

The intersection of complemented submodules is not complemented

Addendum to: The Friedrichs angle and alternating projections in Hilbert C^* -modules

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Abstract

We show by (counter)example that the intersection of complemented submodules in a Hilbert C^* -module is not necessarily complemented, answering an open question from [MR].

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The intersection of complemented submodules

Let X be a right Hilbert C^* -module over the C^* -algebra B and $M \subset X$ a closed B -submodule. The *orthogonal complement* of M is the submodule

$$M^\perp := \{x \in X : \forall m \in M \langle x, m \rangle = 0\}.$$

Recall that M is *complemented* if the map

$$M \oplus M^\perp \rightarrow X, \quad (m, x) \mapsto m + x,$$

is an isomorphism of Hilbert C^* -modules. Given two complemented submodules $M, N \subset X$, their intersection $M \cap N \subset X$ is a closed B -submodule. In [MR] the second and third named authors posed:

“To our knowledge, it is an open question whether the intersection of complemented submodules is again complemented.”

In the present note we settle this question in the negative. The counterexample below was provided by the first named author.

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Example. Consider the Hilbert C^* -module $X := C([-π/2, π/2], \mathbb{C}^2)$ over the C^* -algebra $B := C([-π/2, π/2])$. For $t \in [-π/2, π/2]$ define projection-valued functions

$$P(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q(t) = \begin{cases} \begin{pmatrix} \cos^2 t & \sin t \cos t \\ \sin t \cos t & \sin^2 t \end{pmatrix} & t \geq 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & t < 0 \end{cases}.$$

For the complemented submodules $M := \text{Im}(P)$ and $N := \text{Im}(Q)$ in X we have

$$M \cap N = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in C([-π/2, π/2]), x(t) = 0 \text{ for } t \geq 0 \right\},$$

where the behaviour at $t = 0$ comes from continuity of any x in the images of P, Q . So then continuity again says that the complement is

$$(M \cap N)^\perp = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} : y, z \in C([-π/2, π/2]), y(t) = 0 \text{ for } t \leq 0 \right\}.$$

Hence any vector $\begin{pmatrix} y \\ z \end{pmatrix} \in C([-π/2, π/2], \mathbb{C}^2)$ with $y(0) \neq 0$ is not in $(M \cap N) + (M \cap N)^\perp$. Thus $M \cap N$ is a closed submodule which is not complemented.

References

- [MR] B. Mesland, A. Rennie, *The Friedrichs angle and alternating projections in Hilbert C^* -modules*, J. Math. Anal. Appl., **516** (2022), 126474.