MORPHISMS OF CUNTZ–PIMSNER ALGEBRAS FROM COMPLETELY POSITIVE MAPS

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In memory of Iain Raeburn,
with gratitude for all of his contributions to mathematics and our community.

Abstract. We introduce positive correspondences as right C^*-modules with left actions given by completely positive maps. Positive correspondences form a semi-category that contains the C^*-correspondence (Enchilada) category as a "retract". Kasparov's KSGNS construction provides a semi-functor from this semi-category onto the C^*-correspondence category. The need for left actions by completely positive maps appears naturally when we consider morphisms between Cuntz–Pimsner algebras, and we describe classes of examples arising from projections on C^*-correspondences and Fock spaces, as well as examples from conjugation by bi-Hilbertian bimodules of finite index.

1. Introduction

In this paper we introduce positive correspondences between C^*-algebras and examine their interplay with Cuntz–Pimsner algebras. In particular, we look at positive correspondences arising from both correspondence projections and conjugations by bi-Hilbertian bimodules, and use these to construct C^*-correspondences between Cuntz–Pimsner algebras. In the general framework of noncommutative geometry, C^*-algebras appear as noncommutative topological spaces, making them highly suited to study topological dynamical systems (e.g. symbolic dynamics, directed graphs, k-graphs) as well as more general noncommutative dynamics.

For good and bad, $*$-homomorphisms between C^*-algebras are very rigid, which can make them hard to find in practice. When a right C^*-module (a.k.a. a Hilbert module) X_B over a C^*-algebra B admits an adjointable left action of a possibly different C^*-algebra A, we call ${}_AX_B$ a C^*-correspondence; and we may think of it as a more flexible notion of morphism from A to B. Indeed, every $*$-homomorphism induces a C^*-correspondence. This perspective of C^*-correspondences as morphisms is taken in [MS19, EKQR06, EKQR00]. Accordingly, C^*-correspondences provide a flexible framework comparing the structure of C^*-algebras, and allow us to consider more general classes of dynamics which are not induced by $*$-homomorphisms.

Pimsner [Pim97] pioneered a construction of a C^*-algebra built from a self-correspondence over a (possibly noncommutive) C^*-algebra. Pimsner's C^*-algebras are universal, and generalise both crossed products by $\mathbb{Z}$ and Cuntz–Krieger algebras [CK80]. This construction was later refined by Katsura [Kat04a], and we refer to these C^*-algebras as *Cuntz–Pimsner algebras*. In a sense, these algebras encode the dynamics of a C^*-correspondence, and they have found considerable use in analysing topological dynamical systems [Kat04b, KW05, Nek09, LRRW18, KK14, CDE24].

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Typically, it is difficult to find maps between Cuntz–Pimnser algebras which are induced by maps between their defining C^*-correspondences. Such maps are important, since for commutative algebras they can be considered "dynamically compatible" maps. The coisometric morphisms of [Bre10, Definition 1.3] induce $*$-homomorphisms between Cuntz–Pimsner algebras, but they are often hard to come by. More generally, the covariant correspondences considered in [MS19, Ery22] induce C^*-correspondences between Cuntz–Pimsner algebras, but again these are somewhat rare.

In our previous work [BMR24]—inspired by moves on graphs [Wi73, BP04, ERRS16, MPT08]—we developed notions of splittings for C^*-correspondences and showed how they preserve the isomorphism class or Morita equivalence class of the Cuntz–Pimsner algebras. Here, we take a different approach.

Our starting point is the observation that for a self-correspondence ${}_AX_A$, and a sub-self-correspondence ${}_AY_A \subseteq {}_AX_A$, the inclusion/projection relation does *not*, in general, give rise to a covariant morphism of correspondences in either direction. We do obtain a $*$-homomorphism from the Toeplitz algebra of ${}_AY_A$ to the Toeplitz algebra of ${}_AX_A$, but this $*$-homomorphism does not usually pass to the Cuntz–Pimsner quotients. Instead, we obtain a completely positive map from the Cuntz–Pimsner algebra of ${}_AX_A$ to the Cuntz–Pimsner algebra of ${}_AY_A$. The KSGNS construction [Lan95, Chapter 5] allows us to pass to an honest C^*-correspondence.

A categorical framework was developed in [EKQR00, EKQR06], in which morphisms are isomorphism classes of C^*-correspondences. This correspondence category (a.k.a. the Enchilada category) extends the more traditional category of C^*-algebras with $*$-homomorphisms. The correspondence category was further explored in [MS19, EKQ20, Ery22].

To encapsulate both completely positive maps and C^*-correspondences in a single object we introduce *positive correspondences* as right C^*-modules admitting a left action by a (strict) completely positive map (see Definition 3.10). A similar setup can be found in [Lan95, Ch. 5]. The isomorphism classes of positive correspondences form a semi-category Quesadilla (see Definition 3.15), and this seems to be the right domain for the KSGNS construction. In fact, the semi-category contains Enchilada as a wide subcategory, and KSGNS acts as a semi-functor from our semi-category onto Enchilada, which now appears as a retract of Quesadilla. The only reason why positive correspondences do not form an honest category is a lack of identities: the obvious identity morphisms act as identities on the right, but not on the left (see Lemma 3.14).

There is an abundance of examples giving rise to completely positive maps and thereby positive correspondences, and we outline a few in this paper. We note that this work is different, but related, to the crossed products by completely positive maps considered in [Kwa17, Bre18].

After some general background in Section 2, we recall the KSGNS construction in detail in Section 3, and show how we obtain a semi-category from positive correspondences. In Sections 4 and 5, we show how a range of different constructions on C^*-correspondences and their Fock modules give rise to positive correspondences. The three types of positive correspondences we consider come from projections

(i) on self-correspondences themselves,
(ii) on Fock modules of self-correspondences, and
(iii) on bi-Hilbertian bimodule "amplifications" of the Fock modules of self-correspondences.

In each case we describe the resulting positive maps, and the induced C^*-correspondences we obtain from the KSGNS construction. For the second class of examples we also describe their relation to subproduct systems. A feature of the third class of examples is that we can consider pairs of Cuntz–Pimsner algebras associated to self-correspondences with differing coefficient algebras.

2. Correspondences, Cuntz–Pimsner algebras, and Enchiladas

In this section we collect definitions and basic results, while establishing notation.

2.1. C^*-modules and correspondences.

We refer the reader to [Lan95, RW98] for background on C^*-modules. Throughout this paper, A and B denote separable C^*-algebras. Given a right C^*-module over A (a.k.a. a right Hilbert A-module or right A-module) X—written X_A when we want to remember the coefficient algebra—we denote the C^*-algebra of adjointable operators on X by $\mathrm{End}_A(X)$, the C^*-ideal of compact operators by $\mathrm{End}^0_A(X)$, and the finite-rank operators by $\mathrm{End}^{00}_A(X)$. The finite-rank operators are generated by rank-one operators $\Theta_{x,y}$ with x, $y \in X$ satisfying $\Theta_{x,y}(z) = x \cdot \langle y \mid z\rangle_A$. If A_A is the trivial A-module, then we identify $\mathrm{End}_A(X)$ with the multiplier algebra $\mathrm{Mult}(A)$ of A.

A right C^*-module X_A is *full* if $A = \overline{\mathrm{span}}\langle X \mid X\rangle_A$. We also have occasion to consider left A-modules, which behave analogously.

Definition 2.1. Let X_B be a countably generated right C^*-module, and let $\phi\colon A \to \mathrm{End}_B(X)$ be a $*$-homomorphism. The data $(\phi, {}_AX_B)$ is a *C^*-correspondence* (or an *A–B-correspondence*). If ϕ is understood we just write ${}_AX_B$. If $\overline{\phi(A)X} = X$, we say ${}_AX_B$ is *nondegenerate*. If $A = B$ we say that $(\phi, {}_AX_A)$ is a *C^*-correspondence over A* or a *self-correspondence*. If ϕ is understood we sometimes use the notation $a \cdot x := \phi(a)x$ for $a \in A$ and $x \in X_B$. A C^*-correspondence is *regular* if ϕ is injective and $\phi(A) \subseteq \mathrm{End}^0_B(X)$.

Definition 2.2. A regular correspondence $(\phi, {}_AX_B)$ with X_B full and $\mathrm{End}^0_B(X) = \phi(A)$ is called a *Morita equivalence bimodule.* Morita equivalence bimodules are also left A-modules with inner product ${}_A\langle x, y\rangle = \phi^{-1}(\Theta_{x,y})$.

We discuss the more general class of bi-Hilbertian bimodules in Section 5.

An important technical and computational tool in the theory and applications of C^*-modules are frames. These are as close as we can get to an orthonormal basis in a C^*-module, and serve similar purposes.

Definition 2.3. A (countable) *frame*[1] for a right A-module X is a sequence $(u_j)_{j\geq 1} \subset X$ such that $x = \sum_{j\geq 1} u_j\langle u_j \mid x\rangle_A$ for all $x \in X$, with convergence in norm.

If $(u_j)_{j\geq 1}$ is a frame for X, then X is generated as a right A-module by the u_j, so X is countably generated. Conversely, every countably generated right A-module admits a frame [RT03, Corollary 3.3].

We may compose C^*-correspondences using the internal tensor product. Given correspondences $(\phi, {}_AX_B)$ and $(\psi, {}_BY_C)$, we obtain a new correspondence $(\phi \otimes \mathrm{Id}_Y, {}_A(X \otimes_B Y)_C)$, where $X \otimes_B Y$ is the quotient of $X \otimes_{\mathbb{C}} Y$ by the closed submodule generated by

$$\{xb \otimes y - x \otimes \psi(b)y : x \in X,\ y \in Y,\ b \in B\},$$

[Lan95, Proposition 4.5]. It is a non-trivial result that $X \otimes_B Y$ is a right C^*-module with inner product $\langle x_1 \otimes y_1 \mid x_2 \otimes y_2\rangle_C := \langle y_1 \mid \psi(\langle x_1 \mid x_2\rangle_B)y_2\rangle_C$, for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Definition 2.4 (cf. [Bre10, Definition 1.1])**.** A *correspondence morphism* (π, ψ) from $(\phi_X, {}_AX_A)$ to $(\phi_Y, {}_BY_B)$ consists of a $*$-homomorphism $\pi\colon A \to B$ together with a linear map $\psi\colon X \to Y$ satisfying:

(i) $\pi(\langle \xi \mid \eta\rangle_A) = \langle \psi(\xi) \mid \psi(\eta)\rangle_B$ for all ξ, $\eta \in X$;

(ii) $\phi_Y(\pi(a))\psi(\xi) = \psi(\phi_X(a)\xi)$ for all $a \in A$ and $\xi \in X$; and

[1] In the signal analysis literature our ‘frame’ would be a normalised tight frame.

(iii) $\psi(\xi) \cdot \pi(a) = \psi(\xi \cdot a)$ for all $a \in A$ and $\xi \in X$.

We write $(\pi, \psi)\colon (\phi_X, {}_AX_A) \to (\phi_Y, {}_BY_B)$. If $(\phi_Y, {}_BY_B) = (\mathrm{Id}, {}_BB_B)$, then we call (π, ψ) a *representation* of $(\phi_X, {}_AX_A)$ in B.

Let $(\phi, {}_AX_A)$ be a C^*-correspondence and let $\ker(\phi)^\perp \coloneqq \{a \in A : ab = 0 \text{ for all } b \in \ker(\phi)\}$. Following Katsura [Kat04a], the *covariance ideal* of $(\phi, {}_AX_A)$ is given by

$$J_X \coloneqq \phi^{-1}(\mathrm{End}^0_A(X)) \cap \ker(\phi)^\perp \triangleleft A. \tag{2.1}$$

The covariance ideal is the largest ideal of A on which ϕ is injective with image contained in $\mathrm{End}^0_A(X)$.

We set the scene for a restricted class of *covariant morphisms* which respect the covariance ideal. The following well-known characterisation of the compact operators on X_A will be helpful in formulating covariance.

Notation 2.5. If X_A is a right A-module then we let X^* denote the *conjugate module* of X, which is a left A-module. We identify $\mathrm{End}^0_A(X)$ with $X \otimes_A X^*$ as $\mathrm{End}^0_A(X)$-modules via the map $\Theta_{x,y} \mapsto x \otimes y^*$. We also give $X \otimes_A X^*$ the structure of a C^*-algebra inherited from $\mathrm{End}^0_A(X)$.

Lemma 2.6 ([KPW98, Lemma 2.2])**.** *Let $(\pi, \psi)\colon (\phi_A, {}_AY_A) \to (\phi_B, {}_BX_B)$ be a correspondence morphism. There is an induced $*$-homomorphism $\psi^{(1)}\colon \mathrm{End}^0_A(Y) \to \mathrm{End}^0_B(X)$ satisfying $\psi^{(1)}(y_1 \otimes y_2^*) = \psi(y_1) \otimes \psi(y_2)^*$, for all $y_1, y_2 \in Y$.*

Definition 2.7 (cf. [Bre10, Definition 1.3])**.** We say that $(\pi, \psi)\colon (\phi_A, {}_AY_A) \to (\phi_B, {}_BX_B)$ is *covariant* if for all $a \in J_Y$ we have $\psi^{(1)} \circ \phi_Y(a) = \phi_X \circ \pi(a)$.

Correspondence morphisms between C^*-correspondences induce $*$-homomorphisms between the associated Toeplitz algebras, and covariant morphisms descend to $*$-homomorphisms of the associated Cuntz–Pimsner algebras [Bre10, Proposition 1.4]. We now turn to a description of these algebras.

2.2. **Toeplitz–Pimsner and Cuntz–Pimsner algebras.** Fix a C^*-correspondence $(\phi, {}_AX_A)$. Define $X^{\otimes 0} \coloneqq {}_AA_A$, $X^{\otimes 1} \coloneqq X$, and $X^{\otimes n+1} \coloneqq X^{\otimes n} \otimes_A X$ for $n \geq 1$. The *Fock module* of X is the ℓ^2-direct sum $\mathrm{Fock}_X \coloneqq \bigoplus_{n=0}^\infty X^{\otimes n}$ regarded as a correspondence over A with diagonal left action. If $x \in X^{\otimes n}$ for some $n \geq 0$ we call n the *length* of x and write $|x| = n$.

As in [Pim97], the *Toeplitz algebra* $\mathcal{T}_X$ is the C^*-subalgebra of $\mathrm{End}_A(\mathrm{Fock}_X)$ generated by the creation operators T_x, $x \in X$ which satisfy

$$T_x(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \coloneqq x \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_k$$

on elementary tensors $x_1 \otimes x_2 \otimes \cdots \otimes x_k \in \mathrm{Fock}_X$. The adjoint T_x^* satisfies

$$T_x^*(x_1 \otimes \cdots \otimes x_k) = \langle x \mid x_1\rangle_A \cdot x_2 \otimes \cdots \otimes x_k$$

for $k \geq 1$, and $T_x^*|_{X^{\otimes 0}} = 0$. For $a \in A$ we let T_a denote the operator of left multiplication by a, given on simple tensors by $T_a(x_1 \otimes \cdots \otimes x_k) = a \cdot x_1 \otimes \cdots \otimes x_k$.

Following [Kat04a], with J_X the covariance ideal of X, the algebra of compact operators $\mathrm{End}^0_A(\mathrm{Fock}_X \cdot J_X)$ is an ideal of the Toeplitz-Pimsner algebra $\mathcal{T}_X$. The *Cuntz–Pimsner algebra* $\mathcal{O}_X$ is defined to be the quotient $\mathcal{T}_X / \mathrm{End}^0_A(\mathrm{Fock}_X \cdot J_X)$. Thus, we have an exact sequence

$$0 \to \mathrm{End}^0_A(\mathrm{Fock}_X \cdot J_X) \longrightarrow \mathcal{T}_X \xrightarrow{q} \mathcal{O}_X \to 0. \tag{2.2}$$

The maps $j_X\colon x \mapsto T_x$ and $j_A\colon a \mapsto T_a$ constitute a representation of $(\phi, {}_AX_A)$ whose image generates $\mathcal{T}_X$. This representation is universal: for any representation $(\psi, \pi)\colon (\phi, {}_AX_A) \to B$, there is a unique $*$-homomorphism $\psi\times\pi\colon \mathcal{T}_X \to B$ such that $(\psi\times\pi)\circ j_X = \psi$ and $(\psi\times\pi)\circ j_A = \psi$ (see [Pim97, Theorem 3.4]).

The Cuntz–Pimsner algebra $\mathcal{O}_X$ is generated by the covariant representation of $(\phi, {}_AX_A)$ given by $i_X\colon x \mapsto S_x := q(T_x)$ and $i_A\colon a \mapsto S_a := q(T_a)$. This representation is universal: for every covariant representation $(\psi,\pi)\colon (\phi, {}_AX_A) \to B$ there is a unique $*$-homomorphism $\psi\times\pi\colon \mathcal{O}_X \to B$ such that $(\psi\times\pi)\circ i_X = \psi$ and $(\psi\times\pi)\circ i_A = \psi$.

2.3. **A-C^*-algebras.** For a locally compact Hausdorff space X, the notion of a $C_0(X)$-algebra has been used for many years to handle localisation and limits in homological contexts, e.g. [Kas88]. Here, we relax and extend the notion by allowing a noncommutative algebra in place of $C_0(X)$ and we do not assume a central image.

Definition 2.8. Fix a C^*-algebra A. A C^*-algebra C is an *A-C^*-algebra* (or simply an *A-algebra*) if there is a nondegenerate $*$-homomorphism $\phi\colon A \to \mathrm{Mult}(C)$. If C and D are A-algebras with $\phi_C\colon A \to \mathrm{Mult}(C)$ and $\phi_D\colon A \to \mathrm{Mult}(D)$, then an *$A$-algebra morphism* is a nondegenerate $*$-homomorphism $\alpha\colon C \to D$ such that $\phi_D = \widetilde{\alpha}\circ\phi_C$, where $\widetilde{\alpha}$ is the extension of α to the corresponding multiplier algebras [Lan95, Proposition 2.1]).

When A is commutative, our notion of an A-algebra is more relaxed than the established notion of a $C_0(X)$-algebra. However, the advantage is that a Cuntz–Pimsner algebra over an A-module is an A-algebra in our sense, as we show in the next example.

Example 2.9. Suppose $(\phi, {}_AX_A)$ is a C^*-correspondence and let $j_A\colon A \to \mathcal{T}_X$ be the universal inclusion, which we think of as the diagonal representation of A on Fock_X. Since $\mathrm{End}^0_{J_X}(\mathrm{Fock}_X\cdot J_X)$ is an ideal in $\mathcal{T}_X$, the map j_A induces the structure of an A-C^*-algebra on $\mathrm{End}^0_{J_X}(\mathrm{Fock}_X\cdot J_X)$. In particular, the defining exact sequence for Cuntz–Pimsner algebras,

$$0 \longrightarrow \mathrm{End}^0_{J_X}(\mathrm{Fock}_X\cdot J_X) \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{O}_X \longrightarrow 0\ , \tag{2.3}$$

may be interpreted as an exact sequence of A-C^*-algebras. △

A C^*-algebra C being an A-algebra with $\phi\colon A \to \mathrm{Mult}(C)$ nondegenerate is equivalent to $(\phi, {}_AC_C)$ being a nondegenerate C^*-correspondence. We record the following facts about the relationship between C^*-correspondences and A-algebras.

Lemma 2.10. *Let $(\phi, {}_AF_B)$ be a nondegenerate C^*-correspondence, and let C be a B-algebra. Then the following are true:*

(i) *$F\otimes_B C$ is an $\mathrm{End}^0_C(F\otimes_B C)$–$C$-Morita equivalence bimodule;*
(ii) *$(F\otimes_B C)^* \cong C_B\otimes F^*$, where ${}_B\otimes$ denotes the tensor product of left correspondences;*
(iii) *$\mathrm{End}^0_C(F\otimes_B C) \cong (F\otimes_B C)\otimes_C (C_B\otimes F^*)$ is an A-algebra; and*
(iv) *$(F\otimes_B C)\otimes_C (C_B\otimes F^*)$ can be identified with $F\otimes C\otimes F^*$ via the map*

$$(f_1\otimes c_1)\otimes(c_2\otimes f_2^*) \mapsto f_1\otimes c_1c_2\otimes f_2^*,$$

with the adjoint satisfying $(f_1\otimes c_1\otimes f_2^*)^* = f_2\otimes c_1^*\otimes f_1^*$ and the product satisfying

$$(f_1\otimes c_1\otimes f_2^*)(f_3\otimes c_2\otimes f_4^*) = f_1\otimes c_1\langle f_2|f_3\rangle_B c_2\otimes f_4^*,$$

for all $f_1,f_2,f_3,f_4\in F$ and $c_1,c_2\in C$.

Lemma 2.11. *Let F_B be a right B-module and let C be a B-algebra. Suppose $(\phi, {}_CX_D)$ is a nondegenerate C^*-correspondence with ϕ injective. Then the action of $\mathrm{End}^0_C(F\otimes_B C)$ on $F\otimes_B X_D := F\otimes_B C\otimes X_D$, defined on rank-one operators by*

$$\Theta_{f_1\otimes c_1, f_2\otimes c_2}(f_3\otimes x) = f_1\otimes\phi(c_1\langle f_2\mid f_3\rangle_B c_2^*)x,$$

for all $f_1, f_2, f_3 \in F$ and $c_1, c_2 \in C$, and $x \in X$, is adjointable and injective.

Proof. The left action of $\mathrm{End}^0_C(F \otimes_B C)$ on $F \otimes_B X_D$ is induced by the map $T \mapsto T \otimes \mathrm{Id}_X$ from $\mathrm{End}^0_C(F \otimes_B C)$ to $\mathrm{End}_D(F \otimes_B C \otimes_\phi X_D)$, so the action is adjointable. Since $\mathrm{End}^0_C(F \otimes_B C)$ acts faithfully on $F \otimes_B C$ and ϕ is injective, [Kat04a, Lemma 4.7] implies that $T \mapsto T \otimes \mathrm{Id}_X$ is injective. □

Lemma 2.12. *Let F_B be a right B-module, let C be a B-C^*-algebra, and let I be an ideal in C. Then I is a B-C^*-algebra, $\mathrm{End}^0_I(F \otimes_B I)$ is an ideal in $\mathrm{End}^0_C(F \otimes_B C)$, and the quotient by this ideal is isomorphic to $\mathrm{End}^0_{C/I}(F \otimes_B (C/I))$.*

Proof. Let $\phi\colon B \to \mathrm{Mult}(C)$ be a nondegenerate $*$-homomorphism and let $(e_i)_i$ be an approximate unit for I. For each $d \in I$, we have $\phi(b)d = \lim_i \phi(b)de_i \in I$, so ϕ induces a nondegenerate $*$-homomorphism $\phi_I\colon B \to \mathrm{Mult}(I)$.

Applying [Kat07, Lemma 1.6], we see that $\mathrm{End}^0_I(F \otimes_B I)$ is an ideal of $\mathrm{End}^0_C(F \otimes_B C)$ and that the quotient is isomorphic to $\mathrm{End}^0_{C/I}((F \otimes_B C)/(F \otimes_B I))$, where $(F \otimes_B C)/(F \otimes_B I)$ is the quotient vector space equipped with the structure of a right C/I-module.

The algebra C/I inherits a B-algebra structure from $\phi\colon B \to \mathrm{Mult}(C)$. It remains to show that $(F \otimes_B C)/(F \otimes_B I)$ is isomorphic to $F \otimes_B (C/I)$ as right C/I-modules. A routine computation shows that the map $f \otimes c + F \otimes_B I \mapsto f \otimes (c + I)$ extends to a well-defined isometric linear map from $(F \otimes_B C)/(F \otimes_B I)$ to $F \otimes_B (C/I)$. Right C/I-linearity is routine to check. □

3. Positive Correspondences and Quesadillas

The idea of using some form of correspondences as morphisms has been around for a long time and has appeared in numerous contexts. In the C^*-algebraic setting the correspondence category was formalised in [EKQR06]. We summarise the key points.

Definition 3.1. The *correspondence category* (or *Enchilada category*) Enchilada is the category with objects given by C^*-algebras and such that the set of morphisms from a C^*-algebra A to a C^*-algebra B is the collection of isomorphism classes of nondegenerate A–B-correspondences. Composition is given by the isomorphism class of the balanced tensor product of any representatives.

Remark 3.2. In Definition 3.1 one needs to take isomorphism classes of C^*-correspondences to ensure associativity of composition. Without taking isomorphism classes one instead ends up with a bicategory (see [MS19]) as associativity is only defined up to isomorphism.

We will now set about extending Enchilada by including additional morphisms coming from completely positive maps.

3.1. **The KSGNS construction.**

In this section we recall the details of the KSGNS[2] construction, and we refer to [Kas80] and [Lan95, Chapter 5] who we follow here.

Let X_A and Y_A be right A-modules and let $\mathrm{End}_A(X, Y)$ denote the adjointable A-linear operators from X_A to Y_A. Recall that the *strong-* topology* on $\mathrm{End}_A(X, Y)$ is generated by the seminorms

$$T \mapsto ||Tx||_Y \quad \text{and} \quad T \mapsto ||T^*y||_X,$$

for all $x \in X$ and $y \in Y$. The strong-* topology agrees with the *strict topology* on norm-bounded sets [Lan95, Proposition 8.1]. We only consider these topologies on bounded sets and so only refer

[2]Kasparov–Stinespring–Gelfand–Naimark–Segal

to the strict topology. A bounded net $(T_i)_{i\in\mathbb{N}}$ converges strictly to T in $\mathrm{End}_A(X,Y)$ precisely when

$$||(T_i - T)x||_Y \to 0 \quad \text{and} \quad ||(T_i^* - T^*)y||_X \to 0$$

for every $x \in X$ and $y \in Y$.

Remark 3.3. If X_A is a right A-module, then $(u_j)_{j\geq 1}$ in X_A is a frame if and only if $\sum_{j\geq 1} \Theta_{u_j,u_j}$ converges strictly to Id_X in $\mathrm{End}_A(X)$.

Definition 3.4. If A is a C^*-algebra and X is a B-module, then a completely positive map $\rho\colon A \to \mathrm{End}_B(X)$ is *strict* if $(\rho(a_i))_i$ is strictly Cauchy for some approximate unit $(a_i)_i$ for A. Equivalently, ρ is strict if and only if there is a completely positive map $\overline{\rho}\colon \mathrm{Mult}(A) \to \mathrm{End}_B(X)$ which is strictly continuous on the unit ball and whose restriction to A is ρ, (cf. [Lan95, Corollary 5.7]). A strict map $\rho\colon A \to \mathrm{End}_B(X)$ is *nondegenerate* if $\rho(a_i) \to \mathrm{Id}_X$ strictly.

Lemma 3.5. *If $\rho\colon A \to B$ and $\sigma\colon B \to C$ are strict completely positive maps between C^*-algebras, then $\sigma \circ \rho\colon A \to C$ is a strict completely positive map.*

Proof. Compositions of completely positive maps are completely positive. Since ρ is strict, it lifts to a completely positive map $\overline{\rho}\colon \mathrm{End}_A(A) \to \mathrm{End}_B(B)$ and similarly for σ. Then $\sigma \circ \rho$ is strict since $\overline{\sigma} \circ \overline{\rho}$ is a lift. □

Example 3.6. If $\phi\colon A \to \mathrm{End}_B(X)$ is a $*$-homomorphism, then ϕ is strict if and only if $\overline{\phi(A)X_B}$ is a complemented submodule of X_B [Lan95, Proposition 5.8]. In particular, if $\phi\colon A \to B$ is a surjective $*$-homomorphism, then ϕ is strict. △

Example 3.7. If $B \subseteq A$ and $\rho\colon A \to B$ is contractive and idempotent, then ρ is strict. Indeed, by Tomiyama's theorem [BO08, Theorem 1.5.10], ρ is a conditional expectation, so if (a_i) is an approximate unit for A, then

$$\|\rho(a_i)b - \rho(a_j)b\| = \|\rho(a_i b - a_j b)\| \to 0,$$

for $b \in B$, so ρ is strict. △

We now recall the KSGNS construction.

Construction 3.8 (KSGNS)**.** Let A and B be C^*-algebras, let X_B be a B-module, and let $\rho\colon A \to \mathrm{End}_B(X)$ be a strict completely positive map. The algebraic tensor product $A \odot X$ is a right B-module and a semi inner-product space with respect to the sesquilinear form defined by

$$\langle a_1 \otimes x_1 \mid a_2 \otimes x_2 \rangle := \langle x_1 \mid \rho(a_1^* a_2)x_2 \rangle_B\,, \tag{3.1}$$

for all $a_1, a_2 \in A$ and $x_1, x_2 \in X_B$. The *KSGNS correspondence* $A \otimes_\rho X$ is the completion of the quotient of $A \odot X$ by the ideal of tensors whose norm (coming from the semi-inner product) vanishes. The product $A \otimes_\rho X$ carries the obvious right B-module structure, and there is a left action by a $*$-homomorphism $\pi_\rho\colon A \to \mathrm{End}_B(A \otimes_\rho X)$ satisfying

$$\pi_\rho(a')(a \otimes x) = a'a \otimes x, \tag{3.2}$$

for all $a', a \in A$ and $x \in X$, which makes $(\pi_\rho, A \otimes_\rho X)$ a nondegenerate A–B-correspondence. There is an adjointable map $V_\rho\colon X \to A \otimes_\rho X$ satisfying

$$V_\rho(x) = \lim_i a_i \otimes x, \tag{3.3}$$

where $x \in X$ and $(a_i)_i$ is an approximate unit for A. Note that $V_\rho^*(a \otimes x) = \rho(a)x$ for $a \in A$ and $x \in X$. Then

$$\rho(a) = V_\rho^* \pi_\rho(a) V_\rho, \tag{3.4}$$

for $a \in A$, and $\pi_\rho(A)V_\rho(X) \subseteq A \otimes_\rho X$ is dense.

Moreover, if $(\phi, {}_AY_B)$ is an A–B-correspondence and there exists an adjointable operator $W\colon X \to Y$ such that $\rho(a) = W^*\phi(a)W$ for $a \in A$, and $\phi(A)W(X) \subseteq Y$ is dense, then

$$(\pi_\rho, A \otimes_\rho X) \cong (\phi, Y) \tag{3.5}$$

as correspondences. We refer to this property as the *uniqueness* of the KSGNS construction.

Note that if ρ is nondegenerate, then $V_\rho^* V_\rho(x) = \lim \rho(a_i)x = x$, so V_ρ is an isometry.

Example 3.9. An important case of the KSGNS construction is when $\rho\colon A \to B$ is a conditional expectation. Define a degenerate B-valued inner product on A by $\langle a_1 \mid a_2\rangle_B = \rho(a_1^* a_2)$. After quotienting by vectors of length zero and completing in the induced norm, we arrive at a B-module $L^2_B(A,\rho)$. The identity map Id_A on A induces an adjointable left action on $L^2_B(A,\rho)$ that we denote by Id_A^ρ. Then $(\pi_\rho, A \otimes_\rho B) \cong (\mathrm{Id}_A^\rho, L^2_B(A,\rho))$. △

Our aim is to construct a category similar to the Enchilada category of [EKQR06], where instead of $*$-homomorphisms $\phi\colon A \to \mathrm{End}_B(X)$ we consider strict completely positive maps $\rho\colon A \to \mathrm{End}_B(X)$.

Definition 3.10. Let A and B be C^*-algebras. A *positive A–B-correspondence* is a pair $(\rho, {}_AX_B)$ where X_B is a right B-module and $\rho\colon A \to \mathrm{End}_B(X_B)$ is a completely positive map. The pair $(\rho, {}_AX_B)$ is *strict* if ρ is strict.

Two positive A–B-correspondences $(\rho, {}_AX_B)$ and $(\sigma, {}_AY_B)$ are isomorphic if there is an adjointable unitary map $U\colon X \to Y$ such that $U\rho(a)U^* = \sigma(a)$ for all $a \in A$. We note for later use that the coefficients are equal, not just isomorphic.

Strictness of a positive correspondence ensures that the KSGNS construction yields a nondegenerate C^*-correspondence which is unique up to isomorphism [Lan95, Theorem 5.6].

Example 3.11. Let A be a C^*-algebra. If $\phi\colon A \to \mathbb{C}$ is a state, then $(\phi, {}_A\mathbb{C}_\mathbb{C})$ is a strict positive correspondence. The KSGNS construction applied to this positive correspondence yields the GNS representation associated to ϕ. △

3.2. **The Quesadilla semi-category.** The next lemma is key to defining a semi-category using strict completely positive maps. The reason we do not obtain a category is that the "obvious" identity elements only behave as identities from one side (see Lemma 3.14 below).

Lemma 3.12. *Suppose that $(\rho, {}_AX_B)$ is a strict positive correspondence, and that $(\phi, {}_BY_C)$ is a C^*-correspondence. There is a strict completely positive map $\rho \otimes \mathrm{Id}\colon A \to \mathrm{End}_C(X \otimes_B Y)$ satisfying*

$$(\rho \otimes \mathrm{Id})(a)(x \otimes y) = \rho(a)x \otimes y, \tag{3.6}$$

for all $a \in A$, $x \in X$, and $y \in Y$, and an isomorphism of C^*-correspondences,

$$(\pi_\rho \otimes \mathrm{Id}_Y, (A \otimes_\rho X) \otimes_B Y) \cong (\pi_{\rho\otimes\mathrm{Id}}, A \otimes_{\rho\otimes\mathrm{Id}} (X \otimes_B Y)). \tag{3.7}$$

Proof. Applying the KSGNS construction to ρ, we obtain the A–B-correspondence $A \otimes_\rho X$ with left action $\pi := \pi_\rho\colon A \to \mathrm{End}_B(A \otimes_\rho X)$ and an adjointable operator $V := V_\rho\colon X \to A \otimes_\rho X$ satisfying $\rho(a) = V^*\pi(a)V$ and $\overline{\pi(A)V(X)} = A \otimes_\rho X$, for all $a \in A$. Since the $*$-homomorphism $\rho \otimes \mathrm{Id}$ is the composition of ρ with the nondegenerate $*$-homomorphism $\mathrm{End}_B(X) \to \mathrm{End}_C(X \otimes_B Y)$ given by $T \mapsto T \otimes \mathrm{Id}_Y$, it follows that $\rho \otimes \mathrm{Id}$ is a strict completely positive map. So we also obtain the A–C-correspondence $(\pi_{\rho\otimes\mathrm{Id}}, A \otimes_{\rho\otimes\mathrm{Id}} (X \otimes_B Y))$.

In order to prove the isomorphism (3.7), we consider the adjointable operator $V \otimes \mathrm{Id}_Y\colon X \otimes_B Y \to (A \otimes_\rho X) \otimes_B Y$. Note that $(V \otimes \mathrm{Id}_Y)^* = V^* \otimes \mathrm{Id}_Y$. By the uniqueness of the KSGNS construction (3.5), it suffices to show that for all $a \in A$ we have

$$(\rho \otimes \mathrm{Id})(a) = (V^* \otimes \mathrm{Id}_Y)\pi(a)(V \otimes \mathrm{Id}_Y) \quad\text{and}\quad \overline{\pi(A)(V \otimes \mathrm{Id}_Y)(X \otimes_B Y)} = (A \otimes_\rho X) \otimes_B Y. \tag{3.8}$$

Fix an approximate unit $(a_i)_i$ for A. Let $a \in A$, $x \in X$, and $y \in Y$. For the first equality of (3.8) we compute

$$\begin{aligned}(V^* \otimes \mathrm{Id}_Y)\pi(a)(V \otimes \mathrm{Id}_Y)(x \otimes y) &= \lim_i (V^* \otimes \mathrm{Id}_Y)(aa_i \otimes x) \otimes y = V^*(a \otimes x) \otimes y \\ &= \rho(a)x \otimes y = (\rho \otimes \mathrm{Id})(x \otimes y).\end{aligned}$$

For the second equality of (3.8) we compute

$$\pi(a)(V \otimes \mathrm{Id}_Y)(x \otimes y) = \lim_i (aa_i \otimes x) \otimes y = (a \otimes x) \otimes y.$$

So $\pi(a)(V \otimes \mathrm{Id}_Y)(X \otimes_B Y)$ is dense in $(A \otimes_\rho X) \otimes_B Y$. □

If A and B are C^*-algebras and $(\rho, {}_AX_B)$ is a strict positive A–B-correpondence, then we write $(\rho, {}_AX_B)\colon A \to B$ to indicate that we are thinking of it as a mapping, and below as a morphism, from A to B. Lemma 3.12 allows us to define composition of strict positive correspondences. Given two strict positive correspondences $(\rho, {}_AX_B)\colon A \to B$ and $(\sigma, {}_BY_C)\colon B \to C$, we define

$$X \otimes_\sigma Y \coloneqq X \otimes_B (B \otimes_\sigma Y)$$

so that $(\rho \otimes \mathrm{Id}_{B\otimes_\sigma Y}, X \otimes_\sigma Y)\colon A \to C$ is a strict positive correspondence. The KSGNS space $B \otimes_\sigma Y$ is unique up to isomorphism since σ is strict, and (3.6) implies that

$$\rho \otimes \mathrm{Id}_{B\otimes_\sigma Y}(a)(x \otimes (b \otimes y)) = \rho(a)x \otimes (b \otimes y)$$

for all $a \in A$, $b \in B$, $x \in X$ and $y \in Y$. To simplify notation we write

$$(\rho, {}_AX_B) \otimes (\sigma, {}_BY_C) \coloneqq (\rho \otimes \mathrm{Id}_{B\otimes_\sigma Y}, X \otimes_\sigma Y).$$

The following result gives us associativity of composition of isomorphism classes of positive correspondences.

Lemma 3.13. *Let $(\rho, {}_AX_B)\colon A \to B$, $(\sigma, {}_BY_C)\colon B \to C$, and $(\tau, {}_CZ_D)\colon C \to D$ be strict positive correspondences. Then*

$$\Big((\rho, {}_AX_B) \otimes (\sigma, {}_BY_C)\Big) \otimes (\tau, {}_CZ_D) \cong (\rho, {}_AX_B) \otimes \Big((\sigma, {}_BY_C) \otimes (\tau, {}_CZ_D)\Big).$$

Proof. The compositions

$$\Big((\rho, {}_AX_B) \otimes (\sigma, {}_BY_C)\Big) \otimes (\tau, {}_CZ_D) = \Big((\rho \otimes \mathrm{Id}_{B\otimes_\sigma Y}) \otimes \mathrm{Id}_{C\otimes_T Z}, {}_A(X \otimes_B (B \otimes_\sigma Y)) \otimes_C (C \otimes_T Z)\Big)$$

and

$$(\rho, {}_AX_B) \otimes \Big((\sigma, {}_BY_C) \otimes (\tau, {}_CZ_D)\Big) = \Big(\rho \otimes \mathrm{Id}_{(B\otimes_\sigma Y)\otimes_C (C\otimes_T Z)}, {}_AX \otimes_B ((B \otimes_\sigma Y) \otimes_C (C \otimes_T Z))\Big)$$

are isomorphic by two applications of Lemma 3.12. □

With associative composition of morphisms, it seems obvious that the next step is to define a positive correspondence category. Sadly, not quite.

Lemma 3.14. *Let $(\rho, {}_AX_B)$ be a strict positive correspondence. Then*

$$(\rho, {}_AX_B) \otimes (\mathrm{Id}, {}_BB_B) \cong (\rho, {}_AX_B) \quad \textit{but} \quad (\mathrm{Id}, {}_AA_A) \otimes (\rho, {}_AX_B) \cong (\pi_\rho, A \otimes_\rho X).$$

Lemma 3.14 says that typically we do not have identity morphisms, though we do always have right identities. The usual names for a 'category' without identities is a *semi-category* or a *semi-groupoid* [Til86, Appendix B]. Though, having right identities gives us more structure than a general semi-category. For instance, we can still work just with morphisms by identifying objects with their right identities, though we will not explore the ramifications of the semi-category structure here.

Following the Tex-Mex precedent of [EKQR06] we introduce the following.

Definition 3.15. The *positive correspondence semi-category* (or the *Quesadilla semi-category*) $\mathsf{Quesadilla}$ is defined as follows:

(i) Objects of $\mathsf{Quesadilla}$ are C^*-algebras.
(ii) Morphisms from A to B are isomorphism classes of strict positive A–B-correspondences.
(iii) Given morphisms $(\rho, {}_AX_B)\colon A \to B$ and $(\sigma, {}_BY_C)\colon B \to C$, composition is defined by

$$(\rho, {}_AX_B) \otimes (\sigma, {}_BY_C) := (\rho \otimes \mathrm{Id}_{B\otimes_\sigma Y}, {}_AX \otimes_\sigma Y_C)\colon A \to C.$$

Remark 3.16. Using positive correspondences as morphisms instead of their isomorphism classes, we could perhaps define 2-morphisms to obtain a bi-semi-category analogous to the C^*-correspondence bi-category of [MS19].

Example 3.17. If $\rho\colon A \to B$ is a strict completely positive map, then the isomorphism class of $(\rho, {}_AB_B)$ is a morphism in $\mathsf{Quesadilla}$ from A to B. If ρ is a $*$-homomorphism then $(\rho, {}_AB_B)$ is the isomorphism class of the C^*-correspondence associated with ρ. △

Example 3.18. Let $\rho\colon A \to B$ be a conditional expectation with nondegenerate inclusion $\iota\colon B \to A$. We consider the positive correspondence $(\rho, {}_AB_B)$ and the C^*-correspondence $(\iota, {}_BA_A)$. There are two possible compositions,

$$(\rho \otimes \mathrm{Id}, {}_AB \otimes_\iota A_A) \cong (\iota \circ \rho, {}_AA_A) \quad \text{and} \quad (\iota \otimes \mathrm{Id}_B, {}_BA \otimes_\rho B_B) \cong (\iota, {}_BL^2_B(A, \rho)).$$

The latter correspondence is that of Example 3.9, with the left action restricted to B. As $\iota \circ \rho$ is always an adjointable projection on $L^2_B(A, \rho)$, there is a copy of the identity correspondence $(\mathrm{Id}_B, {}_BB_B)$ contained as a complemented sub-correspondence in $(\iota, {}_BL^2_B(A, \rho))$. △

The positive correspondence semi-category seems to be the right domain of the KSGNS construction. Since every C^*-correspondence is in particular a strict positive correspondence, $\mathsf{Enchilada}$ is a wide subcategory of $\mathsf{Quesadilla}$. Further, we may interpret the proposition below as saying that $\mathsf{Enchilada}$ is a retract of $\mathsf{Quesadilla}$.

Proposition 3.19. *The* KSGNS *construction provides a semi-functor*[3]

$$\mathrm{KSGNS}\colon \mathsf{Quesadilla} \to \mathsf{Enchilada}$$

such that:

(i) $\mathrm{KSGNS}(A) = A$ *for every* C^**-algebra* A*; and*
(ii) *for a positive correspondence* $(\rho, {}_AX_B)$*,* $\mathrm{KSGNS}(\rho, {}_AX_B)$ *is the class of the correspondence* $(\pi_\rho, {}_A(A \otimes_\rho X)_B)$*. In particular, if* ρ *is a nondegenerate* $*$*-homomorphism, then* $\mathrm{KSGNS}(\rho, {}_AX_B) = (\rho, {}_AX_B)$*.*

If $U\colon \mathsf{Enchilada} \to \mathsf{Quesadilla}$ *is the forgetful semi-functor given by considering nondegenerate* $*$*-homomorphims as strict completely positive maps, then*

$$\mathrm{KSGNS} \circ U = \mathrm{Id}_{\mathsf{Enchilada}} \quad \textit{and} \quad U \circ \mathrm{KSGNS} \ \textit{ is idempotent.}$$

Proof. It is clear from our construction that morphisms in $\mathsf{Quesadilla}$ are sent to morphisms in $\mathsf{Enchilada}$, and the fact that the KSGNS respects composition is the content of Lemma 3.12. If $(\rho, {}_AX_B)$ is a strict positive correspondence and ρ is a nondegenerate $*$-homomorphism, then there is an isomorphism $(\pi_\rho, {}_A(A \otimes_\rho X)_B) \cong (\rho, {}_AX_B)$ of C^*-correspondences, so KSGNS acts as the identity. Since morphisms in $\mathsf{Enchilada}$ and $\mathsf{Quesadilla}$ are isomorphism classes, the final assertions are now clear. □

[3] i.e. preserves composition.

In a general semi-category we cannot talk about invertible morphisms, as we lack identities. In Quesadilla, we may talk about invertible morphisms since we have right identities.

Definition 3.20. A morphism $(\rho, {}_AX_B)$ in Quesadilla is *invertible* if there exists a morphism $(\sigma, {}_BY_A)$ such that

$$(\rho, {}_AX_B) \otimes (\sigma, {}_BY_A) = (\mathrm{Id}, {}_AA_A) \quad \text{and} \quad (\sigma, {}_BY_A) \otimes (\rho, {}_AX_B) = (\mathrm{Id}, {}_BB_B).$$

Here, $(\mathrm{Id}, {}_AA_A)$ and $(\mathrm{Id}, {}_BB_B)$ are the right identity morphisms for A and B.

The lemma below should be compared to [EKQR06, Lemma 2.4] which says that the invertible morphisms in Enchilada are the Morita equivalence bimodules.

Lemma 3.21. *The invertible morphisms in* Quesadilla *are exactly the Morita equivalence bimodules.*

Proof. Morita equivalence bimodules are invertible with inverse given by the conjugate module. Conversely, suppose that $(\rho, {}_AX_B)\colon A \to B$ and $(\sigma, {}_BY_A)\colon B \to A$ represent mutually inverse morphisms in Quesadilla. Then

$$\begin{aligned}(\rho, {}_AX_B) &\cong (\rho, {}_AX_B) \otimes (\mathrm{Id}_B, {}_BB_B) \cong (\rho, {}_AX_B) \otimes (\sigma, {}_BY_A) \otimes (\rho, {}_AX_B)\\ &\cong (\mathrm{Id}_A, {}_AA_A) \otimes (\rho, {}_AX_B) \cong (\pi_\rho, A \otimes_\rho X).\end{aligned}$$

Uniqueness of the KSGNS construction implies that ρ must be a $*$-homomorphism so $(\rho, {}_AX_B)$ is a C^*-correspondence. A similar argument shows that $(\sigma, {}_BY_A)$ is also a C^*-correspondence. By [EKQR06, Lemma 2.4], the invertible morphisms in Enchilada are imprimitivity bimodules, so $(\rho, {}_AX_B)$ is an imprimitivity bimodule, and the result follows. □

Under certain conditions, the composition in Quesadilla simplifies significantly.

Lemma 3.22. *Let $\rho\colon A \to B$ and $\sigma\colon B \to C$ be strict completely positive maps between C^*-algebras and assume that ρ is a conditional expectation. There is an inclusion of C-modules*

$$\psi\colon (A \otimes_\rho B) \otimes_B (B \otimes_\sigma C) \to A \otimes_{\sigma\circ\rho} C.$$

satisfying $\psi((a \otimes b_1) \otimes (b_2 \otimes c)) = ab_1b_2 \otimes c$, for all $a \in A$, $b_1, b_2 \in B$, and $c \in C$. If B contains an approximate unit for A, then ψ is an isomorphism.

Proof. Fix $a \in A$, $b_1, b_2 \in B$, and $c \in C$, and observe that

$$\begin{aligned}\langle (a \otimes b_1) \otimes (b_2 \otimes c) \mid (a \otimes b_1) \otimes (b_2 \otimes c)\rangle_C &= c^*\sigma(b_2^*b_1^*\rho(a^*a)b_1b_2)c = c^*\sigma \circ \rho((ab_1b_2)^*ab_1b_2)c\\ &= \langle ab_1b_2 \otimes c \mid ab_1b_2 \otimes c\rangle_C\,,\end{aligned}$$

where the first inner product is on $(A \otimes_\rho B) \otimes_B (B \otimes_\sigma C)$ and the second is on $A \otimes_{\sigma\circ\rho} C$. It follows that there is a well-defined isometric linear map $\psi\colon (A \otimes_\rho B) \otimes_B (B \otimes_\sigma C) \to A \otimes_{\sigma\circ\rho} C$ satisfying $\psi((a \otimes b_1) \otimes (b_2 \otimes c)) = ab_1b_2 \otimes c$. It is straightforward to see that ψ respects the left action of A and right action of C.

For the second statement, choose an approximate unit $(b_i)_i$ for A contained in B. Then for fixed $a \in A$ and $c \in C$,

$$\lim_i \psi((a \otimes b_i^{1/2}) \otimes (b_i^{1/2} \otimes c)) = \lim_i ab_i \otimes c = a \otimes c,$$

and it follows that ψ is surjective. □

Example 3.23. We give two examples of expectations arising from the Cuntz–Pimsner algebra of a C^*-correspondence $(\phi_X, {}_AX_A)$. Let $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_X)$ denote the gauge action and let $\mathcal{O}_X^\gamma$ denote the associated fixed-point algebra. There is a conditional expectation $\Phi\colon \mathcal{O}_X \to \mathcal{O}_X^\gamma$ given by averaging over γ. Applying the KSGNS construction yields the correspondence $(\pi_\Phi, {}_{\mathcal{O}_X}L^2_{\mathcal{O}_X^\gamma}(\mathcal{O}_X, \Phi))$ underlying the Kasparov module encoding the gauge action, [PR06, CNNR11].

If X is a bi-Hilbertian bimodule of finite right Watatani index (see Section 5), there is another expectation $\Phi_\infty : \mathcal{O}_X \to A$. Applying the KSGNS functor yields the correspondence $(\pi_{\Phi_\infty}, {}_{\mathcal{O}_X}\Xi_A)$ underlying the Kasparov module representing the class of the defining extension (2.3), [RRS17, GMR18]. We note that both A and $\mathcal{O}_X^\gamma$ contain an approximate unit for $\mathcal{O}_X$.

Given a densely-defined norm lower semi-continuous trace τ on $\mathcal{O}_X^\gamma$ or on A, we can then compose with the GNS representations $(\pi_\tau, {}_{\mathcal{O}_X^\gamma}L^2(\mathcal{O}_X^\gamma, \tau))$ or $(\pi_\tau, {}_AL^2(A, \tau))$ using Lemma 3.22, and apply the KSGNS construction to obtain

$$(\pi_{\tau\circ\Phi}, {}_{\mathcal{O}_X}L^2(\mathcal{O}_X, \tau\circ\Phi)) \quad \text{or} \quad (\pi_{\tau\circ\Phi_\infty}, {}_AL^2(A, \tau\circ\Phi_\infty)).$$

The process of identifying KMS states of quasi-free real actions on $\mathcal{O}_X$ outlined by Laca and Neshveyev is compatible with these decompositions, [LN04, GRU19, RRS17]. △

To understand the composition of morphisms in Quesadilla to and from Cuntz–Pimsner algebras, we need to understand how composition in Quesadilla interacts with quotient maps. Recall that for a quotient map $q\colon A \to A/I$ with linear splitting $s\colon A/I \to A$ we have $s(ab) - s(a)s(b) \in I$ and $s(a^*) - s(a)^* \in I$, for all $a, b \in A/I$.

Lemma 3.24. *Suppose that we have a commuting diagram of C^*-algebras*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{q_A} & A/I & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \rho|_I} & & \big\downarrow{\scriptstyle \rho} & & & & \\ 0 & \longrightarrow & J & \longrightarrow & B & \xrightarrow{q_B} & B/J & \longrightarrow & 0 \end{array}$$

with exact rows of $*$-homomorphisms. Suppose that the top row has a completely positive splitting, and that ρ is a strict completely positive map satisfying $\rho(I) \subseteq J$. Then the map $\widetilde{\rho}\colon A/I \to B/J$ defined by $\widetilde{\rho}(a + I) = \rho(a) + J$ is completely positive and strict.

Proof. Fix a splitting $s\colon A/I \to A$ of q_A. Then $\tilde{\rho} = q_B \circ \rho \circ s$. Complete positivity of $\tilde{\rho}$ is clear, so we show strictness. Since ρ is strict we can fix an approximate unit (e_i) of A such that $\rho(e_i)$ is strictly Cauchy in $\mathrm{Mult}(B)$. Then $(q_A(e_i))$ is an approximate unit for A/I and $s \circ q_A(e_i) = e_i + k_i$, for some $k_i \in I$. As $\rho(I) \subseteq J$, we have $\widetilde{\rho}(q_A(e_i)) = q_B \circ \rho(e_i)$. Since q_B is a surjective $*$-homomorphism it follows that $\widetilde{\rho}(q_A(e_i))$ is strictly Cauchy, so $\widetilde{\rho}$ is strict. □

In the case where $\rho\colon A \to B$ is a conditional expectation, the KSGNS space of the induced completely positive map $\widetilde{\rho}\colon A/I \to B/J$ enjoys a construction similar to that of Example 3.9.

Lemma 3.25. *Suppose that we have the commuting diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{q_A} & A/I & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \rho|_I} & & {\scriptstyle \alpha}\big\uparrow\big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \widetilde{\rho}} & & \\ 0 & \longrightarrow & J & \longrightarrow & B & \xrightarrow{q_B} & B/J & \longrightarrow & 0 \end{array}$$

as in Lemma 3.24. Suppose further that $\rho\colon A \to B$ is a conditional expectation with nondegenerate inclusion $\alpha\colon B \to A$. Let $(\mathrm{Id}_{A/I}, L^2_{B/J}(A/I, \widetilde{\rho}))$ be the completion (after quotienting by zero vectors) of A/I in the norm coming from the semi-inner product

$$\langle a_1 + I \mid a_2 + I\rangle_{B/J} = \rho(a_1^* a_2) + J, \qquad a_1, a_2 \in A. \tag{3.9}$$

Then $\mathrm{KSGNS}(\widetilde{\rho}, {}_{A/I}(B/J)_{B/J}) \cong (\mathrm{Id}_{A/I}, L^2_{B/J}(A/I, \widetilde{\rho}))$.

Proof. Apart from linearity, it is easy to check that (3.9) defines an inner product. Fix a completely positive splitting s_A of q_A so that $\widetilde{\rho} = q_B \circ \rho \circ s_A$. For the B/J-linearity, let $a + I \in A/I$ and $b + J \in B/J$, and compute

$$\begin{aligned} \widetilde{\rho}(a+I)(b+J) &= \rho \circ s_A(a+I)b + J = \rho(a+i)b + J \quad \text{for some } i \in I \\ &= \rho(a\alpha(b) + i\alpha(b)) + J = \rho(a\alpha(b)) + J. \end{aligned}$$

A straightforward computation shows that

$$\langle a_1 + I \mid a_2 + I\rangle_{B/J}(b+J) = \langle a_1 + I \mid a_2\alpha(b) + I\rangle_{B/J}.$$

For $a \in A$ and $b \in B$ we have the equality

$$\begin{aligned} \langle [a\alpha(b)+I] \mid [a\alpha(b)+I]\rangle_{B/J} &= \rho(\alpha(b^*)a^*a\alpha(b)) + J = (b^*+J)(\rho(a^*a)+J)(b+J) \\ &= \langle b+J \mid \widetilde{\rho}(a^*a+I)(b+J)\rangle_{B/J} \\ &= \langle b+J \mid \widetilde{\rho}(\langle a+I \mid a+I\rangle_{A/I})(b+J)\rangle_{B/J} \\ &= \langle (a+I)\otimes(b+J) \mid (a+I)\otimes(b+J)\rangle_{B/J}, \end{aligned}$$

so there is a well-defined isometric linear map $\Lambda \colon A/I \otimes_{\widetilde{\Psi}} B/J \to L^2_{B/J}(A/I, \widetilde{\rho})$ satisfying $\Lambda((a+I)\otimes(b+J)) \coloneqq [a\alpha(b)+I]$. The right B/J-linearity of Λ is routine and nondegeneracy of α gives the surjectivity of Λ. ☐

The following result is analogous to Lemma 3.22 once we pass to quotients. The result is needed in Section 4.1.

Lemma 3.26. *Suppose that*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{q_A} & A/I & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \rho|_I} & & {\scriptstyle \alpha}\big\uparrow\big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \widetilde{\rho}} & & \\ 0 & \longrightarrow & J & \longrightarrow & B & \xrightarrow{q_B} & B/J & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \sigma|_J} & & \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle \widetilde{\sigma}} & & \\ 0 & \longrightarrow & K & \longrightarrow & C & \xrightarrow{q_C} & C/K & \longrightarrow & 0 \end{array},$$

is a commuting diagram of C^*-algebras with exact rows of $*$-homomorphisms that have completely positive splittings $s_A \colon A/I \to A$ and $s_B \colon B/J \to B$. Let $\rho \colon A \to B$ be a conditional expectation with corresponding inclusion $\alpha \colon B \to A$, and let $\sigma \colon B \to C$ be a strict completely positive map. Further suppose that $\rho(I) \subseteq J$ and $\sigma(J) \subseteq K$, and that the maps $\widetilde{\rho}$ and $\widetilde{\sigma}$ are the strict completely positive maps induced by Lemma 3.24. Then there is an isometric inclusion of correspondences

$$\psi \colon (A/I \otimes_{\widetilde{\rho}} B/J) \otimes_{B/J} (B/J \otimes_{\widetilde{\sigma}} C/K) \to A/I \otimes_{\widetilde{\sigma}\circ\widetilde{\rho}} C/K$$

satisfying

$$\psi((a \otimes b_1) \otimes (b_2 \otimes c)) = a q_A(\alpha(s_B(b_1 b_2))) \otimes c, \tag{3.10}$$

for $a \in A/I$, $b_1, b_2 \in B/J$, and $c \in C/K$. If $q_A \circ \alpha \circ s_B(B/J)$ contains an approximate unit for A/I, then ψ is also surjective.

Proof. We first show that (3.10) yields a well-defined map. Let $s_A \colon A/I \to A$ and $s_B \colon B/J \to J$ be the completely positive splittings of q_A and q_B, respectively. For $a \in A/I$, $b_1, b_2 \in B/J$, and $c \in C/K$, a computation similar to that of Lemma 3.22 shows that

$$\big\langle (a \otimes b_1) \otimes (b_2 \otimes c) \mid (a \otimes b_1) \otimes (b_2 \otimes c)\big\rangle_{C/K} = c^*\widetilde{\sigma}(b_2^* b_1^* \widetilde{\rho}(a^*a) b_1 b_2)c.$$

Recall from Lemma 3.24 that $\widetilde{\rho} = q_B \circ \rho \circ s_A$ and $\widetilde{\sigma} = q_C \circ \sigma \circ s_B$. Since $s_A(a^*)s_A(a) - s_A(a^*a) \in I$ and $\rho(I) \subseteq J$ it follows that

$$\begin{aligned} b_2^* b_1^* \widetilde{\rho}(a^*a) b_1 b_2 &= q_B\big(s_B(b_2^* b_1^*)\rho(s_A(a^*)s_A(a))s_B(b_1 b_2)\big) \\ &= q_B \circ \rho\big(\alpha(s_B(b_1 b_2))^* s_A(a)^* s_A(a)\alpha(s_B(b_1 b_2))\big). \end{aligned}$$

Define $d := a q_A(\alpha(s_B(b_1 b_2))) \in A/I$ and note that

$$s_A(d^*d) - \alpha(s_B(b_1 b_2))^* s_A(a)^* s_A(a)\alpha(s_B(b_1 b_2)) \in I.$$

Consequently,

$$\begin{aligned} \langle (a \otimes b_1) \otimes (b_2 \otimes c) \mid (a \otimes b_1) \otimes (b_2 \otimes c)\rangle_C &= c^* \widetilde{\sigma}(q_B \circ \rho \circ s_A(d^*d))c = c^* \widetilde{\sigma}\widetilde{\rho}(d^*d)c \\ &= \langle d \otimes c \mid d \otimes c\rangle_{C/K}\,, \end{aligned}$$

where the latter inner product is on $A/I \otimes_{\widetilde{\sigma}\circ\widetilde{\rho}} C/K$. This shows that ψ is a well-defined isometric linear map. It is straightforward to see that ψ respects the left action of A/I and the right action of C/K.

For the final statement, suppose $(b_i)_i$ is a net in B/J such that $a_i := q_A \circ \alpha \circ s_B(b_i)$ defines an approximate unit for A/I. Then

$$\lim_i \psi((a \otimes b_i^{1/2}) \otimes (b_i^{1/2} \otimes c)) = \lim_i a a_i \otimes c = a \otimes c,$$

and it follows that ψ is surjective. □

4. Cuntz–Pimsner morphisms from projections

Armed with the theoretical framework of positive correspondences, we now look at applications to Cuntz–Pimsner algebras. We begin by outlining the difficulty with defining correspondence morphisms from complemented sub-correspondences. We then turn to more general projections on Fock modules and the compressions of product systems (over $\mathbb{N}$) which give rise to subproduct systems (over $\mathbb{N}$).

4.1. **Morphisms from correspondence projections.** A *complemented* sub-correspondence is a C^*-correspondence that appears as a direct summand of a C^*-correspondence. This is stronger than being complemented as a right C^*-module since the left action must also be respected. Not every sub-correspondence is complemented. A sub-correspondence is complemented precisely if it is the image of a correspondence projection.

Definition 4.1. A *correspondence projection* on a correspondence $(\phi, {}_AX_B)$ is a projection $P \in \mathrm{End}_B(X)$ such that $\phi(a)P = P\phi(a)$ for all $a \in A$.

If P is a correspondence projection on $(\phi, {}_AX_B)$, then $1 - P$ is a correspondence projection, and $(P\phi, {}_APX_B)$ is a complemented sub-A–B-correspondence determining the direct sum decomposition $X_B \cong PX_B \oplus (1-P)X_B$. Conversely, every complemented sub-A–B-correspondence is the image of a correspondence projection. If $(\phi_Y, {}_AY_A)$ is the image of a correspondence projection P on $(\phi_X, {}_AX_A)$ we will write $P\colon (\phi_X, {}_AX_A) \to (\phi_Y, {}_AY_A)$ where $Y_A = PX_A$. We will also write $Y_A^\perp := (1-P)X_A$.

If $Y = PX$, then the left action ϕ_X acts diagonally on X with respect to the direct sum decomposition. This follows from the computation

$$\begin{aligned} \phi_X(a)x &= P\phi_X(a)Px + (1-P)\phi_X(a)Px + P\phi_X(a)(1-P)x + (1-P)\phi_X(a)(1-P)x \\ &= \phi_Y(a)Px + \phi_{Y^\perp}(a)(1-P)x, \end{aligned}$$

which holds for all $a \in A$ and $x \in X$. In particular, $\phi_X(a) \in \operatorname{End}_A^0(X)$ if and only if $\phi_Y(a) \in \operatorname{End}_A^0(Y)$ and $\phi_{Y^\perp}(a) \in \operatorname{End}_A^0(Y^\perp)$.

Also observe that $\ker(\phi_X) = \ker(\phi_Y) \cap \ker(\phi_{Y^\perp})$. Since $\ker(\phi_Y)^\perp \cap \ker(\phi_{Y^\perp})^\perp$ is a subset of $(\ker(\phi_Y) \cap \ker(\phi_{Y^\perp}))^\perp$ the covariance ideals satisfy $J_Y \cap J_{Y^\perp} \subseteq J_X$. In general, it is not true that $J_Y \subseteq J_X$ nor that $J_X \subseteq J_Y$. However, in the case that $(\phi_{Y^\perp}, Y_A^\perp)$ is regular, we have the following immediate result.

Lemma 4.2. *If $(\phi_Y, {}_AY_A)$ is a complemented sub-correspondence of $(\phi_X, {}_AX_A)$, and $(\phi_{Y^\perp}, Y_A^\perp)$ is regular, then $J_X = J_Y$.*

Let $(\phi_Y, {}_AY_A)$ is a complemented sub-correspondence of $(\phi_X, {}_AX_A)$, and let $\iota \colon Y \to X$ denote the inclusion. Then (Id_A, ι) is a correspondence morphism, so the universal property of $\mathcal{T}_Y$ yields an injective $*$-homomorphism

$$\alpha := \mathrm{Id}_A \times \iota \colon \mathcal{T}_Y \to \mathcal{T}_X \subseteq \operatorname{End}_A(\mathrm{Fock}_X) \tag{4.1}$$

satisfying $\alpha(T_\xi T_\eta^*) = T_{\iota(\xi)} T_{\iota(\eta)}^*$ for all $\xi, \eta \in Y$. The correspondence morphism (Id_A, ι) is typically not covariant as the following example shows, so $\alpha \colon \mathcal{T}_Y \to \mathcal{T}_X$ does not typically descend to a $*$-homomorphism between Cuntz–Pimsner algebras.

Example 4.3. Let ${}_AX_A = {}_{\mathbb{C}}\mathbb{C}^2_{\mathbb{C}}$ with the left action given by scalar multiplication. Let ${}_AY_A = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$ be the complemented sub-$\mathbb{C}$–$\mathbb{C}$-correspondence spanned by the basis vector e_1 of $\mathbb{C}^2$. Let T denote the generating isometry $j_Y(e_1)$ of $\mathcal{T}_Y \cong \mathcal{T}$ and let $T_1 = j_X(e_1)$ and $T_2 = j_X(e_2)$ be the generating isometries of $\mathcal{T}_X \cong \mathcal{T}_2$. Then $\alpha(T) = T_1$.

In the quotient Cuntz–Pimsner algebra, T descends to the unitary $U \colon z \mapsto z$ that generates $\mathcal{O}_Y \cong C(S^1)$, while T_1 and T_2 descend to isometries S_1 and S_2 generating $\mathcal{O}_X \cong \mathcal{O}_2$. Consequently, α cannot descend to a $*$-homomorphism between Cuntz–Pimsner algebras as such a map would take the unitary U to the non-unitary isometry S_1. △

Aside from α, there is a second representation of $\mathcal{T}_Y$ in $\operatorname{End}_A(\mathrm{Fock}_X)$ arising from the correspondence projection P that we will take a moment to explain. First, observe that if $(\phi_Y, {}_AY_A)$ is a sub-A–A-correspondence of $(\phi_X, {}_AX_A)$, then the Fock module Fock_Y may be identified as a sub-A–A-correspondence of Fock_X. The following lemma shows that complementability is also preserved.

Lemma 4.4. *Suppose that $P_0 \colon (\phi_X, {}_AX_A) \to (\phi_Y, {}_AY_A)$ is a correspondence projection. Then P_0 extends to a correspondence projection $P_n \colon (\phi_X, {}_AX_A^{\otimes n}) \to (\phi_Y, {}_AY_A^{\otimes n})$ by*

$$P_n := P_0 \otimes \cdots \otimes P_0. \tag{4.2}$$

The sum $\sum_{n \geq 0} P_n$ converges strictly to a projection $P \colon (\phi_X, \mathrm{Fock}_X) \to (\phi_Y, \mathrm{Fock}_Y)$, so Fock_Y is complemented in Fock_X.

Proof. The A-bilinearity of P_0 ensures that (4.2) is a well-defined correspondence projection. Since Fock_X is a direct sum, the second statement follows from the first. □

If Fock_Y is complemented in Fock_X, then there is an inclusion $\bar{\iota} \colon \operatorname{End}_A(\mathrm{Fock}_Y) \to \operatorname{End}_A(\mathrm{Fock}_X)$ defined by $\bar{\iota}(T)(x \oplus x^\perp) = Tx$ for $x \in \mathrm{Fock}_Y$ and $x^\perp \in \mathrm{Fock}_Y^\perp$. Since $\mathcal{T}_Y$ is a subalgebra of $\operatorname{End}_A(\mathrm{Fock}_Y)$, the map $\bar{\iota}$ restricts to an injective $*$-homomorphism

$$\beta \colon \mathcal{T}_Y \to \operatorname{End}_A(\mathrm{Fock}_X). \tag{4.3}$$

The representations α and β of $\mathcal{T}_Y$ in $\operatorname{End}_A(\mathrm{Fock}_X)$ do not usually coincide. Indeed, if A were unital then $\alpha(1_A) = \mathrm{Id}_{\mathrm{Fock}_X}$ while $\beta(1_A) = \bar{\iota}(\mathrm{Id}_{\mathrm{Fock}_Y})$ is typically a proper subprojection of

$\mathrm{Id}_{\mathrm{Fock}_X}$. Moreover, if $T_y \in \mathcal{T}_Y$ is a creation operator on Fock_Y for some $y \in Y_A$, then $\beta(T_y)$ is typically not a creation operator on Fock_X since it is zero on $\mathrm{Fock}_Y^\perp$.

Despite neither α nor β inducing a $*$-homomorphism from $\mathcal{O}_Y$ to $\mathcal{O}_X$, we can use the projection P of Lemma 4.4 to build a conditional expectation from $\mathcal{T}_X$ to $\mathcal{T}_Y$ which respects both inclusions α and β. By then applying Lemma 3.25 we will construct a positive map from $\mathcal{O}_X$ to $\mathcal{O}_Y$ which gives morphisms in both Quesadilla and Enchilada.

Lemma 4.5. *Let $P\colon (\phi_X, \mathrm{Fock}_X) \to (\phi_Y, \mathrm{Fock}_Y)$ be the correspondence projection of Lemma 4.4. The map $\Psi_P\colon \mathrm{End}_A(\mathrm{Fock}_X) \to \mathrm{End}_A(\mathrm{Fock}_Y)$ defined by*

$$\Psi_P(T) = PTP \tag{4.4}$$

is a conditional expectation. It restricts to a conditional expectation $\Psi_P\colon \mathcal{T}_X \to \mathcal{T}_Y$ such that

$$\Psi_P(\alpha(b_1)a\alpha(b_2)) = b_1\Psi_P(a)b_2 = \Psi_P(\beta(b_1)a\beta(b_2)) \tag{4.5}$$

for all $a \in \mathcal{T}_X$ and $b_1, b_2 \in \mathcal{T}_Y$.

Proof. There are multiple Fock spaces and Toeplitz algebras around, so if $T \in \mathcal{T}_X$, then we write T^X for the corresponding operator on Fock_X, and if $T \in \mathcal{T}_Y$ we write T^Y for the corresponding operator on Fock_Y. We also use $\iota\colon \mathrm{Fock}_Y \to \mathrm{Fock}_X$ to identify Fock_Y as a submodule of Fock_X. In particular, if $\xi \in Y$, then $\alpha(T_\xi^Y) = T_\xi^X$.

That (4.4) defines an expectation for the inclusion $\bar\iota\colon \mathrm{End}_A(\mathrm{Fock}_Y) \to \mathrm{End}_A(\mathrm{Fock}_X)$ is clear. It remains to show that Ψ_P restricts to Toeplitz algebras. Let $y = y_1 \otimes \cdots \otimes y_k \in \mathrm{Fock}_Y \subseteq \mathrm{Fock}_X$ and $\xi, \eta \in \mathrm{Fock}_X$ be elementary tensors. It follows from the A-bilinearity of P that for $k \geq 1$

$$\begin{aligned}\Psi_P(T_\xi^X T_\eta^{X*})y &= PT_\xi^X T_\eta^{X*} Py\\ &= \begin{cases} P(\xi \cdot \langle \eta \mid Py_1 \otimes \cdots \otimes y_{|\eta|}\rangle_A \otimes y_{|\eta|+1} \otimes \cdots \otimes y_k) & \text{if } k \geq |\eta| \\ 0 & \text{if } k < |\eta| \end{cases}\\ &= \begin{cases} (P\xi) \cdot \langle P\eta \mid y_1 \otimes \cdots \otimes y_{|\eta|}\rangle_A \otimes y_{|\eta|+1} \otimes \cdots \otimes y_k & \text{if } k \geq |\eta| \\ 0 & \text{if } k < |\eta| \end{cases} = T_{P\xi}^Y T_{P\eta}^{Y*} y.\end{aligned}$$

Similarly, if $y \in A$, then $\Psi_P(T_\xi^X T_\eta^{X*})y = T_{P\xi}^Y T_{P\eta}^{Y*} y$, so $\Psi_P(\mathcal{T}_X) = \mathcal{T}_Y$.

Recall that for elementary tensors $\xi, \eta \in \mathrm{Fock}_Y$ we have $\alpha(T_\xi^Y T_\eta^{Y*}) = T_\xi^X T_\eta^{X*}$. The previous calculation shows that $\Psi_P(\alpha(T_\xi^Y T_\eta^{Y*})) = T_\xi^Y T_\eta^{Y*}$, so $\Psi_P \circ \alpha = \mathrm{Id}_{\mathcal{T}_Y}$. Since $\Psi_P \circ \bar\iota = \Psi_P \circ \beta = \mathrm{Id}_{\mathcal{T}_Y}$ we also have $\Psi_P \circ \beta = \Psi_P \circ \alpha$.

For the first equality of (4.5), fix elementary tensors $y \in Y_A^{\otimes n}$ and $\xi \in X_A^{\otimes m}$. Then

$$\begin{aligned}T_y^{X*} T_\xi^X &= \begin{cases} \langle y \mid \xi_1 \otimes \cdots \otimes \xi_n\rangle_A T^X_{\xi_{n+1}\otimes\cdots\otimes\xi_m} & \text{if } m \geq n \\ T^{X*}_{y_{m+1}\otimes\cdots\otimes y_n} \langle y_1 \otimes \cdots \otimes y_m \mid \xi\rangle_A & \text{if } n > m \end{cases}\\ &= \begin{cases} T^X_{\phi_X(\langle y | P(\xi_1\otimes\cdots\otimes\xi_n)\rangle_A)\xi_{n+1}\otimes\cdots\otimes\xi_m} & \text{if } m \geq n \\ T^{X*}_{\phi_X(\langle P\xi | y_1\otimes\cdots\otimes y_m\rangle_A) y_{m+1}\otimes\cdots\otimes y_n} & \text{if } n > m. \end{cases}\end{aligned}$$

So, for elementary tensors $x \in \mathrm{Fock}_Y$ and $\eta \in \mathrm{Fock}_X$,

$$\begin{aligned}\Psi_P(\alpha(T^Y_x T^{Y*}_y)T^X_\xi T^{X*}_\eta) &= PT^X_x T^{X*}_y T^X_\xi T^{X*}_\eta P\\ &= \begin{cases} PT^X_{x\otimes\phi_X(\langle y|P_0\xi_1\otimes\cdots\otimes P_0\xi_n\rangle_A)\xi_{n+1}\otimes\cdots\otimes\xi_m} T^{X*}_\eta P & \text{if } m\geq n\\ PT^X_x T^{X*}_{\eta\otimes\phi_X(\langle P\xi|y_1\otimes\cdots\otimes y_m\rangle_A)y_{m+1}\otimes\cdots\otimes y_n} P & \text{if } n>m\end{cases}\\ &= \begin{cases} T^Y_{x\otimes\phi_Y(\langle y|P_0\xi_1\otimes\cdots\otimes P_0\xi_n\rangle_A)P_0\xi_{n+1}\otimes\cdots\otimes P_0\xi_m} T^{Y*}_\eta & \text{if } m\geq n\\ T^Y_x T^{Y*}_{P\eta\otimes\phi_Y(\langle P\xi|y_1\otimes\cdots\otimes y_m\rangle_A)y_{m+1}\otimes\cdots\otimes y_n} & \text{if } n>m\end{cases}\\ &= T^Y_x T^{Y*}_y T^Y_{P\xi}T^{Y*}_{P\eta} = T^Y_x T^{Y*}_y \Psi_P(T^X_\xi T^{X*}_\eta).\end{aligned}$$

Similarly, $\Psi_P(T^X_\xi T^{X*}_\eta \alpha(T^X_x T^{X*}_y)) = \Psi_P(T^X_\xi T^{X*}_\eta)T^Y_x T^{Y*}_y$.

Since $\Psi_P\colon \mathrm{End}_A(\mathrm{Fock}_X) \to \mathrm{End}_A(\mathrm{Fock}_Y)$ is an expectation for the inclusion $\bar\iota$, it follows that Ψ_P descends to an expectation $\Psi_P\colon \mathcal{T}_X \to \mathcal{T}_Y$ for the inclusion α. The second equality of (4.5) follows because Ψ_P is an expectation for $\bar\iota$ and $\beta = \bar\iota|_{\mathcal{T}_Y}$. ☐

With the expectation $\Psi_P\colon \mathcal{T}_X \to \mathcal{T}_Y$ in hand, we aim to appeal to Lemma 3.24 to induce a strict completely positive map between the corresponding Cuntz–Pimsner algebras. Since $\mathcal{O}_X$ is the quotient $\mathcal{T}_X/\mathrm{End}^0_A(\mathrm{Fock}_X\cdot J_X)$ we need to show that Ψ_P takes $\mathrm{End}^0_A(\mathrm{Fock}_X\cdot J_X)$ to $\mathrm{End}^0_A(\mathrm{Fock}_Y\cdot J_Y)$.

Lemma 4.6. *Let $\Psi_P\colon \mathcal{T}_X \to \mathcal{T}_Y$ be the conditional expectation of Lemma 4.5. If $J_X = J_Y$, then $\Psi_P(\mathrm{End}^0_A(\mathrm{Fock}_X\cdot J_X)) = \mathrm{End}^0_A(\mathrm{Fock}_Y\cdot J_Y)$.*

Proof. Since P is right J_X-linear, we have $\Psi_P(\Theta_{\xi,\eta}) = \Theta_{P\xi,P\eta} \in \mathrm{End}^0_A(\mathrm{Fock}_Y\cdot J_Y)$ for all $\xi,\eta \in \mathrm{Fock}_X\cdot J_X$. Since P is surjective $\Psi_P(\mathrm{End}^0_A(\mathrm{Fock}_X\cdot J_X)) = \mathrm{End}^0_A(\mathrm{Fock}_Y\cdot J_Y)$. ☐

To use Lemma 3.24, we need the existence of a completely positive splitting $\mathcal{O}_X \to \mathcal{T}_X$. By the Choi–Effros lifting theorem [BO08, Theorem C.3.] such a splitting exists whenever $\mathcal{O}_X$ is nuclear. A sufficient condition for nuclearity of $\mathcal{O}_X$ is that the coefficient algebra A is nuclear [Kat04a, Corollary 7.4]. The resulting strict completely positive map $\widetilde{\Psi}_P$ is typically not a conditional expectation itself since there is no obvious choice for an inclusion of $\mathcal{O}_Y$ as a subalgebra of $\mathcal{O}_X$.

By applying Lemma 3.24 the map $\Psi_P\colon \mathcal{T}_X \to \mathcal{T}_Y$ passes to the Cuntz–Pimsner quotients.

Theorem 4.7. *Let $(\phi_Y, {}_AY_A)$ be a complemented sub-A–A-correspondence of $(\phi_X, {}_AX_A)$ with $J_X = J_Y$ (for example if $(\phi_{Y^\perp}, {}_AY^\perp_A)$ is regular), and let $\Psi_P\colon \mathcal{T}_X \to \mathcal{T}_Y$ be the conditional expectation of Lemma 4.5. Further suppose that there is a completely positive splitting $\mathcal{O}_X \to \mathcal{T}_X$ (for example if A is nuclear). Then there is a strict completely positive map $\widetilde{\Psi}_P\colon \mathcal{O}_X \to \mathcal{O}_Y$.*

Corollary 4.8. *With the setup of Theorem 4.7 we have the following:*

(i) *$(\widetilde{\Psi}_P, {}_{\mathcal{O}_X}(\mathcal{O}_Y)_{\mathcal{O}_Y})$ is a positive correspondence, defining a morphism in* Quesadilla *from $\mathcal{O}_X$ to $\mathcal{O}_Y$;*

(ii) *the KSGNS space $(\pi_{\widetilde{\Psi}_P}, \mathcal{O}_X \otimes_{\widetilde{\Psi}_P} \mathcal{O}_Y)$ defines a morphism in* Enchilada *from $\mathcal{O}_X$ to $\mathcal{O}_Y$; and*

(iii) *Lemma 3.25 implies that $(\pi_{\widetilde{\Psi}_P}, \mathcal{O}_X \otimes_{\widetilde{\Psi}_P} \mathcal{O}_Y) \cong (\mathrm{Id}_{\mathcal{O}_X}, L^2_{\mathcal{O}_Y}(\mathcal{O}_X, \widetilde{\Psi}_P))$.*

A direct application of Lemma 3.26 gives the following description of composition.

Corollary 4.9. *Suppose that $(\phi_X, {}_AX_A) \xrightarrow{P_0} (\phi_Y, {}_AY_A) \xrightarrow{Q_0} (\phi_Z, {}_AZ_A)$ are correspondence projections that induce correspondence projections $(\phi_X, \mathrm{Fock}_X) \xrightarrow{P} (\phi_Y, \mathrm{Fock}_Y) \xrightarrow{Q} (\phi_Z, \mathrm{Fock}_Z)$.*

Suppose that $J_Z = J_Y = J_Z$ and that there are completely positive splittings $\mathcal{O}_X \to \mathcal{T}_X$ and $\mathcal{O}_Y \to \mathcal{T}_Y$. Then there is an isometric inclusion of correspondences

$$\psi\colon (\mathcal{O}_X \otimes_{\widetilde{\Psi}_P} \mathcal{O}_Y) \otimes (\mathcal{O}_Y \otimes_{\widetilde{\Psi}_Q} \mathcal{O}_Z) \to \mathcal{O}_X \otimes_{\widetilde{\Psi}_{QP}} \mathcal{O}_Z.$$

Moreover, if there is a completely positive splitting $s_Y\colon \mathcal{O}_Y \to \mathcal{T}_Y$ with the additional property that the subspace $\alpha \circ s_Y(\mathcal{O}_Y) + \mathrm{End}^0_A(\mathrm{Fock}_X \cdot J_X)$ contains an approximate unit for $\mathcal{O}_X$, then ψ is also surjective.

Proof. It follows from from Theorem 4.7 and Lemma 3.26 that there is an isometric embedding

$$\psi\colon (\mathcal{O}_X \otimes_{\widetilde{\Psi}_P} \mathcal{O}_Y) \otimes (\mathcal{O}_Y \otimes_{\widetilde{\Psi}_Q} \mathcal{O}_Z) \to \mathcal{O}_X \otimes_{\widetilde{\Psi}_{PQ}} \mathcal{O}_Z.$$

If $s\colon \mathcal{O}_X \to T_X$ is a nondegenerate splitting of the canonical quotient map $\mathcal{T}_X \to \mathcal{O}_X$ (if the C^*-algebras are unital, then s can be chosen to be unital), then there is an approximate unit $(e_i)_i$ in $\mathcal{O}_X$ such that $(s(e_i))_i$ converges strictly to the identity in $\mathrm{Mult}(\mathcal{T}_Y)$. Then $(s(e_i))_i$ is an approximate unit for $\mathcal{T}_X$ and it follows from the last part of Lemma 3.26 that ψ is an isomorphism. ☐

Remark 4.10. The isometric inclusion ψ of Corollary 4.9 is an isomorphism when $\mathcal{T}_Y$ and $\mathcal{O}_Y$ are unital, because the Choi–Effros lifting s_Y can be chosen to be unital, and we suspect that ψ is always an isomorphism.

Since the KSGNS construction is fairly explicit, it is often possible to get one's hands on the module $\mathcal{O}_X \otimes_{\widetilde{\Psi}_P} \mathcal{O}_Y$ in examples. One class of examples arises from graph C^*-algebras.

Example 4.11. Let $E = (E^0, E^1, r, s)$ be a directed graph. We refer to [Rae05, Chapter 8] for the description of the *graph C^*-algebra* $C^*(E)$ as the Cuntz–Pimsner $\mathcal{O}_{X(E)}$ of the graph correspondence $X(E)$.

Suppose that $F = (F^0, F^1, r, s)$ and $E \setminus F := (E^0, E^1 \setminus F^1, r, s)$ are subgraphs of E such that $E^0 = F^0$. Further, suppose that $E \setminus F$ is a *regular* subgraph in the sense that it contains no sources or infinite receivers. In particular, $E \setminus F$ must be a "large" subgraph of E since isolated vertices are sources. Then the graph correspondence $X(E)$ splits into a direct sum $X(E) = X(F) \oplus X(E\setminus F)$ of correspondences over $C_0(E^0)$. Since $X(E\setminus F)$ is regular, Theorem 4.7 provides a $C^*(E)$–$C^*(F)$-correspondence.

To be more explicit, let $\{S_\mu\colon \mu \in E^*\}$ be the generating partial isometries of $C^*(E)$ and let $\{W_\mu\colon \mu \in F^*\}$ be the generating partial isometries of $C^*(F)$. For $\mu, \nu \in E^*$ we have

$$\widetilde{\Psi}_P(S_\mu S_\nu^*) = \begin{cases} W_\mu W_\nu^* & \text{if } \mu, \nu \in F^* \\ 0 & \text{otherwise.} \end{cases}$$

In the KSGNS module $C^*(E) \otimes_{\widetilde{\Psi}_P} C^*(F)$, the inner product satisfies

$$\langle S_\mu S_\nu^* \otimes W_\xi W_\eta^* \mid S_\alpha S_\beta^* \otimes W_\rho W_\sigma^* \rangle_{C^*(F)} = W_\eta W_\xi^* \widetilde{\Psi}_P(S_\nu S_\mu^* S_\alpha S_\beta^*) W_\rho W_\sigma^*.$$

In particular, if $\mu = \alpha$, $\nu = \beta$, $\xi = \rho$, and $\eta = \sigma$, then

$$\begin{aligned} \langle S_\mu S_\nu^* \otimes W_\xi W_\eta^* \mid S_\mu S_\nu^* \otimes W_\xi W_\eta^* \rangle_{C^*(F)} &= W_\eta W_\xi^* \widetilde{\Psi}_P(S_\nu S_\mu^* S_\mu S_\nu^*) W_\xi W_\eta^* \\ &= \begin{cases} W_\eta W_\xi^* W_\nu W_\nu^* W_\xi W_\eta^* & \text{if } \nu \in F^* \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} W_{\eta\nu'} W_{\eta\nu'}^* & \text{if } \nu = \xi\nu' \text{ and } \nu \in F^* \\ W_\eta W_\eta^* & \text{if } \xi = \nu\xi' \text{ and } \nu \in F^* \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, a necessary condition for $S_\mu S_\nu^* \otimes W_\xi W_\eta^* \in C^*(E) \otimes_{\widetilde{\Psi}_P} C^*(F)$ to be nonzero is that $\nu \in F^*$ and that ξ extends ν (or vice versa).

We claim that $C^*(E) \otimes_{\widetilde{\Psi}_P} C^*(F)$ is isomorphic to $\mathrm{Fock}_{X(E)} \otimes_{C_0(E^0)} C^*(F)$ as right $C^*(F)$-modules. Recall that E^k denotes the paths of length k in E, and each $k \geq 0$ we can identify $X(E)^{\otimes k}$ with a completion of $C_c(E^k)$. With this identification, for each $\mu \in E^*$ we let $\delta_\mu \in \mathrm{Fock}_{X(E)}$ denote the point mass at μ in the completion. In $\mathrm{Fock}_{X(E)} \otimes_{C_0(E^0)} C^*(F)$ we calculate the inner product

$$\begin{aligned}\langle \delta_\mu \otimes W_\nu^* W_\xi W_\eta^* \mid \delta_\mu \otimes W_\nu^* W_\xi W_\eta^* \rangle_{C^*(F)} &= \langle W_\nu^* W_\xi W_\eta^* \mid P_{s(\mu)} W_\nu^* W_\xi W_\eta^* \rangle_{C^*(F)} \\ &= \begin{cases} W_\eta W_\xi^* W_\nu W_\nu^* W_\xi W_\eta^* & \text{if } s(\mu) = s(\nu) \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

This means that there is an isometric linear map $\kappa\colon C^*(E) \otimes_{\tilde{\Psi}} C^*(F) \to \mathrm{Fock}_{X(E)} \otimes_{C_0(F^0)} C^*(F)$ given by $\kappa(S_\mu S_\nu \otimes W_\xi W_\eta^*) = \delta_\mu \otimes W_\nu^* W_\xi W_\eta^*$, for every $\mu \in E^*$ and $\nu, \xi, \eta \in F^*$. The target $\mathrm{Fock}_{X(E)} \otimes_{C_0(F^0)} C^*(F)$ is generated by elements of the form $\delta_\mu \otimes W_\xi W_\eta^*$, and this is nonzero only if $r(\xi) = s(\mu)$. Choosing $\nu = s(\mu)$ so that $S_\nu = P_{s(\mu)}$, we see that $\kappa(S_\mu S_\nu \otimes W_\xi W_\eta^*) = \delta_\mu \otimes W_\xi W_\eta^*$. This shows that κ is surjective.

The map κ clearly preserves the right action of $C^*(F)$ and a routine computation shows that κ preserves inner products. Define $\rho\colon C^*(E) \to \mathrm{End}_{C^*(F)}(\mathrm{Fock}_{X(E)} \otimes_{C_0(F^0)} C^*(F))$ by

$$\rho(S_\alpha S_\beta^*)(\delta_\mu \otimes W_\nu^* W_\xi W_\eta^*) = \begin{cases} \delta_{\alpha\mu'} \otimes W_\nu^* W_\xi W_\eta^* & \text{if } \mu = \beta\mu' \\ \delta_\alpha \otimes W_{\nu\beta'}^* W_\xi W_\eta^* & \text{if } \beta = \mu\beta' \text{ and } \beta' \in F^* \\ 0 & \text{otherwise.} \end{cases}$$

Comparing with the (positive) left action on the KSGNS module completes the isomorphism of correspondences.

For a particular instance, suppose that $E^0 = \{v\}$ and that there are n edges (which must have source and range v). Let F be the subgraph consisting of $m \leq n$ edges. Then $E \setminus F$ is regular, $X(E) \cong \mathbb{C}^n$, and $X(F) \cong \mathbb{C}^n$. The graph algebra $C^*(E)$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$, and $C^*(F) \cong \mathcal{O}_m$. Our construction provides a positive $\mathcal{O}_n$–$\mathcal{O}_m$-correspondence with underlying $\mathcal{O}_m$-module isomorphic to $\mathrm{Fock}_{\mathbb{C}^n} \otimes_{\mathbb{C}} \mathcal{O}_m$. Note that $\mathcal{O}_n$ typically does not act by adjointable operators on $\mathrm{Fock}_{\mathbb{C}^n}$, but it does act on the module $\mathrm{Fock}_{\mathbb{C}^n} \otimes_{\mathbb{C}} \mathcal{O}_m$. △

4.2. **Fock projections and subproduct systems.** The previous subsection started from a correspondence projection, then built a correspondence projection on the associated Fock module, and finally constructed a positive map. In certain circumstances we can start directly with a projection on the Fock module and obtain sensible morphisms on the associated Cuntz–Pimsner algebras.

Definition 4.12. Let $(\phi, {}_AX_A)$ be a correspondence, and let $Q_n\colon \mathrm{Fock}_X \to X^{\otimes n}$ be the projection onto the n-th summand. A *Fock projection* $P \in \mathrm{End}_A(\mathrm{Fock}_X)$ is a projection which commutes with the left action of A on Fock_X, and Q_n for all $n \geq 0$. We write $P_n := Q_n P$ and note that the projection P on Fock_X is equal to the strict sum $\sum_{n=0}^{\infty} P_n$.

Example 4.13. A correspondence projection $P_0\colon (\phi_X, {}_AX_A) \to (\phi_Y, {}_AY_A)$ induces a Fock projection as in Lemma 4.4. △

Lemma 4.14. *Let $(\phi, {}_AX_A)$ be a correspondence and let $P \in \mathrm{End}_A(\mathrm{Fock}_X)$ be a Fock projection. The map $\Psi_P\colon \mathrm{End}_A(\mathrm{Fock}_X) \to \mathrm{End}_A(P\,\mathrm{Fock}_X)$ given by $\Psi_P(T) = PTP$ is a conditional expectation and restricts to a surjective completely positive map from $\mathrm{End}_A^0(\mathrm{Fock}_X \cdot J_X)$ to $\mathrm{End}_A^0(P\,\mathrm{Fock}_X \cdot J_X)$.*

Proof. This result follows from the proof of Lemmas 4.5 and 4.6 with minor modification. □

If we restrict Ψ_P to a subalgebra of $\mathrm{End}_A(\mathrm{Fock}_X)$, such as $\mathcal{T}_X$, we get a completely positive map onto *some* subalgebra of $\mathrm{End}_A(P\,\mathrm{Fock}_X)$.

Definition 4.15. Let $(\phi, {}_AX_A)$ be a correspondence, and let $P \in \mathrm{End}_A(\mathrm{Fock}_X)$ be a Fock projection. For $\xi \in P\,\mathrm{Fock}_X$ we define a *projected creation operator* $T_\xi^P := PT_\xi$, where $T_\xi \in \mathcal{T}_X$. Then T_ξ^P is adjointable with adjoint $(T_\xi^P)^* = T_\xi^* P$.

We define the *projected Toeplitz algebra* $\mathcal{T}_X^P$ of $P\,\mathrm{Fock}_X$ to be the C^*-algebra generated by $\{T_\xi^P \mid \xi \in P\,\mathrm{Fock}_X\}$. We define the *projected Cuntz–Pimsner algebra* $\mathcal{O}_X^P$ of $P\,\mathrm{Fock}_X$ to be the quotient of $\mathcal{T}_X^P$ by the ideal

$$\mathrm{End}_A^0(P\,\mathrm{Fock}_X \cdot J_X) \cap \mathcal{T}_X^P.$$

By construction, $\Psi_P\colon \mathcal{T}_X \to \mathcal{T}_X^P$ is a conditional expectation. Building on similar ideas to the previous section, we use Lemma 3.24 to construct a strict completely positive map between a Cuntz–Pimsner algebra and its projected relative.

By applying Lemma 3.24 the map $\Psi_P\colon \mathcal{T}_X \to \mathcal{T}_X^P$ passes to the Cuntz–Pimsner quotients.

Proposition 4.16. *Let $(\phi, {}_AX_A)$ be a C^*-correspondence such that there is a completely positive splitting $\mathcal{O}_X \to \mathcal{T}_X$ (for example if A is nuclear). Let $P \in \mathrm{End}_A(\mathrm{Fock}_X)$ be a Fock projection. Then there is a surjective strict completely positive map $\widetilde{\Psi}_P : \mathcal{O}_X \to \mathcal{O}_X^P$.*

The construction of $\mathcal{O}_X^P$ is closely related to the construction of subproduct systems over $\mathbb{N}$. The various definitions found in [SS09, Vis12, DM14, AK21] (for instance) can all be seen to be equivalent using [SS09, Lemma 6.1].

Definition 4.17. Let A be a separable and nuclear C^*-algebra, and let $(X_m)_{m\in\mathbb{N}_0}$ be a sequence of nondegenerate A–A-correspondences with $X_0 = A$. We say that $(X_m)_{m\in\mathbb{N}_0}$ is a *subproduct system* if for all $n, m \in \mathbb{N}_0$ the correspondence X_{n+m} is a complemented sub-correspondence of $X_n \otimes_A X_m$.

One can now deduce the existence of projections $P_k\colon X_1^{\otimes_A k} \to X_k$, cf. [SS09, Lemma 6.1]. Setting $P_0 = \mathrm{Id}_A$, the strict sum

$$P := \sum_{k\in\mathbb{N}_0} P_k \tag{4.6}$$

is a Fock projection on Fock_{X_1}.

Given a subproduct system we can construct the associated Toeplitz and Cuntz–Pimsner algebras. We use the definition of [AK21], which is derived from [Vis12].

Definition 4.18. Let $X = (X_m)_{m\in\mathbb{N}_0}$ be a subproduct system over a separable and nuclear C^*-algebra A, and let P be the Fock projection (4.6). The *Toeplitz algebra* of X is defined to be $\mathcal{T}_{X_1}^P$. Setting $\mathbb{I} = \{T \in \mathcal{T}_X^P : \lim_{m\to\infty} \|Q_m T\| = 0\}$, the *Cuntz–Pimsner* of X is $\mathbb{O}_X := \mathcal{T}_{X_1}^P/\mathbb{I}$.

Given $X = (X_m)_m$, the distinction between $\mathcal{O}_{X_1}^P$ and $\mathbb{O}_X$ is the distinction between the ideals $\mathrm{End}_A^0(P\,\mathrm{Fock}_{X_1} \cdot J_{X_1})$ and $\mathbb{I}$, see [Vis12].

Proposition 4.19. *Let $X = (X_m)_{m\in\mathbb{N}_0}$ be a subproduct system over a separable and nuclear C^*-algebra A, and let P be the Fock projection (4.6). Then there is a strict completely positive map $\Phi_P\colon \mathcal{O}_{X_1} \to \mathbb{O}_X$.*

Proof. We have an expectation $\Psi_P\colon \mathcal{T}_{X_1} \to \mathcal{T}_{X_1}^P$ which, by Proposition 4.16, descends to a strict completely positive map $\widetilde{\Psi}_P\colon \mathcal{O}_X \to \mathcal{O}_X^P$. Since $\mathrm{End}_A^0(P\,\mathrm{Fock}_{X_1} \cdot J_{X_1})P \subseteq \mathcal{T}_{X_1}^P$ and as $\|Q_m T\| \to 0$ for every $T \in \mathrm{End}_A^0(P\,\mathrm{Fock}_{X_1} \cdot J_{X_1})P$, we have $\mathrm{End}_A^0(P\,\mathrm{Fock}_{X_1} \cdot J_{X_1}) \subset \mathbb{I}$.

Consequently, the map $\lambda\colon \mathcal{O}_X^P \to \mathbb{O}_X$ defined by $\lambda(T + \mathrm{End}_A^0(P\,\mathrm{Fock}_{X_1} \cdot J_{X_1})) = T + \mathbb{I}$ is a well-defined surjective $*$-homomorphism. Then $\Phi_P := \lambda \circ \widetilde{\Psi}_P$ is the desired strict completely positive map. □

Corollary 4.20. *Let $X = (X_m)_{m\in\mathbb{N}_0}$ be a subproduct system over a separable nuclear C^*-algebra A, and let P be the Fock projection (4.6). Then*

(i) *$(\Phi_P, {}_{\mathcal{O}_{X_1}}(\mathbb{O}_X)_{\mathbb{O}_X})$ is a positive correspondence, defining a morphism in* Quesadilla *from $\mathcal{O}_{X_1}$ to $\mathbb{O}_X$, and*

(ii) *the KSGNS space $(\pi_{\Phi_P}, \mathcal{O}_{X_1} \otimes_{\Phi_P} \mathbb{O}_X)$ defines a morphism in* Enchilada *from $\mathcal{O}_{X_1}$ to $\mathbb{O}_X$.*

5. Morphisms from bi-Hilbertian bimodules

In this section we change tack and consider another class of examples of positive correspondence which arise by trying to generalise the covariant correspondences of [MS19].

5.1. Bi-Hilbertian bimodules.

Amongst nondegenerate C^*-correspondences there is a useful subclass more general than Morita equivalences, but much tamer than general C^*-correspondences. The properties of these *bi-Hilbertian bimodules* were developed by Watatani and coauthors (we refer to [KPW04] for most facts) as a C^*-algebraic version of the setting of Jones index theory, and have since arisen in work of [CCH18] on representation theory, [RRS17, RRS19, GMR18, AR19] on the Kasparov theory of Cuntz–Pimsner algebras, and in the characterisation of "non-commutative vector bundles" [RS17].

In this section we assume that all C^*-modules are countably generated.

Definition 5.1. Let A and B be C^*-algebras. Following [KPW04], a *bi-Hilbertian A–B-bimodule* F is a right B-module with inner product $\langle \cdot \mid \cdot \rangle_B$ which is also a left A-module with inner product ${}_A\langle \cdot \mid \cdot \rangle$ such that

(i) the left action of A on F is adjointable with respect to $\langle \cdot \mid \cdot \rangle_B$;

(ii) the right action of B on F is adjointable with respect to ${}_A\langle \cdot \mid \cdot \rangle$; and

(iii) the left and right inner products induce equivalent norms on F.

We usually do not refer to the left and right action $*$-homomorphisms and simply write ${}_AF_B$ for a bi-Hilbertian A–B-bimodule since the definition is symmetric, unlike that of a C^*-correspondence. By [KPW04, Proposition 2.16] the actions on ${}_AF_B$ are automatically nondegenerate. Many examples can be found in [RRS17, Section 2.1].

Before we examine bi-Hilbertian bimodules, we address the question of when a correspondence has a compatible left inner product. A less general version of the following lemma appears in [LRV12, Lemma 3.23].

Lemma 5.2 (cf. [KPW04, Lemma 2.6])**.** *Let F_B be a countably generated right B-module, and let $A \subseteq \mathrm{End}_B(F)$ be a C^*-subalgebra. Suppose that ${}_A\langle \cdot \mid \cdot \rangle$ is a left A-valued inner product on F_B for which the right action of B is adjointable. Then there is a A-bilinear faithful positive map $\Upsilon\colon \mathrm{End}_B^{00}(F) \to A$ such that $\Upsilon(\Theta_{e,f}) := {}_A\langle e \mid f \rangle$ for all $e, f \in F_B$. For any frame (f_i) for F_B, we have*

$$\Upsilon(T) = \sum_i {}_A\langle Tf_i \mid f_i \rangle \quad \text{for all } T \in \mathrm{End}_B^{00}(F).$$

On the other hand if there is an A-bilinear faithful positive map $\Upsilon\colon \mathrm{End}_B^0(F) \to A$, then ${}_A\langle e \mid f \rangle := \Upsilon(\Theta_{e,f})$ defines a left A-valued inner product on F_B for which the right action of B is adjointable.

Definition 5.3 ([KPW04, Definition 2.8])**.** Let ${}_AF_B$ be a bi-Hilbertian A–B-bimodule. We say that ${}_AF_B$ has *finite right numerical index* if there exists $\lambda > 0$ such that

$$\Big\| \sum_{i=1}^{n} {}_A\langle f_i \mid f_i\rangle \Big\| \le \lambda \Big\| \sum_{i=1}^{n} \Theta_{f_i,f_i} \Big\| \quad \text{for all } n \in \mathbb{N} \text{ and all } f_1, \dots, f_n \in F. \tag{5.1}$$

The *right numerical index* of ${}_AF_B$ is the infimum of the numbers λ satisfying (5.1).

Remark 5.4. Following [KPW04, Corollary 2.11], if ${}_AF_B$ has finite right numerical index, then the map $\Upsilon\colon \operatorname{End}_B^{00}(F) \to A$ of Lemma 5.2 extends to an A-bilinear $*$-homomorphism $\Upsilon\colon \operatorname{End}_B^0(F) \to A$ satisfying

$$\Upsilon(\Theta_{e,f}) = {}_A\langle e \mid f\rangle \quad \text{for all } e \in E,\ f \in F.$$

If $T \in \operatorname{End}_B^0(F)$ commutes with all $a \in A$, then $a\Upsilon(T) = \Upsilon(aT) = \Upsilon(Ta) = \Upsilon(T)a$, so $\Upsilon(T)$ is in the centre of A. In particular, $\Upsilon(\mathrm{Id}_F)$ is central in A.

In [KPW04, Proposition 2.13] it is shown that tensor products of bi-Hilbertian bimodules with finite left and right numerical indices are again bi-Hilbertian bimodules.

Let ${}_AF_B$ be a bi-Hilbertian bimodule, countably generated as a right module and with finite right numerical index. By [KPW04, Theorem 2.22], the left action of A on ${}_AF_B$ is by compact endomorphisms with respect to $\langle \cdot \mid \cdot \rangle_B$ if and only if, for every frame (f_j) for the right B-module F_B, the series

$$\sum_{j\ge 1} {}_A\langle f_j \mid f_j\rangle \tag{5.2}$$

converges strictly in $\operatorname{Mult}(A)$. In this case we denote the strict limit by $r\text{-}\operatorname{Ind}(F)$. We note that $r\text{-}\operatorname{Ind}(F)$ is independent of the frame (f_j), [KPW04, Theorem 2.22].

Definition 5.5. When it exists we call $r\text{-}\operatorname{Ind}(F)$ the *right Watatani index* of ${}_AF_B$.

Definition 5.6. We say that ${}_AF_B$ has *finite right Watatani index* if it has finite right numerical index and the left action is by compacts. The left numerical index and left index are defined similarly. If ${}_AF_B$ has finite left and right index, we say that ${}_AF_B$ has *finite index*.

Example 5.7 ([KPW04, Corollary 4.14])*.* A Morita equivalence A–B-bimodule is a finite index bi-Hilbertian A–B-bimodule with right index $1 \in \operatorname{Mult}(A)$ and left index $1 \in \operatorname{Mult}(B)$. On the other hand, if a finite index bi-Hilbertian A–B-bimodule ${}_AF_B$ has right index $1 \in \operatorname{Mult}(A)$ and left index $1 \in \operatorname{Mult}(B)$, then F_B can be equipped with a left inner product making F_B into a Morita equivalence A–B-bimodule. △

We record the following facts about $r\text{-}\operatorname{Ind}(F)$.

Lemma 5.8 ([KPW04, Corollary 2.28])**.** *Let ${}_AX_B$ be a bi-Hilbertian bimodule with finite right Watatani index. Then $r\text{-}\operatorname{Ind}(F)$ is a positive central element of $\operatorname{Mult}(A)$. Moreover, the following are equivalent:*

(i) *$r\text{-}\operatorname{Ind}(F)$ is invertible,*
(ii) *the left action $A \to \operatorname{End}_A^0(F)$ is injective, and*
(iii) *the left inner product ${}_A\langle \cdot \mid \cdot \rangle$ is full.*

Lemma 5.8 motivates the following definition.

Definition 5.9. A bi-Hilbertian bimodule is *regular* if it has finite index and both the left and right inner products are full (equivalently the left and right actions are injective).

If ${}_AF_B$ is a regular bi-Hilbertian bimodule then the left and right Watatani indices are central, positive and invertible elements of $\mathrm{Mult}(A)$ and $\mathrm{Mult}(B)$, respectively. This allows us to write $e^{\beta} \in \mathrm{Mult}(A)$ for the right Watatani index, where $\beta = \beta_R$ is central and self-adjoint. The notation e^{β} will always mean the *right* Watatani index unless stated otherwise. When needed, the left Watatani index is denoted e^{β_L}.

The next result gives useful relationships between frames for the different module structures on a bi-Hilbertian bimodule, its conjugate module, and its algebra of compact operators. Recall from Notation 2.5 that if ${}_AF_B$ is a bi-Hilbertian bimodule then $\mathrm{End}^0_B(F)$ is identified with $F \otimes_B F^*$. Since ${}_AF_B$ comes equipped with a left A-valued inner product, $\mathrm{End}^0_B(F)$ inherits the structure of a bi-Hilbertian A-bimodule, [KPW04, Corollary 2.29].

Lemma 5.10. *Let ${}_AF_B$ be a bi-Hilbertian bimodule. Given a right frame $(u_j)_j \subseteq F_B$ and left frame $(\tilde{u}_k)_k \subseteq {}_AF$, a right frame for ${}_AF \otimes_B F^*_A$ is given by $(u_j \otimes \tilde{u}_k^*)_{j,k}$, and a left frame is given by $(\tilde{u}_j \otimes u_k^*)_{j,k}$. If ${}_AF_B$ has finite index then so does ${}_AF \otimes_B F^*_A$, and if ${}_AF_B$ is regular then so is ${}_AF \otimes_B F^*_A$. Analogous statements also hold for ${}_BF^* \otimes_A F_B$.*

Proof. The statement about frames follows from [BMR24, Proposition 2.16]. If ${}_AF_B$ has finite index, then so does ${}_AF \otimes_B F^*_A$, by [KPW04, Theorem 5.1]. If the left action $\phi\colon A \to \mathrm{End}^0_B(F)$ is injective, then so is $\phi \otimes \mathrm{Id}_{F^*}$ by [Kat04a, Lemma 4.7]. Similarly, if the right action on ${}_AF_B$ is injective, then the right action on ${}_AF \otimes_B F^*_A$ is injective. □

The following property was discovered and described in [KPW04, Section 4.2] using the language of intertwiners.

Lemma 5.11. *Let ${}_AF_B$ be a regular bi-Hilbertian bimodule. Then $P_0\colon F \otimes_B F^* \to F \otimes_B F^*$ defined by*
$$P_0(T) = e^{-\beta}\Upsilon(T)\,\mathrm{Id}_F$$
is an A-bilinear adjointable projection. In particular, $P(F \otimes_B F^*) = A\,\mathrm{Id}_F$ is a complemented bi-Hilbertian sub-bimodule of ${}_AF \otimes_B F^*_B$, which is isomorphic to ${}_AA_A$.

Proof. Recall that if $e^{\beta} \in \mathrm{Mult}(A)$ has finite index then $\Upsilon(T) \in A \subseteq F \otimes_B F^*$ for all $T \in F \otimes_B F^*$. Since $F \otimes_B F^*$ is an ideal in $\mathrm{End}_B(F)$, we have $P_0(T) \in F \otimes_B F^*$. The A-bilinearity of Υ and centrality of e^{β} imply that P_0 is A-bilinear. Fullness of the left inner product implies that P_0 surjects onto $A\,\mathrm{Id}_F$.

To see that P_0 is idempotent, fix a right frame $(u_i)_i$ for F_B. Then for $a \in A$ we have
$$P_0(a\,\mathrm{Id}_F) = \lim_i \sum_i P_0(au_i \otimes u_i^*) = a\,\mathrm{Id}_F\,.$$

Since $e^{-\beta}\Psi(T) \in A$ it follows that $P^2(T) = P_0(e^{-\beta}\Upsilon(T)\,\mathrm{Id}_F) = e^{-\beta}\Upsilon(T)\,\mathrm{Id}_F$.

For adjointability of P_0, fix $e_1 \otimes e_2^*,\ f_1 \otimes f_2^* \in F \otimes_B F^*$. We first observe that
$$\sum_l \langle u_l \otimes u_l^* \mid f_1 \otimes f_2^*\rangle_A = \sum_l \langle u_l^* \mid \langle u_l \mid f_1\rangle_A \cdot f_2^*\rangle_A = \sum_l \big\langle (\langle f_1 \mid u_l\rangle_A \cdot u_l)^* \mid f_2^*\big\rangle_A = {}_A\langle f_1 \mid f_2\rangle.$$

Then, using centrality of $e^{-\beta}$ at the last equality,
$$\begin{aligned}\langle P_0(e_1 \otimes e_2^*) \mid f_1 \otimes f_2^*\rangle_A &= \sum_k \langle e^{-\beta}{}_A\langle e_1 \mid e_2\rangle u_k \otimes u_k^* \mid f_1 \otimes f_2^*\rangle_A \\ &= \sum_k \langle u_k \otimes u_k^* | e^{-\beta}{}_A\langle e_2 \mid e_1\rangle f_1 \otimes f_2^*\rangle_A = {}_A\langle e^{-\beta}{}_A\langle e_2 \mid e_1\rangle f_1 \mid f_2\rangle \\ &= e^{-\beta}{}_A\langle e_2 \mid e_1\rangle {}_A\langle f_1 \mid f_2\rangle = {}_A\langle e_2 \mid e_1\rangle e^{-\beta}{}_A\langle f_1 \mid f_2\rangle.\end{aligned}$$

A symmetric calculation shows $\langle e_1 \otimes e_2^* \mid P_0(f_1 \otimes f_2^*)\rangle_A = {}_A\langle e_2 \mid e_1\rangle e^{-\beta}{}_A\langle f_1 \mid f_2\rangle$, from which it follows that $P_0^* = P_0$. □

A bi-Hilbertian bimodule ${}_AF_B$ does not typically give a covariant correspondence between a C^*-correspondence ${}_AX_A$ and $F^* \otimes_A X \otimes_A F$ in the sense of [MS19, Definition 2.21]. This is because $X \otimes_A F \cong A \otimes_A X \otimes_A F \subseteq F \otimes_B F^* \otimes_A X \otimes_A F$, where the inclusion is as a complemented submodule. The only time this inclusion is an isomorphism is when F is a Morita equivalence.

Although they do not typically induce covariant correspondences, in the next section we show that conjugation of a C^*-correspondence by a bi-Hilbertian bimodule yields a strict completely positive map between corresponding Cuntz–Pimsner algebras.

5.2. **From bi-Hilbertian bimodules to positive maps.** Suppose that ${}_AF_B$ is a bi-Hilbertian bimodule of finite index, and ${}_AX_A$ is a C^*-correspondence. Lemma 5.11 provides natural projections on the Fock module of the B–B-correspondence $F^*XF := {}_B(F^* \otimes_A X \otimes_A F)_B$. We use these projections to induce correspondences between their associated Cuntz–Pimsner algebras. We start on Fock space and build expectations on the associated Toeplitz algebras.

Let $P_0 \colon F \otimes_B F^* \to A\,\mathrm{Id}_F$ be the projection of Lemma 5.11, so

$$P_0(f \otimes g^*) = e^{-\beta}{}_A\langle f \mid g\rangle\,\mathrm{Id}_F,$$

where e^β is the right Watatani index of F_B. Consider the element

$$Z := e^{-\beta/2}\,\mathrm{Id}_F \in \mathrm{End}_B(F). \tag{5.3}$$

Since F is finite index, A acts compactly and so $aZ \in \mathrm{End}_B^0(F) = F \otimes_B F^*$ for all $a \in A$. Then for each $a \in A$, and right frame (u_i) we have

$$P_0(aZ) = e^{-\beta/2}\sum_i P_0(au_i \otimes u_i^*) = e^{-\beta/2}\sum_i e^{-\beta}a_A\langle u_i \mid u_i\rangle\,\mathrm{Id}_F = aZ.$$

Since $e^\beta \in \mathrm{Mult}(A)$ commutes with elements of A, Z commutes with elements of $A \subseteq \mathrm{End}_B^0(F)$. We record how inner products with aZ work in $\mathrm{End}_B^0(F)$.

Lemma 5.12. *Let ${}_AF_B$ be a regular bi-Hilbertian bimodule. Then for $f_1 \otimes f_2^* \in F \otimes_B F^*$ and $a \in A$, we have*

$$\langle f_1 \otimes f_2^* \mid aZ\rangle_A = e^{-\beta/2}{}_A\langle f_1 \mid f_2\rangle a.$$

In particular, for $a_1, a_2 \in A$, we have $\langle a_1Z \mid a_2Z\rangle_A = a_1^*a_2$.

Proof. We compute,

$$\begin{aligned}\langle f_1 \otimes f_2^* \mid aZ\rangle_A &= \sum_i \langle f_2^* \mid \langle f_1 \mid u_i\rangle_B \cdot u_i^*\rangle_A e^{-\beta/2}a = \sum_i \langle (f_2 \cdot \langle f_1 \mid u_i\rangle_B)^* \mid u_i^*\rangle_A e^{-\beta/2}a \\ &= \sum_i {}_A\langle f_2 \mid u_i \cdot \langle u_i \mid f_1\rangle_B\rangle e^{-\beta/2}a = e^{-\beta/2}{}_A\langle f_2 \mid f_1\rangle a.\end{aligned}$$

The second statement follows from the first. □

Let ${}_AF_B$ be a regular bi-Hilbertian bimodule and let $(\phi, {}_AX_A)$ be a C^*-correspondence. Since $e^{-\beta}$ is positive and invertible we have $P_0(F \otimes F^*) = AZ$. Using A-bilinearity of P_0 we can define a projection P_n on $F \otimes_B (F^*XF)^{\otimes_A n} \otimes_B F^*$ by

$$P_n := P_0 \otimes \mathrm{Id}_X \otimes P_0 \otimes \mathrm{Id}_X \otimes P_0 \otimes \cdots \otimes P_0 \otimes \mathrm{Id}_X \otimes P_0,$$

where P_0 appears $n+1$ times. Since the image of P_0 is isomorphic to ${}_AA_A$, the image of P_n can be identified with $X^{\otimes n}$. We make this more precise.

Lemma 5.13. *Let $(\phi, {}_AX_A)$ be a C^*-correspondence and let ${}_AF_B$ a regular bi-Hilbertian bimodule. For all $n \in \mathbb{N}_0$, there is an adjointable A-bilinear isometric inclusion $W_n\colon X^{\otimes_A n} \to F \otimes_B (F^*XF)^{\otimes_A n} \otimes_B F^*$ such that for $n \geq 1$,*

$$W_n(x_1 \otimes \cdots \otimes x_n) = Z \otimes x_1 \otimes Z \otimes \cdots \otimes Z \otimes x_n \otimes Z$$

and $W_0(a) = aZ$ for $a \in A \cong X^{\otimes 0}$. For $f, g \in F$ and $\bigotimes_{i=1}^n (f_i \otimes x_i \otimes g_i^*) \in \mathrm{Fock}_{F^*XF}$, the adjoint satisfies

$$W_n^*\Big(f \otimes \bigotimes_{i=1}^n (f_i^* \otimes x_i \otimes g_i) \otimes g^*\Big) = e^{-\beta/2}{}_A\langle f \mid f_1\rangle \cdot x_1 \otimes e^{-\beta/2} \cdot {}_A\langle g_1 \otimes f_2^*\rangle \cdot x_2 \otimes \cdots$$
$$\otimes\, e^{-\beta/2}{}_A\langle g_{n-1} \mid f_n\rangle \cdot x_n \cdot e^{-\beta/2}{}_A\langle g_n \mid g\rangle$$

Moreover, $W_n(X^{\otimes_A n}) = P_n(F \otimes_B (F^*XF)^{\otimes_A n} \otimes_B F^*)$ is a complemented sub-correspondence of $F \otimes_B (F^*XF)^{\otimes_A n} \otimes_B F^*$. If ${}_AF_B$ is a Morita equivalence bimodule, then conjugation by ${}_AF_B$ yields $X^{\otimes_A n} \cong F \otimes_B (F^*XF)^{\otimes_A n} \otimes_B F^*$.

Proof. For each $x \in {}_AX_A$ we can write $x = a_1 \cdot x' \cdot a_2$ for some $a_1, a_2 \in A$ and $x' \in {}_AX_A$. So $Z \otimes x \otimes Z = Za_1 \otimes x' \otimes a_2Z$ belongs to $F \otimes_B F^* \otimes_A X \otimes_A F \otimes_B F^*$. In particular, the formula defining W_n makes sense.

The case where $n = 0$ follows from the fact that $P_0(F \otimes F^*)$ is the A-span of Z. For $n \geq 1$ we begin by showing that W_n preserves inner products, and is therefore isometric. Let (u_i) be a right frame for F and observe that for any $f \in F$ we have $\sum_i \langle f \mid u_i\rangle_B \cdot u_i^* = f^*$. Fix elementary tensors $x = x_1 \otimes \cdots \otimes x_n$ and $y = y_1 \otimes \cdots \otimes y_n$ in $X^{\otimes n}$. Then

$$\langle Wx \mid Wy\rangle_A = \langle x_n \otimes Z \mid \langle W_{n-1}x' \mid W_{n-1}y'\rangle_A \cdot y_n \otimes Z\rangle_A,$$

where $x' = x_1 \otimes \cdots \otimes x_{n-1}$ and $y' = y_1 \otimes \cdots \otimes y_{n-1}$. Using the frame relation at the fourth equality and the definition of e^β at the fifth equality,

$$\begin{aligned}\langle W_nx \mid W_ny\rangle_A &= \sum_{i,j} \big\langle x_n \otimes e^{-\beta/2} \cdot u_i \otimes u_i^* \mid \langle W_{n-1}x' \mid W_{n-1}y'\rangle_A \cdot y_n \otimes e^{-\beta/2} \cdot u_j \otimes u_j^*\big\rangle_A \\ &= \sum_{i,j} \big\langle u_i^* \mid e^{-\beta/2} \cdot u_i \mid \langle x_n \mid \langle W_{n-1}x' \mid W_{n-1}y'\rangle_A \cdot y_n\rangle_A e^{-\beta/2} \cdot u_j\rangle_B \cdot u_j^*\big\rangle_A \\ &= \sum_{i,j} {}_A\langle u_i \cdot \langle e^{-\beta/2} \cdot u_i \mid \langle x_n \mid \langle W_{n-1}x' \mid W_{n-1}y'\rangle_A \cdot y_n\rangle_A e^{-\beta/2} \cdot u_j\rangle_B \mid u_j\rangle \\ &= \sum_{j} {}_A\langle e^{-\beta}\langle x_n \mid \langle W_{n-1}x' \mid W_{n-1}y'\rangle_A \cdot y_n\rangle_A \cdot u_j \mid u_j\rangle \\ &= \langle x_n \mid \langle W_{n-1}x' \mid W_{n-1}y'\rangle_A \cdot y_n\rangle_A = \langle W_{n-1}x' \otimes x_n \mid W_{n-1}y' \otimes y_n\rangle_A.\end{aligned}$$

An inductive argument now shows that $\langle W_nx \mid W_ny\rangle_A = \langle x \mid y\rangle_A$. The preceding argument extends to spans of elementary tensors, and so W_n extends to an isometric linear map from $X^{\otimes_A n}$ to $F \otimes_B (F^*XF)^{\otimes_A n} \otimes_B F^*$. The A-bilinearity of W_n follows from the fact that Z commutes with elements of A.

The formula for W_n^* can be seen to hold by using Lemma 5.12 to check that $W_n^*W_n = \mathrm{Id}$ on elementary tensors. We also have

$$W_n(X^{\otimes_A n}) = AZ \otimes_A X \otimes_A AZ \otimes \cdots \otimes AZ \otimes_A X \otimes AZ = P_n(F \otimes_B (F^*XF)^{\otimes_A n} \otimes_B F^*),$$

which is complimented in $F \otimes_B (F^*XF)^{\otimes_A n} \otimes_B F^*$.

For the final statement observe that if ${}_AF_B$ is a Morita equivalence, then $F \otimes_B F^* \cong A$. □

The following is an immediate consequence of Lemma 5.13.

Corollary 5.14. *Let $(\phi, {}_AX_A)$ be a C^*-correspondence and let ${}_AF_B$ be a regular bi-Hilbertian bimodule. The universal property of direct sums yields an A-bilinear isometry*

$$W\colon \operatorname{Fock}_X \to F \otimes_B \operatorname{Fock}_{F^*XF} \otimes_B F^*$$

such that $Wx = W_n x$ for all $x \in X^{\otimes_A n}$, and $W^*y = W_n^* y$ for all $y \in F \otimes_B \operatorname{Fock}_{F^*XF} \otimes_B F$. In $\operatorname{End}_A(F \otimes_B \operatorname{Fock}_{F^*XF^*} \otimes_B F^*)$, the strict sum $P = \sum_{n\geq 0} P_n$ defines an A-bilinear projection, and $W(\operatorname{Fock}_X)$ is isomorphic to the complemented A–A-correspondence $P(F^* \otimes_B \operatorname{Fock}_{F^*XF} \otimes_B F^*)$. If ${}_AF_B$ is a Morita equivalence, then $\operatorname{Fock}_X \cong F \otimes_B \operatorname{Fock}_{F^*XF} \otimes_B F^*$.

Remark 5.15. We make the following observations before stating the next result. Let ${}_AF_B$ be a nondegenerate C*-correspondence. For a C^*-correspondence $(\phi, {}_AX_A)$, the Toeplitz algebra $\mathcal{T}_X$ is an A-algebra, so Lemma 2.10 allows us to form the B-algebra

$$F^* \otimes_A \mathcal{T}_X \otimes_A F \cong \operatorname{End}^0_{\mathcal{T}_X}(F^* \otimes_A \mathcal{T}_X).$$

Similarly, Lemma 2.10 gives us the A-algebra

$$F \otimes_B \mathcal{T}_{F^*XF} \otimes_B F^* \cong \operatorname{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_A \mathcal{T}_{F^*XF}).$$

Lemma 2.11 implies that $F \otimes_B \mathcal{T}_{F^*XF} \otimes_B F^*$ acts faithfully by adjointable operators on $F \otimes_B \operatorname{Fock}_{F^*XF}$ with the left action satisfying

$$(f \otimes T_x T_y^* \otimes g^*)(h \otimes z) = f \otimes T_x T_y^* \langle g \mid h\rangle_B z.$$

The action extends to a faithful action on $F \otimes_B \operatorname{Fock}_{F^*XF} \otimes_B F^*$ by acting trivially on F^*, [Kat04a, Lemma 4.7].

Lemma 5.16. *Let $(\phi, {}_AX_A)$ be a C^*-correspondence and let ${}_AF_B$ be regular bi-Hilbertian bimodule. Let π denote the left action of A on $F \otimes_B \mathcal{T}_{F^*XF}$, and let $(u_i)_i$ be a right frame for F_B. Define $\psi\colon X \to \operatorname{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$ by*

$$\psi(x) = \sum_{i,j} u_i \otimes T_{u_i^* \otimes e^{-\beta/2} \cdot x \otimes u_j} \otimes u_j^*. \tag{5.4}$$

Then (π, ψ) is a faithful representation of $(\phi, {}_AX_A)$ in $\operatorname{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$ which induces an injective $*$-homomorphism $\alpha\colon \mathcal{T}_X \to \operatorname{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$.

Remark 5.17. If $\mathcal{T}_{F^*XF}$ is nonunital, we can replace $(u_j \otimes 1)$ with $(u_j \otimes b_i)$ where $b_i = (a_i - a_{i-1})^{1/2}$ for some approximate unit $(a_i) \in \mathcal{T}_{F^*XF}$. This is true because $(u_j \otimes b_i)$ is then a frame for $F \otimes \mathcal{T}_{F^*XF}$, by [LN04, Proposition 1.2].

Proof of Lemma 5.16. First we define ψ by the formula (5.4), so that ψ takes values in the adjointable operators $\operatorname{End}_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$. We show that $\psi(x)^*\psi(y) = \pi(\langle x \mid y\rangle_A)$. We compute,

$$\begin{aligned}
\psi(x)^*\psi(y) &= \sum_{i,j,k,\ell} u_i \otimes \langle u_j \otimes T_{u_j^* \otimes e^{-\beta/2} \cdot x \otimes u_i} \mid u_k \otimes T_{u_k^* \otimes e^{-\beta/2} \cdot y \otimes u_\ell}\rangle_B \otimes (u_\ell \otimes 1)^* \\
&= \sum_{i,j,k,\ell} u_i \otimes T^*_{u_j^* \otimes e^{-\beta/2} \cdot x \otimes u_i} \langle u_j \mid u_k\rangle_B T_{u_k^* \otimes e^{-\beta/2} \cdot y \otimes u_\ell} \otimes (u_\ell \otimes 1)^* \\
&= \sum_{i,j,\ell} u_i \otimes T^*_{u_j^* \otimes e^{-\beta/2} \cdot x \otimes u_i} T_{u_j^* \otimes e^{-\beta/2} \cdot y \otimes u_\ell} \otimes (u_\ell \otimes 1)^* \\
&= \sum_{i,j,\ell} u_i \otimes \langle u_j^* \otimes e^{-\beta/2} \cdot x \otimes u_i \mid u_j^* \otimes e^{-\beta/2} \cdot y \otimes u_\ell\rangle_B \otimes (u_\ell \otimes 1)^*.
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_j \langle u_j^* \otimes e^{-\beta/2} \cdot x \otimes u_i \mid u_j^* \otimes e^{-\beta/2} \cdot y \otimes u_\ell \rangle_B \\
&\quad = \sum_j \langle u_i \mid \pi(\langle e^{-\beta/2} \cdot x \mid {}_A\langle u_j \mid u_j \rangle e^{-\beta/2} \cdot y \rangle_A) \cdot u_\ell \rangle_B \\
&\quad = \langle u_i \mid \pi(\langle x \mid y \rangle_A) \cdot u_\ell \rangle_B,
\end{aligned}$$

so in our original calculation,

$$\begin{aligned}
\psi(x)^* \psi(y) &= \sum_{i,\ell} u_i \otimes \langle u_i \mid \pi(\langle x \mid y \rangle_A) u_\ell \rangle_B \otimes (u_\ell \otimes 1)^* \\
&= \sum_\ell \pi(\langle x \mid y \rangle_A) u_\ell \otimes 1 \otimes (u_\ell \otimes 1)^* = \pi(\langle x \mid y \rangle_A).
\end{aligned}$$

Since the left action on ${}_AF_B$ is faithful, it follows that ψ is an isometric linear map. To see that $\pi(a)\psi(x) = \psi(a \cdot x)$ we compute

$$\begin{aligned}
\pi(a) \sum_{i,j} u_i \otimes T_{u_i^* \otimes e^{\beta/2} x \otimes u_j} \otimes (u_j \otimes 1)^* &= \sum_{i,j,k} u_k \langle u_k \mid \pi(a) u_i \rangle_B \otimes T_{u_i^* \otimes e^{\beta/2} x \otimes u_j} \otimes (u_j \otimes 1)^* \\
&= \sum_{i,j,k} u_k \otimes T_{\langle u_k | \pi(a) u_i \rangle_B u_i^* \otimes e^{\beta/2} x \otimes u_j} \otimes (u_j \otimes 1)^* \\
&= \sum_{i,j,k} u_k \otimes T_{\langle u_k | \pi(a) u_i \rangle_B u_i^* \otimes e^{\beta/2} x \otimes u_j} \otimes (u_j \otimes 1)^* \\
&= \sum_{i,j,k} u_k \otimes T_{(u_i \langle u_i | \pi(a^*) u_k \rangle_B)^* \otimes e^{\beta/2} x \otimes u_j} \otimes (u_j \otimes 1)^* \\
&= \sum_{j,k} u_k \otimes T_{u_k^* \pi(a) \otimes e^{\beta/2} x \otimes u_j} \otimes (u_j \otimes 1)^* = \psi(a \cdot x).
\end{aligned}$$

A similar argument shows that $\psi(x)\pi(a) = \psi(x \cdot a)$.

With this observation, the density of $A \cdot X \cdot A$ in X and the compactness of $a\,\mathrm{Id}_F$ shows that ψ takes values in $\mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$. As π is an injective $*$-homomorphism, the representation (π, ψ) induces an injective $*$-homomorphism $\alpha \colon \mathcal{T}_X \to \mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$. □

For the next result, recall the map $W \colon \mathrm{Fock}_X \to F \otimes_B \mathrm{Fock}_{F^*XF} \otimes_B F^*$ and $P \in \mathrm{End}_A(F \otimes_B \mathrm{Fock}_{F^*XF^*} \otimes_B F^*)$ from Corollary 5.14, and the injection $\alpha \colon \mathcal{T}_X \to \mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$ from Lemma 5.16. Up to identifications, the conditional expectation defined below is simply compression by the projection P.

Proposition 5.18. *Let $(\phi, {}_AX_A)$ be a C^*-correspondence and let ${}_AF_B$ be a regular bi-Hilbertian bimodule. As in Remark 5.15, identify $\mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$ with its faithful representation as operators on $F \otimes_B \mathrm{Fock}_{F^*XF} \otimes_B F^*$. The linear map $\Phi_P \colon \mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF}) \to \mathcal{T}_X$ given by*

$$\Phi_P(T) = W^* PTPW$$

is a conditional expectation onto $\mathcal{T}_X$ for the inclusion α. If ${}_AF_B$ is a Morita equivalence bimodule, then $\mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF}) \cong \mathcal{T}_X$.

Proof. Compression by P defines an expectation from $\mathrm{End}_A(F \otimes_B \mathrm{Fock}_{F^*XF} \otimes_B F^*)$ onto $\mathrm{End}_A(W\,\mathrm{Fock}_X)$, which may be identified with $\mathrm{End}_A(\mathrm{Fock}_X)$ via W. Since we can faithfully represent $\mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$ on $F \otimes_B \mathrm{Fock}_{F^*XF} \otimes_B F^*$, we show that (up to identification) compression of an operator in $\mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$ by P yields an element of $\mathcal{T}_X$.

Fix $f, g \in F$, and $z = \bigotimes_{i=1}^{m}(f_i^* \otimes x_i \otimes g_i)$ and $w = \bigotimes_{i=m+1}^{m+n}(f_i^* \otimes x_i \otimes g_i)$ in Fock_{F^*XF}. Suppose that $\ell > 0$ (the $\ell = 0$ case follows from a similar, but simpler, argument) and fix $y = y_1 \otimes \cdots \otimes y_\ell \in \mathrm{Fock}_X$. We claim that $P(f \otimes T_z T_w^* \otimes g^*)PWy$ belongs to the image of W, from which it follows that $W^*P(f \otimes T_z T_w^* \otimes g^*)PW$ is an operator on Fock_X.

Fix a right frame (u_i) for F_B and observe that for any $f \in F$ we have $\sum_i \langle f \mid u_i\rangle_B \cdot u_i^* = f^*$. We compute,

$$\begin{aligned} P(f \otimes T_z T_w^* \otimes g^*)PWy &= \sum_j Pf \otimes T_z T_w^*(\langle g \mid u_j\rangle_B \cdot u_j^* \otimes e^{-\beta/2} \cdot y_1 \otimes Z \otimes \cdots \otimes Z \otimes y_\ell \otimes Z) \\ &= Pf \otimes T_z T_w^*(g^* \otimes e^{-\beta/2} \cdot y_1 \otimes Z \otimes \cdots \otimes Z \otimes y_\ell \otimes Z). \end{aligned}$$

If $n > \ell$ then this is 0, so assume $n \leq \ell$. For the computation below, we first observe that for $f, g, h \in F$ and $x, y \in X$,

$$\begin{aligned} \sum_j \langle f^* \otimes x \otimes g \mid h^* \otimes y \otimes u_j\rangle_B \cdot u_j^* &= \sum_j \langle g \mid \langle x \mid {}_A\langle f \mid h\rangle \cdot y\rangle_A \cdot u_j\rangle_B \cdot u_j^* \\ &= g^* \cdot \langle x \mid {}_A\langle f \mid h\rangle \cdot y\rangle_A. \end{aligned}$$

Using this at the third equality, we see that

$$\begin{aligned} &T_w^*(g^* \otimes e^{-\beta/2} \cdot y_1 \otimes Z \otimes \cdots \otimes Z \otimes y_\ell \otimes Z) \\ &\quad= \sum_{j,t} \Big\langle \bigotimes_{i=m+1}^{n+m}(f_i^* \otimes x_i \otimes g_i) \,\Big|\, g^* \otimes e^{-\beta/2} \cdot y_1 \otimes u_j \otimes u_j^* \otimes e^{-\beta/2} \cdot y_2 \otimes \cdots \\ &\qquad \otimes Z \otimes y_n \otimes u_t \Big\rangle_B \cdot u_t^* \otimes e^{-\beta/2} \cdot y_{n+1} \otimes Z \otimes \cdots \otimes y_\ell \otimes Z \\ &\quad= \sum_{j,t} \Big\langle \bigotimes_{i=m+2}^{n+m}(f_i^* \otimes x_i \otimes g_i) \,\Big|\, \langle f_{m+1}^* \otimes x_{m+1} \otimes g_{m+1} \mid g^* \otimes e^{-\beta/2} \cdot y_1 \otimes u_j\rangle_B \cdot u_j^* \otimes e^{-\beta/2} \cdot y_2 \\ &\qquad \otimes \cdots \otimes Z \otimes y_n \otimes u_t \Big\rangle_B \cdot u_t^* \otimes e^{-\beta/2} \cdot y_{n+1} \otimes Z \otimes \cdots \otimes y_\ell \otimes Z \\ &\quad= \sum_{t} \Big\langle \bigotimes_{i=m+2}^{n+m}(f_i^* \otimes x_i \otimes g_i) \,\Big|\, g_{m+1}^* \cdot \langle x_{m+1} \mid {}_A\langle f_{m+1} \mid g\rangle e^{-\beta/2} \cdot y_1\rangle_A \otimes y_2 \otimes \cdots \\ &\qquad \otimes Z \otimes y_n \otimes u_t \Big\rangle_B \cdot u_t^* \otimes e^{-\beta/2} \cdot y_{n+1} \otimes Z \otimes \cdots \otimes y_\ell \otimes Z. \end{aligned}$$

Let $a_1 = \langle x_{m+1} \mid {}_A\langle f_{m+1} \mid g\rangle e^{-\beta/2} \cdot y_1\rangle_A$ and for $1 < k \leq n$ inductively define

$$\begin{aligned} a_k &= \big\langle x_{m+k} \mid {}_A\langle f_{m+k} \mid g_{m+k-1}\rangle a_{k-1} e^{-\beta/2} \cdot y_k \big\rangle_A \\ &= \big\langle e^{-\beta/2}{}_A\langle g_{m+k-1} \mid f_{m+k}\rangle \cdot x_{m+k} \mid a_{k-1} \cdot y_k \big\rangle_A. \end{aligned}$$

An inductive argument shows that

$$\begin{aligned} &T_w^*(g^* \otimes e^{-\beta/2} \cdot y_1 \otimes Z \otimes \cdots \otimes Z \otimes y_\ell \otimes Z) \\ &\quad= g_{n+m}^* \cdot a_n \otimes e^{-\beta/2} \cdot y_{n+1} \otimes Z \otimes \cdots \otimes y_\ell \otimes Z. \end{aligned}$$

Observe that $a_k = T^*_{e^{-\beta/2}{}_A\langle g_{m+k-1}|f_{m+k}\rangle \cdot x_{m+k}} a_{k-1} \cdot y_k$, where $T^*_{e^{-\beta/2}{}_A\langle g_{m+k-1}|f_{m+k}\rangle \cdot x_{m+k}} \in \mathcal{T}_X$. Let

$$\eta \coloneqq e^{-\beta/2}{}_A\langle g \mid f_{m+1}\rangle \cdot x_{m+1} \otimes e^{-\beta/2}{}_A\langle g_{m+1} \mid f_{m+2}\rangle \cdot x_{m+2} \otimes \cdots \otimes e^{-\beta/2}{}_A\langle g_{m+n-1} \mid f_{m+n}\rangle \cdot x_{m+n}.$$

Proceeding inductively, we find that $a_n = T_\eta^*(y_1 \otimes \cdots \otimes y_n)$.

Returning to the original calculation, we have

$$\begin{aligned}
&P(f \otimes T_z T_w^* \otimes g^*)PWy \\
&\quad = Pf \otimes T_z(g_{n+m}^* \cdot a_n \otimes e^{-\beta/2} \cdot y_{n+1} \otimes Z \otimes y_{n+2} \otimes Z \otimes \cdots \otimes Z \otimes y_\ell \otimes Z) \\
&\quad = Pf \otimes f_1^* \otimes x_1 \otimes g_1^* \otimes \cdots \otimes f_m^* \otimes x_m \otimes g_m^* \otimes g_{n+m}^* \cdot a_n \\
&\qquad \otimes e^{-\beta/2} \cdot y_{n+1} \otimes Z \otimes y_{n+2} \otimes Z \otimes \cdots \otimes Z \otimes y_\ell \otimes Z \\
&\quad = P_0(f \otimes f_1^*) \otimes x_1 \otimes P_0(g_1 \otimes f_2^*) \otimes x_2 \otimes \cdots \otimes P_0(g_{m-1} \otimes f_m^*) \otimes x_m \otimes P_0(g_m \otimes g_{n+m}^*) \cdot a_n \\
&\qquad \otimes e^{-\beta/2} \cdot y_{n+1} \otimes Z \otimes y_{n+2} \otimes Z \otimes \cdots \otimes Z \otimes y_\ell \otimes Z.
\end{aligned}$$

Recall that $P_0(f \otimes f_1^*) = e^{-\beta/2}{}_A\langle f \mid f_1\rangle Z$ and let

$$\xi := e^{-\beta/2}{}_A\langle f \mid f_1\rangle \cdot x_1 \otimes e^{-\beta/2}{}_A\langle g_1 \mid f_2\rangle \cdot x_2 \otimes \cdots \otimes e^{-\beta/2}{}_A\langle g_{m-1} \mid f_m\rangle \cdot x_m.$$

Then $P(f \otimes T_z T_w^* \otimes g^*)PWy = W(T_\xi e^{-\beta}{}_A\langle g_m \mid g_{n+m}\rangle T_\eta^* y)$, so

$$\Phi_P(f \otimes T_z T_w^* \otimes g^*) = T_\xi e^{-\beta}{}_A\langle g_m \mid g_{n+m}\rangle T_\eta^* = T_\xi P_0(g_m \otimes g_{n+m}^*) T_\eta^*.$$

With $\alpha\colon \mathcal{T}_X \to \mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$ as in Lemma 5.16 and $x \in X_A$, the preceding calculation shows that

$$\Phi_P(\alpha(T_x)) = W^* P \sum_{i,j} u_i \otimes T_{u_i^* \otimes e^{-\beta/2} x \otimes u_j} \otimes u_j^* PW = T_x.$$

Extending the argument to elementary tensors $x, y \in \mathrm{Fock}_X$ we find that $\Phi_P(\alpha(T_x T_y^*)) = T_x T_y^*$ from which it follows that $\Phi_P \circ \alpha = \mathrm{Id}_{\mathcal{T}_X}$. Tomiyama's Theorem [BO08, Theorem 1.5.10] implies that Φ_P is a conditional expectation

If F is a Morita equivalence, Corollary 5.14 implies that $P = \mathrm{Id}$. Consequently, Φ_P defines an isomorphism between $\mathrm{End}^0_{\mathcal{T}_{F^*XF}}(F \otimes_B \mathcal{T}_{F^*XF})$ and $\mathcal{T}_X$. □

Proposition 5.18 shows that conjugating a C^*-correspondence ${}_AX_A$ by a bi-Hilbertian bimodule ${}_AF_B$ of finite index yields a conditional expectation $\Phi_P\colon F \otimes_B \mathcal{T}_{F^*XF} \otimes_B F^* \to \mathcal{T}_X$.

Now suppose that ${}_AX_A$ is a regular C^*-correspondence. If ${}_AF_B$ is regular, then so is the B–B-correspondence F^*XF. Since $\mathrm{End}^0_B(\mathrm{Fock}_{F^*XF})$ is a B-ideal in $\mathcal{T}_{F^*XF}$, Lemma 2.12 implies that

$$\frac{F \otimes_B \mathcal{T}_{F^*XF} \otimes_B F^*}{F \otimes_B \mathrm{End}^0_B(\mathrm{Fock}_{F^*XF}) \otimes_B F^*} \cong F \otimes_B \mathcal{O}_{F^*XF} \otimes_B F^*.$$

In order to apply Lemma 3.24 to Φ_P we require the following lemma.

Lemma 5.19. *Let $\Phi_P\colon F \otimes_B \mathcal{T}_{F^*XF} \otimes_B F^* \to \mathcal{T}_X$ be the expectation of Proposition 5.18. Then $\Phi_P(F \otimes_B \mathrm{End}^0_B(\mathrm{Fock}_{F^*XF}) \otimes_B F^*) \subseteq \mathrm{End}^0_A(\mathrm{Fock}_X)$.*

Proof. For $f, f_1, f_2, g, g_1, g_2 \in F$ and $x_1, x_2 \in X$ we have

$$\begin{aligned}
&\Phi_P(f \otimes (f_1^* \otimes x_1 \otimes g_1) \otimes (f_2^* \otimes x_2 \otimes g_2^*)^* \otimes g^*) \\
&\quad = e^{-\beta}{}_A\langle f \mid f_1\rangle \cdot x_1 \cdot e^{-\beta}{}_A\langle g_1 \mid g_2\rangle \otimes x_2^* \cdot e^{-\beta}{}_A\langle f_2 \mid g\rangle,
\end{aligned}$$

which is a rank-1 operator on X_A. The result follows. □

Theorem 5.20. *Let $(\phi, {}_AX_A)$ be a regular C^*-correspondence and let ${}_AF_B$ be a regular bi-Hilbertian bimodule. Then,*

(i) *the expectation Φ_P of Proposition 5.18 descends to a strict completely positive map $\widetilde{\Phi}_P\colon F \otimes_B \mathcal{O}_{F^*XF} \otimes_B F^* \to \mathcal{O}_X$;*

(ii) *Lemma 3.25 implies that the associated KSGNS correspondence satisfies*

$$(\pi_{\widetilde{\Phi}_P}, \mathrm{End}^0_{\mathcal{O}_{F^*XF}}(F \otimes_B \mathcal{O}_{F^*XF}) \otimes_{\widetilde{\Phi}_P} \mathcal{O}_X) \cong (\mathrm{Id}, L^2_{\mathcal{O}_X}(\mathrm{End}^0_{\mathcal{O}_{F^*XF}}(F \otimes_B \mathcal{O}_{F^*XF}), \widetilde{\Phi}_P)).$$

Proof. Together, Lemma 5.19 and Lemma 3.24 show that Φ_P descends to a strict completely positive map $\widetilde{\Phi}_P \colon F \otimes_B \mathcal{O}_{F^*XF} \otimes_B F^* \to \mathcal{O}_X$. □

Since $\mathcal{O}_{F^*XF} \otimes_B F^*$ induces a Morita equivalence between $\mathcal{O}_{F^*XF}$ and $F \otimes \mathcal{O}_{F^*XF} \otimes_B F^*$ we can also produce an $\mathcal{O}_{F^*XF}$–$\mathcal{O}_X$-correspondence.

Corollary 5.21. *Let $(\phi, {}_AX_A)$ be a regular C^*-correspondence and let ${}_AF_B$ be a regular bi-Hilbertian bimodule. Then*

$$\begin{aligned}
&(\mathcal{O}_{F^*XF} \otimes_B F^*) \otimes_{\widetilde{\Phi}_P} \mathcal{O}_X \\
&\quad := (\mathcal{O}_{F^*XF} \otimes_B F^*) \otimes_{\mathrm{End}^0_{\mathcal{O}_{F^*XF}}(F \otimes_B \mathcal{O}_{F^*XF})} (\mathrm{End}^0_{\mathcal{O}_{F^*XF}}(F \otimes_B \mathcal{O}_{F^*XF}) \otimes_{\widetilde{\Phi}_P} \mathcal{O}_X) \\
&\quad \cong (\mathcal{O}_{F^*XF} \otimes_B F^*) \otimes L^2_{\mathcal{O}_X}(\mathrm{End}^0_{\mathcal{O}_{F^*XF}}(F \otimes_B \mathcal{O}_{F^*XF}), \widetilde{\Phi}_P)
\end{aligned}$$

is a nondegenerate $\mathcal{O}_{F^*XF}$–$\mathcal{O}_X$-correspondence.

Concretely describing $\mathcal{O}_{F^*XF}$—let alone $(\mathcal{O}_{F^*XF} \otimes_B F^*) \otimes_{\widetilde{\Phi}_P} \mathcal{O}_X$—in examples tends to be an involved process. We finish with an example from covering spaces.

Example 5.22. Let M be a compact Hausdorff space with a homeomorphism $\gamma \colon M \to M$ and dual automorphism $\gamma^* \colon C(M) \to C(M)$. Then ${}_{\gamma^*}C(M)_{C(M)}$ is a Morita equivalence bimodule. For $g_1, g_2, g_3 \in C(M)$ the left and right module structures are given by

$$(g_1 \cdot g_2 \cdot g_3)(x) = g_1(\gamma(x))g_2(x)g_3(x), \quad x \in M.$$

The right inner product is the obvious one, and the left inner product is

$${}_{C(M)}\langle g_1 \mid g_2\rangle(x) = (\overline{g_1}g_2)(\gamma^{-1}(x)), \quad x \in M.$$

Suppose that $\pi \colon \widetilde{M} \to M$ is a finite-to-one covering map and let $F_{C(\widetilde{M})} := C(\widetilde{M})_{C(\widetilde{M})}$. Then ${}_{\pi^*}F_{C(\widetilde{M})}$ is a bi-Hilbertian bimodule with left $C(M)$-valued inner product given by

$${}_{C(M)}\langle f_1 \mid f_2\rangle(x) = \sum_{\widetilde{x} \in \pi^{-1}(x)} f_1(\widetilde{x})\,\overline{f_2(\widetilde{x})} \qquad f_1, f_2 \in C(\widetilde{M}).$$

We can form the bi-Hilbertian $C(\widetilde{M})$-bimodule $F^* \otimes_{\gamma^*} C(M) \otimes_{C(M)} F$ and consider the resulting noncommutative dynamics on $C(\widetilde{M})$. An energetic exercise shows that

$$F^* \otimes_{\gamma^*} C(M) \otimes_{C(M)} F \cong C(\widetilde{M}_\pi \times_{\gamma\circ\pi} \widetilde{M}) = C(\{(\widetilde{x}, \widetilde{y}) \in \widetilde{M} \times \widetilde{M} \colon \pi(\widetilde{x}) = \gamma \circ \pi(\widetilde{y})\})$$

as bi-Hilbertian $C(\widetilde{M})$-bimodules. The right inner product on $F^* \otimes_{\gamma^*} C(M) \otimes_{C(M)} F$ is given by

$$\begin{aligned}
\langle f_1^* \otimes h \otimes f_2 \mid g_1^* \otimes k \otimes g_2\rangle_{C(\widetilde{M})}(\tilde{x}) &= \Big\langle f_2 \,\Big|\, \pi^*(\langle h \mid \gamma^*\langle f_1^* \mid g_1^*\rangle_{C(M)}k\rangle_{C(M)})g_2\Big\rangle_{C(\widetilde{M})}(\widetilde{x}) \\
&= \overline{f_2(\widetilde{x})}(\langle h \mid \gamma^*\langle f_1^* \mid g_1^*\rangle_{C(M)}k\rangle_{C(M)})(\pi(\widetilde{x}))g_2(\widetilde{x}) \\
&= \overline{f_2(\widetilde{x})h(\pi(\widetilde{x}))} \sum_{\widetilde{y} \in \pi^{-1}(\gamma(\pi(\widetilde{x})))} (f_1\overline{g_1})(\widetilde{y})k(\pi(\widetilde{x}))g_2(\widetilde{x}).
\end{aligned} \tag{5.5}$$

We use the notion of topological graphs from [Kat04b]. Regarding ${}_{\gamma^*}C(M)_{C(M)}$ as the graph correspondence of the topological graph $M \overset{\gamma}{\leftarrow} M \overset{\mathrm{Id}}{\rightarrow} M$, the correspondence $F^* \otimes_{\gamma^*} C(M) \otimes_{C(M)} F$ is isomorphic to the graph correspondence of the fibre product $\widetilde{M} \overset{p_1}{\leftarrow} \widetilde{M}_\pi \times_{\gamma\circ\pi} \widetilde{M} \overset{p_2}{\rightarrow} \widetilde{M}$, where

p_1 and p_2 are the projections onto the first and second components of $\widetilde{M}_\pi \times_{\gamma\circ\pi} \widetilde{M}$. The correspondence between Cuntz–Pimsner algebras that we obtain from Corollary 5.21 is

$$\mathcal{O}_{F^*C(M)F} \otimes F^* \otimes_{\tilde{\Phi}_P} \mathcal{O}_X.$$

Treatment of the algebra $\mathcal{O}_{F^*C(M)F}$ is difficult, even when the dynamics on M is trivial, and we leave a detailed description of this algebra and the resulting correspondence to another place. △

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