Topological Levinson's theorem in presence of embedded thresholds and discontinuities of the scattering matrix: a quasi-1D example

V. Austen[1], D. Parra[2], A. Rennie[3]*, S. Richard[4]†

1 Department of chemistry, Graduate School of Science, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan

2 Departamento de Matemática y Estadística, Universidad de La Frontera, Av. Francisco Salazar 01145, Temuco, Chile

3 School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, Australia

4 Institute for Liberal Arts and Sciences & Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8601, Japan

E-mail: austen.vic.a9@s.thers.ac.jp, daniel.parra.v@usach.cl, renniea@uow.edu.au, richard@math.nagoya-u.ac.jp

Abstract

A family of quasi-1D Schrödinger operators is investigated through scattering theory. The continuous spectrum of these operators exhibit changes of multiplicity, and some of these operators possess resonances at thresholds. It is shown that the corresponding wave operators belong to an explicitly constructed C^*-algebra. The quotient of this algebra by the ideal of compact operators is studied, and an index theorem is deduced from these investigations. This result corresponds to a topological version of Levinson's theorem in presence of embedded thresholds, resonances, and changes of multiplicity of the scattering matrices. In the last two sections of the paper, the K-theory of the main C^*-algebra and the dependence on an external parameter are carefully analysed. In particular, a surface of resonances is exhibited, probably for the first time. The content of these two sections is of independent interest, and the main result does not depend on them.

2010 Mathematics Subject Classification: 81U10, 47A40

Keywords: Scattering matrix, wave operators, thresholds, index theorem, Levinson's theorem

Contents

*Supported by ARC Discovery grant DP220101196.

†Supported by JSPS Grant-in-Aid for scientific research C no 21K03292.

1 Introduction

In this paper we prove a topological version of Levinson's theorem for scattering systems with non-constant multiplicity of the continuous spectrum. Levinson's theorem relates the number of bound states of a quantum mechanical system to the scattering part of the system. The original formulation was by N. Levinson in [26] in the context of a Schrödinger operator with a spherically symmetric potential. In [21, 22, 23], a topological interpretation of Levinson's theorem was proposed. The topological approach consists in constructing a C^*-algebraic framework for the scattering system involving not only the scattering operator, but also new operators that describe the system at thresholds energies. The number of bound states is the Fredholm index of one wave operator, evaluated via a winding number.

The review paper [30] describes the topological approach and applications to several classical models, see also [2, 6, 16, 17, 18, 28, 31, 34] for similar investigations. A common feature of the examples considered so far is that the continuous spectrum of the underlying self-adjoint operators is made of one connected set, with no change of multiplicity. The non-constancy of the multiplicity of continuous spectrum considered in this paper has various drastic effects: the spectral representation of the underlying unperturbed operator takes place in a direct integral with fibres of non-constant dimension; the scattering matrices are acting on different Hilbert spaces with jumps in their dimension; the C^*-algebras containing the wave operators and the scattering operator have a non-trivial internal structure. The notion of continuity of the scattering operator, which has played a crucial role in all investigations on Levinson's theorem, has also to be suitably adapted.

These interesting new features arise in a family of systems that were introduced and partially studied in [27] as parts of a discrete model of scattering theory in a half-space, see also [8, 9, 12, 13, 19, 32, 33]. Independently, these systems correspond to magnetic systems on the graph $\mathbb{N} \times (\mathbb{Z}/N\mathbb{Z})$, for some $N \in \mathbb{N}$, with a magnetic field constant in the $\mathbb{N}$ direction, and with a perturbation supported on $\{0\} \times (\mathbb{Z}/N\mathbb{Z})$, see [10, 25, 35] for seminal references on discrete magnetic operators. These systems have changes of multiplicity in the continuous spectrum, and accordingly embedded thresholds. The set of thresholds is defined by the set $\{\lambda_j \pm 2\}_{j=1}^N$, where $\{\lambda_j\}_{j=1}^N$ correspond to the eigenvalues of a magnetic Laplace operator on $\mathbb{Z}/N\mathbb{Z}$. The resulting scattering matrices act on some finite dimensional Hilbert spaces with their dimension depending on the energy parameter. Even if a global notion of continuity can not be defined in this setting, it has been shown in [27, Sec. 3.3] that a local notion of continuity holds, together with the existence of left or right limits at the points corresponding to a change of multiplicity (or equivalently to a change of dimension).

Let us now describe more precisely the content of this paper and state its main result. A comparison with the existing literature is provided subsequently. In Section 2 we introduce the model we shall consider and recall the main properties exhibited in [27]. As already mentioned, the unperturbed system H_0 is a discrete magnetic Schrödinger operator acting on a cylinder of the form $\mathbb{N} \times (\mathbb{Z}/N\mathbb{Z})$ for any integer $N \geq 2$, with a constant magnetic field in the $\mathbb{N}$ direction, whose strength is encoded by $\theta \in (0, \pi)$. The perturbed system H is a finite rank perturbation with perturbation supported on $\{0\} \times (\mathbb{Z}/N\mathbb{Z})$. The full spectral and scattering theory for the pair of operators (H, H_0) has been investigated in [27]. After introducing the present precise formulas for the spectral representation of H_0, we recall the main properties of the scattering operator S and of its representation $\{S(\lambda)\}_{\lambda \in \sigma(H_0)}$ in the spectral representation of H_0. Note that the changes of multiplicity clearly appear in the spectral decomposition of H_0 and in the family of scatting matrices $S(\lambda)$. They will persist throughout the subsequent analysis.

In Section 3 we start by recalling an explicit formula for the wave operator W_-, see Theorem 3.1. Then we define two C^*-algebras: the C^*-algebra $\mathcal{A}$ which contains (after a unitary transformation) the scattering operator S, and the C^*-algebra $\mathcal{E}$ which contains (after a unitary transformation) the wave operator W_-. These algebras are constructed in the spectral representation of H_0. Note that the C^*-algebra $\mathcal{E}$ is built from functions of energy (with values in matrices of varying dimension) as well as functions of the appropriate derivative on the energy spectrum. This section also contains a first description of the quotient algebra $\mathcal{Q}$ of $\mathcal{E}$ by the ideal of compact operators. In particular, the proof that $\mathcal{Q}$ consists of matrix-valued functions on a compact Hausdorff space uses techniques of "twisted commutators" [11] which have not appeared before in the scattering literature. The quotient algebra $\mathcal{Q}$ can be thought of as *an upside down comb* with $2N$ teeth: the first N of them support the operators corresponding to the opening of new channels of scattering, while the last N teeth support the operators closing channels of scattering. The structure of the quotient algebra allows us to analyse the various possible behaviours of the system at embedded thresholds.

Section 4 contains the newly developed topological Levinson's theorem, in presence of embedded thresholds and discontinuities of the scattering matrix. The number of bound states $\#\sigma_{\mathrm{p}}(H^\theta)$ is given by

$$\operatorname{Var}\big(\lambda \to \det S(\lambda)\big) + N - \frac{\#\big\{j \mid \mathfrak{s}_{jj}(\lambda_j \pm 2) = 1\big\}}{2} + = \#\sigma_{\mathrm{p}}(H) \tag{1.1}$$

where $\mathfrak{s}_{jj}(\lambda_j \pm 2)$ corresponds to a distinguished entry of the scattering matrix at the threshold energy $\lambda_j \pm 2$, and where $\operatorname{Var}\big(\lambda \mapsto \det S(\lambda)\big)$ is the total variation of the argument of the piecewise continuous function $\lambda \mapsto \det S(\lambda)$. We refer to Theorem 4.2 for the details and for more explanations. The proof is based on the usual winding number argument. Again, its striking feature is its topological nature, despite the non-continuity of the scattering matrix, and even the change of dimension of these matrices. In this section, we also describe the behaviour of the scattering matrix at the opening or at the closing of a new channel of scattering. For example, it is shown in Lemma 4.3 that the distinguished entry of the scattering matrix takes generically the value -1 at thresholds, and therefore the contribution $s_{jj}(\lambda_j \pm 2) = 1$ in (1.1) is exceptional. Because each exceptional contribution would be of $-\frac{1}{2}$, this situation is usually referred to as the appearance of a half-bound state.

The remaining two sections are of independent interest. In Section 5, the K-theory for the algebra $\mathcal{Q}$ is investigated. The algebra is efficiently described by assembling matrix-valued functions on intervals by gluing them at boundaries. This allows the Mayer-Vietoris theorem to be used to compute the K-theory inductively. The final result is rather simple, see Corollary 5.7, but the construction for getting this result is rather instructive. We suspect that the algebra $\mathcal{Q}$ will be used again for other scattering systems with opening and closing of new channels of scattering, its structure does not really depend on the specificity of the model considered in this paper. In fact, our new understanding for the change of multiplicity opens the door for various new applications, such as the N-body problem or highly anisotropic systems.

Finally, we gather in Section 6 some results about the dependence of various quantities upon the parameter θ. For simplicity, the results are stated for $N = 2$, for which all computations can be performed explicitly. For example, for fixed values of the perturbation, we provide an illustration of the non-trivial dependence on θ for the number of bound states of H^θ (we introduced the θ-dependence for clarity), see Figure 3. We also exhibit necessary and sufficient conditions on the perturbation for having a resonance at the lowest value of the essential spectrum for some $\theta \in (0, \pi)$, More precisely, for the perturbations in an open set, there exists a unique $\theta \in (0, \pi)$ for which the scattering system exhibit a resonance at the lowest value of its continuous spectrum, see Figure 4. By collecting these exceptional values of θ one gets the surface of resonances presented in Figure 5. Note that the study of a family of scattering systems (H^θ, H_0^θ) and the dependence on θ of the wave operators $W_\pm(H^\theta, H_0^\theta)$ and of the corresponding scattering operator S^θ, has probably never been studied. Usually, the reference operator H_0 does not depend on an external parameter, while a family of perturbed operator H_κ (depending on a parameter κ) have already been investigated and the resulting wave operators and scattering operators are known to depend very weakly on κ, see for example [7]

Let us now come to a comparison with the existing literature, and provide additional explanations about our results. Discrete Schrödinger operators on $\mathbb{Z}$ or on $\mathbb{N}$ and taking values in $\mathbb{C}^N$ have already been extensively investigated. When $N = 1$, let us mention the seminal work [15] in which Levinson's theorem is studied, and half-bound stated are also considered. More recently, the general case (N arbitrary) has been extensively analysed in [1] and in the series of papers [3, 4, 5]. In particular, long-range type perturbations are considered in [4, 5], necessitating a regularisation process for the statement of Levinson's theorem. Since our perturbation is compactly supported, and closer to the content of [3], such a regularisation process is not necessary for topological version of Levinson's theorem. However, in these references the spectrum of the operator H_0 does not exhibit a change of multiplicity, which means that no internal thresholds appear. Accordingly, half-bound states can only occur at the two ends of the continuous spectrum and not inside the continuous spectrum. In the framework of these investigations, the absence of embedded eigenvalue has also been proved.

Our setting is slightly different. Even if the model is also quasi-1-dimensional, both operators H_0 and H exhibit changes of multiplicity in their spectrum. Internal thresholds are therefore inherent. However, it has not been proved (or disproved) that half-bound states can appear at internal thresholds. It has also not been proved yet that embedded eigenvalues do not exist, even if they are not likely. Such investigations are currently taking place, and further results are expected in the future. Note however that our current statements hold no matter if these internal singularities exist or not.

Let us finally comment on the external fixed parameter θ which corresponds to the intensity of a magnetic field constant in the $\mathbb{N}$ direction. For simplicity, its value have been fixed by the condition $\theta \in (0, \pi)$. First of all, values in $(\pi, 2\pi)$ could have been considered as well, but can be deduced by some symmetry arguments. The values $\theta \in \{0, \pi, 2\pi\}$ are more singular: For these values, some λ_j have multiplicity 2, which does not really make the analysis much harder, but does make the description of the quotient algebra $\mathcal{Q}$ more intricate, with sometimes a change of multiplicity 1, and sometimes a change of multiplicity 2. In addition, for $\theta = 0$, an exceptional case has been left unsolved in [27, Sec. 4.3], and it has not been shown in that reference that a remainder term is really compact. For these reasons and for simplicity, it has been decided that the special cases $\theta \in \{0, \pi, 2\pi\}$ would be deferred to a subsequent work.

Acknowledgements
A. R. acknowledges the support of the ARC Discovery grant DP220101196.

2 The model

In the Hilbert space $\mathcal{H} := \ell^2(\mathbb{N})$, with $\mathbb{N} := \{0, 1, 2, \dots\}$, we consider the discrete Neumann adjacency operator whose action on $\phi \in \ell^2(\mathbb{N})$ is given by

$$\big(\Delta\,\phi\big)(n) = \begin{cases} 2^{1/2}\,\phi(1) & \text{if } n = 0 \\ 2^{1/2}\,\phi(0) + \phi(2) & \text{if } n = 1 \\ \phi(n+1) + \phi(n-1) & \text{if } n \geq 2. \end{cases}$$

For any fixed $N \in \mathbb{N}$ with $N \geq 2$ and for any fixed $\theta \in (0, \pi)$ we also consider the $N \times N$ Hermitian matrix

$$A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & \mathrm{e}^{-i\theta} \\ 1 & 0 & 1 & \ddots & & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & & \ddots & 1 & 0 & 1 \\ \mathrm{e}^{i\theta} & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{2.1}$$

These two ingredients lead to the self-adjoint operator h_0 acting on $\ell^2\big(\mathbb{N};\mathbb{C}^N\big)$ as

$$h_0 := \Delta \otimes 1_N + A$$

with 1_N the $N \times N$ identity matrix. This operator can be viewed as a discrete magnetic adjacency operator, see Remark 2.1.

The perturbed operator h describing the discrete quantum model is then given by

$$h := h_0 + V,$$

where V is the multiplication operator by a nonzero, matrix-valued function v with support on $\{0\} \in \mathbb{N}$. In other words, there exists a nonzero function $v : \{1, \dots, N\} \to \mathbb{R}$ such that for $\psi \in \ell^2\big(\mathbb{N};\mathbb{C}^N\big)$, $n \in \mathbb{N}$, and $j \in \{1, \dots, N\}$ one has

$$(h\psi)_j(n) = (h_0\psi)_j(n) + \delta_{0,n}\, v(j)\, \psi_j(0),$$

with $\delta_{0,n}$ the Kronecker delta function.

Remark 2.1. *The operator h_0 can be interpreted as a magnetic adjacency operator on the Cayley graph of the semi-group $\mathbb{N} \times (\mathbb{Z}/N\mathbb{Z})$ with respect to a magnetic field constant in the $\mathbb{N}$ direction. In particular, it corresponds to choosing a magnetic potential supported only on edges of the form $\big((n, N), (n, 1)\big)$ for any $n \in \mathbb{N}$. In this representation, the perturbation V corresponds to a multiplicative perturbation with support on $\{0\} \times (\mathbb{Z}/N\mathbb{Z})$.*

We now set

$$\mathfrak{h} := \mathrm{L}^2\left([0, \pi), \tfrac{\mathrm{d}\omega}{\pi}; \mathbb{C}^N\right).$$

By a suitable unitary transform introduced in [27, Sec. 2], it can be shown that the operator h_0 is unitarily equivalent to the operator

$$H_0 := 2\cos(\Omega) \otimes 1_N + A$$

in $\mathfrak{h}$ with the operator $2\cos(\Omega)$ of multiplication by the function $[0, \pi) \ni \omega \mapsto 2\cos(\omega) \in \mathbb{R}$. Through the same unitary transform, the operator h is unitarily equivalent to

$$H := 2\cos(\Omega) \otimes 1_N + A + \mathrm{diag}(v)P_0$$

with

$$\big(\mathrm{diag}(v)\mathfrak{f}\big)_j := v(j)\mathfrak{f}_j \quad \text{and} \quad \big(P_0\mathfrak{f}\big)_j := \int_0^\pi \mathfrak{f}_j(\omega)\tfrac{\mathrm{d}\omega}{\pi}$$

for $\mathfrak{f} \in \mathfrak{h}$, $j \in \{1, \dots, N\}$. Observe that the term $\mathrm{diag}(v)P_0$ corresponds to a finite rank perturbation, and therefore H and H_0 differ only by a finite rank operator.

Let us now move to spectral results. A direct inspection shows that the matrix A has eigenvalues

$$\lambda_j := 2\cos\left(\frac{\theta + 2\pi j}{N}\right), \quad j \in \{1, \dots, N\},$$

with corresponding eigenvectors $\xi_j \in \mathbb{C}^N$ having components $\big(\xi_j\big)_k := \mathrm{e}^{i(\theta+2\pi j)k/N}$, $j, k \in \{1, \dots, N\}$. Using the notation $\mathcal{P}_j \equiv \frac{1}{N}|\xi_j\rangle\langle\xi_j|$ for the orthogonal projection associated to ξ_j, one has $A = \sum_{j=1}^N \lambda_j \mathcal{P}_j$.

The next step consists in exhibiting the spectral representation of H_0. For that purpose, we first define for $j \in \{1, \dots, N\}$ the sets

$$I_j := \big(\lambda_j - 2, \lambda_j + 2\big) \quad \text{and} \quad I := \cup_{j=1}^N I_j,$$

with λ_j the eigenvalues of A. Also, we consider for $\lambda \in I$ the fiber Hilbert space

$$\mathscr{H}(\lambda) := \mathrm{span}\left\{\mathcal{P}_j\mathbb{C}^N \mid j \in \{1, \dots, N\} \text{ such that } \lambda \in I_j\right\} \subset \mathbb{C}^N,$$

and the corresponding direct integral Hilbert space

$$\mathscr{H} := \int_I^\oplus \mathscr{H}(\lambda)\,\mathrm{d}\lambda.$$

Then, the map $\mathscr{F} : \mathfrak{h} \to \mathscr{H}$ acting on $\mathfrak{f} \in \mathfrak{h}$ and for a.e. $\lambda \in I$ as

$$\big(\mathscr{F}\mathfrak{f}\big)(\lambda) := \pi^{-1/2} \sum_{\{j \mid \lambda \in I_j\}} \big(4 - (\lambda - \lambda_j)^2\big)^{-1/4} \mathcal{P}_j\mathfrak{f}\left(\arccos\left(\tfrac{\lambda - \lambda_j}{2}\right)\right).$$

is unitary. In addition, $\mathscr{F}$ diagonalises the Hamiltonian H_0, namely for all $\zeta \in \mathscr{H}$ and a.e. $\lambda \in I$ one has

$$\big(\mathscr{F} H_0 \mathscr{F}^* \zeta\big)(\lambda) = \lambda \zeta(\lambda) = \big(\mathbf{X}\zeta\big)(\lambda),$$

with $\mathbf{X}$ the (bounded) operator of multiplication by the variable in $\mathscr{H}$. One infers that H_0 has purely absolutely continuous spectrum equal to

$$\sigma(H_0) = \overline{\operatorname{Ran}(\mathbf{X})} = \overline{I} = \big[\min_j \lambda_j - 2, \max_j \lambda_j + 2\big] \subset [-4, 4].$$

Note that we use the notation $\overline{I}$ for the closure of a set $I \subset \mathbb{R}$.

The spectral representation of H_0 leads also naturally to the notion of thresholds: these real values correspond to a change of multiplicity of the spectrum. Clearly, the set $\mathcal{T}$ of thresholds for the operator H_0 is given by

$$\mathcal{T} := \big\{\lambda_j \pm 2 \mid j \in \{1, \dots, N\}\big\}. \tag{2.2}$$

The main spectral result for H has been presented in [27, Prop. 1.2].??? About scattering theory, since the difference $H - H_0$ is a finite rank operator we observe that the wave operators

$$W_\pm := \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH}\, \mathrm{e}^{-itH_0}$$

exist and are complete, see [20, Thm X.4.4]. As a consequence, the scattering operator

$$S := W_+^* W_-$$

is a unitary operator in $\mathfrak{h}$ commuting with H_0, and thus S is decomposable in the spectral representation of H_0, that is for $\zeta \in \mathscr{H}$ and a.e. $\lambda \in I$, one has

$$\big(\mathscr{F} S \mathscr{F}^* \zeta\big)(\lambda) = S(\lambda)\zeta(\lambda),$$

with the scattering matrix $S(\lambda)$ a unitary operator in $\mathscr{H}(\lambda)$.

For $j, j' \in \{1, \dots, N\}$ and for $\lambda \in \big(I_j \cap I_{j'}\big) \setminus \big(\mathcal{T} \cup \sigma_{\rm p}(H)\big)$ let us define the channel scattering matrix $S(\lambda)_{jj'} := \mathcal{P}_j S(\lambda) \mathcal{P}_{j'}$ and consider the map

$$\big(I_j \cap I_{j'}\big) \setminus \big(\mathcal{T} \cup \sigma_{\rm p}(H)\big) \ni \lambda \mapsto S(\lambda)_{jj'} \in \mathscr{B}\big(\mathcal{P}_{j'}\mathbb{C}^N; \mathcal{P}_j\mathbb{C}^N\big).$$

The continuity of the scattering matrix at embedded eigenvalues has been shown in [27, Thm 3.10], while its behaviour at thresholds has been studied in [27, Thm 3.9]. This latter result can be summarised as follows: For each $\lambda \in \mathcal{T}$, a channel can already be opened at the energy λ (in which case the existence and the equality of the limits from the right and from the left is proved), it can open at the energy λ (in which case only the existence of the limit from the right is proved), or it can close at the energy λ (in which case only the existence of the limit from the left is proved).

3 The C^*-algebraic framework

In this section we construct a C^*-algebra which will contain the wave operator W_-. We also compute the quotient of this C^*-algebra by the ideal of compact operators.

Let us firstly recall the specific form of the wave operators. If we set $\mathscr{K}(\mathfrak{h})$ for the set of compact operators on $\mathfrak{h}$, then the following statement has been proved in [27, Thm 1.5]. Let us emphasise that this result is at the root of the subsequent investigations.

Theorem 3.1. *The following equality holds:*

$$W_- - 1 = \tfrac{1}{2}\big(1 - \tanh(\pi\mathfrak{D}) - i\cosh(\pi\mathfrak{D})^{-1}\tanh(\mathfrak{X})\big)(S-1) + \mathfrak{K},$$

with $\mathfrak{K} \in \mathscr{K}(\mathfrak{h})$ and where $\mathfrak{X}$ and $\mathfrak{D}$ are representations of the canonical position and momentum operators in the Hilbert space $\mathfrak{h}$.

We now exhibit an algebra $\mathcal{A}$ of multiplication operators acting on $\mathscr{H}$.

Definition 3.2. *Let $\mathcal{A} \subset \mathscr{B}(\mathscr{H})$ with $a \in \mathcal{A}$ if for any $\zeta \in \mathscr{H}$ and $\lambda \in I$ one has*

$$\big[a\zeta\big](\lambda) \equiv \big[a(\mathbf{X})\zeta\big](\lambda) = a(\lambda)\zeta(\lambda)$$

and for $j, j' \in \{1, \dots, N\}$ the maps

$$I_j \cap I_{j'} \ni \lambda \mapsto a_{jj'}(\lambda) := \mathcal{P}_j a(\lambda) \mathcal{P}_{j'} \in \mathscr{B}\big(\mathscr{H}(\lambda)\big) \subset \mathscr{B}(\mathbb{C}^N) \tag{3.1}$$

belongs to $C_0\left(I_j \cap I_{j'}; \mathscr{B}(\mathbb{C}^N)\right)$ if $j \neq j'$ while it belongs to $C\left(\overline{I_j}; \mathscr{B}(\mathbb{C}^N)\right)$ if $j = j'$.

In other words, the function defined in (3.1) is a continuous function vanishing at the boundaries of the intersection $I_j \cap I_{j'}$ when these two intervals are not equal, and has limits at the boundaries of these intervals when they coincide.

Remark 3.3. *In the sequel, we shall use the notation $a_{jj'}$ for $\mathcal{P}_j a \mathcal{P}_{j'}$ even if this notation is slightly misleading. Indeed, $a_{jj'}(\lambda)$ exists if and only if $\lambda \in \overline{I_j \cap I_{j'}}$. If $\lambda \notin \overline{I_j}$ or if $\lambda \notin \overline{I_{j'}}$, then the expression containing $a_{jj'}(\lambda)$ should be interpreted as 0. More precisely, with the standard bra-ket notation, we shall write $a_{jj'} = \mathfrak{a}_{jj'} \otimes \frac{1}{N}|\xi_j\rangle\langle\xi_{j'}|$, where $\mathfrak{a}_{jj'}$ denotes the multiplication operator by a function belonging to $C_0\left(I_j \cap I_{j'}; \mathbb{C}\right)$ if $j \neq j'$, while it belongs to $C\big(\overline{I_j}; \mathbb{C}\big)$ if $j = j'$. Thus, with these notations and for any $\lambda \in I$ one has*

$$a(\lambda) = \sum_{\{j,j' \mid \lambda \in I_j, \lambda \in I_{j'}\}} \mathfrak{a}_{jj'}(\lambda) \otimes \tfrac{1}{N}|\xi_j\rangle\langle\xi_{j'}|$$

By the above abuse of notation, we shall conveniently write $a = \sum_{j,j'} \mathfrak{a}_{jj'} \otimes \frac{1}{N}|\xi_j\rangle\langle\xi_{j'}|$.

The interest of the algebra $\mathcal{A}$ comes from the following statement, which can be directly inferred from [27, Thms 3.9 & 3.10]:

Proposition 3.4. *The multiplication operator defined by the map $\lambda \to S(\lambda)$ belongs to $\mathcal{A}$.*

For subsequent constructions, we need to know that some specific functions also belong to $\mathcal{A}$. For this, we recall the definition of a useful unitary map, introduced in [27, Sec. 4.1]. For $j \in \{1, \dots, N\}$ we define the unitary map $\mathcal{V}_j : \mathrm{L}^2(I_j) \to \mathrm{L}^2(\mathbb{R})$ given on $\zeta \in \mathrm{L}^2(I_j)$ and for $s \in \mathbb{R}$ by

$$\big[\mathcal{V}_j\zeta\big](s) := \tfrac{2^{1/2}}{\cosh(s)}\, \zeta\big(\lambda_j + 2\tanh(s)\big). \tag{3.2}$$

Its adjoint $\mathcal{V}_j^* : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(I_j)$ is then given on $f \in \mathrm{L}^2(\mathbb{R})$ and for $\lambda \in I_j$ by

$$[\mathcal{V}_j^* f](\lambda) = \left(\tfrac{2}{4-(\lambda-\lambda_j)^2}\right)^{1/2} f\left(\operatorname{arctanh}\left(\tfrac{\lambda-\lambda_j}{2}\right)\right). \tag{3.3}$$

Based on this, we define the unitary map $\mathcal{V} : \mathscr{H} \to \mathrm{L}^2(\mathbb{R};\mathbb{C}^N)$ acting on $\zeta \in \mathscr{H}$ as

$$\mathcal{V}\zeta := \sum_{j=1}^{N} \big(\mathcal{V}_j \otimes \mathcal{P}_j\big)\zeta|_{I_j},$$

with adjoint $\mathcal{V}^* : \mathrm{L}^2(\mathbb{R};\mathbb{C}^N) \to \mathscr{H}$ acting on $f \in \mathrm{L}^2(\mathbb{R};\mathbb{C}^N)$ and for $\lambda \in I$ as

$$[\mathcal{V}^* f](\lambda) = \sum_{\{j|\lambda\in I_j\}} \big[\big(\mathcal{V}_j^* \otimes \mathcal{P}_j\big) f\big](\lambda).$$

Let X denote the self-adjoint operator of multiplication by the variable in $\mathrm{L}^2(\mathbb{R})$.

Lemma 3.5. *For $\zeta \in \mathscr{H}$ and $\lambda \in I$ one has*

$$\big[\mathcal{V}^*\big(\tanh(X)\otimes 1_N\big)\mathcal{V}\zeta\big](\lambda) = \sum_{\{j|\lambda\in I_j\}} \Big[\tfrac{\mathbf{X}-\lambda_j}{2} \otimes \mathcal{P}_j\zeta\Big](\lambda).$$

Proof. From the definitions of the unitary maps one infers that

$$\begin{aligned}
&\big[\mathcal{V}^*\big(\tanh(X)\otimes 1_N\big)\mathcal{V}\zeta\big](\lambda)\\
&= \sum_{\{j|\lambda\in I_j\}} \big[\big(\mathcal{V}_j^*\otimes\mathcal{P}_j\big)\big(\tanh(X)\otimes 1_N\big)\mathcal{V}\zeta\big](\lambda)\\
&= \sum_{\{j|\lambda\in I_j\}} \Big(\tfrac{2}{4-(\lambda-\lambda_j)^2}\Big)^{1/2}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\mathcal{P}_j\big[\mathcal{V}\zeta\big]\Big(\operatorname{arctanh}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\Big)\\
&= \sum_{\{j|\lambda\in I_j\}} \Big(\tfrac{2}{4-(\lambda-\lambda_j)^2}\Big)^{1/2}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\big[\mathcal{V}_j\otimes\mathcal{P}_j\zeta\big]\Big(\operatorname{arctanh}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\Big)\\
&= \sum_{\{j|\lambda\in I_j\}} \frac{2^{-1/2}}{\Big(1-\big(\tfrac{\lambda-\lambda_j}{2}\big)^2\Big)^{1/2}}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\frac{2^{1/2}}{\cosh\Big(\operatorname{arctanh}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\Big)}\big[\mathcal{P}_j\zeta\big](\lambda)\\
&= \sum_{\{j|\lambda\in I_j\}} \Big(\tfrac{\lambda-\lambda_j}{2}\Big)\big[\mathcal{P}_j\zeta\big](\lambda),
\end{aligned}$$

where the relation $\frac{1}{\cosh(s)^2} = 1-\tanh(s)^2$ has been used for the last equality. This leads directly to the statement. ☐

As a consequence of this lemma one easily infers that $\mathcal{V}^*\big(\tanh(X)\otimes 1_N\big)\mathcal{V}$ belongs to the algebra $\mathcal{A}$ introduced above. In fact, in the matrix formulation mentioned in Remark 3.3, this operator corresponds to a diagonal multiplication operator.

Our next aim is to introduce a C^*-algebra containing the wave operator W_-. First let D be the self-adjoint realization of the operator $-i\frac{\mathrm{d}}{\mathrm{d}x}$ in $\mathrm{L}^2(\mathbb{R})$, so that X and D satisfy the canonical commutation relations. For $\eta \in C_0\big([-\infty,+\infty)\big)$, the algebra of continuous functions on $\mathbb{R}$ having a limit at $-\infty$ and vanishing at $+\infty$, we define $\eta(D)$ by functional calculus. Then the C^*-algebra $\mathcal{C} = \{\eta(D)\otimes 1_N : \eta \in C_0\big([-\infty,+\infty)\big)\}$ is isomorphic to $C_0\big([-\infty,+\infty)\big)$, and contains the operators $\frac{1}{2}\big(1-\tanh(\pi D)\big)\otimes 1_N$ and $\cosh(\pi D)^{-1}\otimes 1_N$. These two operators will subsequently play an important role.

For $\eta \in C_0\big([-\infty,+\infty)\big)$, let us now look at the image of $\eta(D)\otimes 1_N$ in $\mathscr{H}$. For this, we recall that $\mathcal{H}^1(\mathbb{R})$ denotes the first Sobolev space on $\mathbb{R}$.

Lemma 3.6. *(i) For $j \in \{1, \dots, N\}$ the operator $\mathbf{D}_j := \mathcal{V}_j^* D \mathcal{V}_j$ is self-adjoint on $\mathcal{V}_j^* \mathcal{H}^1(\mathbb{R})$, and the following equality holds:*

$$\mathbf{D}_j = 2\bigg(1 - \Big(\tfrac{\mathbf{X}_j - \lambda_j}{2}\Big)^2\bigg)\left(-i\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\right) + i\Big(\tfrac{\mathbf{X}_j - \lambda_j}{2}\Big)$$

where $\mathbf{X}_j$ denotes the operator of multiplication by the variable in $\mathrm{L}^2(I_j)$,

(ii) For any $\eta \in C_0\big([-\infty, +\infty)\big)$ we set

$$\eta(\mathbf{D}) := \mathcal{V}^*\big(\eta(D) \otimes 1_N\big)\mathcal{V}, \tag{3.4}$$

and for $\zeta \in \mathscr{H}$ and for $\lambda \in I$ one has

$$\big[\eta(\mathbf{D})\zeta\big](\lambda) = \sum_{\{j \mid \lambda \in I_j\}} \big[\big(\eta(\mathbf{D}_j) \otimes \mathcal{P}_j\big)\zeta\big](\lambda).$$

Proof. i) The self-adjointness of $\mathbf{D}_j$ on $\mathcal{V}_j^* \mathcal{H}^1(\mathbb{R})$ directly follows from the self-adjointness of D on $\mathcal{H}^1(\mathbb{R})$. Then, for $\zeta \in C_c^\infty(I_j)$ one has

$$\begin{aligned}
&\big[\mathcal{V}_j^* D \mathcal{V}_j \zeta\big](\lambda) \\
&= \Big(\tfrac{2}{4-(\lambda-\lambda_j)^2}\Big)^{1/2} \big[D\mathcal{V}_j\zeta\big]\Big(\operatorname{arctanh}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\Big) \\
&= -i\Big(\tfrac{2}{4-(\lambda-\lambda_j)^2}\Big)^{1/2} \big[\mathcal{V}_j\zeta\big]'\Big(\operatorname{arctanh}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\Big) \\
&= -i\Big(\tfrac{2}{4-(\lambda-\lambda_j)^2}\Big)^{1/2} \Big[\tfrac{2^{1/2}}{\cosh(\cdot)}\zeta\big(\lambda_j + 2\tanh(\cdot)\big)\Big]'\Big(\operatorname{arctanh}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\Big) \\
&= -i\frac{1}{\Big(1-\Big(\frac{\lambda-\lambda_j}{2}\Big)^2\Big)^{1/2}}\Big[\tfrac{-\tanh(\cdot)}{\cosh(\cdot)}\zeta\big(\lambda_j + 2\tanh(\cdot)\big) \\
&\qquad\qquad + 2\tfrac{1-\tanh(\cdot)^2}{\cosh(\cdot)}\zeta'\big(\lambda_j + 2\tanh(\cdot)\big)\Big]\Big(\operatorname{arctanh}\Big(\tfrac{\lambda-\lambda_j}{2}\Big)\Big) \\
&= i\tfrac{\lambda-\lambda_j}{2}\zeta(\lambda) - 2i\bigg(1 - \Big(\tfrac{\lambda-\lambda_j}{2}\Big)^2\bigg)\zeta'(\lambda),
\end{aligned}$$

which leads directly to the statement.

ii) For $\zeta \in \mathscr{H}$ and $\lambda \in I$ one has

$$\begin{aligned}
\big[\mathcal{V}^*\big(\eta(D) \otimes 1_N\big)\mathcal{V}\zeta\big](\lambda) &= \sum_{\{j \mid \lambda \in I_j\}} \big[\big(\mathcal{V}_j^* \otimes \mathcal{P}_j\big)\big(\eta(D) \otimes 1_N\big)\mathcal{V}\zeta\big](\lambda) \\
&= \sum_{\{j \mid \lambda \in I_j\}} \big[\big(\mathcal{V}_j^*\eta(D) \otimes \mathcal{P}_j\big)\mathcal{V}\zeta\big](\lambda) \\
&= \sum_{\{j \mid \lambda \in I_j\}} \big[\big(\mathcal{V}_j^*\eta(D)\mathcal{V}_j \otimes \mathcal{P}_j\big)\zeta\big](\lambda) \\
&= \sum_{\{j \mid \lambda \in I_j\}} \big[\big(\eta(\mathbf{D}_j) \otimes \mathcal{P}_j\big)\zeta\big](\lambda),
\end{aligned}$$

which corresponds to the statement. $\square$

Let us now introduce our main C^*-algebra. We recall that $\eta(\mathbf{D})$ has been introduced in (3.4). We set

$$\mathcal{E} := C^*\Big(\eta(\mathbf{D})\, a + b \mid \eta \in C_0\big([-\infty,+\infty)\big), a \in \mathcal{A}, b \in \mathscr{K}(\mathscr{H})\Big)^+$$

which is a C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ containing the ideal $\mathscr{K}(\mathscr{H})$ of compact operators on $\mathscr{H}$. Here, the exponent $+$ means that $\mathbb{C}$ times the identity of $\mathscr{B}(\mathscr{H})$ have been added to the algebra, turning it into a unital C^*-algebra. Observe that for a typical element $\eta(\mathbf{D})\, a$ with $\eta \in C_0\big([-\infty,+\infty)\big)$ and $a \in \mathcal{A}$, and for $\zeta \in \mathscr{H}$ and $\lambda \in I$, one has

$$\begin{aligned}
\big[\eta(\mathbf{D})\, a\zeta\big](\lambda) &= \sum_{\{j|\lambda\in I_j\}} \big[\big(\eta(\mathbf{D}_j)\otimes \mathcal{P}_j\big)a\zeta\big](\lambda) \\
&= \sum_{\{j|\lambda\in I_j\}} \sum_{j'=1}^{N} \big[\eta(\mathbf{D}_j)\, a_{jj'}\mathcal{P}_{j'}\zeta\big](\lambda) \\
&= \sum_{\{j|\lambda\in I_j\}} \sum_{j'=1}^{N} \big[\big(\eta(\mathbf{D}_j)\, \mathfrak{a}_{jj'}\big)\otimes \tfrac{1}{N}|\xi_j\rangle\langle\xi_{j'}|\zeta\big](\lambda) \\
&= \sum_{j,j'} \big[\big(\eta(\mathbf{D}_j)\, \mathfrak{a}_{jj'}\big)\otimes \tfrac{1}{N}|\xi_j\rangle\langle\xi_{j'}|\zeta\big](\lambda), \qquad (3.5)
\end{aligned}$$

where the notation introduced in Remark 3.3 has been used.

Our main interest in this C^*-algebra is based on the following result, which is a direct consequence of Theorem 3.1.

Proposition 3.7. *The wave operator $\mathscr{F}W_-\mathscr{F}^*$ belongs to $\mathcal{E}$.*

Proof. Let us start by recalling the formula for the wave operator,

$$W_- - 1 = \tfrac{1}{2}\big(1-\tanh(\pi\mathfrak{D}) - i\cosh(\pi\mathfrak{D})^{-1}\tanh(\mathfrak{X})\big)(S-1) + \mathfrak{K},$$

with $\mathfrak{K} \in \mathscr{K}(\mathfrak{h})$. By looking at the r.h.s. through the unitary conjugation by $\mathscr{F}$, one ends up with the expression in $\mathscr{H}$:

$$\tfrac{1}{2}\Big(1-\tanh(\pi\mathbf{D}) - i\cosh(\pi\mathbf{D})^{-1}\mathcal{V}^*\big(\tanh(X)\otimes 1_N\big)\mathcal{V}\Big)(S(\mathbf{X})-1) + k, \qquad (3.6)$$

with $k \in \mathscr{K}(\mathscr{H})$.

The functions $1-\tanh(\pi\cdot)$ and $\cosh(\pi\cdot)^{-1}$ belong to $C_0\big([-\infty,+\infty)\big)$. From Proposition 3.4, the map $\lambda \mapsto S(\lambda)$ belongs to $\mathcal{A}$, as does the same is true for the constant function 1. As a consequence, the map $\lambda \mapsto S(\lambda)-1$ belongs to $\mathcal{A}$, or equivalently the multiplication operator $S(\mathbf{X})-1$ belongs to $\mathcal{A}$. By Lemma 3.5, the operator $\mathcal{V}^*\big(\tanh(X)\otimes 1_N\big)\mathcal{V}$ also belongs to the algebra $\mathcal{A}$. Thus, the leading term in (3.6) belongs to $\mathcal{E}$, and the remainder term k also belongs to $\mathcal{E}$ because it is a compact. □

Our next task is to compute the quotient of the algebra $\mathcal{E}$ by the set of compact operators. For that purpose, we firstly show that if some restrictions are imposed on the functions $\mathfrak{a}_{jj'}$ and η appearing in (3.5), then the corresponding operator is compact. For this we introduce an ideal in $\mathcal{A}$, namely

$$\mathcal{A}_0 := \Big\{a \in \mathcal{A} \mid a_{jj} \in C_0\left(I_j; \mathscr{B}(\mathbb{C}^N)\right) \text{ for all } j \in \{1,\dots,N\}\Big\}.$$

Lemma 3.8. *For $a \in \mathcal{A}_0$ and $\eta \in C_0(\mathbb{R})$ the operator $\eta(\mathbf{D})a$ belongs to $\mathscr{K}(\mathscr{H})$.*

Proof. Let us first observe that the condition $a \in \mathcal{A}_0$ means that all multiplication operators $\mathfrak{a}_{jj'}$ introduced in Remark 3.3 are defined by functions in $C_0\big(I_j \cap I_{j'};\mathbb{C}\big)$, for all $j,j' \in \{1,\dots,N\}$. Thus, according to (3.5) it is enough to show that each operator $\eta(\mathbf{D}_j)\mathfrak{a}_{jj'} : \mathrm{L}^2(I_{j'}) \to \mathrm{L}^2(I_j)$ is compact. However, since $\operatorname{supp}\mathfrak{a}_{jj'} \subset \overline{I_j \cap I_{j'}}$, it is simpler to show that $\eta(\mathbf{D}_j)\mathfrak{a}_{jj'} \in \mathscr{K}\big(\mathrm{L}^2(I_j)\big)$. Now, if we apply the unitary transforms defined in (3.2) and (3.3) one gets

$$\mathcal{V}_j\,\eta(\mathbf{D}_j)\mathfrak{a}_{jj'}\,\mathcal{V}_j^* = \eta(D)\,\mathfrak{a}_{jj'}\big(\lambda_j + 2\tanh(X)\big). \tag{3.7}$$

Since η and $\mathfrak{a}_{jj'}\big(\lambda_j + 2\tanh(\cdot)\big)$ belong to $C_0(\mathbb{R})$, the r.h.s. of (3.7) is known to be a compact operator in $\mathrm{L}^2(\mathbb{R})$. Thus, each summand in (3.5) is unitarily equivalent to a compact operator, which means that $\eta(\mathbf{D})a$ is compact. □

Let us supplement the previous result with a compactness result about commutators.

Lemma 3.9. *For any $\eta \in C_0\big([-\infty,\infty)\big)$ and $a \in \mathcal{A}$, the commutator $[\eta(\mathbf{D}), a]$ belongs to $\mathscr{K}(\mathscr{H})$.*

Proof. It suffices to show that $F := \mathbf{D}\big(1+\mathbf{D}^2\big)^{-1/2}$ has compact commutator with elements of $\mathcal{A}$. For if this is true, then any polynomial in F compactly commutes with $\mathcal{A}$. If $\eta \in C_0\big([-\infty,\infty)\big)$ then there exists a unique $h \in C_0\big([-1,1)\big)$ such that $\eta(\mathbf{D}) = h(F)$. By Stone-Weierstrass theorem, any $h \in C_0\big([-1,1)\big)$ can be approximated by a polynomial on $[-1,1)$. Then, one concludes the proof by recalling that the set of compact operators is norm closed.

Now, with the notation of Remark 3.3 and for any $a \in \mathcal{A}$, let us define $\operatorname{diag}(a)$ by

$$\big[\operatorname{diag}(a)\big](\lambda) := \sum_{\{j \mid \lambda \in I_j\}} \mathfrak{a}_{jj}(\lambda) \otimes \mathcal{P}_j, \qquad \forall \lambda \in I.$$

We also consider the diagonal multiplication operator $\Lambda \in \mathcal{A}_0$ given for any $\lambda \in I$ by

$$\Lambda(\lambda) := \sum_{\{j \mid \lambda \in I_j\}} \Lambda_j(\lambda) \otimes \mathcal{P}_j$$

with $\Lambda_j(\lambda) := 1 - \Big(\frac{\lambda-\lambda_j}{2}\Big)^2$. For $\epsilon > 0$ small, let $\mathcal{T}_\epsilon$ be an open ϵ-neighbourhood of $\mathcal{T}$, where the set of thresholds $\mathcal{T}$ has been defined in (2.2). We then set $\mathcal{A}_\epsilon$ for the set of $a \in \mathcal{A}$ satisfying $\operatorname{supp}\big(a - \operatorname{diag}(a)\big) \subset I \setminus \mathcal{T}_\epsilon$. For $a \in \mathcal{A}_\epsilon$ we define

$$\rho(a) := \operatorname{diag}(a) + \Lambda\big(a - \operatorname{diag}(a)\big)\Lambda^{-1}.$$

With the notations introduced before it means that for any $\lambda \in I$ one has

$$\begin{aligned}\big[\rho(a)\big](\lambda) = &\sum_{\{j \mid \lambda \in I_j\}} \mathfrak{a}_{jj}(\lambda) \otimes \mathcal{P}_j \\ &+ \sum_{\{j \neq j' \mid \lambda \in I_j, \lambda \in I_{j'}\}} \Lambda_j(\lambda)\mathfrak{a}_{jj'}(\lambda)\big(\Lambda_{j'}(\lambda)\big)^{-1} \otimes \tfrac{1}{N}|\xi_j\rangle\langle\xi_{j'}|.\end{aligned}$$

One observes that ρ is an algebra homomorphism on the subalgebra $\mathcal{A}_\epsilon$, but it is not a $*$-homomorphism. With a grain of salt, the expression $\rho(a)$ could simply be written $\Lambda a \Lambda^{-1}$, but the diagonal elements have to be suitably interpreted (the cancellation of two factors before the evaluation at thresholds).

Let us also consider the subalgebra $\mathcal{A}_{\epsilon,1}$ of $\mathcal{A}_\epsilon$ of those a whose derivatives a' exist, are continuous and satisfy $\operatorname{supp}(a') \subset I \setminus \mathcal{T}_\epsilon$. Elements of the algebra $\mathcal{A}_{\epsilon,1}$ preserve the domain of $\mathbf{D}$, and any element of $\mathcal{A}$ is a norm limit of elements in $\mathcal{A}_{\epsilon,1}$ as $\epsilon \searrow 0$. If we define the additional diagonal multiplication operator

$$\Omega(\lambda) = \sum_{\{j|\lambda\in I_j\}} i\Big(\tfrac{\lambda-\lambda_j}{2}\Big) \otimes \mathcal{P}_j, \qquad \forall \lambda \in I,$$

then one readily checks that for $a \in \mathcal{A}_{\epsilon,1}$, on the domain of $\mathbf{D}$ we have

$$\begin{aligned}
\mathbf{D}a - \rho(a)\mathbf{D} &= -2i\Lambda a' + \Omega a - \rho(a)\Omega \\
&= -2i\Lambda a' + \Omega a - \rho\big(a - \operatorname{diag}(a)\big)\Omega - \operatorname{diag}(a)\Omega \\
&= -2i\Lambda a' + \Omega\big(a - \operatorname{diag}(a)\big) - \rho\big(a - \operatorname{diag}(a)\big)\Omega.
\end{aligned}$$

Clearly, the resulting expression is bounded, and the same notation is kept for the bounded operator extending it continuously. One also observes that this operator corresponds to a multiplication operator with compact support. By adapting the argument of Connes-Moscovici [11, Prop. 3.2] we finally write

$$\begin{aligned}
\big[F, a\big] =&\big(1+\mathbf{D}^2\big)^{-1/2}\mathbf{D}a - a\big(1+\mathbf{D}^2\big)^{-1/2}\mathbf{D} \\
=&\big(1+\mathbf{D}^2\big)^{-1/2}\big(\mathbf{D}a - \rho(a)\mathbf{D}\big) \qquad (3.8)\\
&+ \big(1+\mathbf{D}^2\big)^{-1/2}\Big(\rho(a) - \big(1+\mathbf{D}^2\big)^{1/2}a\big(1+\mathbf{D}^2\big)^{-1/2}\Big)\mathbf{D}. \qquad (3.9)
\end{aligned}$$

In order to analyse these terms, let us firstly observe that $\mathbf{D}^2$ is a second order uniformly elliptic pseudodifferential operator on $I\setminus\mathcal{T}_\epsilon$, and so the inverse of $1+\mathbf{D}^2$ is a pseudodifferential operator of order -2. By using the formula

$$\big(1+\mathbf{D}^2\big)^{-1/2} = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\big(1+\lambda+\mathbf{D}^2\big)^{-1}\,\mathrm{d}\lambda$$

one then infers that $\big(1+\mathbf{D}^2\big)^{-1/2}$ is a pseudodifferential operator of order -1, and accordingly that $\big(1+\mathbf{D}^2\big)^{1/2}$ is a pseudodifferential operator of order 1. Note that these properties hold on the domain $I \setminus \mathcal{T}_\epsilon$.

Now, the term in (3.8) is a compact operator since the second factor is a multiplication operator with support in $I \setminus \mathcal{T}_\epsilon$ and the first factor is a pseudodifferential operator of order -1. For the term (3.9), we show below that

$$\big(1+\mathbf{D}^2\big)^{1/2}a\big(1+\mathbf{D}^2\big)^{-1/2} = \rho(a) + R,$$

where R is a pseudodifferential operator of order -1 with compact support. The compactness of (3.9) follows then directly. One finishes the proof with the density of $\mathcal{A}_{\epsilon,1}$ in $\mathcal{A}$ as $\epsilon \searrow 0$ and the fact that the set of compact operators is norm closed.

For the last argument, observe that the principal symbols of the operators $\big(1+\mathbf{D}^2\big)^{1/2}$ and $\big(1+\mathbf{D}^2\big)^{-1/2}$ are given by $I\times\mathbb{R}\ni(\lambda,\xi)\mapsto 2\Lambda(\lambda)|\xi|$ and $I\times\mathbb{R}\ni(\lambda,\xi)\mapsto\frac{1}{2}\Lambda^{-1}(\lambda)|\xi|^{-1}$ respectively. It then follows that

$$\begin{aligned}(1+\mathbf{D}^2)^{1/2}a(1+\mathbf{D}^2)^{-1/2} =&\Lambda\big(a-\mathrm{diag}(a)\big)\Lambda^{-1}+\mathrm{diag}(a)+R\\ =&\rho(a)+R\end{aligned}$$

with R a pseudodifferential operator of order -1 with compact support. □

For the computation of the quotient algebra $\mathcal{Q}:=\mathcal{E}/\mathscr{K}(\mathscr{H})$, and thanks to the previous result, we can focus on elements of the form $\eta(\mathbf{D})a$ with $\eta\in C_0\big([-\infty,+\infty)\big)$ and $a\in\mathcal{A}$. The starting point is again the decomposition of such elements provided in (3.5). As introduced in (3.5) and as explained in the proof of Lemma 3.8, it is enough to study the operator $\eta(\mathbf{D}_j)\mathfrak{a}_{jj'}$ acting on $\mathrm{L}^2(I_j)$. If we set $q_j:\mathscr{B}\big(\mathrm{L}^2(I_j)\big)\to\mathscr{B}\big(\mathrm{L}^2(I_j)\big)/\mathscr{K}\big(\mathrm{L}^2(I_j)\big)$ then the image of $\eta(\mathbf{D}_j)\mathfrak{a}_{jj'}$ through q_j falls into two distinct situations.

1. If $j\neq j'$, then
$$q_j\big(\eta(\mathbf{D}_j)\,\mathfrak{a}_{jj'}\big)=\eta(-\infty)\,\mathfrak{a}_{jj'}\in C_0\big(I_j\cap I_{j'}\big), \tag{3.10}$$

2. If $j=j'$, then
$$\begin{aligned}q_j\big(\eta(\mathbf{D}_j)\,\mathfrak{a}_{jj}\big)=&\Big(\eta\,\mathfrak{a}_{jj}(\lambda_j-2),\ \eta(-\infty)\,\mathfrak{a}_{jj},\ \eta\,\mathfrak{a}_{jj}(\lambda_j+2)\Big)\\ &\in C_0\big((+\infty,-\infty]\big)\oplus C\big(\overline{I_j}\big)\oplus C_0\big([-\infty,+\infty)\big).\end{aligned} \tag{3.11}$$

These statement can be obtained as in the proof of Lemma 3.8 by looking at these operators in $\mathrm{L}^2(\mathbb{R})$ through the conjugation by $\mathcal{V}_j$. Then, one ends up with operators of the form $\eta(D)\varphi(X)$ for $\eta\in C_0\big([-\infty,+\infty)\big)$ and for $\varphi\in C_0(\mathbb{R})$ in the first case, and $\varphi\in C\big([-\infty,+\infty]\big)$ in the second case. The image of such operators by the quotient map (defined by the compact operators) have been extensively studied in [30, Sec. 4.4], from which we infer the results presented above.

Remark 3.10. *Observe that in the first component of* (3.11)*, the interval* $[-\infty,+\infty)$ *has been oriented in the reverse direction. The reason is that the function introduced in* (3.11) *can be seen as a continuous function on the union of the three intervals*

$$(+\infty,-\infty]\cup\overline{I_j}\cup[-\infty,+\infty)$$

once their endpoints are correctly identified. This observation and this trick will be used several times in the sequel. Note also that (3.10) *could be expressed as* (3.11) *by considering the triple* $\big(0,\,\eta(-\infty)\mathfrak{a}_{jj'},\,0\big)$.

In the next statement we collect the various results obtained so far. However, some notations have to be slightly updated. More precisely, recall that for $j\in\{1,\dots,N\}$ one has $\lambda_j:=2\cos\Big(\frac{\theta+2\pi j}{N}\Big)$. For the following statement it will be useful to have a better understanding of the sets $\{\lambda_j\}_{j=1}^N$. Namely, let us observe that for N even, we have

$$\lambda_{\frac{N}{2}}<\lambda_{\frac{N}{2}-1}<\lambda_{\frac{N}{2}+1}<\lambda_{\frac{N}{2}-2}<\lambda_{\frac{N}{2}+2}<\ldots<\lambda_1<\lambda_{N-1}<\lambda_N,$$

while for N odd we have

$$\lambda_{\frac{N-1}{2}} < \lambda_{\frac{N+1}{2}} < \lambda_{\frac{N-1}{2}-1} < \lambda_{\frac{N+1}{2}+1} < \lambda_{\frac{N-1}{2}-2} < \ldots < \lambda_1 < \lambda_{N-1} < \lambda_N.$$

We now rename these eigenvalues and set $\tilde\lambda_1 < \tilde\lambda_2 < \ldots < \tilde\lambda_N$ for these N distinct and ordered values. Accordingly, we define the eigenvector $\tilde\xi_j$ and the orthogonal projection $\tilde{\mathcal{P}}_j$, based on the eigenvector ξ_k and the projection $\mathcal{P}_k$ corresponding to the eigenvalue $\lambda_k = \tilde\lambda_j$. Finally we set

$$\tilde{\mathscr{H}_j} := \operatorname{span}\{\tilde{\mathcal{P}}_i\mathbb{C}^N \mid i \leq j\}.$$

Proposition 3.11. *The $\mathcal{Q} := \mathcal{E}/\mathscr{K}(\mathscr{H})$ has the shape of an upside down comb, with $2N$ teeth, and more precisely:*

$$\mathcal{Q} \subset C\left(\Big(\bigoplus_{j=1}^{N-1} \downarrow_j \oplus \rightarrow_j\Big) \oplus \Big(\downarrow_N \oplus \rightarrow_N \oplus \uparrow^1\Big) \oplus \Big(\bigoplus_{j=2}^{N} \rightarrow^j \oplus \uparrow^j\Big); \mathscr{B}(\mathbb{C}^N)\right)^+, \qquad (3.12)$$

with

$$\begin{aligned}
\downarrow_j &:= [+\infty, -\infty] \qquad \forall j \in \{1,\ldots,N\}\\
\rightarrow_j &:= [\tilde\lambda_j - 2, \tilde\lambda_{j+1} - 2) \qquad \forall j \in \{1,\ldots,N-1\}\\
\rightarrow_N &:= [\tilde\lambda_N - 2, \tilde\lambda_1 + 2],\\
\rightarrow^j &:= (\tilde\lambda_{j-1} + 2, \tilde\lambda_j + 2] \qquad \forall j \in \{2,\ldots,N\}\\
\uparrow^j &:= [-\infty, +\infty] \qquad \forall j \in \{1,\ldots,N\}.
\end{aligned}$$

Moreover, if ϕ_* denotes the restriction to the edge $*$ of any $\phi \in \mathcal{Q}$, then these restrictions satisfy the conditions:

$$\begin{aligned}
\phi_{\rightarrow_j} &\in C\big([\tilde\lambda_j - 2, \tilde\lambda_{j+1} - 2); \mathscr{B}(\tilde{\mathscr{H}_j})\big) \quad \forall j \in \{1,\ldots,N-1\}\\
\phi_{\rightarrow_N} &\in C\big([\tilde\lambda_N - 2, \tilde\lambda_1 + 2]; \mathscr{B}(\mathbb{C}^N)\big)\\
\phi_{\rightarrow^j} &\in C\big((\tilde\lambda_{j-1} + 2, \tilde\lambda_j + 2]; \mathscr{B}((\tilde{\mathscr{H}}_{j-1})^\perp)\big) \quad \forall j \in \{2,\ldots,N\},
\end{aligned}$$

together with

$$\begin{aligned}
\phi_{\downarrow_j} &\in C\big([+\infty, -\infty]; \mathscr{B}(\tilde{\mathcal{P}}_j\mathbb{C}^N)\big)\\
\phi_{\uparrow^j} &\in C\big([-\infty, +\infty]; \mathscr{B}(\tilde{\mathcal{P}}_j\mathbb{C}^N)\big),
\end{aligned}$$

for all $j \in \{1,\ldots,N\}$. In addition, the following continuity properties hold:

$$\begin{aligned}
\phi_{\rightarrow_1}(\tilde\lambda_1 - 2) &= \phi_{\downarrow_1}(-\infty),\\
\phi_{\rightarrow_j}(\tilde\lambda_j - 2) &= \lim_{\lambda \nearrow \tilde\lambda_j - 2} \phi_{\rightarrow_{j-1}}(\lambda) \oplus \phi_{\downarrow_j}(-\infty) \quad \forall j \in \{2,\ldots,N\}\\
\phi_{\rightarrow_N}(\tilde\lambda_1 + 2) &= \phi_{\uparrow^1}(-\infty) \oplus \lim_{\lambda \searrow \tilde\lambda_1 + 2} \phi_{\rightarrow^2}(\lambda)\\
\phi_{\rightarrow_j}(\tilde\lambda_j + 2) &= \phi_{\uparrow^j}(-\infty) \oplus \lim_{\lambda \searrow \tilde\lambda_j + 2} \phi_{\rightarrow^{j+1}}(\lambda) \quad \forall j \in \{2,\ldots,N-1\}\\
\phi_{\rightarrow^N}(\tilde\lambda_N + 2) &= \phi_{\uparrow^N}(-\infty),
\end{aligned} \qquad (3.13)$$

and there exists $c \in \mathbb{C}$ such that for all $j \in \{1,\ldots,N\}$,

$$c = \operatorname{Tr}\big(\phi_j(+\infty)\big) = \operatorname{Tr}\big(\phi^j(+\infty)\big). \qquad (3.14)$$

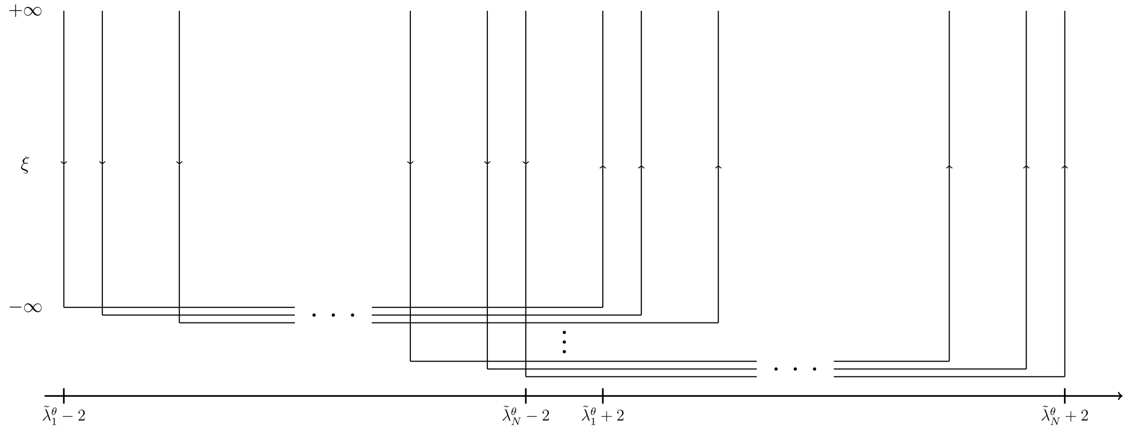

Figure 1: A representation of the quotient algebra $\mathcal{Q}$.

A representation for the support of the quotient algebra $\mathcal{Q}$ is provided in Figure 1. In the previous description of the quotient algebra, note that the condition (3.14) is related to the addition of the unit to $\mathcal{E}$. Indeed, if no unit is added to $\mathcal{E}$, then one has $c = 0$. Let us also denote by q the quotient map

$$q : \mathcal{E} \to \mathcal{Q} \equiv \mathcal{E}/\mathscr{K}(\mathscr{H}).$$

Proof. The proof consists in looking carefully at the expressions provided in (3.10) and (3.11), and in keeping track of the matricial form of the matrix-valued function a. Let us consider a generic element of $\mathcal{E}$ given by $\eta(\mathbf{D})a + c1$ for $\eta \in C_0\big([-\infty, +\infty)\big)$, $a \in \mathcal{A}$, and $c \in \mathbb{C}$. By taking (3.5) into account, and the specific form of $\mathscr{H}$, one has

$$\eta(\mathbf{D})a + c1 = \sum_{j,j'} \big(\eta(\tilde{\mathbf{D}}_j)\,\tilde{\mathfrak{a}}_{jj'} + c\delta_{jj'}\big) \otimes \tfrac{1}{N}|\tilde{\xi}_j\rangle\langle\tilde{\xi}_{j'}|$$

with $\delta_{jj'}$ the Kronecker delta function. According to (3.10) and (3.11) one has

$$\begin{aligned}
& q\big(\eta(\mathbf{D})a + c1\big) \\
&= \sum_{\{j,j' \mid j' \neq j\}} \eta(-\infty)\,\tilde{\mathfrak{a}}_{jj'} \otimes \tfrac{1}{N}|\tilde{\xi}_j\rangle\langle\tilde{\xi}_{j'}| \\
&\quad + \sum_j \Big(\eta\,\tilde{\mathfrak{a}}_{jj}(\tilde{\lambda}_j - 2) + c,\ \eta(-\infty)\,\tilde{\mathfrak{a}}_{jj} + c,\ \eta\,\tilde{\mathfrak{a}}_{jj}(\tilde{\lambda}_j + 2) + c\Big) \otimes \tfrac{1}{N}|\tilde{\xi}_j\rangle\langle\tilde{\xi}_j| \\
&= \sum_{j,j'} \Big(\big(\eta\,\tilde{\mathfrak{a}}_{jj}(\tilde{\lambda}_j - 2) + c\big)\delta_{jj'},\ \eta(-\infty)\,\tilde{\mathfrak{a}}_{jj'} + c\delta_{jj'},\ \big(\eta\,\tilde{\mathfrak{a}}_{jj}\tilde{\lambda}_j + 2) + c\big)\delta_{jj'}\Big) \otimes \tfrac{1}{N}|\tilde{\xi}_j\rangle\langle\tilde{\xi}_{j'}|.
\end{aligned}$$

In order to fully understand the previous expression, it is necessary to remember the special structure of the underlying Hilbert space $\mathscr{H} := \int_I^\oplus \mathscr{H}(\lambda)\,\mathrm{d}\lambda$ with

$$\mathscr{H}(\lambda) = \begin{cases} \tilde{\mathscr{H}_j} & \text{if } \tilde{\lambda}_j - 2 \leq \lambda < \tilde{\lambda}_{j+1} - 2 \\ \mathbb{C}^N & \text{if } \tilde{\lambda}_N - 2 \leq \lambda \leq \tilde{\lambda}_1 + 2 \\ (\tilde{\mathscr{H}_j})^\perp & \text{if } \tilde{\lambda}_j + 2 < \lambda \leq \tilde{\lambda}_{j+1} + 2 \end{cases}.$$

Note that compared with the original definition of $\mathscr{H}(\lambda)$ we have changed the fiber at a finite number of points, which does not impact the direct integral, but simplify our argument subsequently. Thus, the changes of dimension of the fibers take place at all $\tilde\lambda_j - 2$ and $\tilde\lambda_j + 2$, for $j \in \{1, \dots, N\}$. By taking this into account, the interval I has to be divided into $2N-1$ subintervals, firstly of the form $[\tilde\lambda_j - 2, \tilde\lambda_{j+1} - 2)$ for $j \in \{1, \dots, N-1\}$, then the special interval $[\tilde\lambda_N - 2, \tilde\lambda_1 + 2]$, and finally the intervals $(\tilde\lambda_{j-1} + 2, \tilde\lambda_j + 2]$ for $j \in \{2, \dots, N\}$. By using this partition of I, one gets

$$\begin{aligned} & q\big(\eta(\mathbf{D})a + c1\big) \\ & = \sum_{j=1}^{N-1} \Big(\big(\eta\, \tilde{\mathfrak{a}}_{jj}(\tilde\lambda_j - 2) + c\big) \otimes \tilde{\mathcal{P}}_j,\ \chi_{[\tilde\lambda_j - 2, \tilde\lambda_{j+1} - 2)}\big(\eta(-\infty)\, a + c1_{\tilde{\mathscr{H}}_j}\big)\Big) \\ & \quad + \Big(\big(\eta\, \tilde{\mathfrak{a}}_{NN}(\tilde\lambda_N - 2) + c\big) \otimes \tilde{\mathcal{P}}_N,\ \chi_{[\tilde\lambda_N - 2, \tilde\lambda_1 + 2]}\big(\eta(-\infty)\, a + c1\big),\ \big(\eta\, \tilde{\mathfrak{a}}_{11}(\tilde\lambda_1 + 2) + c\big) \otimes \tilde{\mathcal{P}}_1\Big) \\ & \quad + \sum_{j=2}^{N} \Big(\chi_{(\tilde\lambda_{j-1} + 2, \tilde\lambda_j + 2]}\big(\eta(-\infty)\, a + c1_{(\tilde{\mathscr{H}}_{j-1})^\perp}\big),\ \big(\eta\, \tilde{\mathfrak{a}}_{jj}(\tilde\lambda_j + 2) + c\big) \otimes \tilde{\mathcal{P}}_j\Big). \end{aligned}$$

The description obtained so far leads directly to the structure of (3.12).

The properties stated in (3.13) follow from the continuity of $a \in \mathcal{A}$ and from its properties at thresholds. The final property (3.14) comes from the unit and the fact that η vanishes at $+\infty$. ◻

Since $\mathscr{F} W_- \mathscr{F}^* \in \mathcal{E}$, by Proposition 3.7, one can look at the image of this operator in the quotient algebra. The following statement contains a description of this image, using the notations introduced in Proposition 3.11. The functions $\eta_\pm : \mathbb{R} \to \mathbb{C}$ defined for any $s \in \mathbb{R}$ by

$$\eta_\pm(s) := \tanh(\pi s) \pm i \cosh(\pi s)^{-1}. \tag{3.15}$$

will also be used. Finally, based on the projections $\{\tilde{\mathcal{P}}_j\}_{j=1}^N$ introduced before Proposition 3.11, we define the channel scattering matrix $\tilde S_{jj}(\lambda) := \tilde{\mathcal{P}}_j S(\lambda) \tilde{\mathcal{P}}_j$.

Lemma 3.12. *Let $\phi := q\big(\mathscr{F} W_- \mathscr{F}^*\big)$ denote the image of $\mathscr{F} W_- \mathscr{F}^*$ in the quotient algebra. Then, the restrictions of ϕ on the various parts of $\mathcal{Q}$ are given by:*

for $j \in \{1, \dots, N-1\}$ and $\lambda \in [\tilde\lambda_j - 2, \tilde\lambda_{j+1} - 2)$, $\qquad \phi_{\to_j}(\lambda) = S(\lambda) \ \in \mathscr{B}(\tilde{\mathscr{H}}_j)$

for $\lambda \in [\tilde\lambda_N - 2, \tilde\lambda_1 + 2]$, $\qquad \phi_{\to_N}(\lambda) = S(\lambda) \ \in \mathscr{B}(\mathbb{C}^N)$

for $j \in \{2, \dots, N\}$ and $\lambda \in (\tilde\lambda_{j-1} + 2, \tilde\lambda_j + 2]$, $\qquad \phi_{\to^j}(\lambda) = S(\lambda) \ \in \mathscr{B}\big((\tilde{\mathscr{H}}_{j-1})^\perp\big)$,

and for $s \in \mathbb{R}$

$$\phi_{\downarrow_j}(s) = 1 + \tfrac{1}{2}\big(1 - \eta_-(s)\big)\big(\tilde S_{jj}(\tilde\lambda_j - 2) - 1\big) \ \in \mathscr{B}(\tilde{\mathcal{P}}_j \mathbb{C}^N) \tag{3.16}$$

$$\phi_{\uparrow^j}(s) = 1 + \tfrac{1}{2}\big(1 - \eta_+(s)\big)\big(\tilde S_{jj}(\tilde\lambda_j + 2) - 1\big) \ \in \mathscr{B}(\tilde{\mathcal{P}}_j \mathbb{C}^N). \tag{3.17}$$

Proof. Let us start by looking at the expression for $\mathscr{F}\big(W_- - 1\big)\mathscr{F}^*$, as provided for example in (3.6), and by taking the content of Lemma 3.5 into account. It then follows that $\mathscr{F}\big(W_- - 1\big)\mathscr{F}^*$ can be rewritten as

$$\tfrac{1}{2}\big(1 - \tanh(\pi \mathbf{D})\big)\big(S(\mathbf{X}) - 1\big) - i\tfrac{1}{2}\cosh(\pi \mathbf{D})^{-1} \sum_j \tfrac{\mathbf{X} - \lambda_j}{2} \mathcal{P}_j \big(S(\mathbf{X}) - 1\big) + k \tag{3.18}$$

with $k \in \mathscr{K}(\mathscr{H})$. Thus, one ends up with two generic elements of $\mathcal{E}$ which have been carefully studied in the proof of Proposition 3.11. The only additional necessary tricky observation is that

$$\tfrac{\lambda-\lambda_j}{2}\Big|_{\lambda=\lambda_j\pm 2} = \pm 1$$

which explains the appearance of the two functions $\eta_\pm$. Note also that

$$\lim_{s\to-\infty} \tfrac{1}{2}\big(1-\tanh(\pi s)\big) = 1 \quad \text{and} \quad \lim_{s\to\infty} \tfrac{1}{2}\big(1-\tanh(\pi s)\big) = 0,$$

while $\lim_{s\to\pm\infty}\cosh(\pi s)^{-1} = 0$. The rest of the proof is just a special instance of the proof of Proposition 3.11 for the two main terms exhibited in (3.18), since the compact term verifies $q(k) = 0$. □

4 Topological Levinson's theorem

In this section we provide the topolgical version of Levinson's theorem, linking the number of bound states to an expression involving the image of W_- in the quotient algebra. First of all, we provide a statement about the behavior of the scattering matrix at thresholds. It shows that the expression obtained for the wave operator W_- is rather rigid and imposes a strict behaviour at thresholds. For its statement we define the scalar valued function $\tilde{\mathfrak{s}}_{jj}$ by the relation $\tilde{S}(\lambda)_{jj} =: \tilde{\mathfrak{s}}_{jj}(\lambda)\tilde{\mathcal{P}}_j$.

Lemma 4.1. *For any $j \in \{1,\ldots,N\}$ one has $\tilde{\mathfrak{s}}_{jj}(\tilde{\lambda}_j \pm 2) \in \{-1,1\}$.*

Proof. Since W_- is a Fredholm operator of norm 1, the image $q\big(\mathscr{F}W_-\mathscr{F}^*\big)$ is a unitary operator, and therefore its restrictions on all components of $\mathcal{Q}$ must be unitary. The restrictions on $\to_j$, $\to_N$, and $\to^j$ do not impose any conditions, since the scattering operator is unitary valued, see Lemma 3.12. On the other hand, by checking that the restrictions on $\downarrow_j$ and on $\uparrow^j$, the conditions appearing in the statement have to be imposed.

Starting with (3.16), the restriction on $\downarrow_j$, we can rewrite this operator as a scalar function multiplying a rank one projection. By imposing that the operator is unitary valued, one infers that

$$\Big(1+\tfrac{1}{2}\big(1-\eta_-(s)\big)\big(\tilde{\mathfrak{s}}_{jj}(\tilde{\lambda}_j-2)-1\big)\Big)\Big(1+\tfrac{1}{2}\big(1-\eta_-(s)\big)\big(\tilde{\mathfrak{s}}_{jj}(\tilde{\lambda}_j-2)-1\big)\Big)^* = 1$$

for all $s \in \mathbb{R}$. Some direct computations lead then to the condition $\Im\big(\eta_-(s)\big)\Im\big(\tilde{\mathfrak{s}}_{jj}(\tilde{\lambda}_j-2)\big) = 0$ for any $s \in \mathbb{R}$, meaning that $\tilde{\mathfrak{s}}_{jj}(\tilde{\lambda}_j-2) \in \mathbb{R}$. Since $\tilde{\mathfrak{s}}_{jj}(\tilde{\lambda}_j-2)$ is also unitary valued, the only solutions are the ones given in the statement. A similar argument holds for $\tilde{\mathfrak{s}}_{jj}(\tilde{\lambda}_j+2)$, starting with the restriction on $\uparrow^j$. □

The topological Levinson's theorem corresponds to an index theorem in scattering theory. By considering the C^*-algebras introduced in Section 3 we can consider the short exact sequence of C^*-algebras

$$0 \longrightarrow \mathscr{K}(\mathscr{H}) \longrightarrow \mathcal{E} \xrightarrow{\;q\;} \mathcal{Q} \longrightarrow 0.$$

Since $\mathscr{F}W_-\mathscr{F}^*$ belongs to $\mathcal{E}$ and is a Fredholm operator we infer the equality

$$\operatorname{ind}\Big([q\big(\mathscr{F}W_-\mathscr{F}^*\big)]_1\Big) = -\big[E_{\rm p}(H)\big]_0, \tag{4.1}$$

where ind denotes the index map from $K_1(\mathcal{Q})$ to $K_0(\mathscr{K}(\mathscr{H}))$ and where $E_{\rm p}(H)$ corresponds to the projection on the subspace spanned by the eigenfunctions of H. Note that this projection appears from the standard relation

$$[1 - W_-^* W_-]_0 - [1 - W_- W_-^*]_0 = -[E_{\rm p}(H)]_0.$$

The equality (4.1) can be directly deduced from [30, Prop. 4.3]. Let us emphasise that this equality corresponds to the topological version of Levinson's theorem: it is a relation (by the index map) between the equivalence class in K_1 of quantities related to scattering theory, as described in Lemma 3.12, and the equivalence class in K_0 of the projection on the bound states of H. However, the standard formulation of Levinson's theorem is an equality between numbers. Thus, our final task is to extract a numerical equality from (4.1).

In the next statement, the notation $\mathrm{Var}(\lambda \mapsto \det S(\lambda))$ should be understood as the total variation of the argument of the piecewise continuous function

$$I \ni \lambda \mapsto \det S(\lambda) \in \mathbb{S}^1.$$

where we compute the argument increasing with increasing λ. Our convention is also that the increase of the argument is counted clockwise.

Theorem 4.2. *The following equality holds:*

$$\mathrm{Var}(\lambda \to \det S(\lambda)) + N - \frac{\#\{j \mid \mathfrak{s}_{jj}(\lambda_j \pm 2) = 1\}}{2} = \#\sigma_{\rm p}(H) \tag{4.2}$$

Proof. The proof starts by evaluating both sides of (4.1) with the operator trace to obtain a numerical equation. For the left hand side, we obtain (minus) the number of bound states, or more precisely $\mathrm{Tr}(E_{\rm p}(H)) = \#\sigma_{\rm p}(H)$ if the multiplicity of the eigenvalues is taken into account.

For the right hand side, it is shown in Section 5 that $\mathcal{Q}$ is isomorphic to Q_N^+ and that this algebra is naturally embedded (after rescaling) in $C_0(\mathbb{R}; \mathscr{B}(\mathbb{C}^N))^+$. Thus, the index of W_- is computed by the winding number of the pointwise determinant of $q(W_-)$, see for example [24, Prop. 7]. The various contributions for this computation can be inferred either from Figure 1 or from Figure 2, but the functions to be considered are provided by Lemma 3.12. If we use the representation of the quotient algebra provided in Figure 1, then all horizontal contributions $\phi_{\to j}$ can be encoded in the expression $\mathrm{Var}(\lambda \to \det S(\lambda))$. For $\lambda \mapsto S(\lambda)$ piecewise C^1 these contributions can be computed analytically by the formula

$$\begin{aligned} &-\frac{1}{2\pi i}\sum_{k=1}^{N-1}\int_{\tilde\lambda_k-2}^{\tilde\lambda_{k+1}-2} \mathrm{Tr}\big(S(\lambda)^* S'(\lambda)\big)\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_{\tilde\lambda_N-2}^{\tilde\lambda_1+2} \mathrm{Tr}\big(S(\lambda)^* S'(\lambda)\big)\,\mathrm{d}\lambda \\ &-\frac{1}{2\pi i}\sum_{k=1}^{N-1}\int_{\tilde\lambda_k+2}^{\tilde\lambda_{k+1}+2} \mathrm{Tr}\big(S(\lambda)^* S'(\lambda)\big)\,\mathrm{d}\lambda \end{aligned}$$

where S is C^1 on each interval.

For the vertical contributions, recall that the functions $\eta_\pm$ have been introduced in (3.15). These contributions have to be computed from $+\infty$ to $-\infty$ for the intervals with one endpoint at $\tilde\lambda_j - 2$, while they have to be computed from $-\infty$ to $+\infty$ for the intervals having an endpoint at $\tilde\lambda_j + 2$.

In both cases, if $\tilde{S}_{jj}(\tilde{\lambda}_j \pm 2) = 1$, then the contribution is 0, as a straightforward consequence of (3.16) and (3.17). On the other hand, if $\tilde{S}_{jj}(\tilde{\lambda}_j - 2) = -1$, then $\phi_{\downarrow j} = \eta_-$, and this leads to a contribution of $\frac{1}{2}$, with our clockwise convention for the increase of the variation. Similarly, if $\tilde{S}_{jj}(\tilde{\lambda}_j + 2) = -1$, then $\phi_{\uparrow j} = \eta_+$ and this leads again to a contribution of $\frac{1}{2}$, because of the change of orientation of the path. As a consequence, if $\tilde{S}_{jj}(\tilde{\lambda}_j \pm 2) = -1$, the corresponding contribution is $\frac{1}{2}$, no matter if it is the opening or the closing of a channel of scattering. Our presentation in (4.2), which takes the content of Lemma 4.1 into account, reflects the genericity of the value -1 at thresholds over the value 1. ☐

In the previous proof and for the statement of (4.2) we have used the genericity of the value -1 over the value 1 for the special entry of the scattering matrix at thresholds. In the next lemma, we justify this statement, and therefore complement the information provided in Lemma 4.1. Unfortunately, its proof uses extensively the notations introduced in [27], and reintroducing all of them would be rather heavy and out of the scope of the present paper. For that reason, we refer as precisely as possible to the notation introduced in this reference, but do not recall all precise definitions.

Lemma 4.3. *For any $j \in \{1, \dots, N\}$ the equality $\tilde{\mathfrak{s}}_{jj}(\tilde{\lambda}_j \pm 2) = -1$ holds generically.*

Proof. By borrowing the expression provided in [27, Thm. 3.9] one has for the new entry at thresholds

$$S(\lambda_j - 2)_{jj} = \mathcal{P}_j - \mathcal{P}_j \mathfrak{v}\big(I_0(0) + S_0\big)^{-1} \mathfrak{v}\,\mathcal{P}_j + \mathcal{P}_j \mathfrak{v}\, C'_{10}(0) S_1 \big(I_2(0) + S_2\big)^{-1} S_1 C'_{10}(0) \mathfrak{v}\,\mathcal{P}_j, \quad (4.3)$$

while at the closure of thresholds one has

$$S(\lambda_j + 2)_{jj} = \mathcal{P}_j - i\,\mathcal{P}_j \mathfrak{v}\big(I_0(0) + S_0\big)^{-1} \mathfrak{v}\,\mathcal{P}_j + i\,\mathcal{P}_j \mathfrak{v}\, C'_{10}(0) S_1 \big(I_2(0) + S_2\big)^{-1} S_1 C'_{10}(0) \mathfrak{v}\,\mathcal{P}_j, \quad (4.4)$$

where each operator is computed explicitly at the corresponding threshold. At the opening of a channel of scattering, the operator $I_0(0)$ is given by $\frac{1}{2}\mathfrak{v}\mathcal{P}_j\mathfrak{v}$ while at the closure of a channel of scattering the operator one has $I_0(0) = \frac{i}{2}\mathfrak{v}\mathcal{P}_j\mathfrak{v}$. Since S_0 corresponds to the projection on $\ker\big(I_0(0)\big)$ one infers that $\mathcal{P}_j \mathfrak{v}\big(I_0(0) + S_0\big)^{-1} \mathfrak{v}\,\mathcal{P}_j = 2\mathcal{P}_j$ in the first case, while $i\mathcal{P}_j \mathfrak{v}\big(I_0(0) + S_0\big)^{-1} \mathfrak{v}\,\mathcal{P}_j = 2\mathcal{P}_j$ in the second case. As a consequence, equations (4.3) and (4.4) read

$$S(\lambda_j - 2)_{jj} = -\mathcal{P}_j + \mathcal{P}_j \mathfrak{v}\, C'_{10}(0) S_1 \big(I_2(0) + S_2\big)^{-1} S_1 C'_{10}(0) \mathfrak{v}\,\mathcal{P}_j,$$

while at the closure of thresholds one has

$$S(\lambda_j + 2)_{jj} = -\mathcal{P}_j + i\,\mathcal{P}_j \mathfrak{v}\, C'_{10}(0) S_1 \big(I_2(0) + S_2\big)^{-1} S_1 C'_{10}(0) \mathfrak{v}\,\mathcal{P}_j.$$

Then, by looking at the definition of the orthogonal projection S_1 as provided in the proof [27, Prop. 3.4], one easily infers that $S_1 = 0$ generically. Here, generically means for almost every θ and almost all perturbations V. The statement follows then directly, by oversing that $S(\lambda_j \pm 2)_{jj} = \tilde{S}(\tilde{\lambda}_j \pm 2)_{kk}$ for some $k \in \{1, \dots, N\}$. ☐

5 K-theory for the quotient algebra

In this section, we compute the K-groups of the quotient algebra. Since this computation is of independent interest and does not rely on the details of the model, we provide a self-contained proof with simpler notations. The main difficulty in the quotient algebra is the appearance of continuous functions with values in matrices of different sizes. However, the continuity conditions are very strict, and allow us to determine the K-theory for this algebra.

Proposition 3.11 tells us that the quotient algebra is matrix-valued functions on the union of the boundaries of the squares $I_j \cap I_{j'} \times [-\infty, \infty]$. Proposition 3.11 also describes how the rank of the matrices increases with λ until we reach $\tilde{\lambda}_1 + 2$ when it decreases again. The changes of rank happen continuously as the "extra rank" appears from (or disappears to) infinity via the "vertical" functions $\phi_{\downarrow j}$ and $\phi_{\uparrow^j}$. This section repeats this construction in a more induction friendly way, so that the K-theory can be computed. Interval by interval the isomorphism of the two constructions will be clear, and that the gluing of the pieces is the same can be checked using Proposition 3.11.

Before starting the construction, let us recall a result which will be constantly used, see [14, Ex. 4.10.22]. Consider three C^*-algebras A, B, and C, two surjective $*$-homomorphisms $\pi_1 : A \to C$ and $\pi_2 : B \to C$, and the pullback diagram

$$\begin{array}{ccc} A \oplus_C B & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle \pi_1} \\ B & \xrightarrow[\pi_2]{} & C \end{array}$$

with the pullback algebra $A \oplus_C B = \big\{(a, b) \in A \oplus B \mid \pi_1(a) = \pi_2(b)\big\}$. Then, the Mayer-Vietoris sequence

$$\begin{array}{ccccc} K_0(A \oplus_C B) & \longrightarrow & K_0(A) \oplus K_0(B) & \xrightarrow{\pi_{1*} - \pi_{2*}} & K_0(C) \\ \uparrow & & & & \downarrow \\ K_1(C) & \xleftarrow[\pi_{1*} - \pi_{2*}]{} & K_1(A) \oplus K_1(B) & \longleftarrow & K_1(A \oplus_C B) \end{array} \tag{5.1}$$

allows us to compute the K-theory of the pullback algebra in terms of the building blocks. The map $\pi_{1*} - \pi_{2*}$ is often referred to as the difference homomorphism.

Let us also introduce two easy lemmas, whose proofs are elementary exercises in K-theory. For shortness, we shall use the notation I for $(0, 1]$, the half-open interval.

Lemma 5.1. *One has* $K_0\big(C_0(I)\big) = K_1\big(C_0(I)\big) = 0$.

For the next statement, we introduce the algebra D_j for $j \geq 2$ by

$$D_j := \big\{f : [0, 1] \to \mathscr{B}(\mathbb{C}^j) \mid f(0) \in \mathscr{B}(\mathbb{C}^{j-1}) \oplus \mathbb{C}\big\}.$$

In other words, D_j is made of function on $[0, 1]$ with values in $\mathscr{B}(\mathbb{C}^j)$ with the only condition that $f(0)$ is block diagonal, with one block of size $(j-1) \times (j-1)$ and one block of size 1×1. Note that we identity $\mathbb{C}$ with $\mathscr{B}(\mathbb{C})$.

Lemma 5.2. *The short exact sequence*

$$0 \to C_0\big((0,1)\big) \otimes \mathscr{B}(\mathbb{C}^j) \to D_j \to \mathscr{B}(\mathbb{C}^{j-1}) \oplus \mathbb{C} \oplus \mathscr{B}(\mathbb{C}^j) \to 0$$

gives $K_0(D_j) = \mathbb{Z}^2$, $K_1(D_j) = 0$.

Let us now construct inductively some algebras. Again, I denotes $(0,1]$, and let $\{P_j\}_{j=1}^N$ be the rank one projections based on the standard basis of $\mathbb{C}^N$. For $j \in \{1,\dots,N\}$ set $A_j := C_0\big(I;\mathscr{B}(P_j\mathbb{C}^N)\big)$, and let us also fix $A(1) := A_1$. For $j \geq 2$ let $A(j)$ be the algebra obtained by gluing $A(j-1) \oplus A_j$ to D_j, namely

$$A(j) = \{(f \oplus g, h) \in A(j-1) \oplus A_j \oplus D_j : \, f(1) \oplus g(1) = h(0)\}.$$

Note that once the algebras $A(j-1)\oplus A_j$ and D_j are glued together, the resulting algebra $A(j)$ is considered again as matrix-valued functions defined on $[0,1]$. It means that a reparametrization of the interval is performed at each step of the iterative process. Note also that the algebra $A(j)$ can be thought as an algebra of functions with values in a set of matrices of increasing size. However, the change of dimension from $k-1$ to k is performed concomitantly with the addition of the new path A_k linking 0 to the new diagonal entry of the $k \times k$ matrix.

Proposition 5.3. *For any* $j \in \{1,\dots,N\}$ *one has* $K_0\big(A(j)\big) = 0$ *and* $K_1\big(A(j)\big) = 0$.

Proof. For $j = 1$, the statement corresponds to Lemma 5.1, since

$$A(1) \equiv A_1 = C_0\big(I;\mathscr{B}(P_1\mathbb{C}^N)\big) = C_0(I) \otimes \mathscr{B}(P_1\mathbb{C}^N).$$

We then look at the result for $j = 2$. In this case the construction leads to the pullback diagram

$$\begin{array}{ccc} (A_1 \oplus A_2) \oplus_{\mathbb{C}^2} D_2 & \longrightarrow & A_1 \oplus A_2 \\ \downarrow & & \downarrow \\ D_2 & \longrightarrow & \mathbb{C}^2 \end{array}$$

Using the computations of the K-theory of $A_1 \oplus A_2$ and D_2 from Lemmas 5.1 and 5.2, the Mayer-Vietoris sequence gives

$$0 \to K_0\big(A(2)\big) \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to K_1\big(A(2)\big) \to 0.$$

We will check that the difference homomorphism $K_0(D_2) = \mathbb{Z}^2 \to \mathbb{Z}^2 = K_0(\mathbb{C}^2)$ is a bijection, and therefore $K_0\big(A(2)\big) = 0$ and $K_1\big(A(2)\big) = 0$. For that purpose, we need the generators of $K_0(D_2)$. One choice consists in the equivalence classes of projections

$$\left[t \mapsto \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}\right] - \left[t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right], \qquad \left[t \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

for $t \in [0,1]$. By evaluating them at $t = 0$ one gets

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right], \qquad \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

and therefore we deduce the expected surjection $K_0(D_2) \to K_0(\mathbb{C}^2)$.

By induction, let us now suppose that the result is true for some j, and prove it for $j+1$. Then we consider the pullback algebra diagram determined by

$$\begin{array}{ccc} (A(j)\oplus A_{j+1})\oplus_{\mathscr{B}(\mathbb{C}^j)\oplus\mathbb{C}} D_{j+1} & \longrightarrow & D_{j+1} \\ \downarrow & & \downarrow \\ A(j)\oplus A_{j+1} & \longrightarrow & \mathscr{B}(\mathbb{C}^j)\oplus\mathbb{C} \end{array}$$

and get the Mayer-Vietoris sequence

$$0 \to K_0\big(A(j+1)\big) \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to K_1\big(A(j+1)\big) \to 0.$$

We again show that the difference homomorphism

$$K_0(D_{j+1}) = \mathbb{Z}^2 \to \mathbb{Z}^2 = K_0\big(\mathscr{B}(\mathbb{C}^j)\oplus\mathbb{C}\big)$$

is a bijection. For that purpose, we need the generators of $K_0(D_{j+1})$. Let us set $E_{1,j} \in \mathbb{C}^j$ with $E_{1,j} = (1,0,\ldots,0)^T$, and let $E_{j,1}$ denote its transpose. Then we build representative equivalence classes of projections. These projections are block matrices (the first block being of size $j \times j$) and the choice of representatives are the maps

$$\left[t \mapsto \begin{pmatrix} tP_1 & \sqrt{t(1-t)}E_{1,j} \\ \sqrt{t(1-t)}E_{j,1} & 1-t \end{pmatrix}\right] - \left[t \mapsto \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}\right], \qquad \left[t \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

for $t \in [0,1]$. By evaluating them at $t=0$ one gets

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}\right], \qquad \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

and therefore we deduce the expected surjection $K_0(D_{j+1}) \to K_0\big(\mathscr{B}(C^j)\oplus\mathbb{C}\big)$. This completes the proof. □

Note that the same result holds for the matrices $B(j)$ obtained by reversing the construction (and by taking care of the subspaces of $\mathbb{C}^N$ involved). For this reverse construction, we firstly set for $j \in \{1,\ldots,N\}$

$$E_j := \big\{f : [0,1] \to \mathscr{B}\big((\mathbb{C}^{N-j})^\perp\big) \mid f(1) \in \mathbb{C} \oplus \mathscr{B}\big((\mathbb{C}^{N-j+1})^\perp\big)\big\}$$

and $B_j := C_0\big([0,1);\mathscr{B}(P_{N-j+1}\mathbb{C}^N)\big)$. We also fix $B(1) := B_1$. For $j \geq 2$ let $B(j)$ be the algebra obtained by gluing E_j to $B_j \oplus B(j-1)$, namely for $h \in E_j$ and for $f \oplus g \in B_j \oplus B(j-1)$ we set $h(1) = f(0) \oplus g(0)$. Note that once the algebras E_j and $B_j \oplus B(j-1)$ are glued together, the resulting algebra $B(j)$ is considered again as matrix-valued functions defined on $[0,1]$. It means that a reparametrization of the basis is performed at each step of the iterative process. The algebra $B(j)$ can be thought as an algebra of functions with values in a set of matrices of decreasing size. However, the change of dimension from k to $k-1$ is performed concomitantly with the addition of the new path B_k linking the removed entry to 0.

By exactly the same proof, we get the same result:

Proposition 5.4. *For any $j \in \{1,\ldots,N\}$ one has $K_0\big(B(j)\big) = 0$ and $K_1\big(B(j)\big) = 0$.*

The next step will be to glue together an algebra of the family $A(j)$ with an algebra of the family $B(j)$. Before this, we need another lemma, whose proof is again an elementary exercise.

Lemma 5.5. *For*

$$C_N := \{f : [0,1] \to \mathscr{B}(\mathbb{C}^N) \mid f(0) \in \mathscr{B}(\mathbb{C}^{N-1}) \oplus \mathbb{C},\ f(1) \in \mathbb{C} \oplus \mathscr{B}\big((\mathbb{C}^1)^\perp\big)\},$$

the exact sequence

$$0 \to C_0\big((0,1)\big) \otimes \mathscr{B}(\mathbb{C}^N) \to C_N \to \mathscr{B}(\mathbb{C}^{N-1}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathscr{B}\big((\mathbb{C}^1)^\perp\big) \to 0$$

gives $K_0(C_N) = \mathbb{Z}^3$ *and* $K_1(C_N) = 0$.

We can now collect the information obtained so far, and provide the K-theory for the main algebra of this section.

Proposition 5.6. *For*

$$Q_N = \big(A(N-1) \oplus A_N\big) \oplus_{\mathscr{B}(\mathbb{C}^{N-1})\oplus\mathbb{C}} C_N \oplus_{\mathbb{C}\oplus\mathscr{B}((\mathbb{C}^1)^\perp)} \big(B_N \oplus B(N-1)\big)$$

one has $K_0\big(Q_N^+\big) = \mathbb{Z}$ *and* $K_1\big(Q_N^+\big) = \mathbb{Z}$.

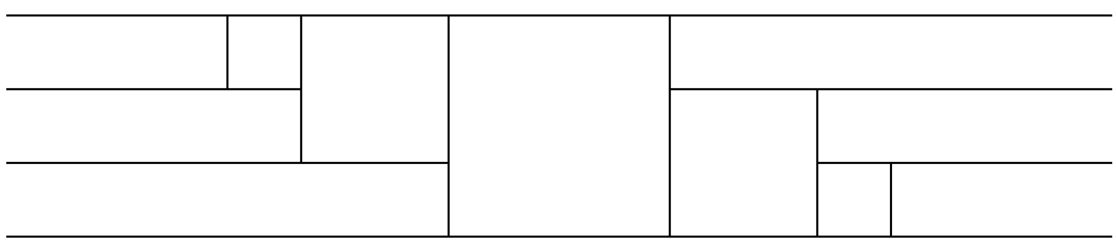

Figure 2: Representation of the algebra Q_4, with the biggest square representing C_4, while the part of the left of this square corresponds to $A(3) \oplus A_4$, and the part on the right corresponds to $B_4 \oplus B(3)$. For comparison with the algebra $\mathcal{Q}$, the dashed lines represent the support of the functions $\phi_{\downarrow_j}$ while the dotted lines represent the support of the functions $\phi_{\uparrow^j}$. The remaining lines and squares correspond to the support of the scattering matrix.

Proof. We shall ignore the unit in the following proof, since adding the unit will only add a copy of $\mathbb{Z}$ to $K_0(Q_N)$. Let us consider the pullback diagram

$$\begin{array}{ccc} Q_N & \longrightarrow & \big(A(N-1) \oplus A_N\big) \oplus \big(B_N \oplus B(N-1)\big) \\ \downarrow & & \downarrow \\ C_N & \longrightarrow & \mathscr{B}(\mathbb{C}^{N-1}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathscr{B}\big((\mathbb{C}^1)^\perp\big) \end{array}$$

and the corresponding Mayer-Vietoris sequence

$$0 \to K_0(Q_N) \to \mathbb{Z}^3 \to \mathbb{Z}^4 \to K_1(Q_N) \to 0$$

where the content of Propositions 5.3 and 5.4 and of Lemma 5.5 have been used for the computation of the various K-groups. By examining the generators, we shall show that the difference homomorphism

$$K_0(C_N) = \mathbb{Z}^3 \to \mathbb{Z}^4 = K_0\Big(\mathscr{B}(\mathbb{C}^{N-1}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathscr{B}\big((\mathbb{C}^1)^\perp\big)\Big)$$

is one-to-one, leading to $K_0(Q_N) = 0$ and $K_1(Q_N) = \mathbb{Z}$. In the construction of the generators of $K_0(C_N)$ we use the notation already introduced in the proof of Proposition 5.3. These generators are equivalence classes of projections. These projections are made of block matrices (the first block being of size $(N-1) \times (N-1)$) and one choice of representatives are the maps defined by

$$\left[t \mapsto \begin{pmatrix} tP_1 & \sqrt{t(1-t)}E_{1,N-1} \\ \sqrt{t(1-t)}E_{N-1,1} & 1-t \end{pmatrix}\right] - \left[t \mapsto \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

$$\left[t \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \qquad \text{and} \qquad \left[t \mapsto \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

for $t \in [0,1]$. By evaluating them at $t = 0$ one gets

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] - \left[\begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}\right], \quad \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right], \quad \left[\begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

and evaluating at $t = 1$ gives (with the first block of size 1×1)

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & P_N \end{pmatrix}\right], \quad \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

The image under the combined evaluation map evidently contains three independent generators of $\mathbb{Z}^4 = K_0\Big(\mathscr{B}(\mathbb{C}^{N-1}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathscr{B}\big((\mathbb{C}^1)^\perp\big)\Big)$, and that suffices to complete the proof. $\square$

By construction, the algebras $\mathcal{Q}$ and Q_N^+ are isomorphic, therefore they share the same K-theory. Note that the isomorphism can also be inferred by comparing Figures 1 and 2: the vertical lines of Figure 1 are symbolised horizontally in Figure 2, with dashed or dotted lines. Furthermore, since these lines represent the support of a block diagonal matrix-valued function, they can be rescaled independently. Thanks to this isomorphism, the K-theory of $\mathcal{Q}$ can be directly deduced.

Corollary 5.7. *One has $K_0\big(\mathcal{Q}\big) = \mathbb{Z}$ and $K_1\big(\mathcal{Q}\big) = \mathbb{Z}$.*

6 The θ-dependence: explicit computations for $N = 2$

In the analysis performed so far, the parameter $\theta \in (0, \pi)$ appearing in (2.1) has been kept constant. In this final section, we would like to illustate the dependence of some spectral quantities as a function of θ. For simplicity we stick to the case $N = 2$.

We firstly recall some notations restricted to the case $N = 2$ and add a θ-dependence for clarity. For the analysis of H^θ, the main spectral result is a necessary and sufficient condition for the existence of an eigenvalue for the operator H^θ. For that purpose, we decompose the matrix $\operatorname{diag}(v) := \operatorname{diag}\big(v(1), v(2)\big)$ as a product $\operatorname{diag}(v) = \mathfrak{u}\mathfrak{v}^2$, where $\mathfrak{v} := |\operatorname{diag}(v)|^{1/2}$ and $\mathfrak{u}$ is the diagonal matrix with components

$$\mathfrak{u}_{jj} = \begin{cases} +1 & \text{if } v(j) \geq 0 \\ -1 & \text{if } v(j) < 0, \end{cases}$$

for $j \in \{1,2\}$. For simplicity, we set $v(1) = u_1 a^2$, $v(2) = u_2 b^2$ and assume that $a \geq b \geq 0$ and $a > 0$. We also impose the following condition: if $u_1 = u_2$, then $a > b$, since otherwise the function v would be 1–periodic and not 2–periodic. With these notations one has $\mathfrak{v} = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ and $\mathfrak{u} = \left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)$. For $z \in \mathbb{C}$ and $j \in \{1,2\}$ we also introduce the expression

$$\beta_j^\theta(z) := \big|\big(z - \lambda_j^\theta\big)^2 - 4\big|^{1/4}.$$

We finally observe that $\lambda_1^\theta = -2\cos(\frac{\theta}{2})$ and $\lambda_2^\theta = 2\cos(\frac{\theta}{2})$, $\xi_1^\theta = \left(\begin{smallmatrix} -\mathrm{e}^{i\theta/2} \\ \mathrm{e}^{i\theta} \end{smallmatrix}\right)$ and $\xi_2^\theta = \left(\begin{smallmatrix} \mathrm{e}^{i\theta/2} \\ \mathrm{e}^{i\theta} \end{smallmatrix}\right)$.

With an emphasize on the θ-dependence, let us also recall the main statement about eigenvalues proved in [27, Prop. 1.2] and valid for any N :

Proposition 6.1. *A value $\lambda \in \mathbb{R} \setminus \mathcal{T}^\theta$ is an eigenvalue of H^θ if and only if*

$$\mathcal{K} := \ker\left(\mathfrak{u} + \sum_{\{j \mid \lambda < \lambda_j^\theta - 2\}} \frac{\mathfrak{v}\,\mathcal{P}_j^\theta \mathfrak{v}}{\beta_j^\theta(\lambda)^2} - \sum_{\{j \mid \lambda > \lambda_j^\theta + 2\}} \frac{\mathfrak{v}\,\mathcal{P}_j^\theta \mathfrak{v}}{\beta_j^\theta(\lambda)^2}\right) \bigcap \Big(\cap_{\{j \mid \lambda \in I_j^\theta\}} \ker\big(\mathcal{P}_j^\theta \mathfrak{v}\big)\Big) \neq \{0\},$$

in which case the multiplicity of λ equals the dimension of $\mathcal{K}$.

Based on this statement, we now illustrate the θ-dependence of the number of eigenvalues located below the essential spectrum of H^θ. More precisely, the sign of the determinant of the matrix

$$\mathfrak{u} + \frac{\mathfrak{v}\mathcal{P}_1^\theta \mathfrak{v}}{\beta_1^\theta(\lambda)^2} + \frac{\mathfrak{v}\mathcal{P}_2^\theta \mathfrak{v}}{\beta_2^\theta(\lambda)^2}.$$

is computed, as a function of λ and θ. Depending on its value, either a red colour or a blue colour is assigned to the point (λ, θ) in Figure 3. For this figure, we chose $u_1 = u_2 = -1$, $a = 1$ and $b = \frac{1}{\sqrt{2}}$. Thus, interfaces between a blue region and a red region coincide with eigenvalues below the essential spectrum, according to Proposition 6.1. In Figure 3, the black curve represents the bottom of the essential spectrum of H^θ. For most fixed θ, there exists only one interface between the red region and the blue region, meaning that the corresponding operator H^θ possesses only one eigenvalue. However, for θ close to π, a second interface appears, as emphasised in the magnified part. Then the corresponding operators H^θ possess two eigenvalues below the essential spectrum. The minimal value θ_0 above which a second eigenvalue appears can be determined by solving the equation

$$\big(\cos(\tfrac{\theta_0}{2}) + \cos^2(\tfrac{\theta_0}{2})\big)^{\frac{1}{2}} = \frac{1}{6}.$$

Let us now turn to scattering theory, and in particular to the behavior of the scattering operator at thresholds. For that purpose, we set

$$\Xi_+(\theta) := \big(\cos(\tfrac{\theta}{2}) + \cos^2(\tfrac{\theta}{2})\big)^{\frac{1}{2}} > 0 \quad \text{and} \quad \varrho := u_2 a^2 + u_1 b^2$$

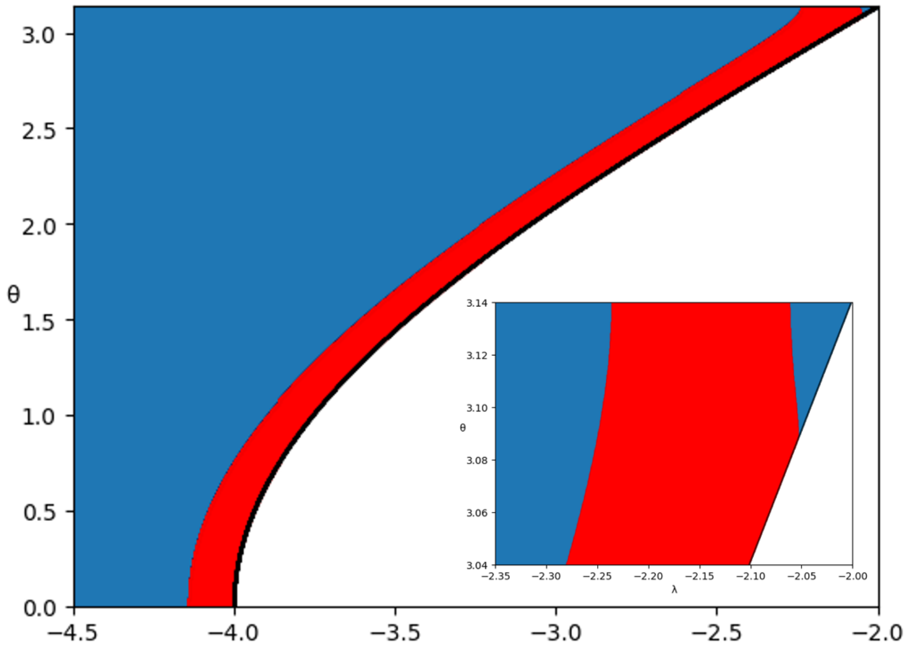

Figure 3: Visual representation of the number of eigenvalues below the essential spectrum of H^θ in the case $u_1 = u_2 = -1$, $a = 1$ and $b = \frac{1}{\sqrt{2}}$: For each fixed θ, each interface between a blue and a red region corresponds to an eigenvalue. The magnified picture represents the region close to $\theta = \pi$, where a second interface appears.

since these expressions will appear a few times in the sequel. For conciseness, we restrict our attention to $\lambda \in (-4, 0)$. In this setting, only two thresholds appear: one at $\lambda_1^\theta - 2$ and one at $\lambda_2^\theta - 2$. The first one corresponds to the threshold at the bottom of the essential spectrum, while the second one corresponds to an embedded threshold. Since we already know from [27, Thm. 3.9] that S_{11}^θ is continuous at $\lambda_2^\theta - 2$, we concentrate on S_{22}^θ at $\lambda_2^\theta - 2$.

The next statement provides the behavior of the scattering operator at the two opening channels. The proof, consisting of an explicit computation based on expressions available in [27, Sec. X], will be given elsewhere.

Proposition 6.2. *Let $\theta \in (0, \pi)$ and consider $a \geq b \geq 0$ with $a > 0$. Then the following equalities hold:*

$$\mathfrak{s}^\theta(\lambda_1^\theta - 2)_{11} = 1 \quad \textit{if } \; 2\varrho\Xi_+(\theta) + a^2b^2 = 0 \quad \textit{(resonant case)}, \tag{6.1}$$

$$\mathfrak{s}^\theta(\lambda_1^\theta - 2)_{11} = -1 \quad \textit{otherwise} \qquad \textit{(generic case)}, \tag{6.2}$$

and

$$\mathfrak{s}^\theta(\lambda_2^\theta - 2)_{22} = -1.$$

Let us briefly comment on this statement with the help of Figures 4 and 5. In Figure 4, the x-axis corresponds to the values of $v(1)$ while the y-axis corresponds to the values of $v(2)$. The red zones (2 open cones + the diagonal line in quadrants I and III) are the ones we can disregard since either $a < b$ or the system is 1–periodic. The magenta zones (including the diagonal line in quadrants II and IV, the x-axis and the curved boundaries) correspond

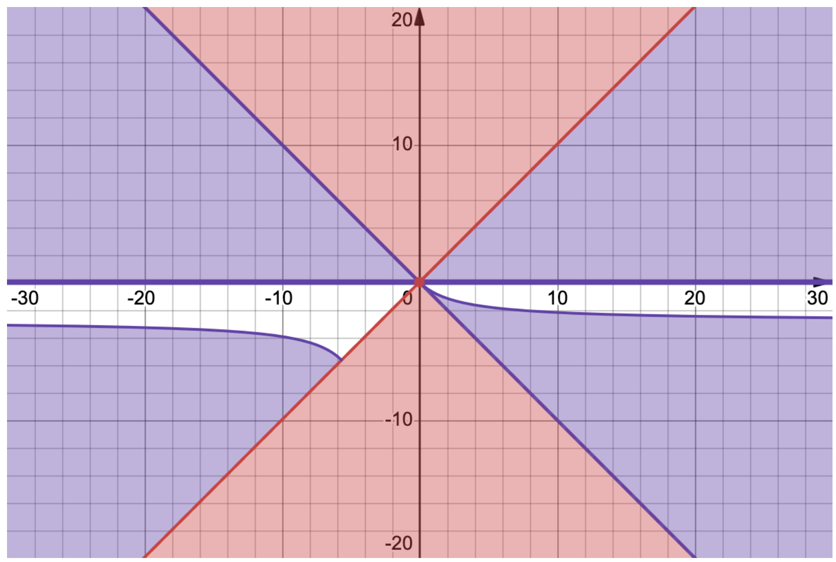

Figure 4: The horizontal axis corresponds to $v(1)$ while the vertical axis corresponds to $v(2)$. The two white regions correspond to points $\big(v(1), v(2)\big)$ for which there exists a unique $\theta_0 \in (0, \pi)$ leading to a resonant case.

to combinations of $v(1)$ and $v(2)$ which lead to the generic case, namely to (6.2). The open white region has two connected components and is given by points $\big(v(1), v(2)\big)$ such that there exists a unique $\theta_0 \in (0, \pi)$ which verifies $v(2) = -\frac{2\Xi_+(\theta_0)v(1)}{v(1)+2\Xi_+(\theta_0)}$. This condition is equivalent to the condition leading to the resonant case given by (6.1). In particular, it follows from the equality $2\varrho\Xi_+(\theta_0) = -a^2b^2$ that $\varrho < 0$, which means that either $u_1 = u_2 = -1$, corresponding to the left white region, or to $u_2 = -1$ and $u_1 = 1$, corresponding to the right white region.

In order to understand the dependence of θ_0 on $v(1)$ and $v(2)$ for the resonant case, we have represented in Figure 5 the value θ_0 on the z–axis as function of $v(1)$ on the x–axis and of $v(2)$ on the y–axis. Note that θ_0 is only shown on the left white region of Figure 4 which is bounded on the y–axis and unbounded on the x–axis.

References

[1] T. Aktosun, R. Weder, *High-energy analysis and Levinson's theorem for the selfadjoint matrix Schrödinger operator on the half line*, J. Math. Phys. 54 no.1, 012108, 27 pp, 2013.

[2] A. Alexander, A. Rennie, *Levinson's theorem as an index pairing*, J. Funct. Anal. 28 no. 5, paper No. 110287, 2024.

[3] M. Ballesteros, G. Franco, H. Schulz-Baldes, *Analyticity properties of the scattering matrix for matrix Schrödinger operators on the discrete line*, J. Math. Anal. Appl. 497 no. 1, Paper No. 124856, 27 pp, 2021.

[4] M. Ballesteros, G. Franco, G. Garro, H. Schulz-Baldes, *Band edge limit of the scattering matrix for quasi-one-dimensional discrete Schrödinger operators*, Complex Anal. Oper. Theory 16 no. 2, Paper No. 23, 31 pp, 2022.

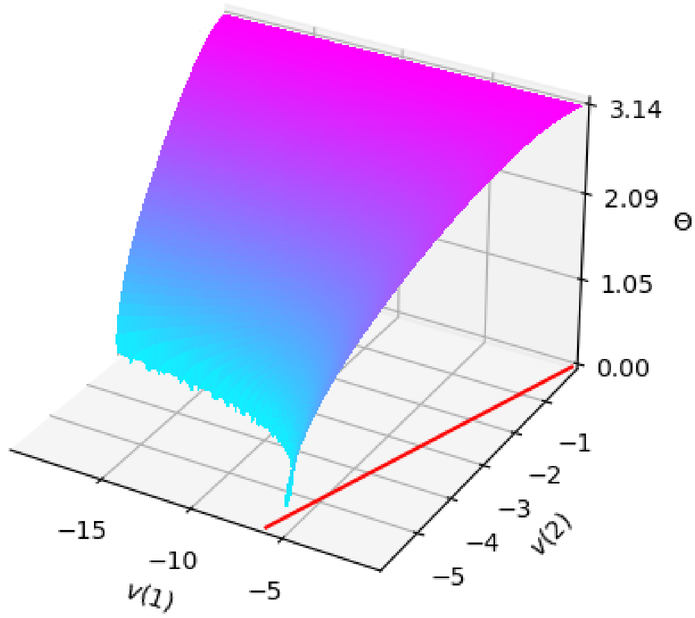

Figure 5: The values θ_0 that satisfy the resonant case (in the left white region of Figure 4) is represented, with the value $v(1)$ and $v(2)$ on the x and y axes. The 2 scales are different, and the red line represent the diagonal on this quadrant. The z–axis corresponds to θ in $(0, \pi)$, and the surface represents the values of θ_0, as a function of $v(1)$ and $v(2)$.

[5] M. Ballesteros, G. Franco Córdova, I. Naumkin, H. Schulz-Baldes, *Levinson theorem for discrete Schrödinger operators on the line with matrix potentials having a first moment*, Commun. Contemp. Math. 26 no. 7, Paper No. 2350017, 2024.

[6] J. Bellissard, H. Schulz-Baldes, *Scattering theory for lattice operators in dimension $d \geq 3$*, Rev. Math. Phys. 24 no. 8, 1250020, 51 pp., 2012.

[7] E. Brüning, F. Gesztesy, *Continuity of wave and scattering operators with respect to interactions*, J. Math. Phys. 24 no.6, 1516–1528, 1983.

[8] A. Chahrour, *On the spectrum of the Schrödinger operator with periodic surface potential*, Lett. Math. Phys. 52 no. 3, 197–209, 2000.

[9] A. Chahrour, J. Sahbani, *On the spectral and scattering theory of the Schrödinger operator with surface potential*, Rev. Math. Phys. 12 no. 4, 561–573, 2000.

[10] Y. Colin de Verdière, N. Torki-Hamza, Nabila, F. Truc, *Essential self-adjointness for combinatorial Schrödinger operators III—Magnetic fields*, Ann. Fac. Sci. Toulouse Math. (6) 20 no. 3, 599–611, 2011.

[11] A. Connes, H. Moscovici, *Type III and spectral triples*, Aspects Math. E38, Friedr. Vieweg & Sohn, Wiesbaden, 57–71, 2008.

[12] R. Frank, *On the scattering theory of the Laplacian with a periodic boundary condition. I. Existence of wave operators*, Doc. Math. 8, 547–565, 2003.

[13] R. Frank, R. Shterenberg, *On the scattering theory of the Laplacian with a periodic boundary condition. II. Additional channels of scattering*, Doc. Math. 9, 57–77, 2004.

[14] N. Higson, J. Roe, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[15] D.B. Hinton, M. Klaus, J.K. Shaw, *Half-bound states and Levinson's theorem for discrete systems*, SIAM J. Math. Anal. 22, 754–768, 1991.

[16] H. Inoue, S. Richard, *Index theorems for Fredholm, semi-Fredholm, and almost periodic operators: all in one example*, J. Noncommut. Geom. 13 no. 4, 1359–1380, 2019.

[17] H. Inoue, S. Richard, *Topological Levinson's theorem for inverse square potentials: complex, infinite, but not exceptional*, Rev. Roumaine Math. Pures Appl. 64 no. 2-3, 225–250, 2019.

[18] H. Inoue, N. Tsuzu, *Schrödinger Wave Operators on the Discrete Half-Line*, Integr. Equ. Oper. Theory 91 no. 42, 2019.

[19] V. Jakšić, Y. Last, *Corrugated surfaces and a.c. spectrum*, Rev. Math. Phys. 12 no. 11, 1465–1503, 2000.

[20] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[21] J. Kellendonk, S. Richard, *Levinson's theorem for Schrödinger operators with point interaction: a topological approach*, Phys. A 39 no. 46, 14397–14403, 2006.

[22] J. Kellendonk, S. Richard, *Topological boundary maps in physics*, in Perspectives in operator algebras and mathematical physics, pages 105–121, Theta Ser. Adv. Math. 8, Theta, Bucharest, 2008.

[23] J. Kellendonk, S. Richard, *The topological meaning of Levinson's theorem, half-bound states included*, J. Phys. A 41 no. 29, 295207, 2008.

[24] J. Kellendonk, S. Richard, *On the structure of the wave operators in one-dimensional potential scattering*, Mathematical Physics Electronic Journal 14, 1–21, 2008.

[25] E. Korotyaev, N. Saburova, *Magnetic Schrödinger operators on periodic discrete graphs*, J. Funct. Anal. 272 no. 4, 1625–1660, 2017.

[26] N. Levinson, *On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase*, Danske Vid. Selsk. Mat.-Fys. Medd. 25 no. 9, 29 pp., 1949.

[27] H.S. Nguyen, S. Richard, R. Tiedra de Aldecoa, *Discrete Laplacian in a half-space with a periodic surface potential I: Resolvent expansions, scattering matrix, and wave operators*, Math. Nachr. 295 no. 5, 912–949, 2022.

[28] F. Nicoleau, D. Parra, S. Richard, *Does Levinson's theorem count complex eigenvalues?* J. Math. Phys. 58, 102101-1 - 102101-7, 2017.

[29] S. Richard, *Discrete Laplacian in a half-space with a periodic surface potential: an overview of analytical investigations*, RIMS Kokyuroku 2195, 18 - 26, Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 2021.

[30] S. Richard, *Levinson's theorem: an index theorem in scattering theory*, in Spectral Theory and Mathematical Physics, volume 254 of Operator Theory: Advances and Applications, pages 149–203, Birkhäuser, Basel, 2016.

[31] S. Richard, R. Tiedra de Aldecoa, *New formulae for the wave operators for a rank one interaction*, Integral Equations and Operator Theory 66, 283–292, 2010.

[32] S. Richard, R. Tiedra de Aldecoa, *Resolvent expansions and continuity of the scattering matrix at embedded thresholds: the case of quantum waveguides*, Bull. Soc. Math. France 144 no. 2, 251–277, 2016.

[33] S. Richard, R. Tiedra de Aldecoa, *Spectral and scattering properties at thresholds for the Laplacian in a half-space with a periodic boundary condition*, J. Math. Anal. Appl. 446 no. 2, 1695–1722, 2017.

[34] H. Schulz-Baldes, *The density of surface states as the total time delay*, Lett. Math. Phys. 106 no. 4, 485–507, 2016.

[35] T. Sunada, *A discrete analogue of periodic magnetic Schrödinger operators*, in Geometry of the spectrum (Seattle, WA, 1993), 283–299, Contemp. Math. 173, American Mathematical Society, Providence, RI, 1994.