho.$$

□

Corollary 3.5. *The Watatani index $e^\beta := \sum_j {}_{\mathcal{B}}\langle \omega_j \mid \omega_j\rangle$ [KPW] of $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is $e^\beta := q^2 + q^{-2}$ and the line element (or quantum metric [BM2]) is*

$$G = \sum_j \omega_j \otimes \omega_j^\dagger = \sum_j q\begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} + q^{-1}\begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix}.$$

Normalising gives $Z := e^{-\beta/2}G$ and $z := m(Z) = e^{-\beta/2}\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$.

Proof. The first statement comes from Equations (3.11) and (3.12) for the frame elements, and the orthogonality relations (3.8). For instance, the cross-term vanishes because for any $j, k = 1, 2, 3$, we have $t^1_{2-k,-1}(t^1_{2-j,-1})^* \in \mathcal{B}$ and so

$$\sum_j \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} = \sum_{j,k} \begin{pmatrix} 0 & (t^1_{2-k,-1})^*t^1_{2-k,-1}(t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix}$$

$$= \sum_{j,k} \begin{pmatrix} 0 & (t^1_{2-k,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-k,-1}(t^1_{2-j,-1})^* t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} = 0, \tag{3.13}$$

the last line following from the orthogonality relations. The other cross-term vanishes similarly. Applying the multiplication map gives

$$\sum_j \omega_j \omega_j^\dagger = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix},$$

and applying Φ gives $\sum_j {}_{\mathcal{B}}\langle \omega_j \mid \omega_j \rangle = q^2 + q^{-2}$. □

Observe that $\langle Z \mid Z \rangle^{T^2_{\mathcal{D}}}_{\mathcal{B}} = 1$, so that $|Z\rangle\langle Z|$ is the orthogonal projection onto span G.

Using the orthogonality relations as in (3.13), we see that

$$\sum_i t^l_{j1}(t^l_{i1})^* [\mathcal{D}, t^l_{i0}] = q^{-2+i-1/2} \begin{pmatrix} 0 & t^l_{j1} \\ 0 & 0 \end{pmatrix}, \tag{3.14}$$

and similarly

$$\sum_i t^l_{j,-1}(t^l_{i,-1})^* [\mathcal{D}, t^l_{i0}] = q^{-2+i+1/2} \begin{pmatrix} 0 & 0 \\ t^l_{j,-1} & 0 \end{pmatrix}. \tag{3.15}$$

Remark 3.6. One can now deduce the well-known result that $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is a direct sum of two noncommutative line bundles given by the ± 1 spectral subspaces of the circle action defining $\mathcal{B}$ (see [AAL, AKL]). Indeed, $\mathcal{O}(SU_q(2)) = \oplus_{k \in \frac{1}{2}\mathbb{Z}} \mathcal{L}_k$ is a sum of line bundles and we have shown that $\Omega^1_{\mathcal{D}}(\mathcal{B}) = \mathcal{L}_1 \oplus \mathcal{L}_{-1}$, and by definition $S_\pm = \mathcal{L}_{\pm 1/2}$. We deduce from Equations (3.14) and (3.15) that the inner product (3.6) is a linear combination of the standard inner products on the line bundles $\mathcal{L}_1, \mathcal{L}_{-1}$ where $\Omega^1_{\mathcal{D}}(\mathcal{B}) = \mathcal{L}_1 \oplus \mathcal{L}_{-1}$. Therefore, the metric is an $SU_q(2)$-comodule map.

3.3 Existence of an Hermitian torsion-free connection

Since our frame consists of exact one-forms (i.e. of the form $[\mathcal{D}, b]$ for $b \in \mathcal{B}$), Theorem 2.5 applies to $\Omega^1_{\mathcal{D}}(\mathcal{B})$. Hence the junk two-tensors are given by the bimodule generated by $\overrightarrow{\nabla}^G \circ \mathrm{d}(\mathcal{B})$ where $\overrightarrow{\nabla}^G$ is the Grassmann connection of the exact frame. Moreover, since $\overrightarrow{\nabla}^G$ is Hermitian, in order to establish that it is also torsion-free, it suffices to show that $JT^2_{\mathcal{D}} \subset T^2_{\mathcal{D}}$ is a complemented submodule. Thus our immediate aim is to compute the Grassmann connection on exact forms, and determine the junk bimodule and its complement.

Since $\Omega^1_{\mathcal{D}} \cong \mathcal{L}_1 \oplus \mathcal{L}_{-1}$ we find that $T^2_{\mathcal{D}} \cong \mathcal{B} \oplus \mathcal{B} \oplus \mathcal{L}_2 \oplus \mathcal{L}_{-2}$. Thus a two-tensor of the form $\overrightarrow{\nabla}^G([\mathcal{D}, b])$ will be a sum of degree 0, 2 and -2 components. In fact the splitting of $\mathcal{B} \oplus \mathcal{B}$ is not the most obvious one, and the next result picks out a component proportional to G.

Lemma 3.7. *The Grassmann connection* $\overrightarrow{\nabla}^G$ *of the frame from Lemma 3.4 applied to an exact form* $[\mathcal{D}, b]$ *is*

$$\overrightarrow{\nabla}^G([\mathcal{D},b]) = G\partial_e\partial_f(b) + \sum_{j,k}\begin{pmatrix}0 & (t^1_{2-j,-1})^*\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & t^1_{2-j,-1}(t^2_{k,-2})^*\\ 0 & 0\end{pmatrix}t^2_{k,-2}\partial_e^2(b)$$
$$+\sum_{j,k}\begin{pmatrix}0 & 0\\ (t^1_{2-j,1})^* & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ t^1_{2-j,1}(t^2_{k,2})^* & 0\end{pmatrix}t^2_{k,2}\partial_f^2(b). \tag{3.16}$$

Proof. From the proof of Lemma 3.4

$$\langle\omega_j \mid [\mathcal{D},b]\rangle_{\mathcal{B}} = q^{-2+j}\big(q^2t^{1*}_{-2+j,-1}\partial_f(b) + q^{-2}t^{1*}_{-2+j,1}\partial_e(b)\big),$$

or after computing adjoints of matrix elements

$$\langle\omega_j \mid [\mathcal{D},b]\rangle_{\mathcal{B}} = (-1)^{1-j}\big(qt^1_{2-j,1}\partial_f(b) + q^{-1}t^1_{2-j,-1}\partial_e(b)\big).$$

Using Equation (3.4), the relations $\partial_e(t^1_{m,1}) = \partial_f(t^1_{m,-1}) = 0$, and the formulae for adjoints, we get

$$\partial_e(\langle\omega_j \mid [\mathcal{D},b]\rangle_{\mathcal{B}}) = (-1)^{1-j}t^1_{2-j,1}\partial_e\partial_f(b) + (-1)^{1-j}\kappa^1_0t^1_{2-j,0}\partial_e(b) + (-1)^{1-j}t^1_{2-j,-1}\partial_e^2(b)$$

and

$$\partial_f(\langle\omega_j \mid [\mathcal{D},b]\rangle_{\mathcal{B}}) = (-1)^{1-j}t^1_{2-j,-1}\partial_f\partial_e(b) + (-1)^{1-j}\kappa^1_1t^1_{2-j,0}\partial_f(b) + (-1)^{1-j}t^1_{2-j,1}\partial_f^2(b).$$

Thus

$$[\mathcal{D},\langle\omega_j \mid [\mathcal{D},b]\rangle_{\mathcal{B}}] = \omega_j^*\partial_e\partial_f(b) + (-1)^{1-j}\kappa^1_1t^1_{2-j,0}[\mathcal{D},b] + \begin{pmatrix}0 & q^{-1/2}(-1)^{1-j}t^1_{2-j,-1}\partial_e^2(b)\\ q^{1/2}(-1)^{1-j}t^1_{2-j,1}\partial_f^2(b) & 0\end{pmatrix}.$$

Next observe that

$$(-1)^{1-j}\omega_j = \begin{pmatrix}0 & q^{-1/2}(-1)^{1-j}q^{-2+j}t^1_{-2+j,1}\\ q^{1/2}(-1)^{1-j}q^{-2+j}t^1_{-2+j,-1} & 0\end{pmatrix} = \begin{pmatrix}0 & q^{1/2}t^{1*}_{2-j,-1}\\ q^{-1/2}t^{1*}_{2-j,1} & 0\end{pmatrix}$$

and so the orthogonality relations (3.8) yield

$$\sum_j(-1)^{1-j}\omega_jt^1_{2-j,0} = 0.$$

As $\partial_e\partial_f(b) = \partial_f\partial_e(b)$ for $b\in\mathcal{B}$, and using orthogonality to remove cross terms as in (3.13) we find that $\overrightarrow{\nabla}^G([\mathcal{D},b])$ is

$$\sum_j\omega_j\otimes[\mathcal{D},\langle\omega_j \mid [\mathcal{D},b]\rangle_{\mathcal{B}}] = \sum_j\omega_j\otimes\omega_j^*\partial_e\partial_f(b) + \omega_j\otimes\begin{pmatrix}0 & q^{-1/2}(-1)^{1-j}t^1_{2-j,-1}\partial_e^2(b)\\ q^{1/2}(-1)^{1-j}t^1_{2-j,1}\partial_f^2(b) & 0\end{pmatrix}$$
$$= G\partial_e\partial_f(b) + \sum_{j,k}\begin{pmatrix}0 & (t^1_{2-j,-1})^*\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & t^1_{2-j,-1}(t^2_{k,-2})^*\\ 0 & 0\end{pmatrix}t^2_{k,-2}\partial_e^2(b)$$
$$+\sum_{j,k}\begin{pmatrix}0 & 0\\ (t^1_{2-j,1})^* & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ t^1_{2-j,1}(t^2_{k,2})^* & 0\end{pmatrix}t^2_{k,2}\partial_f^2(b).$$

the last equality following from the orthogonality relations (3.8), with $k = -2,-1,0,1,2$. □

The two families

$$\sum_j \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,-1}(t^2_{k,-2})^* \\ 0 & 0 \end{pmatrix}, \qquad \sum_j \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,1}(t^2_{k,2})^* & 0 \end{pmatrix}$$

are mutually orthogonal and are contained in $\ker(m : T^2_{\mathcal{D}} \to \Omega^2_{\mathcal{D}}(\mathcal{B}))$. Both families of two-tensors are orthogonal to the line element G as well, and we will now use them to construct a frame for $JT^2_{\mathcal{D}}$.

We may identify $\mathrm{End}_{\mathcal{B}}(\mathcal{S})$ with a subset of $M_2(\mathcal{A})$, where $\mathcal{A} \subset C(SU_q(2))$ is the coordinate algebra of $SU_q(2)$. This is done by identifying $\Omega^1_{\mathcal{D}}(\mathcal{B})$ with off-diagonal matrices, with the E_{12} component of degree 1 with respect to the circle action defining $\mathcal{B}$, and the E_{21} component of degree -1. We can also identify $\Omega^1_{\mathcal{D}} \otimes_{\mathcal{B}} \Omega^1_{\mathcal{D}}$ with sums of tensor products of off-diagonal matrices in $M_2(\mathcal{A}) \otimes_{\mathcal{B}} M_2(\mathcal{A})$, but now the degrees can be $-2, 0, 2$.

Lemma 3.8. *The $\mathcal{B}$-bimodules $X, Y \subset T^2_{\mathcal{D}}$ of degree $-2, 2$ elements respectively have frames*

$$Y_k = q \sum_j \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,-1}(t^2_{k,-2})^* \\ 0 & 0 \end{pmatrix}$$

$$X_k = q^{-1} \sum_j \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,1}(t^2_{k,2})^* & 0 \end{pmatrix}.$$

For any two-tensor $\rho \otimes \eta = \left(\begin{smallmatrix} 0 & \rho_+ \\ \rho_- & 0 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 0 & \eta_+ \\ \eta_- & 0 \end{smallmatrix}\right) \in T^2_{\mathcal{D}}$ we have

$$\langle Y_k \mid \rho \otimes \eta \rangle_{\mathcal{B}} = q t^2_{k,-2} \langle E_{12} \otimes E_{12} \mid \rho \otimes \eta \rangle_{\mathcal{B}} = q^{-1} t^2_{k,-2} \rho_+ \eta_+ \quad \text{and}$$
$$\langle X_k \mid \rho \otimes \eta \rangle_{\mathcal{B}} = q^{-1} t^2_{k,2} \langle E_{21} \otimes E_{21} \mid \rho \otimes \eta \rangle_{\mathcal{B}} = q t^2_{k,2} \rho_- \eta_- \tag{3.17}$$

where E_{ij} are standard matrix units, and on the right we take the inner product on $M_2(\mathcal{A}) \otimes_{\mathcal{B}} M_2(\mathcal{A})$ to be the inner product arising from the inner product on $M_2(\mathcal{A})$ defined by Equation (3.6). Moreover $\langle Y_k \mid X_l \rangle_{\mathcal{B}} = 0$ for each k, l. Hence the junk bimodule $JT^2_{\mathcal{D}}$ is the $\mathcal{B}$-span of G and the X_k, Y_k.

Proof. The vectors X_k, Y_k are in $T^2_{\mathcal{D}}$ by Equation (3.16), and the inner product calculations (3.17) are straightforward, as is $\langle Y_k \mid X_l \rangle = 0$. The Clebsch-Gordan relations tell us that $T^2_{\mathcal{D}}$ breaks up as homogenous components of degrees $-2, 0, 2$. Let $\rho \otimes \eta = \rho_- E_{21} \otimes \eta_- E_{21}$ be an element of degree -2. Then since $t^1_{2-j,1}\rho_- \in \mathcal{B}$ we have

$$\sum_{k=-2}^{2} X_k \langle X_k \mid \rho \otimes \eta \rangle_{\mathcal{B}}$$
$$= q^{-2} \sum_{k=-2}^{2} \sum_{j=1}^{3} \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,1}(t^2_{k,2})^* & 0 \end{pmatrix} t^2_{k,2} \langle E_{21} \otimes E_{21} \mid \rho \otimes \eta \rangle_{\mathcal{A}}$$
$$= \sum_{k=-2}^{2} \sum_{j=1}^{3} \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,1}(t^2_{k,2})^* & 0 \end{pmatrix} t^2_{k,2} \rho_- \eta_-$$

$$= \sum_{j=1}^{3} \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,1}\rho_-\eta_- & 0 \end{pmatrix}$$

$$= \sum_{j=1}^{3} \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* t^1_{2-j,1}\rho_- & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ \eta_- & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \rho_- & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ \eta_- & 0 \end{pmatrix}.$$

A similar calculation proves the result for Y_k. □

The next Proposition identifies the orthogonal complement of $JT^2_{\mathcal{D}}$. In particular, we produce a two-tensor C such that the copy (up to isomorphism) of $\mathcal{B} \oplus \mathcal{B}$ in $T^2_{\mathcal{D}}$ is provided by $\mathcal{B}$-span$(G) \oplus \mathcal{B}$-span(C). Our method for obtaining C utilises the non-trivial modular structure of the Podleś sphere and its Haar state.

The spectral triple for the Podleś sphere we are using has spectral dimension 0, and the Podleś sphere has Hochschild and cyclic homological dimensions zero. When considered as a modular spectral triple, the spectral dimensions is 2 [KW, RS], and likewise when the modular automorphism of the Haar state is used to define twisted Hochschild and cyclic homologies of the Podleś sphere, the homological dimension is 2, [H, HK, K].

We will use Chern character techniques, due to Wagner in this setting [W], to find a two-tensor which will have non-trivial twisted Hochschild class, and so provide a sensible starting point for finding a non-trivial two-form.

Proposition 3.9. *The two-tensors $T^2_{\mathcal{D}}$ can be decomposed as an orthogonal direct sum*

$$T^2_{\mathcal{D}} = X \oplus Y \oplus \mathrm{span}(G) \oplus \mathrm{span}(C)$$

where the two-tensor C satisfies that $\langle C \mid C \rangle_{\mathcal{B}} = \alpha$ is a scalar. The differential d_Ψ is defined using the projection $1 - \Psi = \frac{1}{\alpha}|C\rangle\langle C|$, and is given by

$$\mathrm{d}_\Psi(a[\mathcal{D}, b]) = \frac{q}{2} C\big(q^{-1}\partial_e(a)\partial_f(b) - q\partial_f(a)\partial_e(b)\big), \qquad a, b \in \mathcal{B}.$$

In particular, $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger, \langle \cdot \mid \cdot \rangle, \Psi)$ is an Hermitian differential structure.

Proof. To find a non-trivial non-junk two-form, we follow Wagner [W] and construct a non-trivial two-form from the twisted (by the modular group action) Chern character $\mathrm{Ch}_2(P_+)$ of the projection defining positive spinors P_+. According to [S, Section 4.3],

$$\mathrm{Ch}_2(P_+) = -2 \sum_{k_0,k_1,k_2=0}^{1} q^{-2k_0}$$
$$\Big(t^{1/2}_{1/2-k_0,1/2} t^{1/2*}_{1/2-k_1,1/2} - \tfrac{1}{2}\delta_{k_0,k_1}\Big) \otimes t^{1/2}_{1/2-k_1,1/2} t^{1/2*}_{1/2-k_2,1/2} \otimes t^{1/2}_{1/2-k_2,1/2} t^{1/2*}_{1/2-k_0,1/2}.$$

Using the definitions we can compute

$$\partial_b(t^{1/2}_{1/2-k,1/2} t^{1/2*}_{1/2-h,1/2}) = \begin{cases} -q^{1/2} t^{1/2}_{1/2-k,1/2} t^{1/2*}_{1/2-h,-1/2} & b = e \\ q^{-1/2} t^{1/2}_{1/2-k,-1/2} t^{1/2*}_{1/2-h,1/2} & b = f \end{cases}. \tag{3.18}$$

A computation using the orthogonality relations (3.8) and Equation (3.5) shows that

$$\pi_{\mathcal{D}}(\mathrm{Ch}_2(P_+)) = \sum_{k_0,k_2=0}^{1} q^{-2k_0} \begin{pmatrix} 0 & t^{1/2}_{1/2-k_0,1/2} t^{1/2*}_{1/2-k_2,-1/2} \\ t^{1/2}_{1/2-k_0,-1/2} t^{1/2*}_{1/2-k_2,1/2} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -t^{1/2}_{1/2-k_2,1/2} t^{1/2*}_{1/2-k_0,-1/2} \\ t^{1/2}_{1/2-k_2,-1/2} t^{1/2*}_{1/2-k_0,1/2} & 0 \end{pmatrix}. \tag{3.19}$$

Since, for example, $t^{1/2}_{1/2-k_2,-1/2} t^{1/2*}_{1/2-k_0,1/2} t^{1*}_{2-j,-1} \in \mathcal{B}$, we can use the orthogonality (3.8) and adjoint (3.7) relations to see that

$$\pi_{\mathcal{D}}(\mathrm{Ch}_2(P_+)) = \begin{pmatrix} 0 & t^{1*}_{2-j,-1} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^{1}_{2-j,-1} & 0 \end{pmatrix} - q^{-2} \begin{pmatrix} 0 & 0 \\ t^{1*}_{2-j,1} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^{1}_{2-j,1} \\ 0 & 0 \end{pmatrix}. \tag{3.20}$$

To obtain a non-trivial two-form orthogonal to Z, we define

$$C := \pi_{\mathcal{D}}(\mathrm{Ch}_2(P_+)) - Z\langle Z \mid \pi_{\mathcal{D}}(\mathrm{Ch}_2(P_+))\rangle_{\mathcal{B}} = \pi_{\mathcal{D}}(\mathrm{Ch}_2(P_+)) - \frac{1}{q^2+q^{-2}} \omega_l \otimes \omega_l^\dagger \langle \rho_{(1)}^\dagger \mid \rho_{(2)}\rangle_{\mathcal{B}},$$

where we have written $\pi_{\mathcal{D}}(\mathrm{Ch}_2(P_+)) = \sum \rho_{(1)} \otimes \rho_{(2)}$ in Sweedler notation. Since we have

$$\langle C \mid C\rangle_{\mathcal{B}} = \langle \rho_{(2)} \mid \langle \rho_{(1)} \mid \rho_{(1)}\rangle_{\mathcal{B}} \rho_{(2)}\rangle_{\mathcal{B}} - \frac{1}{q^2+q^{-2}} \langle \rho_{(2)} \mid \rho_{(1)}^\dagger\rangle_{\mathcal{B}} \langle \rho_{(1)}^\dagger \mid \rho_{(2)}\rangle_{\mathcal{B}}$$

we need to compute some inner products. Using (3.20) the first is given by

$$\langle \rho_{(1)}^\dagger \mid \rho_{(2)}\rangle_{\mathcal{B}} = \mathrm{Tr}\left(\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \rho_{(1)}\rho_{(2)}\right) = q^{-1}(q^2 - q^{-2}),$$

which yields

$$\begin{aligned} C &= \pi_{\mathcal{D}}(\mathrm{Ch}_2(P_+)) + q^{-1}\frac{q^{-2}-q^2}{q^{-2}+q^2} \omega_l \otimes \omega_l^\dagger \\ &= \frac{2q^{-1}}{q^{-2}+q^2}\left(q^{-1} \begin{pmatrix} 0 & t^{1*}_{2-j,-1} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^{1}_{2-j,-1} & 0 \end{pmatrix} - q \begin{pmatrix} 0 & 0 \\ t^{1*}_{2-j,1} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^{1}_{2-j,1} \\ 0 & 0 \end{pmatrix}\right). \end{aligned} \tag{3.21}$$

The remaining inner products are computed similarly, and

$$\alpha = \langle C \mid C\rangle_{\mathcal{B}} = q^{-2}(q^2+q^{-2}) - \frac{1}{q^2+q^{-2}} q^{-2}(q^2-q^{-2})^2 = \frac{4q^{-2}}{q^2+q^{-2}} \tag{3.22}$$

is a scalar. Hence

$$\frac{1}{\alpha}|C\rangle\langle C|$$

is a rank one projection, and of course C is orthogonal to Z. Note that $\alpha \to 2$ as $q \to 1$.

To complete the proof it suffices to show that C is orthogonal to the kernel of the multiplication map. With X_k, Y_k as in Lemma 3.8 a direct computation shows that

$$\langle Y_k \mid C\rangle_{\mathcal{A}} = \langle X_k \mid C\rangle_{\mathcal{A}} = 0.$$

The formula for d_Ψ follows from the definition (2.3). □

When computing the inner product, we can apply the orthogonality relations and perform the sums to see that inner products with C are the same, in the sense of Lemma 3.8, as inner products with

$$\tilde{C} = \frac{2q^{-1}}{q^2+q^{-2}}\big(q^{-1}E_{12}\otimes E_{21} - qE_{21}\otimes E_{12}\big). \tag{3.23}$$

Likewise inner products with G are the same as inner products with

$$\tilde{G} = qE_{12}\otimes E_{21} + q^{-1}E_{21}\otimes E_{12} \tag{3.24}$$

Neither $\tilde{C}$ nor $\tilde{G}$ is in $T^2_{\mathcal{D}}$, though the shorthand is useful for computations of inner products with C and G.

Combining Theorem 2.5 and Proposition 3.9 proves

Theorem 3.10. *The right Grassmann connection $\overrightarrow{\nabla}^G$ of the frame from Lemma 3.4 is Hermitian and torsion-free.*

To prove uniqueness, we need to prove concordance and †-concordance, as well as have a suitable bimodule connection. We will leave most of the uniqueness proof to the Appendix, but the braiding which yields a †-bimodule connection is presented next.

3.4 Braiding and bimodule connection

Next we show that the left and right Grassmann connections are a †-bimodule connection.

Definition 3.11. Define $\sigma : T^2_{\mathcal{D}} \to T^2_{\mathcal{D}}$ by

$$\sigma(Y_k) = q^{-2}Y_k, \quad \sigma(X_k) = q^2 X_k,$$
$$\sigma\left(\begin{pmatrix}0 & (t^1_{2-j,-1})^*\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ t^1_{2-j,-1} & 0\end{pmatrix}\right) = q^{-2}\begin{pmatrix}0 & 0\\ (t^1_{2-j,1})^* & 0\end{pmatrix}\otimes\begin{pmatrix}0 & t^1_{2-j,1}\\ 0 & 0\end{pmatrix},$$
$$\sigma\left(\begin{pmatrix}0 & 0\\ (t^1_{2-j,1})^* & 0\end{pmatrix}\otimes\begin{pmatrix}0 & t^1_{2-j,1}\\ 0 & 0\end{pmatrix}\right) = q^{2}\begin{pmatrix}0 & (t^1_{2-j,-1})^*\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ t^1_{2-j,-1} & 0\end{pmatrix},$$

and extend as a bimodule map. Observe that $\sigma^2 \neq 1$ on $X\oplus Y$.

Remark 3.12. In [KLS, Lemma 3.6] a braiding on tensor powers of holomorphic line bundles/modules on the Podleś sphere is introduced. It would be of interest to see if restricting their braiding to these particular modules coincides with the braiding σ of Definition 3.11.

Using Equation (3.16) we determine that

$$\begin{aligned}\overrightarrow{\nabla}^G([\mathcal{D},b]a) &= G\partial_e\partial_f(b)a + q^{-1}Y_k t^2_{2-k,-2}\partial_e^2(b)a + qX_k t^2_{2-k,2}\partial_f^2(b)a + [\mathcal{D},b]\otimes[\mathcal{D},a]\\ &= G\partial_e\partial_f(b)a + q^{-1}Y_k t^2_{2-k,-2}\partial_e^2(b)a + qX_k t^2_{2-k,2}\partial_f^2(b)a\\ &\quad + q^{-2}Y_k t^2_{2-k,-2}\partial_e(b)\partial_e(a) + q^2X_k t^2_{2-k,2}\partial_f(b)\partial_f(a)\end{aligned} \tag{3.25}$$

$$+\begin{pmatrix}0 & (t^1_{2-j,-1})^*\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ t^1_{2-j,-1} & 0\end{pmatrix}\partial_e(b)\partial_f(a)+\begin{pmatrix}0 & 0\\ (t^1_{2-j,1})^* & 0\end{pmatrix}\otimes\begin{pmatrix}0 & t^1_{2-j,1}\\ 0 & 0\end{pmatrix}\partial_f(b)\partial_e(a) \tag{3.26}$$

using the frame $\{X_k, Y_k, Z\}$ described in the last section.

Proposition 3.13. *The conjugate left connection* $\overleftarrow{\nabla}^G := -\dagger\circ\overrightarrow{\nabla}^G\circ\dagger$ *is given by*

$$\begin{aligned}\overleftarrow{\nabla}^G([\mathcal{D},b]a) &= [\mathcal{D},{}_{\mathcal{B}}\langle[\mathcal{D},b]a\mid\omega_j^*\rangle]\otimes\omega_j^*\\ &= G\partial_f\partial_e(b)a+q^3X_kt^2_{2-k,2}\partial_f^2(b)a+q^{-3}Y_kt^2_{2-k,-2}\partial_e^2(b)a+q^4X_kt^2_{2-k,2}\partial_f(b)\partial_f(a)\\ &\quad+q^{-4}Y_kt^2_{2-k,-2}\partial_e(b)\partial_e(a)+q^2\begin{pmatrix}0 & (t^1_{2-j,-1})^*\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ t^1_{2-j,-1} & 0\end{pmatrix}\partial_f(b)\partial_e(a)\\ &\quad+q^{-2}\begin{pmatrix}0 & 0\\ (t^1_{2-j,1})^* & 0\end{pmatrix}\otimes\begin{pmatrix}0 & t^1_{2-j,1}\\ 0 & 0\end{pmatrix}\partial_e(b)\partial_f(a).\end{aligned} \tag{3.27}$$

Proof. We start with computing $\overleftarrow{\nabla}^G$ on exact forms. First, the definition yields

$${}_{\mathcal{B}}\langle[\mathcal{D},b]\mid\omega_j^\dagger\rangle=(-1)^{1-j}\big(\partial_e(b)(t^1_{2-j,1})^*+\partial_f(b)(t^1_{2-j,-1})^*\big).$$

Next using the twisted derivation rule (3.4) and the orthogonality relations (3.8) in the same way as in Section 3.3 we find

$$\begin{aligned}&[\mathcal{D},{}_{\mathcal{B}}\langle[\mathcal{D},b]\mid\omega_j^\dagger\rangle]\otimes\omega_j^\dagger\\ &=\begin{pmatrix}0 & q^{-3/2}\partial_e^2(b)\\ q^{-1/2}\partial_f\partial_e(b) & 0\end{pmatrix}\otimes\begin{pmatrix}0 & q^{-1/2}\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & q^{1/2}\partial_e\partial_f(b)\\ q^{3/2}\partial_f^2(b) & 0\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ q^{1/2} & 0\end{pmatrix}\\ &=G\partial_f\partial_e(b)+q^3X_kt^2_{2-k,2}\partial_f^2(b)+q^{-3}Y_kt^2_{2-k,-2}\partial_e^2(b),\end{aligned}$$

where we also used $\partial_e\partial_f(b)=\partial_f\partial_e(b)$ for $b\in\mathcal{B}$. Now use the Leibniz rule to find

$$\begin{aligned}\overleftarrow{\nabla}^G([\mathcal{D},b]a)&=\overleftarrow{\nabla}^G([\mathcal{D},ba])-\overleftarrow{\nabla}^G(b[\mathcal{D},a])\\ &=\overleftarrow{\nabla}^G([\mathcal{D},ba])-b\overleftarrow{\nabla}^G([\mathcal{D},a])-[\mathcal{D},b]\otimes[\mathcal{D},a]\\ &=G\big(\partial_f\partial_e(ba)-b\partial_f\partial_e(a)\big)+q^3X_kt^2_{2-k,2}\big(\partial_f^2(ba)-b\partial_f^2(a)\big)\\ &\quad+q^{-3}Y_kt^2_{2-k,-2}\big(\partial_e^2(ba)-b\partial_e^2(a)\big)-[\mathcal{D},b]\otimes[\mathcal{D},a].\end{aligned}$$

Using the twisted derivation rule (3.4) yields the three relations

$$\begin{aligned}\partial_f\partial_e(ba)-b\partial_f\partial_e(a)&=\partial_f\partial_e(b)a+q\partial_f(b)\partial_e(a)+q^{-1}\partial_e(b)\partial_f(a)\\ \partial_f^2(ba)-b\partial_f^2(a)&=\partial_f^2(b)a+(q+q^{-1})\partial_f(b)\partial_f(a)\\ \partial_e^2(ba)-b\partial_e^2(a)&=\partial_e^2(b)a+(q+q^{-1})\partial_e(b)\partial_e(a).\end{aligned}$$

Together with expanding the $[\mathcal{D},b]\otimes[\mathcal{D},a]$ term

$$[\mathcal{D},b]\otimes[\mathcal{D},a]=q^{-2}Y_kt^2_{2-k,-2}\partial_e(b)\partial_e(a)+q^2X_kt^2_{2-k,2}\partial_f(b)\partial_f(a)$$

$$+ \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} \partial_e(b)\partial_f(a) + \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} \partial_f(b)\partial_e(a),$$

we find that

$$\begin{aligned}
\overleftarrow{\nabla}^G([\mathcal{D}, b]a) &= G\partial_f\partial_e(b)a + q^3 X_k t^2_{2-k,2}\partial_f^2(b)a + q^{-3}Y_k t^2_{2-k,-2}\partial_e^2(b)a \\
&+ G(q\partial_f(b)\partial_e(a) + q^{-1}\partial_e(b)\partial_f(a)) + q^3 X_k t^2_{2-k,2}(q+q^{-1})\partial_f(b)\partial_f(a) \\
&+ q^{-3}Y_k t^2_{2-k,-2}(q+q^{-1})\partial_e(b)\partial_e(a) - q^{-2}Y_k t^2_{2-k,-2}\partial_e(b)\partial_e(a) - q^2 X_k t^2_{2-k,2}\partial_f(b)\partial_f(a) \\
&- \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} \partial_e(b)\partial_f(a) - \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} \partial_f(b)\partial_e(a) \\
&= G\partial_f\partial_e(b)a + q^3 X_k t^2_{2-k,2}\partial_f^2(b)a + q^{-3}Y_k t^2_{2-k,-2}\partial_e^2(b)a + q^4 X_k t^2_{2-k,2}\partial_f(b)\partial_f(a) \\
&+ q^{-4}Y_k t^2_{2-k,-2}\partial_e(b)\partial_e(a) + q^2 \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} \partial_f(b)\partial_e(a) \\
&+ q^{-2} \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} \partial_e(b)\partial_f(a).
\end{aligned}$$

This completes the proof. □

Comparing Equations (3.26) and (3.27) we obtain

Proposition 3.14. *The conjugate pair of connections* $\overrightarrow{\nabla}^G$ *and* $\overleftarrow{\nabla}^G$ *satisfy* $\sigma \circ \overrightarrow{\nabla}^G = \overleftarrow{\nabla}^G$.

Remark 3.15. It is worth noting that $\sigma(G) = G$, but

$$\sigma(C) = -C + 2q^{-1}\frac{q^{-2} - q^2}{q^{-2} + q^2}G. \tag{3.28}$$

Since we only require that $(1-\Psi)\sigma\overrightarrow{\nabla}^G = -(1-\Psi)\overrightarrow{\nabla}^G$ for compatibility with the torsion-free condition. To see that (3.28) is compatible with the torsion-free condition, we need only check that $\frac{1}{\alpha}|C\rangle\langle C|\sigma(C) = -C$. This in turn follows from the definition of the inner product, and the orthogonality of C and G. Moreover on $\mathrm{span}(C, G)$ we have $\sigma^2 = \mathrm{Id}$.

In Theorem A.12 of the Appendix, we prove that in fact the pair $(\overrightarrow{\nabla}^G, \sigma)$ is the unique Hermitian torsion-free σ-†-bimodule connection.

4 Curvature of the Podleś sphere

We now have all the requisite structure to compute the curvature of the unique Hermitian torsion-free connection $\overrightarrow{\nabla}^G$ on the Podleś sphere.

4.1 The Riemann tensor

We first compute the full Riemann tensor following Definition 2.8 in Section 2.3. Since we are dealing with a Grassmann connection, we will use the explicit formulae of Lemma 2.9.

Theorem 4.1. *The Riemann tensor of the Podleś sphere is*

$$R = \frac{[2]_q}{2} q\, \omega_i \otimes C \otimes \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \omega_i^\dagger = \frac{[2]_q}{2} q\, \omega_i \otimes C \otimes \omega_i^\dagger \begin{pmatrix} -q^2 & 0 \\ 0 & q^{-2} \end{pmatrix}.$$

Proof. We start by computing inner products of frame elements and their differentials. First

$$\begin{aligned} &\langle \omega_j \mid \omega_k \rangle_{\mathcal{B}} \\ &= q^{-4+j+k} \operatorname{Tr}\left(\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \begin{pmatrix} 0 & q^{1/2} t^{1*}_{-2+j,-1} \\ q^{-1/2} t^{1*}_{-2+j,1} & 0 \end{pmatrix} \begin{pmatrix} 0 & q^{-1/2} t^{1}_{-2+k,1} \\ q^{1/2} t^{1}_{-2+k,-1} & 0 \end{pmatrix} \right) \\ &= q^{-4+j+k} (q^2 t^{1*}_{-2+j,-1} t^1_{-2+k,-1} + q^{-2} t^{1*}_{-2+j,1} t^1_{-2+k,1}) \\ &= (-1)^{1-j} q^{-1+k} t^1_{2-j,1} t^1_{-2+k,-1} + (-1)^{1-j} q^{-3+k} t^1_{2-j,-1} t^1_{-2+k,1}. \end{aligned} \tag{4.1}$$

Then we apply the twisted Leibniz rule and $\partial_e(t^1_{x,1}) = \partial_f(t^1_{x,-1}) = 0$ to find

$$\begin{aligned} [\mathcal{D}, \langle \omega_j \mid \omega_k \rangle_{\mathcal{B}}] &= (-1)^{1-j} q^{-1+k} [\mathcal{D}, t^1_{2-j,1} t^1_{-2+k,-1}] + (-1)^{1-j} q^{-3+k} [\mathcal{D}, t^1_{2-j,-1} t^1_{-2+k,1}] \\ &= (-1)^{1-j} q^{-1+k} \begin{pmatrix} 0 & q^{-1/2} \partial_e (t^1_{2-j,1} t^1_{-2+k,-1}) \\ q^{1/2} \partial_f (t^1_{2-j,1} t^1_{-2+k,-1}) & 0 \end{pmatrix} \\ &\quad + (-1)^{1-j} q^{-3+k} \begin{pmatrix} 0 & q^{-1/2} \partial_e (t^1_{2-j,-1} t^1_{-2+k,1}) \\ q^{1/2} \partial_f (t^1_{2-j,-1} t^1_{-2+k,1}) & 0 \end{pmatrix} \\ &= (-1)^{1-j} \begin{pmatrix} 0 & q^{-5/2+k} \kappa^1_0 t^1_{2-j,1} t^1_{-2+k,0} \\ q^{-3/2+k} \kappa^1_1 t^1_{2-j,0} t^1_{-2+k,-1} & 0 \end{pmatrix} \\ &\quad + (-1)^{1-j} \begin{pmatrix} 0 & q^{-5/2+k} \kappa^1_0 t^1_{2-j,0} t^1_{-2+k,1} \\ q^{-3/2+k} \kappa^1_1 t^1_{2-j,-1} t^1_{-2+k,0} & 0 \end{pmatrix} \\ &= (-1)^{1-j} \kappa^1_1 q^{-2+k} \begin{pmatrix} 0 & q^{-1/2} (t^1_{2-j,1} t^1_{-2+k,0} + t^1_{2-j,0} t^1_{-2+k,1}) \\ q^{1/2} (t^1_{2-j,0} t^1_{-2+k,-1} + t^1_{2-j,-1} t^1_{-2+k,0}) & 0 \end{pmatrix} \\ &= (-1)^{1-j} \kappa^1_1 t^1_{2-j,0} \omega_k + \kappa^1_1 q^{-2+k} \omega_j^\dagger t^1_{-2+k,0}. \end{aligned}$$

Thus we find that

$$\begin{aligned} &[\mathcal{D}, \langle \omega_i \mid \omega_j \rangle_{\mathcal{B}}] \otimes [\mathcal{D}, \langle \omega_j \mid \omega_k \rangle_{\mathcal{B}}] \\ &= (\kappa^1_1)^2 \Big((-1)^{1-i} t^1_{2-i,0} \omega_j + q^{-2+j} \omega_i^\dagger t^1_{-2+j,0} \Big) \otimes \Big((-1)^{1-j} t^1_{2-j,0} \omega_k + q^{-2+k} \omega_j^\dagger t^1_{-2+k,0} \Big) \\ &= (\kappa^1_1)^2 \Big((-1)^{i+j} t^1_{2-i,0} \omega_j \otimes t^1_{2-j,0} \omega_k + (-1)^i t^1_{2-i,0} \omega_j \otimes q^{-2+k} \omega_j^\dagger t^1_{-2+k,0} \\ &\qquad + (-1)^j q^{-2+j} \omega_i^\dagger t^1_{-2+j,0} \otimes t^1_{2-j,0} \omega_k + q^{-4+j+k} \omega_i^\dagger t^1_{-2+j,0} \otimes \omega_j^\dagger t^1_{-2+k,0} \Big). \end{aligned}$$

Summing over j, using the $\mathcal{B}$-centrality of $G = \sum_j \omega_j \otimes \omega_j^\dagger$ and Equation (3.11) gives

$$\begin{aligned}
&\sum_j [\mathcal{D}, \langle \omega_i \mid \omega_j \rangle_{\mathcal{B}}] \otimes [\mathcal{D}, \langle \omega_j \mid \omega_k \rangle_{\mathcal{B}}] \\
&= (\kappa_1^1)^2 \Big(\sum_j (-1)^{i+j} t^1_{2-i,0} \omega_j t^1_{2-j,0} \otimes \omega_k + (-1)^i q^{-2+k} t^1_{2-i,0} t^1_{-2+k,0} G \\
&\qquad + \omega_i^\dagger \otimes \omega_k + q^{-2+k} \omega_i^\dagger \otimes \sum_j (-1)^j (\omega_j t^1_{2-j,0})^\dagger t^1_{-2+k,0} \Big) \\
&= (\kappa_1^1)^2 \Big((-1)^i q^{-2+k} t^1_{2-i,0} t^1_{-2+k,0} G + \omega_i^\dagger \otimes \omega_k \Big),
\end{aligned}$$

where the explicit form of ω_j and the orthogonality relations Equation (3.8) give the vanishing of the two terms. From Equation (4.1) we also have

$$\langle \omega_i \mid \omega_k \rangle_{\mathcal{B}} = \delta_{ik} - (-1)^{i+k} t^1_{2-i,0} t^{1*}_{2-k,0}$$

so

$$\sum_j [\mathcal{D}, \langle \omega_i \mid \omega_j \rangle_{\mathcal{B}}] \otimes [\mathcal{D}, \langle \omega_j \mid \omega_k \rangle_{\mathcal{B}}] = (\kappa_1^1)^2 (\delta_{ik} \sum_l \omega_l \otimes \omega_l^\dagger - \langle \omega_i \mid \omega_k \rangle_{\mathcal{B}} \sum_l \omega_l \otimes \omega_l^\dagger + \omega_i^\dagger \otimes \omega_k),$$

and writing

$$\begin{aligned}
\tilde{R} :&= \omega_i \otimes \sum_j [\mathcal{D}, \langle \omega_i \mid \omega_j \rangle_{\mathcal{B}}] \otimes [\mathcal{D}, \langle \omega_j \mid \omega_k \rangle_{\mathcal{B}}] \otimes \omega_k^\dagger \\
&= (\kappa_1^1)^2 (\delta_{ik} \omega_i \otimes \sum_l \omega_l \otimes \omega_l^\dagger - \omega_k \otimes \sum_l \omega_l \otimes \omega_l^\dagger + \omega_i \otimes \omega_i^\dagger \otimes \omega_k) \otimes \omega_k^\dagger \\
&= (\kappa_1^1)^2 (\omega_k \otimes \sum_l \omega_l \otimes \omega_l^\dagger - \omega_k \otimes \sum_l \omega_l \otimes \omega_l^\dagger + \omega_i \otimes \omega_i^\dagger \otimes \omega_k) \otimes \omega_k^\dagger \\
&= (\kappa_1^1)^2 \omega_i \otimes \omega_i^\dagger \otimes \omega_k \otimes \omega_k^\dagger = [2]_q \omega_i \otimes \omega_i^\dagger \otimes \omega_k \otimes \omega_k^\dagger,
\end{aligned}$$

we have $R = (1 \otimes (1 - \Psi) \otimes 1)(\tilde{R})$. to obtain the curvature tensor

$$\begin{aligned}
R = \frac{[2]_q}{\alpha} \omega_i \otimes C \langle C \mid \omega_i^\dagger \otimes \omega_k \rangle_{\mathcal{B}} \otimes \omega_k^\dagger &= \frac{[2]_q}{\alpha} \omega_i \otimes C \otimes \big(\langle \omega_i^\dagger \mid C_{(1)} \rangle_{\mathcal{B}} C_{(2)} \big)^\dagger \\
&= \frac{[2]_q}{\alpha} \omega_i \otimes C \otimes C_{(2)}^\dagger \langle C_{(1)} \mid \omega_i^\dagger \rangle_{\mathcal{B}} \\
&= \frac{[2]_q}{2} q \, \omega_i \otimes C \otimes \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \omega_i^\dagger. \qquad \square
\end{aligned}$$

4.2 The Ricci and scalar curvature

We now have the pieces in place to compute the Ricci and scalar curvature of S_q^2.

Proposition 4.2. *The Ricci curvature is given by*

$$\mathrm{Ric} = {}_{\mathcal{B}}\langle R \mid G \rangle = \frac{[2]_q}{q^2 + q^{-2}} \omega_i \otimes \begin{pmatrix} q^{-4} & 0 \\ 0 & q^4 \end{pmatrix} \omega_i^\dagger = \frac{[2]_q}{q^2 + q^{-2}} \omega_i \otimes \omega_i^\dagger \begin{pmatrix} q^4 & 0 \\ 0 & q^{-4} \end{pmatrix}.$$

Proof. We start by expanding the inner product $\mathrm{Ric} = {}_{\mathcal{B}}\langle R\,|\,G\rangle$ defining the Ricci curvature to obtain

$$\begin{aligned}
\mathrm{Ric} &= \frac{[2]_q}{2} q\,\omega_i \otimes C_{(1)}{}_{\mathcal{B}}\langle C_{(2)} \otimes \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \omega_i^\dagger \mid qE_{12}\otimes E_{21} + q^{-1}E_{21}\otimes E_{12}\rangle \\
&= \frac{[2]_q}{q^2+q^{-2}}\,\omega_i \Big(q^{-1}E_{12}\,{}_{\mathcal{B}}\langle E_{21}\,{}_{\mathcal{B}}\langle \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \omega_i^\dagger \mid qE_{21}\rangle \mid E_{12}\rangle \\
&+ q^{-1}E_{12}\,{}_{\mathcal{B}}\langle E_{21}\,{}_{\mathcal{B}}\langle \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \omega_i^\dagger \mid q^{-1}E_{12}\rangle \mid E_{21}\rangle \\
&- qE_{21}\,{}_{\mathcal{B}}\langle E_{12}\,{}_{\mathcal{B}}\langle \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \omega_i^\dagger \mid qE_{21}\rangle \mid E_{12}\rangle \\
&- qE_{21}\,{}_{\mathcal{B}}\langle E_{12}\,{}_{\mathcal{B}}\langle \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \omega_i^\dagger \mid q^{-1}E_{12}\rangle \mid E_{21}\rangle \Big).
\end{aligned}$$

where we have abbreviated the computation by taking inner products using

$$\tilde{G} = qE_{12}\otimes E_{21} + q^{-1}E_{21}\otimes E_{12}.$$

First, two of the four terms in our expression for Ric are zero as E_{12} and E_{21} are orthogonal for the left inner product as well. Next

$$\begin{aligned}
{}_{\mathcal{B}}\langle \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \omega_i^\dagger \mid E_{21}\rangle &= (-1)^i q^{3/2} t^1_{2-i,-1} \\
{}_{\mathcal{B}}\langle \begin{pmatrix} q^{-2} & 0 \\ 0 & -q^2 \end{pmatrix} \omega_i^\dagger \mid E_{12}\rangle &= (-1)^{i+1} q^{-3/2} t^1_{2-i,1}
\end{aligned}$$

and so

$$\mathrm{Ric} = \frac{[2]_q}{q^2+q^{-2}}\,\omega_i \otimes \Big((-1)^{1+i} q^{-7/2} t^1_{2-i,1} E_{12}\,{}_{\mathcal{B}}\langle E_{21} \mid E_{21}\rangle + (-1)^{i+1} q^{7/2} t^1_{2-i,-1} E_{21}\,{}_{\mathcal{B}}\langle E_{12} \mid E_{12}\rangle \Big).$$

Since

$${}_{\mathcal{B}}\langle E_{12} \mid E_{12}\rangle = q, \qquad {}_{\mathcal{B}}\langle E_{21} \mid E_{21}\rangle = q^{-1}$$

we find that

$$\begin{aligned}
\mathrm{Ric} &= \frac{[2]_q}{q^2+q^{-2}}\,\omega_i \otimes \left(q^{-9/2}(-1)^{1+i} t^1_{2-i,1} E_{12} + q^{9/2}(-1)^{1+i} t^1_{2-i,-1} E_{21} \right) \\
&= \frac{[2]_q}{q^2+q^{-2}}\,\omega_i \otimes \begin{pmatrix} q^{-4} & 0 \\ 0 & q^4 \end{pmatrix} \omega_i^\dagger = \frac{[2]_q}{q^2+q^{-2}}\,\omega_i \otimes \omega_i^\dagger \begin{pmatrix} q^4 & 0 \\ 0 & q^{-4} \end{pmatrix}.
\end{aligned}$$

□

Observe that if $q = 1$ then the Ricci curvature is proportional to the line element $G = \sum_i \omega_i \otimes \omega_i^\dagger$. When $q \neq 1$, this relation breaks down and the metric on the Podleś sphere is not Einstein in the classical sense.

Proposition 4.3. *The scalar curvature $r = \langle G\,|\,\mathrm{Ric}\rangle_{\mathcal{B}}$ is constant, given by*

$$r = \frac{[2]_q}{q^2+q^{-2}}(q^{-6}+q^6) = [2]_q(1 + (q^{-2} - q^2)^2),$$

and as $q \to 1$ the scalar curvature converges to 2.

Proof. Again, this is a computation, with

$$
\begin{aligned}
r &= \frac{[2]_q}{q^2+q^{-2}}\Big(q\langle E_{21} \mid \langle E_{12} \mid \omega_i\rangle_{\mathcal{B}} \begin{pmatrix} q^{-4} & 0 \\ 0 & q^4 \end{pmatrix} \omega_i^\dagger\rangle_{\mathcal{B}} + \langle E_{12} \mid \langle E_{21} \mid \omega_i\rangle_{\mathcal{B}} \begin{pmatrix} q^{-4} & 0 \\ 0 & q^4 \end{pmatrix} \omega_i^\dagger\rangle_{\mathcal{B}}\Big) \\
&= \frac{[2]_q}{q^2+q^{-2}} q\langle E_{21} \mid q^{-1/2}(-1)^{1+i}(t^1_{2-i,-1})^* \begin{pmatrix} q^{-4} & 0 \\ 0 & q^4 \end{pmatrix} \omega_i^\dagger\rangle_{\mathcal{B}} \\
&\qquad + \frac{[2]_q}{q^2+q^{-2}} q^{-1}\langle E_{12} \mid q^{1/2}(-1)^{1+i}(t^1_{2-i,1})^* \begin{pmatrix} q^{-4} & 0 \\ 0 & q^4 \end{pmatrix} \omega_i^\dagger\rangle_{\mathcal{B}} \\
&= \frac{[2]_q}{q^2+q^{-2}}\Big(q^6(t^1_{2-i,-1})^* t^1_{2-i,-1} + q^{-6}(t^1_{2-i,1})^* t^1_{2-i,1}\Big) \\
&= \frac{[2]_q}{q^2+q^{-2}}\big(q^6+q^{-6}\big).
\end{aligned}
$$

The last expression follows from $(q^{-6}+q^6)(q^{-2}+q^2)^{-1} = (q^{-4}+q^4)-1 = (q^{-2}-q^2)^2+1$. □

Remark 4.4. One may ask whether there is an inner product on the one-forms for which the scalar curvature is precisely $[2]_q$. In [BM2, Section 8.2.3] such a construction is presented. It depends on a choice of embedding of the abstract two-forms into the two-tensors $T^2_{\mathcal{D}}$, which is necessary to define the Ricci tensor. The choice made by Beggs-Majid yields an Einstein metric with scalar curvature $[2]_q$. However, our constructions depend on the additional constraint that $\Psi : T^2_{\mathcal{D}} \to T^2_{D}$ be a *self-adjoint* idempotent with the two-forms isomorphic to $(1-\Psi)T^2_{\mathcal{D}} \cong \Lambda^2$, and so cannot be modified in a manner similar to [BM2, Section 8.2.3].

5 Weitzenböck formula

5.1 Dirac spectral triple

We will establish that $(\mathcal{B}, L^2(S^+\oplus S^-, h), \mathcal{D})$ is a Dirac spectral triple by verifying conditions 1-4 of Definition 2.11. The first of these conditions is straightforward.

Lemma 5.1. *For $\rho\otimes\eta \in JT^2_{\mathcal{D}}$ we have the equality*

$$m\circ\Psi(\rho\otimes\eta) = e^{-\beta} m(G)\langle \rho^\dagger \mid \eta\rangle_{\mathcal{B}}.$$

Moreover we have $\sigma(G) = G$ and $m\circ\sigma\circ\Psi = m\circ\Psi$.

Proof. We have that $\Psi T^2_{\mathcal{D}} = X\oplus Y\oplus \mathrm{span}(G)$, and $X\oplus Y\subset \ker m$ whereas the projection onto $\mathrm{span}(G)$ is given by $e^{-\beta}|G\rangle\langle G|$. Therefore

$$m\circ\Psi(\rho\otimes\eta) = m\left(e^{-\beta}G\langle G\mid \rho\otimes\eta\rangle_{\mathcal{B}}\right) = e^{-\beta}m(G)\langle G\mid\rho\otimes\eta\rangle_B = e^{-\beta}m(G)\langle\rho^\dagger\mid\eta\rangle,$$

as claimed. The equality $\sigma(G) = G$ was noted in Remark 3.15. By definition of σ, we have $\sigma(X\oplus Y) = X\oplus Y$. Therefore $m\circ\sigma\circ\Psi = m\circ\Psi$. □

We give the spinor bundles $S^\pm$ the left inner products

$$ {}_{\mathcal{B}}\langle s_+ \mid t_+\rangle = \Phi(|s_+\rangle\langle t_+|) = q s_+ t_+^*, \quad {}_{\mathcal{B}}\langle s_- \mid t_-\rangle = \Phi(|s_-\rangle\langle t_-|) = q^{-1} s_- t_-^*, $$

and observe that they restrict to $\mathcal{B}$-valued inner products on $\mathcal{S}^\pm$. Now let $\overleftarrow{\nabla}^{\mathcal{S}^\pm}$ be the left Grassmann connections of the bundles $S^\pm$ for the left frames $s_{\pm 1/2,-} := q^{1/2} t^{1/2}_{\pm 1/2,-1/2}$, $s_{\pm 1/2,+} := q^{-1/2} t^{1/2}_{\pm 1/2,1/2}$, and $\nabla^{\mathcal{S}} := \nabla^{\mathcal{S}^+} \oplus \nabla^{\mathcal{S}^-}$. Moreover the action of the Clifford algebra $\mathcal{C}_{\mathcal{D}}(\mathcal{B})$ on $L^2(S,h)$ restricts to an action

$$ c : \mathcal{C}_{\mathcal{D}}(\mathcal{B}) \otimes_{\mathcal{B}} (\mathcal{S}^+ \oplus \mathcal{S}^-) \to \mathcal{S}^+ \oplus \mathcal{S}^-, $$

of the Clifford algebra on spinors, which establishes condition 2 of Definition 2.11. We now verify condition 3 of Definition 2.11.

Lemma 5.2. *The Dirac operator $\mathcal{D}$ satisfies $\mathcal{D} = c \circ \nabla^{\mathcal{S}} : \mathcal{S} \to L^2(S,h)$.*

Proof. As Grassmann connections are always Hermitian, we use [MRLC, Proposition 2.24] in the computation

$$ \begin{aligned} \mathcal{D}\begin{pmatrix} s \\ 0 \end{pmatrix} &= \sum_i [\mathcal{D}, {}_{\mathcal{B}}\langle s \mid s_{i,+}\rangle] \begin{pmatrix} s_{i,+} \\ 0 \end{pmatrix} + {}_{\mathcal{B}}\langle s \mid s_{i,+}\rangle \mathcal{D}\begin{pmatrix} s_{i,+} \\ 0 \end{pmatrix} \\ &= \sum_i c\Big({}_{\mathcal{B}}\langle \overleftarrow{\nabla}^{\mathcal{S}^+}(s) \mid s_{i,+}\rangle \begin{pmatrix} s_{i,+} \\ 0 \end{pmatrix}\Big) - c\Big({}_{\mathcal{B}}\langle s \mid \overleftarrow{\nabla}^{\mathcal{S}^+}(s_{i,+})\rangle \begin{pmatrix} s_{i,+} \\ 0 \end{pmatrix}\Big) + {}_{\mathcal{B}}\langle s \mid s_{i,+}\rangle \mathcal{D}\begin{pmatrix} s_{i,+} \\ 0 \end{pmatrix} \\ &= \sum_i c \circ \overleftarrow{\nabla}^{\mathcal{S}^+}\begin{pmatrix} s \\ 0 \end{pmatrix} - c\Big({}_{\mathcal{B}}\langle s \mid s_{i,+}\rangle \begin{pmatrix} \overleftarrow{\nabla}^{\mathcal{S}^+} s_{i,+} \\ 0 \end{pmatrix}\Big) + {}_{\mathcal{B}}\langle s \mid s_{i,+}\rangle \mathcal{D}\begin{pmatrix} s_{i,+} \\ 0 \end{pmatrix} \\ &= c \circ \overleftarrow{\nabla}^{\mathcal{S}^+}\begin{pmatrix} s \\ 0 \end{pmatrix} \end{aligned} $$

and one can check the final equality by checking that $c \circ \overleftarrow{\nabla}^{\mathcal{S}}$ and $\mathcal{D}$ agree on frame elements using the formulae (5.3) below. The same argument holds for S^- as well, and we obtain

$$ \mathcal{D} = c \circ (\overleftarrow{\nabla}^{\mathcal{S}^+} + \overleftarrow{\nabla}^{\mathcal{S}^-}) = c \circ \overleftarrow{\nabla}^{\mathcal{S}}. \qquad \square $$

Next we must check compatibility of the connections, meaning that for a one form ω and spinor s we have

$$ \overleftarrow{\nabla}^{\mathcal{S}}(c(\omega \otimes s)) = c \circ m \circ (\sigma \otimes 1)\big(\overrightarrow{\nabla}^{G}(\omega) \otimes s + \omega \otimes \overleftarrow{\nabla}^{\mathcal{S}}(s)\big) $$

where $\overrightarrow{\nabla}^{G}$ is the right Levi-Civita connection on one forms. Here σ is the generalised braiding from Definition 3.11.

We will compute connections and their squares on the module $\mathcal{S}$, prove the compatibility with the connection on $\Omega^1_{\mathcal{D}}$ and then prove the Weitzenböck formula. In all of these tasks we use the formulae

$$ \partial_b(t^{1/2}_{1/2-k,1/2} t^{1/2*}_{1/2-h,1/2}) = \begin{cases} -q^{1/2} t^{1/2}_{1/2-k,1/2} t^{1/2*}_{1/2-h,-1/2} & b = e \\ q^{-1/2} t^{1/2}_{1/2-k,-1/2} t^{1/2*}_{1/2-h,1/2} & b = f \end{cases}, \tag{5.1} $$

$$\partial_b(t^{1/2}_{1/2-k,-1/2}t^{1/2*}_{1/2-h,-1/2}) = \begin{cases} q^{1/2}t^{1/2}_{1/2-k,1/2}t^{1/2*}_{1/2-h,-1/2} & b=e \\ -q^{-1/2}t^{1/2}_{1/2-k,-1/2}t^{1/2*}_{1/2-h,1/2} & b=f \end{cases} \tag{5.2}$$

and

$$\partial_e(t^{1/2}_{i,1/2}) = 0, \quad \partial_e(t^{1/2}_{i,-1/2}) = t^{1/2}_{i,1/2}, \quad \partial_f(t^{1/2}_{i,1/2}) = t^{1/2}_{i,-1/2}, \quad \partial_f(t^{1/2}_{i,-1/2}) = 0. \tag{5.3}$$

$$\partial_e\partial_f(t^{1/2}_{i1/2}) = t^{1/2}_{i1/2}, \qquad \partial_f\partial_e(t^{1/2}_{i,-1/2}) = t^{1/2}_{i,-1/2}$$

$$\partial_e\partial_f(t^{1/2}_{i1/2}(t^{1/2}_{l1/2})^*) = q^{-1}t^{1/2}_{i1/2}(t^{1/2}_{l1/2})^* - qt^{1/2}_{i1/2}(t^{1/2}_{l,-1/2})^*.$$

Using these formulae we can obtain the Grassmann connections on $\mathcal{S}^\pm$.

Lemma 5.3. *The left Grassmann connections* $\overleftarrow{\nabla}^{\mathcal{S}^-}$ *and* $\overleftarrow{\nabla}^{\mathcal{S}^+}$ *are given by*

$$\overleftarrow{\nabla}^{\mathcal{S}^+}\begin{pmatrix} b_is_{i,+} \\ 0\end{pmatrix} = [\mathcal{D}, b_i]\otimes\begin{pmatrix} s_{i,+} \\ 0\end{pmatrix} + b_i[\mathcal{D}, t^{1/2}_{i,1/2}(t^{1/2}_{l,1/2})^*]\otimes\begin{pmatrix} s_{l,+} \\ 0\end{pmatrix}$$
$$= \begin{pmatrix} 0 & q^{-1/2}\partial_e(b_i) \\ q^{1/2}\partial_f(b_i) & 0\end{pmatrix}\otimes\begin{pmatrix} s_{i,+} \\ 0\end{pmatrix} + b_i\begin{pmatrix} 0 & -t^{1/2}_{i,1/2}(t^{1/2}_{l,-1/2})^* \\ t^{1/2}_{i,-1/2}(t^{1/2}_{l,1/2})^* & 0\end{pmatrix}\otimes\begin{pmatrix} s_{l,+} \\ 0\end{pmatrix}$$

and

$$\overleftarrow{\nabla}^{\mathcal{S}^-}\begin{pmatrix} 0 \\ c_ls_{l,-}\end{pmatrix} = [\mathcal{D}, c_l]\otimes\begin{pmatrix} 0 \\ s_{l,-}\end{pmatrix} + c_l[\mathcal{D}, t^{1/2}_{l,-1/2}(t^{1/2}_{p,-1/2})^*]\otimes\begin{pmatrix} 0 \\ s_{p,-}\end{pmatrix}$$
$$= \begin{pmatrix} 0 & q^{-1/2}\partial_e(c_l) \\ q^{1/2}\partial_f(c_l) & 0\end{pmatrix}\otimes\begin{pmatrix} 0 \\ s_{l,-}\end{pmatrix} + c_l\begin{pmatrix} 0 & t^{1/2}_{l,1/2}(t^{1/2}_{p,-1/2})^* \\ -t^{1/2}_{l,-1/2}(t^{1/2}_{p,1/2})^* & 0\end{pmatrix}\otimes\begin{pmatrix} 0 \\ s_{p,-}\end{pmatrix}$$

Using the formulae for the connections we can check condition 4 of Definition 2.11.

Theorem 5.4. *For* $[\mathcal{D}, b]a \in \Omega^1_{\mathcal{D}}$ *we have the compatibility*

$$\mathcal{D}\Big([\mathcal{D}, b]a\begin{pmatrix} b_is_{i,+} \\ c_ls_{l,-}\end{pmatrix}\Big) = c\Big(\overleftarrow{\nabla}^{\mathcal{S}}\Big([\mathcal{D}, b]a\begin{pmatrix} b_is_{i,+} \\ c_ls_{l,-}\end{pmatrix}\Big)\Big)$$
$$= c\Big((m\circ\sigma)\otimes 1\Big([\mathcal{D}, b]a\otimes\overleftarrow{\nabla}^{\mathcal{S}}\begin{pmatrix} b_is_{i,+} \\ c_ls_{l,-}\end{pmatrix}\Big)\Big) + c\Big((m\circ\sigma)\otimes 1\Big(\overrightarrow{\nabla}^{G}([\mathcal{D}, b]a)\otimes\begin{pmatrix} b_is_{i,+} \\ c_ls_{l,-}\end{pmatrix}\Big)\Big). \tag{5.4}$$

Proof. Let $b, a \in \mathcal{B}$ and define a one-form by

$$[\mathcal{D}, b]a = \begin{pmatrix} 0 & q^{-1/2}\partial_e(b)a \\ q^{1/2}\partial_f(b)a & 0\end{pmatrix}.$$

We compute the left hand side of Equation (5.4), ignoring elements of $\ker(m) \subset T^2_{\mathcal{D}}(\mathcal{B})$ as they act by zero on the spinor bundle. We obtain

$$c\left(\overleftarrow{\nabla}^{\mathcal{S}}\left([\mathcal{D}, b]a\begin{pmatrix} b_is_{i,+} \\ c_ls_{l,-}\end{pmatrix}\right)\right)$$

$$= \begin{pmatrix} q\partial_e\partial_f(b)a & 0 \\ 0 & q^{-1}\partial_e\partial_f(b)a \end{pmatrix} \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} + \begin{pmatrix} q^2\partial_f(b)\partial_e(a)b_i s_{i,+} + q^2\partial_f(b)a\partial_e(b_i)s_{i,+} \\ q^{-2}\partial_e(b)\partial_f(a)c_l s_{l,-} + q^{-2}\partial_e(b)a\partial_f(c_l)s_{l,-} \end{pmatrix},$$

the last line following since $\partial_f(t_{l,-1/2}) = \partial_e(t_{i,1/2}) = 0$. Again omitting elements of $\ker(m)$ (which we recall is preserved by σ), the second term on the right hand side of Equation (5.4) is

$$\begin{aligned}
&c \circ (m \circ \sigma) \otimes 1 \left(\overrightarrow{\nabla}^G([\mathcal{D}, b]a) \otimes \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} \right) \\
&= c \circ (m \circ \sigma) \otimes 1 \Big(\Big(G\partial_e\partial_f(b)a \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} \Big) + \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} \partial_e(b)\partial_f(a) \Big) \\
&+ c \circ (m \circ \sigma) \otimes 1 \left(\left(\begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} \partial_f(b)\partial_e(a) \right) \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} \right).
\end{aligned}$$

Applying the braiding σ (recalling that σ maps $\ker(m)$ to itself) and then implementing the action of forms on $\mathcal{S}$ gives

$$\begin{aligned}
&c \left((m \circ \sigma) \otimes 1 \left(\overrightarrow{\nabla}^G([\mathcal{D}, b]a) \otimes \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} \right) \right) \\
&= c\Big(G\partial_e\partial_f(b)a \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} \Big) + c \left(q^{-2} \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} \partial_e(b)\partial_f(a) \right) \\
&+ c \left(q^2 \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} \partial_f(b)\partial_e(a) \right) \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} \\
&= \begin{pmatrix} q\partial_e\partial_f(b)a + q^2\partial_f(b)\partial_e(a) & 0 \\ 0 & q^{-1}\partial_e\partial_f(b)a + q^{-2}\partial_e(b)\partial_f(a) \end{pmatrix} \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix}.
\end{aligned}$$

The first term on the right hand side of Equation (5.4) is

$$\begin{aligned}
&[\mathcal{D}, b]a \otimes \overleftarrow{\nabla}^{\mathcal{S}} \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} \\
&= \begin{pmatrix} 0 & q^{-1/2}\partial_e(b)a \\ q^{1/2}\partial_f(b)a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & q^{-1/2}\partial_e(c_l t_{l,-1/2})t^*_{p,-1/2} \\ q^{1/2}\partial_f(c_l t_{l,-1/2} t^*_{p,-1/2}) & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ s_{p,-} \end{pmatrix} \\
&+ \begin{pmatrix} 0 & q^{-1/2}\partial_e(b)a \\ q^{1/2}\partial_f(b)a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & q^{-1/2}\partial_e(b_i t_{i,1/2} t^*_{k,1/2}) \\ q^{1/2}\partial_f(b_i t_{i,1/2})t^*_{k,1/2} & 0 \end{pmatrix} \otimes \begin{pmatrix} s_{k,+} \\ 0 \end{pmatrix} \\
&= \Bigg(\begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} \partial_f(b)a\partial_e(c_l t_{l,-1/2})t^*_{p,-1/2} \\
&+ \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} \partial_e(b)a\partial_f(c_l)t_{l,-1/2}t^*_{p,-1/2} \Bigg) \otimes \begin{pmatrix} 0 \\ s_{p,-} \end{pmatrix} \\
&+ \Bigg(\begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} \partial_f(b)a\partial_e(b_i)t_{i,1/2}t^*_{k,1/2} \\
&+ \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} \partial_e(b)a\partial_f(b_i t_{i,1/2})t^*_{k,1/2} \Bigg) \otimes \begin{pmatrix} s_{k,+} \\ 0 \end{pmatrix}
\end{aligned}$$

where simplification occurs in two terms due to orthogonality, the derivation rule, and $\partial_f(t_{l,-1/2}) = \partial_e(t_{i,1/2}) = 0$. Applying $\sigma \otimes 1$ yields

$$\sigma \otimes 1 \left([\mathcal{D}, b] a \otimes \overleftarrow{\nabla}^{\mathcal{S}} \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} \right)$$
$$= \Bigg(q^2 \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} q^{1/2} \partial_f(b) a q^{-1/2} \partial_e(c_l t_{l,-1/2}) t^*_{p,-1/2}$$
$$+ q^{-2} \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} q^{-1/2} \partial_e(b) a q^{1/2} \partial_f(c_l) t_{l,-1/2} t^*_{p,-1/2} \Bigg) \otimes \begin{pmatrix} 0 \\ s_{p,-} \end{pmatrix}$$
$$+ \Bigg(q^2 \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0 \end{pmatrix} q^{1/2} \partial_f(b) a q^{-1/2} \partial_e(b_i) t_{i,1/2} t^*_{k,1/2}$$
$$+ q^{-2} \begin{pmatrix} 0 & 0 \\ (t^1_{2-j,1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{2-j,1} \\ 0 & 0 \end{pmatrix} q^{-1/2} \partial_e(b) a \partial_f(b_i t_{i,1/2}) t^*_{k,1/2} \Bigg) \otimes \begin{pmatrix} s_{k,+} \\ 0 \end{pmatrix}$$

and then applying the action gives

$$c\left((m \circ \sigma) \otimes 1 \left([\mathcal{D}, b] a \otimes \overleftarrow{\nabla}^{\mathcal{S}} \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} \right) \right) = \begin{pmatrix} 0 \\ q^{-2} \partial_e(b) a \partial_f(c_l) s_{l,-} \end{pmatrix} + \begin{pmatrix} q^2 \partial_f(b) a \partial_e(b_i) s_{i,+} \\ 0 \end{pmatrix}.$$

Combining these calculations yields the result. □

Theorem 5.5. *The spectral triple* $(\mathcal{B}, L^2(S,h), \mathcal{D})$ *is a Dirac spectral triple relative to* $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger, \langle\cdot\mid\cdot\rangle, \sigma)$. *The connection Laplacian* $\Delta^{\mathcal{S}}$ *of Equation* (2.12) *and the Dirac operator* $\mathcal{D}$ *satisfy the Weitzenböck formula*

$$\mathcal{D}^2 = \Delta^{\mathcal{S}} + c \circ m \circ (\sigma \otimes 1)(R^{\overleftarrow{\nabla}^{\mathcal{S}}}).$$

Proof. We have verified the conditions of Definition 2.11, so $(\mathcal{B}, L^2(S,h), \mathcal{D})$ is a Dirac spectral triple over the braided Hermitian differential structure $(\Omega^1_{\mathcal{D}}(\mathcal{B}), \dagger, \langle\cdot\mid\cdot\rangle, \sigma)$. Therefore the connection Laplacian $\Delta^{\mathcal{S}} : \mathcal{S} \to L^2(S,h)$ of Equation (2.12) is a well-defined operator. By Theorem 2.12, the operators $\mathcal{D}$ and $\Delta^{\mathcal{S}}$ are related via the Weitzenböck formula

$$\mathcal{D}^2 = \Delta^{\mathcal{S}} + c \circ m \circ \sigma \otimes 1(R^{\overleftarrow{\nabla}^{\mathcal{S}}}).$$

□

Remark 5.6. By Theorem 5.5 and [NT, Proposition 3.1], the operator $\mathcal{D}^2$ concides with the action of the Casimir element. Thus, the connection Laplacian $\Delta^{\mathcal{S}}$ differs from the action of the Casimir precisely by the action of the curvature.

5.2 Curvature of the spinor bundle and positivity of the Laplacian

We now further analyse the Weitzenböck formula for the Podleś sphere. Our first goal is to compute the curvature term $c \circ m \circ \sigma \otimes 1(R^{\overleftarrow{\nabla}^{\mathcal{S}}})$.

Proposition 5.7. *The curvature of* $\mathcal{S}^+ \oplus \mathcal{S}^-$ *is given by*

$$R^{\overleftarrow{\nabla}^{\mathcal{S}}}\begin{pmatrix} b_i s_{i,+} \\ c_l s^{1/2}_{l,-}\end{pmatrix} = \frac{q}{2} C \otimes \begin{pmatrix} -q^{-1} & 0 \\ 0 & q\end{pmatrix}\begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-}\end{pmatrix}, \tag{5.5}$$

and its Clifford representation is given by

$$c \circ m \circ \sigma \otimes 1(R^{\overleftarrow{\nabla}^{\mathcal{S}}}) = \frac{1}{q^{-2}+q^2}\begin{pmatrix} q^2 & 0 \\ 0 & q^{-2}\end{pmatrix}.$$

Proof. To compute the curvature of $\overleftarrow{\nabla}^{\mathcal{S}}$ we first observe that the computation

$$\begin{aligned} 1 \otimes \overleftarrow{\nabla}^{\mathcal{S}^-}([\mathcal{D}, c_l t_{l,-1/2} t^*_{m,-1/2}] \otimes t_{m,-1/2}) &= [\mathcal{D}, c_l t_{l,-1/2} t^*_{m,-1/2}] \otimes [\mathcal{D}, t_{m,-1/2} t^*_{p,-1/2}] \otimes t_{p,-1/2} \\ &= c_l [\mathcal{D}, t_{l,-1/2} t^*_{m,-1/2}] \otimes [\mathcal{D}, t_{m,-1/2} t^*_{p,-1/2}] \otimes t_{p,-1/2} \end{aligned}$$

shows that the $[\mathcal{D}, c_l]$ term vanishes because it is multiplied by $v^*[\mathcal{D}, p]v = v^* p[\mathcal{D}, p]pv = 0$ where v is the stabilisation map defined by the frame, and $p = vv^*$. The same applies to the $\overleftarrow{\nabla}^{\mathcal{S}^+}$ part of the computation, and we use this freely below. So we start by computing (throwing away elements of $\ker(m)$ and the $[\mathcal{D}, b_i]$ term at the last step)

$$\begin{aligned} &1 \otimes \overleftarrow{\nabla}^{\mathcal{S}^+} \circ \overleftarrow{\nabla}^{\mathcal{S}^+}\begin{pmatrix} b_i s_{i,+} \\ 0\end{pmatrix} = \begin{pmatrix} 0 & q^{-1/2}\partial_e(b_i) \\ q^{1/2}\partial_f(b_i) & 0\end{pmatrix} \otimes \overleftarrow{\nabla}^{\mathcal{S}^+}\begin{pmatrix} s_{i,+} \\ 0\end{pmatrix} \\ &+ \begin{pmatrix} 0 & -b_i t^{1/2}_{i,1/2}(t^{1/2}_{k,-1/2})^* \\ b_i t^{1/2}_{i,-1/2}(t^{1/2}_{k,1/2})^* & 0\end{pmatrix} \otimes \overleftarrow{\nabla}^{\mathcal{S}^+}\begin{pmatrix} s_{k,+} \\ 0\end{pmatrix} \\ &= \begin{pmatrix} 0 & q^{-1/2}\partial_e(b_i) \\ q^{1/2}\partial_f(b_i) & 0\end{pmatrix} \otimes \begin{pmatrix} 0 & -t^{1/2}_{i,1/2}(t^{1/2}_{m,-1/2})^* \\ t^{1/2}_{i,-1/2}(t^{1/2}_{m,1/2})^* & 0\end{pmatrix} \otimes \begin{pmatrix} s_{m,+} \\ 0\end{pmatrix} \\ &+ \begin{pmatrix} 0 & -b_i t^{1/2}_{i,1/2}(t^{1/2}_{l,-1/2})^* \\ b_i t^{1/2}_{i,-1/2}(t^{1/2}_{l,1/2})^* & 0\end{pmatrix} \otimes \begin{pmatrix} 0 & -t^{1/2}_{l,1/2}(t^{1/2}_{m,-1/2})^* \\ t^{1/2}_{l,-1/2}(t^{1/2}_{m,1/2})^* & 0\end{pmatrix} \otimes \begin{pmatrix} s_{m,+} \\ 0\end{pmatrix} \\ &= \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0\end{pmatrix} \otimes \begin{pmatrix} -b_i s_{i,+} \\ 0\end{pmatrix}. \end{aligned}$$

Ignoring the kernel of m again (which is orthogonal to the image of $1-\Psi$), using $\mathcal{B}$-linearity of the tensor product and the orthogonality relations gives

$$\begin{aligned} \overrightarrow{\nabla}^G \otimes 1 \circ \overleftarrow{\nabla}^{\mathcal{S}^+}\begin{pmatrix} b_i s_{i,+} \\ 0\end{pmatrix} &= \overrightarrow{\nabla}^G([\mathcal{D}, b_i t^{1/2}_{i,1/2}(t^{1/2}_{k,1/2})^*]) \otimes \begin{pmatrix} s_{k,+} \\ 0\end{pmatrix} \\ &= G\partial_e\partial_f(b_i t^{1/2}_{i,1/2}(t^{1/2}_{k,1/2})^*) \otimes \begin{pmatrix} s_{k,+} \\ 0\end{pmatrix}. \end{aligned}$$

Recalling that $\langle C \mid G\rangle = 0$, we see that this term does not contribute to the curvature, and applying $1-\Psi$ yields

$$R^{\overleftarrow{\nabla}^{\mathcal{S}^+}}\begin{pmatrix} b_i s_{i,+} \\ 0\end{pmatrix} = \frac{1}{\alpha} C\left\langle C, \begin{pmatrix} 0 & (t^1_{2-j,-1})^* \\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{2-j,-1} & 0\end{pmatrix}\right\rangle \otimes \begin{pmatrix} -b_i s_{i,+} \\ 0\end{pmatrix}$$

$$= -\frac{1}{2} C \otimes \begin{pmatrix} b_i s_{i,+} \\ 0 \end{pmatrix}.$$

We continue with the negative spinors, computed much as the positive ones, and find

$$1 \otimes \overleftarrow{\nabla}^{\mathcal{S}^-} \circ \overleftarrow{\nabla}^{\mathcal{S}^-} \begin{pmatrix} 0 \\ c_l s_{l,-} \end{pmatrix}$$
$$= c_l \begin{pmatrix} 0 & t^{1/2}_{l,1/2}(t^{1/2}_{m,-1/2})^* \\ -t^{1/2}_{l,-1/2}(t^{1/2}_{m,1/2})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^{1/2}_{m,1/2}(t^{1/2}_{p,-1/2})^* \\ -t^{1/2}_{m,-1/2}(t^{1/2}_{p,1/2})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ s_{p,-} \end{pmatrix}$$

along with

$$\overrightarrow{\nabla}^{G} \otimes 1 \circ \overleftarrow{\nabla}^{\mathcal{S}^-} \begin{pmatrix} 0 \\ c_l s_{l,-} \end{pmatrix} = G \otimes \begin{pmatrix} 0 \\ \partial_f \partial_e(c_l) s_{l,-} + q^{3/2} \partial_f(c_l) s_{l,+} + q^1 c_l s_{l,-} \end{pmatrix}.$$

Applying $1 - \Psi$ and combining the $\mathcal{S}^-$ and $\mathcal{S}^+$ computations yields (5.5). Finally, observe that the action of $\sigma(C)$ is given by

$$\frac{-2q^{-1}}{q^2 + q^{-2}} \begin{pmatrix} q^3 & 0 \\ 0 & -q^{-3} \end{pmatrix}$$

and so

$$c \circ \sigma \otimes 1(R^{\overleftarrow{\nabla}^{\mathcal{S}}}) = c\Big(\sigma(C) \otimes \begin{pmatrix} -1/2 & 0 \\ 0 & q^2/2 \end{pmatrix}\Big) = \frac{1}{q^2 + q^{-2}} \begin{pmatrix} q^2 & 0 \\ 0 & q^{-2} \end{pmatrix},$$

which completes the proof. □

Using that

$$\mathcal{D}^2 \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} = \begin{pmatrix} q \partial_e \partial_f(b_i) s_{i,+} + q^{-3/2} \partial_e(b_i) s_{i,-} + b_i s_{i,+} \\ q^{-1} \partial_f \partial_e(c_l) s_{l,-} + q^{3/2} \partial_f(c_l) s_{l,+} + c_l s_{l,-} \end{pmatrix},$$

the Weitzenböck formula now yields the explicit expression

$$\Delta^{\mathcal{S}} \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} = \frac{1}{q^2 + q^{-2}} \begin{pmatrix} q^{-2} & 0 \\ 0 & q^2 \end{pmatrix} \begin{pmatrix} b_i s_{i,+} \\ c_l s_{l,-} \end{pmatrix} + \begin{pmatrix} q \partial_e \partial_f(b_i) s_{i,+} + q^{-3/2} \partial_e(b_i) s_{i,-} \\ q^{-1} \partial_f \partial_e(c_l) s_{l,-} + q^{3/2} \partial_f(c_l) s_{l,+} \end{pmatrix},$$

for the connection Laplacian. Finally we show that the Laplacian is positive, completing the analogy with the usual Lichnerowicz formula on the sphere.

Proposition 5.8. *The Laplacian $\Delta^{\mathcal{S}}$ is positive.*

Proof. By [MRC, Section 4.4], the result follows provided that for all $x, y \in S$, the divergence term $h(\langle \overrightarrow{\nabla}^{G}({}_{\Omega^1}\langle \overleftarrow{\nabla}^{\mathcal{S}} x \mid y \rangle) \mid G \rangle)$ vanishes where h is the Haar state of $SU_q(2)$.

First we compute

$$\langle \overrightarrow{\nabla}^{G}({}_{\Omega^1}\langle \overleftarrow{\nabla}^{\mathcal{S}} x \mid y \rangle) \mid G \rangle$$

$$= \mathrm{Tr}\Big(\begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}\Big(m\circ \overrightarrow{\nabla}^{G}(\langle \overleftarrow{\nabla}^{\mathcal{S}} x \mid y\rangle) - m\circ \overrightarrow{\nabla}^{G}(\omega_j)\langle \omega_j \mid \langle \overleftarrow{\nabla}^{\mathcal{S}} x \mid y\rangle\rangle_{\mathcal{B}}\Big)\Big),$$

and so it suffices to show that for all one-forms ω we have

$$h\Big(\mathrm{Tr}\Big(\begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}\Big(m\circ \overrightarrow{\nabla}^{G}(\omega)\Big)\Big)\Big) = 0. \tag{5.6}$$

Using (3.26) we find that

$$m\circ \overrightarrow{\nabla}^{G}([\mathcal{D},b]a) = \begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}\partial_e\partial_f(b)a + \begin{pmatrix} \partial_e(b)\partial_f(a) & 0 \\ 0 & \partial_f(b)\partial_e(a)\end{pmatrix}.$$

Since $\partial_e\partial_f(b) = \partial_f\partial_e(b)$ for all $b\in\mathcal{B}$ we have

$$\partial_e\partial_f(b)a = \partial_e(\partial_f(b)a) - q\partial_f(b)\partial_e(a) = \partial_f(\partial_e(b)a) - q^{-1}\partial_e(b)\partial_f(a).$$

Thus

$$m\circ \overrightarrow{\nabla}^{G}([\mathcal{D},b]a) = \begin{pmatrix} q\partial_f(\partial_e(b)a) & 0 \\ 0 & q^{-1}\partial_e(\partial_f(b)a)\end{pmatrix}$$

and so

$$h\Big(\mathrm{Tr}\Big(\begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}\Big(m\circ \overrightarrow{\nabla}^{G}([\mathcal{D},b]a)\Big)\Big)\Big) = q^2 h(\partial_f(\partial_e(b)a)) + q^{-2}h(\partial_e(\partial_f(b)a)).$$

Finally, $h(t^{\ell}_{ij}) = \delta_{\ell,0}$, and so $h(\partial_e(c)) = h(\partial_f(c)) = 0$ for all $c\in\mathcal{B}$, and hence the divergence (5.6) vanishes, and [MRC, Corollary 4.9] gives the positivity of $\Delta^{\mathcal{S}}$. □

Remark 5.9. For this example the coincidence between the action of the spinor curvature and 1/4 of the scalar curvature (which classically is $r = 2$) holds only in the classical limit $q = 1$.

A Uniqueness of Levi-Civita connection

In this appendix we present the uniqueness condition of Hermitian torsion-free σ-†-bimodule connections in the context of Grassmann connections associated to an exact frame. The discussion of [MRLC] simplifies substantially in this setting, and in particular becomes more algebraic.

To prove the uniqueness of the Hermitian torsion-free connection, we need to accomplish two tasks. First we need to prove the differential structure is concordant, which we describe and prove next. Then we need to verify an injectivity condition, stated below in Theorem A.7.

A.1 Extension to the local closure

For the first task, we need to make contact with the analytic context of [MRLC]. For this reason we recall a natural class of dense $*$-subalgebras of C^*-algebras. See [B98, Section 3.1] and [BM2, Section 3.3.1].

Definition A.1. Let B be a unital C^*-algebra. A unital $*$-subalgebra $\mathcal{B} \subset B$ is *local* if $\mathcal{B}$ is dense in B and if for all $n \in \mathbb{N}$ the $*$-subalgebra $M_n(\mathcal{B}) \subset M_n(B)$ is spectral invariant.

The assumption that $\mathcal{B} \subset B$ is local holds for many algebras $\mathcal{B}$, such as smooth functions on a manifold. Given a spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D})$, the completion $\mathcal{B}_1$ of the $*$-algebra $\mathcal{B}$ in the norm $\|b\|_{\mathcal{D}} := \|b\| + \|[\mathcal{D}, b]\|$ is always local, [BC91]. We refer to $\mathcal{B}_1$ as the local closure of $\mathcal{B}$ relative to $\mathcal{D}$.

In this paper we study the polynomial algebra of the Podleś sphere, which is not local. However, it has a local completion constructed from the Dabrowski-Sitarz spectral triple. The notion of local algebra is not necessary for the construction of the Hermitian, torsion-free σ-bimodule connection $\overrightarrow{\nabla}^G$ of Theorem 3.10 and Proposition 3.14. Nonetheless, locality is needed to establish its uniqueness.

Lemma A.2. *Let $(\mathcal{B}, \mathcal{H}, \mathcal{D})$ be the Dabrowski-Sitarz spectral triple for $\mathcal{B} = \mathcal{O}(S^2_q)$ and $(\mathcal{B}_1, \mathcal{H}, \mathcal{D})$ the spectral triple obtained by taking the local closure of $\mathcal{B}$ relative to $\mathcal{D}$. The inner product* (3.6) *on $\Omega^1_{\mathcal{D}}(\mathcal{B})$ extends to $\Omega^1_{\mathcal{D}}(\mathcal{B}_1)$ and the frame (ω_j)* (3.10) *is also a frame for $\Omega^1_{\mathcal{D}}(\mathcal{B}_1)$.*

Proof. This follows directly from the fact any $b \in \mathcal{B}_1$ can be approximated by a sequence $b_n \in \mathcal{B}$ in the norm $\|\cdot\|_{\mathcal{D}}$ and the formula for the inner product (3.6). □

Lemma A.3. *For $n \geq 1$ the map*

$$\mathcal{B}_1 \otimes_{\mathcal{B}} T^n_{\mathcal{D}}(\mathcal{B}) \otimes_{\mathcal{B}} \mathcal{B}_1 \to T^n_{\mathcal{D}}(\mathcal{B}_1), \quad b_1 \otimes \omega \otimes b_2 \mapsto b_1 \omega b_2,$$

are $\dagger$-bimodule isomorphisms. The maps $T^n_{\mathcal{D}}(\mathcal{B}) \to T^n_{\mathcal{D}}(\mathcal{B}_1)$ induced by the inclusion $\Omega^1_{\mathcal{D}}(\mathcal{B}) \to \Omega^1_{\mathcal{D}}(\mathcal{B}_1)$ are injective.

Proof. The only non-trivial point here is the surjectivity, which follows from the fact that for $\omega \in T^n_{\mathcal{D}}(\mathcal{B})$ we have $1 \otimes \omega \otimes 1 \mapsto \omega$, and thus a finite frame for $\mathcal{B}_1 \otimes_{\mathcal{B}} T^n_{\mathcal{D}}(\mathcal{B}) \otimes_{\mathcal{B}} \mathcal{B}_1$ gets mapped to a finite frame for $T^n_{\mathcal{D}}(\mathcal{B}_1)$. □

An immediate corollary is that each $T^n_{\mathcal{D}}(\mathcal{B}_1)$ is generated by elements $b_1 \omega b_2$ with $b_1, b_2 \in \mathcal{B}_1$ and $\omega \in T^n_{\mathcal{D}}(\mathcal{B})$. The projection Ψ and the braiding σ thus extend canonically to bimodule maps on $T^2_{\mathcal{D}}(\mathcal{B}_1)$. The following is verified by direct calculation.

Proposition A.4. *Any σ-bimodule connection $\overrightarrow{\nabla} : \Omega^1_{\mathcal{D}}(\mathcal{B}) \to T^2_{\mathcal{D}}(\mathcal{B})$ extends to a σ-bimodule connection $\overrightarrow{\nabla}_1 : \Omega^1_{\mathcal{D}}(\mathcal{B}_1) \to T^2_{\mathcal{D}}(\mathcal{B}_1)$ via the formula*

$$\overrightarrow{\nabla}_1(b_1 \omega b_2) := b_1 \overrightarrow{\nabla}(\omega) b_2 + b_1 \omega \otimes [\mathcal{D}, b_2] + \sigma^{-1}([\mathcal{D}, b_1] \otimes \omega b_2).$$

If furthermore $\overrightarrow{\nabla}$ is a σ-$\dagger$ bimodule connection then so is $\overrightarrow{\nabla}_1$.

A.2 Uniqueness

Definition A.5. Let $(\Omega^1_{\mathcal{D}}(\mathcal{B}_1), \dagger, \Psi, \langle \cdot \mid \cdot \rangle)$ be an Hermitian differential structure with $\mathcal{B}_1$ local. Define the projections $P := \Psi \otimes 1$ and $Q := 1 \otimes \Psi$ on $T^3_{\mathcal{D}}(\mathcal{B}_1)$. The differential structure is concordant if $T^3_{\mathcal{D}}(\mathcal{B}_1) = (\mathrm{Im}(P) \cap \mathrm{Im}(Q)) \oplus (\mathrm{Im}(1-P) + \mathrm{Im}(1-Q))$. Let Π be the projection onto $\mathrm{Im}(P) \cap \mathrm{Im}(Q)$.

Lemma 3.8 and Proposition 3.9 give us a frame for the two-junk module

$$Z = \frac{1}{\sqrt{q^2 + q^{-2}}} G, \quad X_k, \quad Y_k, \quad k = -2, -1, 0, 1, 2.$$

The three projections on two-forms $P_Z = |Z\rangle\langle Z|$, $P_X = \sum_k |X_k\rangle\langle X_k|$ and $P_Y = \sum_k |Y_k\rangle\langle Y_k|$ are mutually orthogonal and the projection onto the junk submodule is $\Psi = P_Z + P_X + P_Y$.

Lemma A.6. *The projections P_Z, P_X, P_Y satisfy*

$$P_X \otimes 1 \circ 1 \otimes P_Y = P_Y \otimes 1 \circ 1 \otimes P_X = 0,$$

$$P_X \otimes 1 \circ 1 \otimes P_Z \circ P_Z \otimes 1 \circ 1 \otimes P_Y = P_Y \otimes 1 \circ 1 \otimes P_Z \circ P_Z \otimes 1 \circ 1 \otimes P_X = 0$$
$$P_X \otimes 1 \circ 1 \otimes P_X \circ (\Psi \otimes 1 \circ 1 \otimes \Psi - P_X \otimes 1 \circ 1 \otimes P_X - P_Y \otimes 1 \circ 1 \otimes P_Y) = 0$$
$$P_Y \otimes 1 \circ 1 \otimes P_Y \circ (\Psi \otimes 1 \circ 1 \otimes \Psi - P_X \otimes 1 \circ 1 \otimes P_X - P_Y \otimes 1 \circ 1 \otimes P_Y) = 0$$
$$P_Y \otimes 1 \circ 1 \otimes P_Y \circ P_Z \otimes 1 = P_X \otimes 1 \circ 1 \otimes P_X \circ P_Z \otimes 1 = 0$$
$$P_X \otimes 1 \circ 1 \otimes P_Z \circ P_Y \otimes 1 = P_Y \otimes 1 \circ 1 \otimes P_Z \circ P_X \otimes 1 = 0$$
$$P_Z \otimes 1 \circ 1 \otimes P_Z \circ P_X \otimes 1 = P_Z \otimes 1 \circ 1 \otimes P_Z \circ P_Y \otimes 1 = 0$$
$$(P_X \otimes 1 \circ 1 \otimes P_X)^2 = P_X \otimes 1 \circ 1 \otimes P_X, \qquad (P_Y \otimes 1 \circ 1 \otimes P_Y)^2 = P_Y \otimes 1 \circ 1 \otimes P_Y,$$
$$(P_Z \otimes 1 \circ 1 \otimes P_X)^2 = \frac{q^2}{q^2 + q^{-2}} (P_Z \otimes 1 \circ 1 \otimes P_X), \quad (P_Z \otimes 1 \circ 1 \otimes P_Y)^2 = \frac{q^{-2}}{q^2 + q^{-2}} (P_Z \otimes 1 \circ 1 \otimes P_Y)$$
$$(P_Z \otimes 1 \circ 1 \otimes P_Z)^2 = \frac{1}{q^2 + q^{-2}} P_Z \otimes 1 \circ 1 \otimes P_Z.$$

If we write $V_{AB} = P_A \otimes 1 \circ 1 \otimes P_B$ then

$$\Psi \otimes 1 \circ 1 \otimes \Psi = V_{ZZ} + V_{ZX} + V_{XZ} + V_{ZY} + V_{YZ} + V_{XX} + V_{YY}$$

and

$$(\Psi \otimes 1 \circ 1 \otimes \Psi)^n = \frac{1}{(q^2 + q^{-2})^n} \big(V_{ZZ} + q^{2n}(V_{ZX} + V_{XZ}) + q^{-2n}(V_{ZY} + V_{YZ}) \big) + V_{XX} + V_{YY}.$$

Hence

$$\Pi = \lim_{n \to \infty} (\Psi \otimes 1 \circ 1 \otimes \Psi)^n = P_X \otimes 1 \circ 1 \otimes P_X + P_Y \otimes 1 \circ 1 \otimes P_Y$$

and the differential structure of the Podleś sphere is concordant.

Proof. The algebraic relations are all simple if tedious verifications.

To compute the limit $\lim_{n\to\infty}(P_Z\otimes 1\circ 1\otimes P_Z)^n$, we first note that $\langle\rho^\dagger\mid\omega_j\rangle_{\mathcal{B}}\omega_j^\dagger=\rho$, which allows us to compute

$$P_Z\otimes 1\circ 1\otimes P_Z(\rho\otimes\eta\otimes\tau)=Ze^{-\beta/2}\otimes\rho\langle\eta^\dagger\mid\tau\rangle_{\mathcal{B}}e^{-\beta/2}.$$

Repeating we find

$$(P_Z\otimes 1\circ 1\otimes P_Z)^2(\rho\otimes\eta\otimes\tau)=Ze^{-3\beta/2}\otimes\rho\langle\eta^\dagger\mid\tau\rangle_{\mathcal{B}}e^{-3\beta/2},$$

and as $e^{-\beta}=(q^2+q^{-2})^{-1}<1$, we readily see that $\lim_{n\to\infty}(P_Z\otimes 1\circ 1\otimes P_Z)^n=0$. The formula for $(\Psi\otimes 1\circ 1\otimes\Psi)^n$ is then just a consequence of the algebraic relations. The concordance is a consequence of [MRLC, Proposition 4.14]. □

We now turn to the injectivity condition. In this paper we are concerned with the Hermitian torsion-free †-bimodule Grassmann connection $\overset{\rightleftarrows}{\nabla}{}^G$ associated to the exact frame (ω_j). We rephrase the uniqueness condition of [MRLC, Theorem 5.14] for this particular case. In order to do this we require the bimodule isomorphisms

$$\begin{aligned}\overrightarrow{\alpha}:T^3_{\mathcal{D}}\to\overrightarrow{\mathrm{Hom}}{}^*_{\mathcal{B}}(\Omega^1_{\mathcal{D}},T^2_{\mathcal{D}}),\quad &\overrightarrow{\alpha}(\omega\otimes\eta\otimes\tau)(\rho):=\omega\otimes\eta\langle\tau^\dagger\mid\rho\rangle\\ \overleftarrow{\alpha}:T^3_{\mathcal{D}}\to\overleftarrow{\mathrm{Hom}}{}^*_{\mathcal{B}}(\Omega^1_{\mathcal{D}},T^2_{\mathcal{D}}),\quad &\overleftarrow{\alpha}(\eta\otimes\omega\otimes\tau)(\rho):=\langle\rho^\dagger\mid\eta\rangle\omega\otimes\tau,\end{aligned}\tag{A.1}$$

where $\rho,\eta,\tau,\omega\in\Omega^1_{\mathcal{D}}$.

We denote by $\mathcal{Z}(M)$ the centre of a $\mathcal{B}$-bimodule M.

Theorem A.7. *Let $(\Omega^1_{\mathcal{D}}(\mathcal{B}),\dagger,\Psi,\langle\cdot\mid\cdot\rangle)$ be the Hermitian differential structure of the Podleś sphere. If the braiding $\sigma:T^2_{\mathcal{D}}(\mathcal{B})\to T^2_{\mathcal{D}}(\mathcal{B})$ is such that the map*

$$\overrightarrow{\alpha}+\sigma^{-1}\circ\overleftarrow{\alpha}:\mathcal{Z}(\Pi(T^3_{\mathcal{D}}(\mathcal{B})))\to\overleftrightarrow{\mathrm{Hom}}(\Omega^1_{\mathcal{D}}(\mathcal{B}),T^2_{\mathcal{D}}(\mathcal{B}))$$

is injective, then $\overset{\rightleftarrows}{\nabla}{}^G$ is the unique Hermitian torsion-free σ-†-bimodule connection on $\Omega^1_{\mathcal{D}}(\mathcal{B})$.

Proof. Let $\overset{\rightleftarrows}{\nabla}{}^G_1$ be the extension of $\overset{\rightleftarrows}{\nabla}{}^G$ to $\Omega^1_{\mathcal{B}}(\mathcal{B}_1)$ (see Proposition A.4). We first note that †-concordance, [MRLC, Definition 4.26], is automatic in the presence of an exact frame [MRLC, Remark 4.27]. Now suppose that $A\in T^3_{\mathcal{D}}(\mathcal{B})$ is such that $\overset{\rightleftarrows}{\nabla}{}^G+\overrightarrow{\alpha}(A)$ is another Hermitian torsion-free σ-†-bimodule connection on $\Omega^1_{\mathcal{D}}(\mathcal{B})$. Extending to a connection on $\Omega^1_{\mathcal{D}}(\mathcal{B}_1)$, †-concordance gives us that $(1-\Pi)(A)=0$ in $T^3_{\mathcal{D}}(\mathcal{B}_1)$. Hence by Lemma A.3, $(1-\Pi)(A)=0$ in $T^3_{\mathcal{D}}(\mathcal{B})$ as well. Since $T^3_{\mathcal{D}}(\mathcal{B})$ is finite projective, the maps $\overrightarrow{\alpha}$ and $\overleftarrow{\alpha}$ are injective and the argument in the proof of [MRLC, Theorem 5.14] then shows that

$$(\overrightarrow{\alpha}+\sigma^{-1}\circ\overleftarrow{\alpha})(A)=0.$$

The injectivity hypothesis then gives us that $A=0$ and $\overset{\rightleftarrows}{\nabla}{}^G$ is unique. □

To check the injectivity condition of Theorem A.7, we first prove a few simple lemmas that help us identify the centre of the bimodule $\Pi(T^3_{\mathcal{D}}(\mathcal{B}))$.

Lemma A.8. *The commutant of $\mathcal{B}$ in $\mathcal{A} = \mathcal{O}(SU_q(2))$ is the scalar multiples of the identity.*

Proof. Every element of $\mathcal{O}(SU_q(2))$ is a linear combination of $a^i b^j (b^*)^k$ and $(a^*)^i b^j (b^*)^k$ for $i, j, k \in \mathbb{N}$. Up to scalar multiples, the generators of $\mathcal{B}$ are bb^*, ab, b^*a^*. For scalars $\lambda_{ijk}, \rho_{ijk}$ we have

$$\big[\sum_{ijk} \lambda_{ijk} a^i b^j (b^*)^k + \rho_{ijk}(a^*)^i b^j (b^*)^k, bb^*\big] = \sum_{ijk} [\lambda_{ijk} a^i + \rho_{ijk}(a^*)^i, bb^*] b^j (b^*)^k$$
$$= bb^* \sum_{ijk} \big(\lambda_{ijk}(q^{2i} - 1)a^i + \rho_{ijk}(q^{-2i} - 1)(a^*)^i\big) b^j (b^*)^k.$$

The linear independence of the monomials $a^i b^j (b^*)^k$ and $(a^*)^i b^j (b^*)^k$ shows that the sum vanishes only when all coefficients $\lambda_{ijk}, \rho_{ijk}$ vanish except perhaps $i = 0$. Then consider

$$\Big[\sum_{jk} \lambda_{0jk} b^j (b^*)^k, ab\Big] = \sum_{jk} \big[\lambda_{0jk} b^j (b^*)^k, ab\big] = ab \sum_{jk} (q^{-j-k} - 1)\lambda_{0jk} b^j (b^*)^k,$$

and again the commutator vanishes only when all $\lambda_{0jk} = 0$ except perhaps $j = k = 0$. Hence only scalars in $\mathcal{A}$ commute with all of $\mathcal{B}$. □

Since $\Omega^1_{\mathcal{D}}(\mathcal{B}) \subset \mathcal{O}(SU_q(2))^{\oplus 2}$ does not intersect the scalars we have

Corollary A.9. *The centre of $\Omega^1_{\mathcal{D}}(\mathcal{B})$ is $\{0\}$.*

Lemma A.10. *The linear maps $\mathbb{X}, \mathbb{Y} : T^3_{\mathcal{D}}(\mathcal{B}) \to \Omega^1_{\mathcal{D}}(\mathcal{B})$ given on simple tensors $\rho \otimes \eta \otimes \tau \in T^3_{\mathcal{D}}(\mathcal{B})$ by*

$$\mathbb{X}(\rho \otimes \eta \otimes \tau) = \sum_{k=-2,-2,0,1,2} (t^2_{k2})^* \langle X_k \mid \rho \otimes \eta \rangle_{\mathcal{B}}\, \tau$$

and

$$\mathbb{Y}(\rho \otimes \eta \otimes \tau) = \sum_{k=-2,-2,0,1,2} (t^2_{k,-2})^* \langle Y_k \mid \rho \otimes \eta \rangle_{\mathcal{B}}\, \tau$$

are bimodule maps, and so map the centre $\mathcal{Z}T^3_{\mathcal{D}}(\mathcal{B})$ to the centre $\mathcal{Z}\Omega^1_{\mathcal{D}}(\mathcal{B}) = \{0\}$.

Proof. We prove the result for $\mathbb{X}$ as the argument for $\mathbb{Y}$ is the same. Let $\rho \otimes \eta \otimes \tau$ be a simple tensor and write $\rho = \begin{pmatrix} 0 & \rho_+ \\ \rho_- & 0 \end{pmatrix}$ and similarly for η, τ. For $b \in \mathcal{B}$ we have $\mathbb{X}(\rho \otimes \eta \otimes \tau b) = \mathbb{X}(\rho \otimes \eta \otimes \tau)b$ by definition. Using Lemma 3.8 we have

$$\mathbb{X}(b\rho \otimes \eta \otimes \tau) = \sum_{k=-2,-2,0,1,2} (t^2_{k2})^* \langle X_k \mid b\rho \otimes \eta \rangle_{\mathcal{B}}\, \tau = q \sum_{k=-2,-2,0,1,2} (t^2_{k2})^* t^2_{k2} b \rho_- \eta_-\, \tau$$
$$= b\rho_- \eta_-\, \tau = b\mathbb{X}(\rho \otimes \eta \otimes \tau).$$

Hence $\mathbb{X}$ is a bimodule map, and so maps centres to centres. □

Corollary A.11. *For $q \in (0, 1)$ we have $\mathcal{Z}\Pi(T^3_{\mathcal{D}}(\mathcal{B})) = \{0\}$.*

Proof. Let $\rho \otimes \eta \otimes \tau$ be a simple tensor and again write $\rho = \begin{pmatrix} 0 & \rho_+ \\ \rho_- & 0 \end{pmatrix}$ and similarly for η, τ. Then, using the orthogonality relations, the projection onto $\mathrm{Im}(\Pi)$ is

$$\begin{aligned}
\Pi(\rho \otimes \eta \otimes \tau) &= \begin{pmatrix} 0 & 0 \\ \rho_- & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ \eta_- & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ \tau_- & 0 \end{pmatrix} + \begin{pmatrix} 0 & \rho_+ \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \eta_+ \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \tau_+ \\ 0 & 0 \end{pmatrix} \\
&= \sum_{j,k=-1,0,1} \begin{pmatrix} 0 & 0 \\ (t^1_{j1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{j1}\rho_-\eta_-\tau_-(t^1_{k,-1})^* & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ t^1_{k,-1} & 0 \end{pmatrix} \\
&+ \sum_{j,k=-1,0,1} \begin{pmatrix} 0 & (t^1_{j,-1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{j,-1}\rho_+\eta_+\tau_+(t^1_{k1})^* \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & t^1_{k1} \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Hence a linear combination $\Pi(\sum_i \rho^i \otimes \eta^i \otimes \tau^i)$ is zero if and only if both $\sum_i \rho^i_- \eta^i_- \tau^i_- = 0$ and $\sum_i \rho^i_+ \eta^i_+ \tau^i_+ = 0$. These observations together with Lemma A.10 prove the claim. □

Theorem A.12. *The †-bimodule connection* $(\overrightarrow{\nabla}^G, \sigma)$ *is the unique Hermitian torsion-free †-bimodule connection on* $\Omega^1_{\mathcal{D}}(\mathcal{B})$.

Proof. We need to show that for $q \in (0, 1]$ the linear map

$$\overrightarrow{\alpha} + \sigma^{-1} \circ \overleftarrow{\alpha} : \mathcal{Z}(\Pi T^3_{\mathcal{D}}(\mathcal{B})) \to \overleftrightarrow{\mathrm{Hom}}(\Omega^1_{\mathcal{D}}, T^2_{\mathcal{D}}),$$

is injective. For $q \in (0, 1)$ this follows from Corollary A.11. For $q = 1$, [MRLC, Lemma 6.11] shows that $\overrightarrow{\alpha} + \sigma^{-1} \circ \overleftarrow{\alpha}$ is injective. □

Conflict of interest The authors declare that there is no conflict of interest.

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