The Godbillon-Vey invariant in equivariant KK-theory

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Abstract

We construct a groupoid equivariant Kasparov class for transversely oriented foliations in all codimensions. In codimension 1 we show that the Chern character of an associated semifinite spectral triple recovers the Connes-Moscovici cyclic cocycle for the Godbillon-Vey secondary characteristic class.

1 Introduction

In this paper we construct a semifinite spectral triple for codimension 1 foliations whose Chern character is the cyclic cocycle, constructed by Connes and Moscovici [22], representing the Godbillon-Vey class. The construction passes through groupoid equivariant Kasparov theory, and this initial part of the construction works in all codimensions.

Associated to any foliated manifold $(M, \mathcal{F})$ of codimension q is a canonical real rank q vector bundle $N = TM/T\mathcal{F}$ called the normal bundle. One of the foundational results of the theory of foliated manifolds is *Bott's vanishing theorem*, which states that the Pontrjagin classes $p^i(N)$ of the normal bundle N must vanish for all $i > 2q$ [5]. This vanishing theorem guarantees the existence of new characteristic classes for M called *secondary characteristic classes*, which have been studied extensively [6, 8, 39]. It has been shown in particular that all such classes arise under the image of a characteristic map from the Gelfand-Fuchs cohomology of the Lie algebra of formal vector fields [28] to the cohomology of M [7, 8].

The most famous example of a secondary characteristic class is the Godbillon-Vey invariant, first discovered by Godbillon and Vey [29], which arises in the context of transversely orientable foliations and can be constructed explicitly at the level of differential forms. More specifically, transverse orientability of a codimension q foliated manifold $(M, \mathcal{F})$ amounts to the existence of a nonvanishing section of the top degree line bundle $\Lambda^q N^*$ of the conormal bundle N^* over M. Any identification of N^* with a subbundle of T^*M, obtained say by equipping M with a Riemannian metric, identifies such a section with a nonvanishing differential form $\omega \in \Omega^q(M)$ such that

$$\omega(X_1 \wedge \cdots \wedge X_q) = 0 \tag{1}$$

whenever any one of the X_j is contained in the space $\Gamma(T\mathcal{F})$ of vector fields which are tangent to the foliation. Since the subbundle $T\mathcal{F} \subset TM$ is integrable, by the Frobenius theorem one is guaranteed the existence of a 1-form $\eta \in \Omega^1(M)$ for which

$$d\omega = \eta \wedge \omega.$$

The differential form $\eta \wedge (d\eta)^q$ is closed, and its class GV in de Rham cohomology is independent of the choices of ω and η. The Godbillon-Vey invariant has been shown to be closely related to measure theory and dynamics: see [10, 26, 32, 35] for example.

Building on work of Winkelnkemper [53] which associated to any foliated manifold $(M, \mathcal{F})$ its holonomy groupoid $G_{\mathcal{F}}$, Connes [18] initiated the study of foliated manifolds as noncommutative geometries using the convolution algebra $C_c^\infty(G_{\mathcal{F}})$. Connes shows [19] that all Gelfand-Fuchs cohomology classes (hence all secondary characteristic classes) can be represented by cyclic cocycles on $C_c^\infty(G_{\mathcal{F}})$. Connes gives in particular an explicit formula for the cyclic cocycle defined by the Godbillon-Vey invariant on foliations of codimension 1. The differential form $\omega \in \Omega^1(M)$ used in the construction (1) of the Godbillon-Vey invariant can be regarded as a transverse volume form, whose Radon-Nikodym derivative with respect to holonomy transport by an element $u \in G_{\mathcal{F}}$ we denote by

$$\Delta(u) = \frac{d(u_*\omega)}{d\omega}.$$

By regarding the top degree conormal bundle as a trivial line bundle using the transverse orientation, we can regard this Radon-Nikodym derivative as a homomorphism $\Delta : G_{\mathcal{F}} \to \mathbb{R}_+^*$ into the multiplicative group of positive real numbers, and hence its logarithm $\ell = \log \circ \Delta : G_{\mathcal{F}} \to \mathbb{R}$ as an additive homomorphism. Connes shows that the formula

$$\phi_{GV}(a_0, a_1, a_2) := \int_M \int_{u_0u_1u_2 = x \in M} a_0(u_0)a_1(u_1)a_2(u_2)(\ell(u_2)d\ell(u_1) - \ell(u_1)d\ell(u_2)) \tag{2}$$

defines a cyclic 2-cocycle on $C_c^\infty(G_{\mathcal{F}})$, and that the class of this 2-cocycle coincides with that defined by the Godbillon-Vey invariant.

More recently, Connes and Moscovici have used a deep link with Hopf symmetry [24] to construct a characteristic map sending Gelfand-Fuchs cocycles to cyclic cocycles on the convolution algebra $C_c^\infty(\tilde{G}_{\mathcal{F}})$ of the groupoid $\tilde{G}_{\mathcal{F}}$ associated to the lift of $\mathcal{F}$ to the oriented frame bundle F^+N for N. Connes and Moscovici show in [22] that the formula

$$\tilde{\phi}_{GV}(a_0, a_1) := \int_{F^+N} \int_{u_0u_1 = y \in F^+N} a_0(u_0)(\delta_1 a_1)(u_1)\tilde{\omega}(y), \tag{3}$$

where δ_1 is a derivation of $C_c^\infty(\tilde{G}_{\mathcal{F}})$ related to $d\ell$ and where $\tilde{\omega}$ is a G-invariant transverse volume form on F^+N, defines a 1-cocycle on $C_c^\infty(\tilde{G}_{\mathcal{F}})$ that represents the Godbillon-Vey invariant. As will be shown in this paper, the derivation δ_1 in fact arises from a commutator of $C_c^\infty(\tilde{G}_{\mathcal{F}})$ with a dual Dirac operator on a Hilbert space of sections of an exterior algebra bundle. In noncommutative geometry, the Godbillon-Vey invariant has since been further explored in groupoid cohomology [25], cyclic cohomology [30, 31], via its pairing with the indices of longitudinal Dirac operators [45], and in relation to manifolds with boundary [46].

Accompanying his introduction of the formula (2) for the cyclic cocycle ϕ_{GV}, Connes remarks [19, Page 4] that the pairing of ϕ_{GV} with K-theory will not in general be integer-valued, which implies that ϕ_{GV} must not arise as the Chern character of a spectral triple on $C_c^\infty(G_{\mathcal{F}})$. Such constraints do not apply to *semifinite spectral triples*, whose pairings with K-theory need not lie in the integers, [21, 3, 13].

In this paper we will recover the formula (3) from a semifinite spectral triple. Bearing in mind the close relationship between semifinite spectral triples and KK-theory [38], this fact can be seen already in the specific case of the codimension 1 Godbillon-Vey invariant using the formalism of differential forms on jet bundles arising from Gelfand-Fuchs cohomology [22, Proposition 19]. An entirely novel nuance of our constructions, however, is the fact that they rely *only* on the intrinsic dynamics of the holonomy groupoid, and at no point invoke the Gelfand-Fuchs machinery that has been traditionally used. This has the advantage of potentially admitting generalisation to arenas where Gelfand-Fuchs technology either is not available, as is the case for singular foliations, or will not yield spectral triples and so cannot be used to calculate index formulae, as is the case when the Gelfand-Fuchs map to differential forms on jet bundles does not yield volume forms on these bundles.

We now outline the layout of the paper. Section 1 will discuss the background required on Clifford bundles, groupoid actions, semifinite spectral triples and groupoid equivariant KK-theory. Section 2 will detail the constructions of the KK-classes required. The constructions of this section are very natural for foliations of arbitrary codimension, so will be carried out at this level of generality. Section 3 will consist of the proof of an index theorem in codimension 1 which states that the pairing with K-theory of the semifinite spectral triple obtained using the constructions of Section 2 coincides with the pairing coming from the Connes-Moscovici Godbillon-Vey cyclic cocycle. We remark that while the spectral triple itself can be easily constructed for foliations of arbitrary codimension, it is at this stage unclear whether the corresponding index pairing continues to compute the pairing of the higher codimension Godbillon-Vey invariant with K-theory. We leave this question to future work.

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2 Background

Here we recall some basic facts about groupoid actions on spaces, Clifford algebras, semifinite spectral triples, groupoid actions on algebras and the resulting equivariant Kasparov theory.

We will assume that the reader is familiar with locally compact groupoids and their associated convolution algebras [18, 50]. All Hilbert spaces are assumed to be separable. For such a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the bounded operators on $\mathcal{H}$ and by $\mathcal{K}(\mathcal{H})$ the compact operators on $\mathcal{H}$. Inner products on Hilbert modules and Hilbert spaces are assumed to be conjugate-linear in the left variable and linear in the right.

If X, Y and Z are sets with maps $f : Y \to X$ and $g : Z \to X$, we denote by $Y \times_{f,g} Z$ the fibered product $\{(y, z) \in Y \times Z : f(y) = g(z)\}$ of Y and Z.

2.1 Clifford algebras

For our constructions we will need some facts regarding Clifford algebras and their representations on exterior algebra bundles. First, if $(V, \langle\cdot,\cdot\rangle)$ is a real inner product space with nondegenerate inner product, we denote by $\mathbb{C}\mathrm{liff}(V)$ the *complex Clifford algebra* of V, which is the complexification of the real Clifford algebra $\mathrm{Cliff}(V, \langle\cdot,\cdot\rangle)$.

There exists a linear isomorphism $\psi_V : \Lambda^*V \to \mathrm{Cliff}(V, \langle\cdot,\cdot\rangle)$ between the exterior algebra and the Clifford algebra of V defined with respect to any orthonormal basis $\{e_1, \ldots, e_{\mathrm{rank}(V)}\}$ by

$$\psi_V(e_{i_1} \wedge \cdots \wedge e_{i_r}) := e_{i_1} \cdot \cdots \cdot e_{i_r}$$

for any multi-index $(i_1, \ldots, i_r)$ with $r \leq \mathrm{rank}(V)$. The isomorphism ψ_V determines the structure of a Clifford bimodule on $\Lambda^*(V)$, with left action given by

$$c_L(a)w := \psi_V^{-1}(a \cdot \psi_V(w))$$

and right action given by

$$c_R(a)w := \psi_V^{-1}(\psi_V(w) \cdot a)$$

for $a \in \mathbb{C}\mathrm{liff}(V)$ and $w \in \Lambda^*(V)$. We have the following important lemma describing how these representations behave with respect to orthogonal maps.

Lemma 2.1. *Let V and W be finite dimensional inner product spaces and let $\psi_V : \Lambda^*V \to \mathrm{Cliff}(V)$, $\psi_W : \Lambda^*W \to \mathrm{Cliff}(W)$ be the corresponding linear isomorphisms. Then if $A : V \to W$ is an orthogonal transformation with induced algebra isomorphisms $A_\Lambda : \Lambda^*V \to \Lambda^*W$ and $A_{\mathrm{Cliff}} : \mathrm{Cliff}(V) \to \mathrm{Cliff}(W)$, we have*

$$A_{\mathrm{Cliff}} \circ \psi_V = \psi_W \circ A_\Lambda.$$

Proof. Regard V as a subspace of Λ^*V in the usual way, let $\iota : V \to \mathrm{Cliff}(V)$ denote the inclusion map, and consider the map $j := (\psi_W \circ A_\Lambda)|_V : V \to \mathrm{Cliff}(W)$. Since A is orthogonal, we have $j(v)^2 = \|v\|^2 1_{\mathrm{Cliff}(W)}$ and so by the universal property of the Clifford algebra, there is a unique algebra isomorphism $\phi : \mathrm{Cliff}(V) \to \mathrm{Cliff}(W)$ such that $\phi \circ \iota = j$. Given any vector $v \in V$ we see that

$$j(v) = A_{\mathrm{Cliff}} \circ \iota(v)$$

so that $\phi = A_{\mathrm{Cliff}}$. Given an orthonormal basis $\{e_1, \ldots, e_{\dim(V)}\}$ for V, and a multi-index $(i_1, \ldots, i_k)$ we calculate

$$\begin{aligned} A_{\mathrm{Cliff}} \circ \psi_V(e_{i_1} \wedge \cdots \wedge e_{i_k}) &= A_{\mathrm{Cliff}}(\iota(e_{i_1}) \cdots \iota(e_{i_k})) \\ &= A_{\mathrm{Cliff}}(\iota(e_{i_1})) \cdots A_{\mathrm{Cliff}}(\iota(e_{i_k})) \\ &= \psi_W(A_\Lambda(e_{i_1})) \wedge \cdots \wedge \psi_W(A_\Lambda(e_{i_k})) \\ &= \psi_W \circ A_\Lambda(e_{i_1} \wedge \cdots \wedge e_{i_k}), \end{aligned}$$

where the first line is due to the equality $\psi_V|_V = \iota$, and the second is since A_{Cliff} is an algebra homomorphism. By linearity we obtain the required identity. □

By abuse of notation, we have a linear isomorphism $\psi_V : \Lambda^*(V) \otimes \mathbb{C} \to \mathbb{C}\mathrm{liff}(V)$, which gives, by the same formulae as in the real case, commuting actions c_L and c_R of $\mathbb{C}\mathrm{liff}(V)$ on $\Lambda^*(V) \otimes \mathbb{C}$. Any orthogonal map $A : V \to W$ of inner product spaces has the property that the induced maps $A_{\mathbb{C}\mathrm{liff}} : \mathbb{C}\mathrm{liff}(V) \to \mathbb{C}\mathrm{liff}(W)$ and $A_{\Lambda_{\mathbb{C}}} : \Lambda^*(V) \otimes \mathbb{C} \to \Lambda^*(W) \otimes \mathbb{C}$ satisfy $A_{\mathbb{C}\mathrm{liff}} \circ \psi_V = \psi_W \circ A_{\Lambda_{\mathbb{C}}}$.

If Y is a manifold and $E \to Y$ is a Euclidean vector bundle, we obtain a corresponding Clifford algebra bundle $\mathrm{Cliff}(E)$ and exterior bundle $\Lambda^*(E)$, as well as corresponding complexifications $\mathbb{C}\mathrm{liff}(E) = \mathrm{Cliff}(E) \otimes \mathbb{C}$ and $\Lambda^*(E) \otimes \mathbb{C}$. Operating pointwise, we have an isomorphism $\psi_E : \Lambda^*(E) \otimes \mathbb{C} \to \mathbb{C}\mathrm{liff}(E)$ of vector spaces giving $\Lambda^*(E) \otimes \mathbb{C}$ the structure of a $\mathbb{C}\mathrm{liff}(E)$-bimodule, with left and right actions denoted, again by abuse of notation, by c_L and c_R respectively. We will denote by $\mathbb{C}\ell(E)$ the continuous sections vanishing at infinity of the bundle $\mathbb{C}\mathrm{liff}(E)$ over Y. This $\mathbb{C}\ell(E)$ is a C^*-algebra and is $\mathbb{Z}_2$-graded by even and odd elements.

2.2 G-spaces and G-bundles

Let G be a groupoid, with unit space X and range and source maps $r : G \to X$ and $s : G \to X$ respectively. We say that G *acts on (the left of) a set* Y or that Y *is a G-space* if there exists a map $a : Y \to X$ called the *anchor map* and a map $m : G \times_{s,a} Y \to Y$, denoted $m(u, y) := u \cdot y$, such that

1. $a(u \cdot y) = r(u)$ for all $(u, y) \in G \times_{s,a} Y$,
2. $(uv) \cdot y = u \cdot (v \cdot y)$ for all $(v, y) \in G \times_{s,a} Y$ and $(u, v) \in G^{(2)}$,
3. $a(y) \cdot y = y$ for all $y \in Y$.

If G and Y are topological (resp. smooth) spaces we require the maps a and m to be continuous (resp. smooth). The simplest example of a G-space is the unit space X of G.

If G acts on Y, we denote by $Y \rtimes G$ the space $Y \times_{a,r} G$, regarded as a groupoid whose unit space is Y, with range and source maps $r(y,u) := y$ and $s(y,u) := u^{-1} \cdot y$ respectively, and with multiplication defined by

$$(y,u) \cdot (u^{-1} \cdot y, v) := (y, uv)$$

for all $(y,u) \in Y \times_{a,r} G$ and $(u,v) \in G^{(2)}$. If G and Y are topological (resp.) smooth spaces, the groupoid $Y \rtimes G$ is equipped with a topological (resp. smooth) structure from its containment as a subspace of the topological (resp. smooth) space $Y \times G$. While for left G-spaces it is more natural to consider the analogous (and isomorphic) groupoid $G \ltimes Y$ obtained from the set $G \times_{s,a} Y$, it will be easier for our purposes to use $Y \rtimes G$ because, as we will see, our convention in using G-equivariant Kasparov theory consists in forming pullbacks using the range map rather than the source.

We say that a vector bundle $\pi : E \to X$ is *G-equivariant* if E is a G-space, with G-action conventionally denoted $(u,e) \mapsto u_* e$ and with anchor map π, and if for each $u \in G$ the map $(u,e) \mapsto u_* e$ defined on $E_{s(u)} := \pi^{-1}\{s(u)\}$ is a vector space isomorphism $E_{s(u)} \to E_{r(u)}$. More generally, if $\pi : E \to Y$ is a vector bundle over a G-space Y, we say that E is *G-equivariant* if it is $Y \rtimes G$-equivariant as a bundle over Y, in which case we will often denote the map $(Y \rtimes G) \times_{s,\pi} E \to E$, $((y,u),e) \mapsto (y,u)_* e$, by simply $(u,e) \mapsto u_* e$. If $\pi : E \to X$ admits a Euclidean (resp. Hermitian) structure, we say that E is a *G-equivariant Euclidean (resp. Hermitian) bundle* if for all $(y,u) \in Y \rtimes G$ the linear isomorphism $E_{u^{-1}\cdot y} \to E_y$ defined by $(u,e) \mapsto u_* e$ is orthogonal (resp. unitary).

If $\pi : E \to Y$ is a G-equivariant vector bundle over Y, then by functoriality $\Lambda^*(E) \otimes \mathbb{C}$ is also an equivariant bundle over Y, with action of $u \in G$ denoted by $u_* : \Lambda^*(E|_{Y_{s(u)}}) \otimes \mathbb{C} \to \Lambda^*(E|_{Y_{r(u)}}) \otimes \mathbb{C}$. If moreover E is an equivariant Euclidean bundle, then by functoriality $\mathbb{C}\mathrm{liff}(E)$ is also an equivariant bundle, with action of $u \in G$ denoted by $u_\diamond : \mathbb{C}\mathrm{liff}(E|_{Y_{s(u)}}) \to \mathbb{C}\mathrm{liff}(E|_{Y_{r(u)}})$. In this case, by Lemma 2.1 we have

$$u_*(c_L(a)e) = c_L(u_\diamond a)(u_* e) \tag{4}$$

and

$$u_*(c_R(a)e) = c_R(u_\diamond a)(u_* e) \tag{5}$$

for all $u \in G$, $a \in \mathbb{C}\mathrm{liff}(E|_{Y_{s(u)}})$ and $e \in \Lambda^*(E|_{Y_{s(u)}})$.

When $(M, \mathcal{F})$ is a foliated manifold with holonomy groupoid G, the normal bundle $N = TM/T\mathcal{F} \to M$ is a G-equivariant bundle. As this fact is fundamental for our constructions, let us briefly review why it is the case. We assume a countable covering of M by foliated charts $\phi_i : U_i \cong T_i \times P_i$, where $T_i \subset \mathbb{R}^q$ and $P_i \subset \mathbb{R}^p$ are open balls, with change-of-chart maps $\varphi_{i,j} := \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ of the form

$$\varphi_{i,j}(t,p) = (h_{i,j}(t), \tilde{\varphi}_{i,j}(t,p)),$$

such that the $h_{i,j}$ are *compatible* in the sense that they satisfy

$$h_{i,k} = h_{i,j} \circ h_{j,k}$$

whenever $U_i \cap U_j \cap U_k \neq \emptyset$. That such a covering can be chosen can be regarded as the definition of the foliation $\mathcal{F}$ on M [9, Chapter 1.2]. We say that a path $\gamma : [0,1] \to M$ is *leafwise* if its image is entirely contained in a leaf L of M, and we refer to its endpoints $\gamma(0)$ and $\gamma(1)$ as its *source* and *range*, denoted $s(\gamma)$ and $r(\gamma)$ respectively. Any leafwise path γ whose image is contained in a union $U_0 \cup U_1$ of charts such that $U_0 \cap U_1 \neq \emptyset$, and with $s(\gamma) \in U_0$ and $r(\gamma) \in U_1$, determines a local diffeomorphism $h_\gamma := h_{0,1}$ on a small neighbourhood of $T_0 \subset \mathbb{R}^q$. More generally, if the image of a leafwise path γ is covered by a chain of charts $\{U_0, \ldots, U_k\}$ such that for each $0 \leq j < k$ we have $U_j \cap U_{j+1} \neq \emptyset$, on a sufficiently small neighbourhood of T_0 we may define a local diffeomorphism

$$h_\gamma := h_{k,k-1} \circ h_{k-1,k-2} \circ \cdots \circ h_{1,0}$$

mapping onto a small neighbourhood of T_k. Because of the compatibility of the $h_{i,j}$, the germ of h_γ at $s(\gamma)$ does not depend on the chain of charts chosen in its definition. By definition, the holonomy groupoid G consists of equivalence classes of leafwise paths γ for which $\gamma_1 \sim \gamma_2$ if and only if γ_1 and γ_2 have the same source and range and the germ at $s(\gamma_1) = s(\gamma_2)$ of h_{γ_1} is equal to that of h_{γ_2}.

In the coordinates defined by a chart U_j, the fibres of N identify with tangent vectors to the transversal neighbourhood T_j, and via this identification it follows that for any leafwise path γ in M, the derivative of h_γ furnishes a linear isomorphism

$$dh_\gamma : N_{s(\gamma)} \to N_{r(\gamma)}.$$

It can be seen from the definition of h_γ that $dh_{\gamma_1} \circ dh_{\gamma_2} = dh_{\gamma_1\gamma_2}$ whenever the range of γ_2 is equal to the source of γ_1, where $\gamma_1\gamma_2$ is the path obtained by concatenating γ_1 and γ_2. Since local diffeomorphisms with the same germ at a point have the same derivative at that point, to any $u \in G$ corresponds a well-defined linear isomorphism $u_* := h_\gamma : N_{s(u)} \to N_{s(u)}$ for any path γ that represents u. Since $dh_{\gamma_1\gamma_2} = dh_{\gamma_1} \circ dh_{\gamma_2}$, we have $(uv)_* = u_* \circ v_*$ for all $(u, v) \in G^{(2)}$, and so N is indeed a G-equivariant bundle over M.

We remark that in general the normal bundle N of a foliated manifold $(M, \mathcal{F})$ will not admit the structure of a G-equivariant Euclidean bundle. Indeed, the existence of a G-equivariant Euclidean structure for N implies the existence of a G-invariant transverse volume form ω for $(M, \mathcal{F})$, and hence implies the existence of a faithful normal semifinite trace on the von Neumann algebra of $(M, \mathcal{F})$ defined by restricting functions in the weakly dense algebra $C_c(G)$ to M, and then integrating with respect to ω. If the Godbillon-Vey invariant of $(M, \mathcal{F})$ is nonzero, however, then by results of Hurder and Katok [36, Theorem 2] and, in codimension 1, Connes [19, Theorem 7.14], the von Neumann algebra of $(M, \mathcal{F})$ contains a type III factor and so admits no nonzero semifinite normal traces. Examples of foliated manifolds with nonzero Godbillon-Vey invariant are known to be plentiful [51].

2.3 Equivariant KK-theory for locally Hausdorff groupoids

Equivariant KK-theory for Hausdorff topological groupoids was first developed by Le Gall [44]. Since foliated manifolds generally have only locally Hausdorff holonomy groupoids, Le Gall's treatment requires extension for applications to foliation theory. Androulidakis and Skandalis [1] have developed an equivariant KK-theory for the holonomy groupoids arising from singular foliations, whose topologies are generally even worse than the locally Hausdorff topologies on the holonomy groupoids of regular foliations, and which include all regular foliation groupoids as a subclass.

This section will summarise the required results and definitions of Androulidakis and Skandalis in the setting of locally Hausdorff Lie groupoids, as well as giving the unbounded picture in parallel with work of Pierrot [48]. See also Muhly and Williams [47] and Tu [52] for useful perspectives on non-Hausdorff groupoid actions which have further informed the exposition.

Let G be a locally Hausdorff Lie groupoid with locally compact Hausdorff unit space X, and let $\{U_i\}_{i \in I}$ is a countable cover of G by Hausdorff open sets. For each $i \in I$ we let $r_i := r|_{U_i}$ and $s_i := s|_{U_i}$ be the restrictions of range and source respectively to the set U_i.

Definition 2.2. *A $C_0(X)$-**algebra** is a C^*-algebra A together with a homomorphism $\theta : C_0(X) \to \mathcal{M}(A)$ into the multiplier algebra of A such that $\theta(C_0(X))A = A$. For $a \in A$ and $f \in C_0(X)$, we will often denote $\theta(f)a$ by $f \cdot a$.*

*For $x \in X$, the **fibre over** x is the algebra $A_x := A/I_xA$, where I_x is the kernel of the evaluation functional $C_0(X) \ni f \mapsto f(x)$ on $C_0(X)$.*

*If A and B are $C_0(X)$-algebras, a homomorphism $\phi : A \to B$ is said to be a $C_0(X)$-**homomorphism** if $\phi(f \cdot a) = f \cdot \phi(a)$ for all $f \in C_0(X)$ and $a \in A$. Such a homomorphism induces a family $\phi_x : A_x \to B_x$ of homomorphisms between the fibres.*

The simplest nontrivial example of a $C_0(X)$-algebra is $C_0(Y)$, where Y is a locally compact Hausdorff space equipped with a continuous map $p : Y \to X$. The $C_0(X)$-structure of $C_0(Y)$ is given by $\theta(f)g(y) := f(p(y))g(y)$ for all $f \in C_0(X)$ and $g \in C_0(Y)$, and the fibre over $x \in X$ is $C_0(Y)_x = C_0(Y_x)$, where $Y_x := p^{-1}\{x\}$.

Definition 2.3. *Let A be a $C_0(X)$-algebra, and let $p : Y \to X$ be a continuous map of locally compact Hausdorff spaces. Then the* ***pullback*** *of A by p is the $C_0(Y)$-algebra $p^*A := C_0(Y) \otimes_{p,C_0(X)} A$, where we take the balanced tensor product by regarding the $C_0(X)$-algebras $C_0(Y)$ and A as $C_0(X)$-modules. If there is no ambiguity about the map p, it will often be omitted from the notation, so that $p^*A = C_0(Y) \otimes_{C_0(X)} A$.*

It is easy to check that if A is a $C_0(X)$-algebra and $p : Y \to X$ is a continuous map of locally compact Hausdorff spaces, the fibre over $y \in Y$ of p^*A is $A_{p(y)}$. Equipped with the notion of pullbacks, we can define what is meant by a G-algebra.

Definition 2.4. *Let A be a $C_0(X)$-algebra. A* ***G-action*** *on A is a family $\alpha = \{\alpha^i : s_i^*A \to r_i^*A\}_{i\in I}$ of grading-preserving $C_0(U_i)$-isomorphisms, such that $\alpha^i|_{s|^*_{U_i\cap U_j}A} = \alpha^j|_{s|^*_{U_i\cap U_j}A}$ for all $i, j \in I$, and such that the induced homomorphisms $\alpha_u : A_{s(u)} \to A_{r(u)}$ satisfy $\alpha_{uv} = \alpha_u \circ \alpha_v$. If A admits a G-action α, we call (A, α) a* ***G-algebra****.*

The simplest nontrivial example of a G-algebra is $C_0(Y)$, where Y is a G-space with anchor map $p : Y \to X$, and where $C_0(Y)$ is equipped with the G-action

$$\alpha_u(f)(y) := f(u^{-1} \cdot y)$$

for all $u \in G$ and $f \in C_0(Y_{r(u)})$.

Now suppose that E is a Hilbert module over a G-algebra A. For $x \in X$, we can consider the fibre $E_x := E \otimes_A A_x$, which is a Hilbert A_x-module, and if $p : Y \to X$ is a continuous map of locally compact Hausdorff spaces, we can consider the pullback $p^*E := E \otimes_A p^*A$, which is a Hilbert p^*A-module. If T is an A-linear operator on E, we let $p^*T := T \otimes 1_{p^*A}$ be its pullback to a p^*A-linear operator on p^*E.

Definition 2.5. *Let (A, α) be a G-algebra, and let E be a Hilbert A-module. A* ***G-action*** *on E consists of a family $W = \{W^i : s_i^*E \to r_i^*E\}_{i\in I}$ of grading-preserving isometric Banach space isomorphisms, such that $W^i|_{s|^*_{U_i\cap U_j}E} = W^j|_{s|^*_{U_i\cap U_j}E}$ for all $i, j \in I$, and such that the induced isomorphisms $W_u : E_{s(u)} \to E_{r(u)}$ on the fibres satisfy $W_{uv} = W_u \circ W_v$, $\langle W_u\rho_1, W_u\rho_2\rangle_{r(u)} = \alpha_u(\langle\rho_1, \rho_2\rangle_{s(u)})$ and $W_u(\rho\cdot a) = W_u(\rho)\cdot\alpha_u(a)$ for all $(u, v) \in G^{(2)}$, $a \in A_{s(u)}$ and $\rho, \rho_1, \rho_2 \in E_{s(u)}$. If E admits a G-action W, we call (E, W) a* ***G-Hilbert A-module****.*

If $V \to Y$ is a G-equivariant Hermitian vector bundle over a G-space Y, then the continuous sections vanishing at infinity $\Gamma_0(Y; V)$ of V over Y is a G-Hilbert $C_0(Y)$-module, with pointwise inner product and right action by $C_0(Y)$, and with G-action defined by

$$(W_u\rho)(y) := u_*\rho(u^{-1} \cdot y) \tag{6}$$

for all $\rho \in \Gamma_0(Y_{r(u)}; V|_{Y_{r(u)}})$. All G-Hilbert module constructions in this paper will arise from some variant of the action (6).

Definition 2.6. *If B is a G-algebra, and $\pi : A \to \mathcal{L}(E)$ is a representation of a G-algebra (A, α) on a G-Hilbert B-module (E, W), we say that π is* ***equivariant*** *if for all $i \in I$ we have*

$$\mathrm{Ad}_{W^i}(\pi_i^s(a)) = \pi_i^r(\alpha^i(a))$$

for all $a \in A$. Here $\pi_i^s := 1_{C_b(U_i)} \otimes \pi$ and $\pi_i^r := 1_{C_b(U_i)} \otimes \pi$ are respectively the induced homomorphisms $s_i^*A = C_0(U_i) \otimes_{s,C_0(X)} A \to \mathcal{L}(s_i^*E)$ and $r_i^*A = C_0(U_i) \otimes_{r,C_0(X)} A \to \mathcal{L}(r_i^*E)$.

The definition of the equivariant KK-groups now follows in the usual way.

Definition 2.7. *Let (A, α) and (B, β) be G-C^*-algebras. A G**-equivariant Kasparov** A-B-**module** is a triple $(A, {}_\pi E_B, F)$, where (E, W) is a G-equivariant Hilbert B-module carrying an equivariant representation $\pi : A \to \mathcal{L}(E)$, and where $F \in \mathcal{L}(E)$ is homogeneous of degree 1 such that for all $a \in A$ one has*

1. $\pi(a)(F - F^*) \in \mathcal{K}(E)$,
2. $\pi(a)(F^2 - 1) \in \mathcal{K}(E)$,
3. $[F, \pi(a)] \in \mathcal{K}(E)$,

and such that for all $i \in I$

4. $\pi_i^r(r_i^*(a))(r_i^* F - W^i \circ s_i^* F \circ (W^i)^{-1}) \in r_i^* \mathcal{K}(E)$.

We say that two G-equivariant Kasparov A-B-modules $(A, {}_\pi E_B, F)$ and $(A, {}_{\pi'} E'_B, F')$ are ***unitarily equivalent*** *if there exists a G-equivariant unitary $V : E \to E'$ of degree 0 such that $VFV^* = F'$ and $V\pi(a)V^* = \pi'(a)$ for all $a \in A$. We denote by $\mathbb{E}^G(A, B)$ the set of all unitary equivalence classes of G-equivariant Kasparov A-B-modules.*

A ***homotopy*** *in $\mathbb{E}^G(A, B)$ is an element of $\mathbb{E}^G(A, B[0, 1])$, and we define $KK^G(A, B)$ to be the set of homotopy equivalence classes in $\mathbb{E}^G(A, B)$.*

The direct sum of G-equivariant Kasparov A-B-modules makes $KK^G(A, B)$ into an abelian group.

We also need *unbounded representatives* of equivariant KK-classes. The definition for such representatives is the natural extension of that due to Pierrot [48] to the locally Hausdorff case. We remark here that if $\mathcal{A}$ is a dense $*$-subalgebra of a $C_0(X)$-algebra A, then we will assume that $C_0(X) \cdot \mathcal{A} \subset \mathcal{A}$, which will be true in our examples. We will denote by $\mathcal{A}_x := \mathcal{A} / I_x \mathcal{A}$ the fibre over $x \in X$, where as before I_x is the kernel of the evaluation functional $f \mapsto f(x)$ on $C_0(X)$.

Definition 2.8. *Let A and B be G-algebras. An* ***unbounded** G**-equivariant Kasparov** A-B-**module*** *is a triple $(\mathcal{A}, {}_\pi E, D)$, where (E, W) is a G-Hilbert B-module carrying an equivariant representation π of A in $\mathcal{L}(E)$, D is a densely defined, odd, unbounded, self adjoint and regular operator on E commuting with the right action of B, and where $\mathcal{A}$ is a dense $*$-subalgebra of A preserved by the action of G such that for all $a \in \mathcal{A}$ one has:*

1. $\pi(a) \operatorname{dom}(D) \subset \operatorname{dom}(D)$,
2. $[D, \pi(a)]$ *extends to an element of* $\mathcal{L}(E)$,
3. $\pi(a)(1 + D^2)^{-\frac{1}{2}} \in \mathcal{K}(E)$,

and such that for all $i \in I$, $a \in \mathcal{A}$ and $f \in C_c(U_i)$ one has

4. $f \cdot \pi_i^r(r_i^*(a)) \cdot (r_i^* D - W^i \circ s_i^* D \circ (W^i)^{-1})$ *extends to an element of* $\mathcal{L}(r_i^* E)$ *and*
5. $\operatorname{dom}((r_i^* D)f) = W^i \operatorname{dom}((s_i^* D)f)$.

That all unbounded equivariant Kasparov modules define classes in KK^G is an easy consequence of the corresponding result by Pierrot for Hausdorff groupoids.

Proposition 2.9. *Let A and B be G-algebras, and let $(\mathcal{A}, {}_\pi E, D)$ be an unbounded G-equivariant Kasparov A-B-module. Then $(A, {}_\pi E, D(1 + D^2)^{-\frac{1}{2}})$ is a G-equivariant Kasparov A-B-module.*

Proof. That the first three requirements of Definition 2.7 are met by $(A, {}_\pi E, D(1+D^2)^{-\frac{1}{2}})$ is a consequence of the corresponding result in the nonequivariant case [2]. That the fourth requirement is met is a consequence of restricting the corresponding result of Pierrot [48, Théorème 6] to each of the Hausdorff open subsets U_i of G. □

We now come to the descent map in equivariant KK-theory, for which we need to discuss groupoid crossed products. We will assume for this that G comes equipped with a bundle $\Omega^{\frac{1}{2}} \to G$ of leafwise half-densities, as in [20, Chapter 2.8]. Regard a $C_0(X)$-algebra A as the continuous sections vanishing vanishing at infinity $\Gamma_0(X;\mathfrak{A})$ of the upper-semicontinuous bundle $\mathfrak{A} \to X$ of C^*-algebras whose fibre over $x \in X$ is A_x [44, 47]. Thus a G-algebra (A,α) can be regarded as the continuous sections vanishing at infinity of the G-space $\mathfrak{A}$ over X, where $\alpha_u : A_{s(u)} \to A_{r(u)}$ determines the action of G on the bundle $\mathfrak{A}$.

Define $\Gamma_c(G; r^*\mathfrak{A}\otimes\Omega^{\frac{1}{2}})$ to be the space of finite linear combinations of sections of the bundle $r^*\mathfrak{A}\otimes\Omega^{\frac{1}{2}} \to G$ which have compact support and are continuous in one of the U_i. The space $\Gamma_c(G; r^*\mathfrak{A}\otimes\Omega^{\frac{1}{2}})$ is a $*$-algebra equipped with the convolution product

$$(f * g)_u := \int_{v\in G^{r(u)}} f_v\alpha_v(g_{v^{-1}u}) \quad \text{and with involution} \quad (f^*)_u := \alpha_u((f_{u^{-1}})^*).$$

The appropriate completion of $\Gamma_c(G; r^*\mathfrak{A}\otimes\Omega^{\frac{1}{2}})$ to a reduced C^*-algebra $A \rtimes_r G$ has been given in [42, Section 3.7].

In a similar manner, if A is a G-algebra we can regard any G-Hilbert A-module E as the continuous sections vanishing at infinity of an upper-semicontinuous bundle $\mathfrak{E} \to X$ whose fibre over $x \in X$ is E_x. We define $\Gamma_c(G; r^*\mathfrak{E}\otimes\Omega^{\frac{1}{2}})$ to be the space of finite linear combinations of sections of the bundle $r^*\mathfrak{E}\otimes\Omega^{\frac{1}{2}} \to G$ that have compact support and are continuous in one of the U_i. The formulae

$$\langle\rho^1,\rho^2\rangle^G_u := \int_{v\in G^{r(u)}} \alpha_v\langle\rho^1_{v^{-1}},\rho^2_{v^{-1}u}\rangle \quad \text{and} \quad (\rho\cdot f)_u := \int_{v\in G^{r(u)}} \rho_v\alpha_v(f_{v^{-1}u})$$

defined for $\rho^1,\rho^2,\rho \in \Gamma_c(G; r^*\mathfrak{E}\otimes\Omega^{\frac{1}{2}})$ and $f \in \Gamma_c(G; r^*\mathfrak{A}\otimes\Omega^{\frac{1}{2}})$ determine an $A \rtimes_r G$-valued inner product and right action respectively on $\Gamma_c(G; r^*\mathfrak{E}\otimes\Omega^{\frac{1}{2}})$, and we may complete in the norm arising from $\langle\cdot,\cdot\rangle^G$ to obtain a Hilbert $A\rtimes_r G$-module which we denote by $E \rtimes_r G$. If T is an A-linear operator on E, we denote by $\mathfrak{dom}(T)$ the bundle over X whose fibre over $x \in X$ is $\mathrm{dom}(T)\otimes_A A_x$. Then as in [48, Définition 2, Proposition 3] we define $r^*(T)$ on $\Gamma_c(G; r^*\mathfrak{dom}(T)\otimes\Omega^{\frac{1}{2}})$ by

$$(r^*(T)\rho)_u := T_{r(u)}\rho_u.$$

If $T \in \mathcal{L}(E)$ one can use the norm of T to bound that of $r^*(T)$, and then one can use T^* to show that $r^*(T) \in \mathcal{L}(E \rtimes_r G)$.

Lemma 2.10. *For any densely defined A-linear operator $T : \mathrm{dom}(T) \to E$, we have $r^*(T^*) \subset r^*(T)^*$. Moreover $\overline{r^*(T^*)} = r^*(T)^*$.*

Proof. Fix $\xi \in \mathrm{dom}(r^*(T^*)) = \Gamma_c(G; r^*\mathfrak{dom}(T^*)\otimes\Omega^{\frac{1}{2}})$, and assume without loss of generality that ξ has compact support in some Hausdorff open subset U_i of G. For each $u \in G$, use the fact that $\xi_u \in \mathfrak{dom}(T^*)_{r(u)}\otimes\Omega^{\frac{1}{2}}_u$ to define a section η of $r^*\mathfrak{E}\otimes\Omega^{\frac{1}{2}} \to G$ by

$$\eta_u := T^*_{r(u)}\xi_u.$$

Since ξ is continuous with compact support in U_i so too is η, thus $\eta \in \Gamma_c(G, r^*\mathfrak{E}\otimes\Omega^{\frac{1}{2}})$. For any $\rho \in \mathrm{dom}(r^*(T)) = \Gamma_c(G; r^*\mathfrak{dom}(T)\otimes\Omega^{\frac{1}{2}})$ we can then calculate

$$\langle\xi, r^*(T)\rho\rangle^G_u = \int_{v\in G^{r(u)}} \alpha_v(\langle\xi_{v^{-1}}, T_{s(v)}\rho_{v^{-1}u}\rangle) = \int_{v\in G^{r(u)}} \alpha_v(\langle T^*_{s(v)}\xi_{v^{-1}}, \rho_{v^{-1}u}\rangle) = \langle\eta,\rho\rangle^G_u$$

for all $u \in G$, so that $\xi \in \mathrm{dom}(r^*(T)^*)$. The above calculation also shows that $r^*(T)^*\xi = \eta = r^*(T^*)\xi$, so that we indeed have $r^*(T^*) \subset r^*(T)^*$.

Fix $\xi \in \mathrm{dom}(r^*(T)^*)$. We show that $\xi \in \overline{r^*(T^*)}$. Let $\{\xi^n\}_{n\in\mathbb{N}} \subset \Gamma_c(G; r^*\mathfrak{dom}(T^*) \otimes \Omega^{\frac{1}{2}})$ be a sequence converging in $E \rtimes_r G$ to ξ. Then the sequence $\{\langle \xi^n, r^*(T)\rho\rangle^G\}_{n\in\mathbb{N}}$ of elements of $\Gamma_c(G; r^*\mathfrak{A} \otimes \Omega^{\frac{1}{2}})$ defined for $u \in G$ by

$$\langle \xi^n, r^*(T)\rho\rangle^G_u = \int_{v\in G^{r(u)}} \alpha_v(\langle \xi^n_{v^{-1}}, T_{s(v)}\rho_{v^{-1}u}\rangle) = \int_{v\in G^{r(u)}} \alpha_v(\langle T^*_{s(v)}\xi^n_{v^{-1}}, \rho_{v^{-1}u}\rangle) \tag{7}$$

converges in $A \rtimes_r G$ for all $\rho \in \Gamma_c(G; r^*\mathfrak{dom}(T) \otimes \Omega^{\frac{1}{2}})$. For each $v \in G^{r(u)}$ one can on the right hand side of (7) take bump functions ρ with support of decreasing radius about $v^{-1}u$ to show that we have convergence of $\{(r^*(T^*)\xi^n)_{v^{-1}} = T^*_{s(v)}\xi^n_{v^{-1}}\}_{n\in\mathbb{N}}$ to an element of $E_{s(v)}$, and doing this for all $v \in G^{r(u)}$ and all $u \in G$ shows that in fact $\{r^*(T^*)\xi^n\}_{n\in\mathbb{N}}$ converges in $E \rtimes_r G$, implying that $\xi^n \to \xi$ in the graph norm on $\mathrm{dom}(r^*(T^*))$ as claimed. $\square$

Finally, we observe that if A and B are G-algebras, and if (E, W) is a G-Hilbert B-module with an equivariant representation $\pi : A \to \mathcal{L}(E)$, then the formula

$$((\pi \rtimes_r G)(f)\rho)_u := \int_{v\in G^{r(u)}} \pi(f_v)W_v(\rho_{v^{-1}u})$$

defined for $f \in \Gamma_c(G; r^*\mathfrak{A} \otimes \Omega^{\frac{1}{2}})$ and $\rho \in \Gamma_c(G; r^*\mathfrak{E} \otimes \Omega^{\frac{1}{2}})$ determines a representation $\pi \rtimes_r G : A \rtimes_r G \to \mathcal{L}(E \rtimes_r G)$.

Proposition 2.11. *Let A and B be G-algebras, and let $(\mathcal{A}, {}_\pi E, D)$ be a G-equivariant unbounded Kasparov A-B-module. Let $\widetilde{\mathcal{A}}$ denote the bundle of $*$-algebras over X whose fibre over $x \in X$ is $\mathcal{A}_x$. Then*

$$(\Gamma_c(G; r^*\widetilde{\mathcal{A}} \otimes \Omega^{\frac{1}{2}}), {}_{\pi\rtimes_r G}E \rtimes_r G, r^*(D))$$

is an unbounded Kasparov $A \rtimes_r G$-$B \rtimes_r G$-module.

Proof. Since D is odd for the grading of E, $r^*(D)$ is odd for the induced grading of $E \rtimes_r G$. Symmetry of D gives symmetry of $r^*(D)$, so without loss of generality we may assume that $r^*(D)$ is closed. Self adjointness of $r^*(D)$ is then a consequence of the self adjointness of D together with Lemma 2.10.

Regularity of $r^*(D)$ is a consequence of that of D. Indeed, for any $\rho \in \Gamma_c(G; r^*\mathfrak{dom}(D) \otimes \Omega^{\frac{1}{2}})$ we have

$$((1 + r^*(D)^2)\rho)_u = (1_{r(u)} + D^2_{r(u)})\rho_u.$$

Hence the range of the operator $(1 + r^*(D)^2)$ when restricted to $\Gamma_c(G; r^*\mathfrak{dom}(D) \otimes \Omega^{\frac{1}{2}})$ is $\Gamma_c(G; r^*\mathfrak{range}(1 + D^2) \otimes \Omega^{\frac{1}{2}})$, where $\mathfrak{range}(1 + D^2)$ denotes the bundle over X whose fibre over $x \in X$ is $\mathrm{range}(1 + D^2) \otimes_A A_x$, which by regularity of D is dense in $E_x = E \otimes_A A_x$. Thus the range of $(1 + r^*(D)^2)$ contains the dense subspace $\Gamma_c(G; r^*\mathfrak{range}(1 + D^2) \otimes \Omega^{\frac{1}{2}})$ of $E \rtimes_r G$, and it follows that $r^*(D)$ is regular.

Regarding commutators, a simple calculation tells us that

$$([r^*(D), (\pi \rtimes_r G)(f)]\rho)_u = \int_{v\in G^{r(u)}} \pi(f_v)\big(D_{r(v)} - W_v \circ D_{s(v)} \circ W_{v^{-1}}\big)(W_v\rho_{v^{-1}u})$$

for all $\rho \in \Gamma_c(G; r^*\mathfrak{dom}(T) \otimes \Omega^{\frac{1}{2}})$, so Property 4 in Definition 2.8 tells us that $[r^*(D), (\pi \rtimes_r G)(f)]$ is bounded.

The only thing that remains to check is compactness of $(\pi \rtimes_r G)(f)(1 + r^*(D)^2)^{-\frac{1}{2}}$ for $f \in \Gamma_c(G; r^*\tilde{\mathcal{A}} \otimes \Omega^{\frac{1}{2}})$. For any $\rho \in \Gamma_c(G; r^*\mathfrak{E} \otimes \Omega^{\frac{1}{2}})$ the definition of $r^*(D)$ gives

$$\begin{aligned}((1 + r^*(D)^2)^{-\frac{1}{2}}(\pi \rtimes_r G)(f^*)\rho)_u =&(1 + D^2_{r(u)})^{-\frac{1}{2}} \int_{v\in G^{r(u)}} \pi((f)^*_v)W_v(\rho_{v^{-1}u}) \\ =& \int_{v\in G^{r(u)}} (1 + D^2_{r(v)})^{-\frac{1}{2}}\pi((f)^*_v)W_v(\rho_{v^{-1}u}),\end{aligned}$$

and since $(1 + D^2_{r(v)})^{-\frac{1}{2}}\pi((f)^*_v) \in \mathcal{K}(E)_{r(v)}$ for all $v \in G^{r(u)}$ by Property 3 in Definition 2.8, it follows that $(1 + r^*(D)^2)^{-\frac{1}{2}}(\pi \rtimes_r G)(f^*)$ is an element of $\Gamma_c(G; r^*\mathcal{K}(E) \otimes \Omega^{\frac{1}{2}})$. A similar argument to the one used in [41, Page 172] then tells us that $(1 + r^*(D))^{-\frac{1}{2}}(\pi \rtimes_r G)(f^*)$ can be approximated by finite rank operators on $E \rtimes_r G$ so is an element of $\mathcal{K}(E \rtimes_r G)$, and hence so too is its adjoint $(\pi \rtimes_r G)(f)(1 + r^*(D)^2)^{-\frac{1}{2}}$. $\square$

Let us remark finally that if Y is a locally compact Hausdorff G-space, with corresponding bundle $C_0(\mathfrak{Y}) \to X$ whose fibre over $x \in X$ is $C_0(Y_x)$, then we have an inclusion $\Gamma_c(Y \rtimes G; \Omega^{\frac{1}{2}}) \ni f \mapsto \tilde{f} \in \Gamma_c(G; r^*C_0(\mathfrak{Y}) \otimes \Omega^{\frac{1}{2}})$ defined by

$$\tilde{f}_u(y) := f(y, u).$$

For ease of notation we will usually just refer to $\tilde{f}$ as f. By density of $C_c(Y_x)$ in $C_0(Y_x)$ for each $x \in X$, this subalgebra $\Gamma_c(Y \rtimes G; \Omega^{\frac{1}{2}})$ is dense in $C_0(Y) \rtimes_r G$. We will use this fact in the construction of our Godbillon-Vey spectral triple.

2.4 Semifinite spectral triples

One of the defining features of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is that the operators $a(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ are contained in the compact operators $\mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{A}$. These compact operators come equipped with a trace Tr, which is used to measure the rank of projections that appear in the definition of the index, and subsequent index formulae [23, 34].

Semifinite spectral triples are a generalisation of spectral triples for which the rank of projections is measured by a different trace. More precisely we require a faithful normal semifinite trace τ on a semifinite von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$. We denote by $\mathcal{K}_\tau(\mathcal{N})$ the norm-closed ideal in $\mathcal{N}$ generated by projections of finite τ -trace, and refer to $\mathcal{K}_\tau(\mathcal{N})$ as the ideal of *τ-compact operators*, [27].

Definition 2.12. *Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra, regarded as an algebra of operators on a Hilbert space $\mathcal{H}$. A* ***semifinite spectral triple relative to*** *$(\mathcal{N}, \tau)$ is a triple $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ consisting of a $*$-algebra $\mathcal{A}$ represented in $\mathcal{N}$ by $\pi : \mathcal{A} \to \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, and a densely defined, unbounded, self adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that*

1. *$\pi(a)\operatorname{dom}(\mathcal{D}) \subset \operatorname{dom}(\mathcal{D})$ so that $[\mathcal{D}, \pi(a)]$ is densely defined, and moreover extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$,*
2. *$\pi(a)(1 + \mathcal{D}^2)^{-\frac{1}{2}} \in \mathcal{K}_\tau(\mathcal{N})$ for all $a \in \mathcal{A}$.*

We say that $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ is ***even*** *if $\mathcal{A}$ is even and $\mathcal{D}$ is odd for some $\mathbb{Z}_2$-grading on $\mathcal{H}$, and otherwise we call $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$* ***odd****.*

Connes' original notion of spectral triple defines a subclass of semifinite spectral triples, for which $(\mathcal{N}, \tau) = (\mathcal{B}(\mathcal{H}), \mathrm{Tr})$. Just as the bounded transform of a spectral triple $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ defines a Fredholm module (over the C^*-completion A of $\mathcal{A}$), and hence a class in $KK_*(A, \mathbb{C})$, semifinite spectral triples have a close relationship with KK-theory.

To see this, we first suppose that B is a C^*-algebra, X_B is a Hilbert B-module with inner product $\langle \cdot, \cdot \rangle_B$, and τ is a faithful norm lower semicontinuous semifinite trace on B. We can form the GNS space $L^2(B, \tau)$, or $L^2(X, \tau)$ with inner product $(x|y) = \tau(\langle x, y \rangle_B)$. These two Hilbert spaces are related by $X \otimes_B L^2(B, \tau) \cong L^2(X, \tau)$.

Then by results in [43], we obtain a faithful normal semifinite trace Tr_τ, called the *dual trace*, on the weak closure $\mathcal{N} = \mathrm{End}_B(X)'' \subset \mathcal{B}(L^2(X_B, \tau))$ of the adjointable B-linear operators on X_B. The functional Tr_τ satisfies

$$\mathrm{Tr}_\tau(\Theta_{\xi,\eta}) := \tau(\langle \eta, \xi \rangle_B).$$

Proposition 2.13. *Let $(\mathcal{A}, {}_\pi X_B, \mathcal{D})$ be an even (resp. odd) unbounded Kasparov A-B module, and suppose that τ is a faithful norm lower semicontinuous semifinite trace on B. Let $(\mathcal{N}, \mathrm{Tr}_\tau)$ be the semifinite von Neumann algebra obtained from X_B and τ as above. Then (with a slight abuse of notation)*

$$(\mathcal{A}, {}_{\pi \hat{\otimes} 1} X_B \hat{\otimes}_B L^2(B, \tau), \mathcal{D} \hat{\otimes} 1) = (\mathcal{A}, {}_\pi L^2(X_B, \tau), \mathcal{D})$$

is an even (resp. odd) semifinite spectral triple relative to $(\mathcal{N}, \mathrm{Tr}_\tau)$.

Proof. Clearly $\mathcal{A} \subset \mathcal{N}$, and the commutant of $\mathcal{N}$ is just B''. Since $\mathcal{D}$ is B-linear, every unitary in B'' preserves the domain of $\mathcal{D} \hat{\otimes} 1$, whence $\mathcal{D} \hat{\otimes} 1$ is affiliated to $\mathcal{N}$. That $[\mathcal{D} \hat{\otimes} 1, \pi(a) \hat{\otimes} 1]$ is bounded for all $a \in \mathcal{A}$ is a consequence of the corresponding fact for the Kasparov module $(\mathcal{A}, {}_\pi X_B, \mathcal{D})$, and that $(\pi(a) \hat{\otimes} 1)(1 + \mathcal{D} \hat{\otimes} 1^2)^{-\frac{1}{2}}$ is τ-compact is true because the algebra $\mathcal{K}(X_B)$ is contained in $\mathcal{K}_\tau(\mathcal{N})$ by construction. $\square$

In fact, a converse to Proposition 2.13 is also true: namely, every semifinite spectral triple can be factorised into a KK-class and a trace [38]. Although we will not need this converse result, it provides a useful way of thinking about semifinite spectral triples.

One of the most useful features of (nice) spectral triples is that their pairing with K-theory can be computed using the local index formula, [23]. The same is true for (nice) semifinite spectral triples. There are now numerous results generalising the Connes-Moscovici local index formula for spectral triples to semifinite spectral triples [3, 14, 15, 16, 17, 11, 12].

3 Construction of the Kasparov modules

In this section, $(M, \mathcal{F})$ will denote a transversely orientable foliated manifold of codimension q, with holonomy groupoid G and normal bundle $N = TM/T\mathcal{F} \to M$. The normal bundle is a G-equivariant vector bundle, as explained at the end of Section 2.2, and for $u \in G$ we let $u_* : N_{s(u)} \to N_{r(u)}$ be the corresponding map $n \mapsto u_* n$. We assume G to be equipped with a countable cover $\mathcal{U} := \{U_i\}_{i \in I}$ by Hausdorff open subsets. We do not assume K-orientability at any point, working with exterior algebra bundles instead of spinor bundles.

The first of the two constructions, the Connes fibration, will not feature in the index theorem in the final section. The Kasparov module of the Connes fibration provides a Thom-type isomorphism which does not conceptually affect our final index formulae. We include the Connes fibration for the sake of completeness, and to show that the whole construction does indeed factor through groupoid equivariant KK-theory.

3.1 The Connes fibration

We begin this section with a revision of a construction due to Connes [19]. Connes starts with an oriented manifold M of dimension n with an action of a discrete group Γ of orientation-preserving diffeomorphisms. Such a setting provides an étale model of the transverse geometry of a transversely oriented foliation.

Connes shows that if $W \to M$ denotes the "bundle of Euclidean metrics" for the tangent bundle TM over M, then one can construct a dual Dirac class in $KK^{\Gamma}_{\frac{n(n+1)}{2}}(C_0(M), C_0(W))$. The manifold W has the advantage that the pullback of TM to W admits a Γ-invariant Euclidean metric, even though one need not exist on M in general. We show show that Connes' construction can be carried out directly in the groupoid equivariant setting, as it may be useful for future work in constructing the Godbillon-Vey invariant as a semifinite spectral triple in arbitrary codimension.

We let $\pi_F : F^+N \to M$ be the principal $GL^+(q,\mathbb{R})$-bundle of positively oriented frames for the vector bundle $N \to M$, whose fibre $(F^+N)_x$ over $x \in M$ consists of positively oriented linear isomorphisms $\phi : \mathbb{R}^q \to N_x$. Then F^+N is a G-space with anchor map $\pi_F : F^+N \to M$ and action defined by

$$u \cdot \phi := u_* \circ \phi : \mathbb{R}^q \to N_{r(u)} \tag{8}$$

for $\phi : \mathbb{R}^q \to N_{s(u)}$ in $(F^+N)_{s(u)}$. Observe that this action of G commutes with the right action of $GL^+(q,\mathbb{R})$ on the principal $GL^+(q,\mathbb{R})$-bundle $F^+N \to M$.

The vertical subbundle $\ker(d\pi_F) = VF^+N \to F^+N$ of TF^+N admits a trivialisation $VF^+N \to F^+N \times \mathfrak{gl}(q,\mathbb{R})$, where $\mathfrak{gl}(q,\mathbb{R}) = M_q(\mathbb{R})$ is the Lie algebra of $GL^+(q,\mathbb{R})$ consisting of all $q \times q$ real matrices. The trivialisation is given by the formula

$$F^+N \times \mathfrak{gl}(q,\mathbb{R}) \ni (\phi, v) \mapsto v_\phi := \frac{d}{dt}(\phi \cdot \exp(tv))\bigg|_{t=0} \in VF^+N.$$

For $u \in G$, the differential $u_* : VF^+N_{s(u)} \to VF^+N_{r(u)}$ of $u\cdot : F^+N_{s(u)} \to F^+N_{r(u)}$ in the fibres defines on VF^+N the structure of a G-equivariant vector bundle. Since the left action of G commutes with the right action of $GL^+(q,\mathbb{R})$, one has

$$u_* v_\phi = \frac{d}{dt}(u \cdot (\phi \cdot \exp(tv))\bigg|_{t=0} = \frac{d}{dt}((u \cdot \phi) \cdot \exp(tv))\bigg|_{t=0} = v_{u\cdot\phi} \tag{9}$$

for all $\phi \in (F^+N)_{s(u)}$, and so with respect to the trivialisation $F^+N \times \mathfrak{gl}(q,\mathbb{R})$ of VF^+N we have

$$u_*(\phi, v) = (u \cdot \phi, v). \tag{10}$$

for all $\phi \in F^+N$ and $v \in \mathfrak{gl}(q,\mathbb{R})$.

Consider now the quotient $CN := F^+N/SO(q,\mathbb{R})$ of F^+N by the right action of $SO(q,\mathbb{R})$. The projection $\pi_F : F^+N \to M$ descends to a projection $\pi_C : CN \to M$, which defines a fibre bundle with typical fibre $S_q^+ := GL^+(q,\mathbb{R})/SO(q,\mathbb{R})$, the space of positive definite, symmetric $q \times q$ matrices. Moreover, since the action of G on F^+N commutes with the right action of $SO(q,\mathbb{R})$, it follows that CN is a G-space with anchor map $\pi_C : CN \to M$, and with action of $u \in G$ given by

$$u \cdot [\phi] := [u \cdot \phi] = [u_* \circ \phi] \tag{11}$$

for all $[\phi] \in CN_{s(u)}$. Following [4, 54], we refer to $\pi_C : CN \to M$ as the Connes fibration.

Definition 3.1. *The fibre bundle $\pi_C : CN \to M$ is a G-space called the **Connes fibration** for the normal bundle N.*

Let us consider the geometry of the fibres of $CN \to M$. Since $SO(q,\mathbb{R})$ is compact, the pair $(GL^+(q,\mathbb{R}), SO(q,\mathbb{R}))$ is a Riemannian symmetric pair and hence the space S_q^+ can be equipped with a $GL^+(q,\mathbb{R})$-invariant metric under which it is by [33, Proposition 3.4] a globally symmetric Riemannian space. The Riemannian space S_q^+ is moreover of noncompact type, so by [33, Theorem 3.1] has everywhere non-positive sectional curvature. We can find a locally finite open cover $\mathcal{U}$ of M by sets U for which the vertical bundle $VCN|_U \cong U \times TS_q$, and then choosing a partition of unity subordinate to $\mathcal{U}$ allows us to equip the bundle $VCN \to CN$ with a Euclidean structure. We will assume from here on that $VCN \to CN$ is equipped with a Euclidean structure in this way.

Proposition 3.2. *The bundle $VCN \to CN$ is a G-equivariant Euclidean bundle over the G-space CN. Consequently $\mathbb{C}\mathrm{liff}(VCN)$ and $\mathbb{C}\mathrm{liff}(V^*CN)$ are G-equivariant bundles.*

Proof. Fix $u \in G$ and suppose that U_s and U_r are open sets in M containing $s(u)$ and $r(u)$ respectively, such that we have local trivialisations $N|_{U_s} \cong U \times \mathbb{R}^q$ and $N|_{U_r} \cong U \times \mathbb{R}^q$, with respect to which the holonomy action $u_* : N_{s(u)} \to N_{r(u)}$ is the action on $\mathbb{R}^q$ of an element $\tilde{u} \in GL^+(q, \mathbb{R})$.

We obtain corresponding local trivialisations $F^+N|_{U_s} \cong U \times GL^+(q, \mathbb{R})$ and $F^+N|_{U_r} \cong U \times GL^+(q, \mathbb{R})$ of the local frame bundles over U_s and U_r, in which the holonomy action $u\cdot : F^+N_{s(u)} \to F^+N_{r(u)}$ is left multiplication on $GL^+(q, \mathbb{R})$ by $\tilde{u}$, and taking the quotient by $SO(q, \mathbb{R})$ we get local trivialisations $CN|_{U_s} \cong U \times S_q^+$ and $CN|_{U_r} \cong U \times S_q^+$ in which $u\cdot : CN_{s(u)} \to CN_{r(u)}$ is the isometry of $S_q^+ = GL^+(q, \mathbb{R})/SO(q, \mathbb{R})$ defined by left multiplication by $\tilde{u} \in GL^+(q, \mathbb{R})$. Thus G acts by orientation-preserving isometries between the fibres of CN, inducing an action by special orthogonal transformations on the Euclidean bundle $VCN \to CN$ of vectors tangent to the fibres of $CN \to M$, hence making $VCN \to CN$ a G-equivariant Euclidean bundle over the G-space CN. The final statement follows from functoriality of Clifford algebras with respect to orthogonal maps. $\square$

That the fibres have nonpositive sectional curvature allows us to define a dual Dirac class for CN over M in a similar manner to Connes [19]. First, let $\mathbb{C}\ell(V^*CN)$ be equipped with the G-structure arising from the action of G on the equivariant bundle $\mathbb{C}\mathrm{liff}(V^*CN)$ over the G-space CN, denoted for $u \in G$ by $u_\diamond : \mathbb{C}\mathrm{liff}(V^*_{[\phi]}CN) \to \mathbb{C}\mathrm{liff}(V^*_{u\cdot[\phi]}CN)$ for all $[\phi] \in CN_{s(u)}$. That is, we define for any $u \in G$ an isomorphism $\alpha_u^1 : \mathbb{C}\ell(V^*CN|_{CN_{s(u)}}) \to \mathbb{C}\ell(V^*CN|_{CN_{s(u)}})$ by

$$\alpha_u^1(a)([\phi]) := u_\diamond a(u^{-1} \cdot [\phi]) \tag{12}$$

for all $[\phi] \in CN_{r(u)}$. Also let

$$E^1 := \Lambda^*(V^*CN) \otimes \mathbb{C}$$

be the complexified exterior algebra bundle of the bundle of vertical covectors V^*CN over CN. Here we equip V^*CN with the Euclidean structure coming from its dual VCN, which determines a Hermitian structure on $V^*CN \otimes \mathbb{C}$ and hence on E^1. Observe that

$$X_{E^1} := \Gamma_0(CN; E^1)$$

is a Hilbert $\mathbb{C}\ell(V^*CN)$-module under the inner product

$$\langle \rho^1, \rho^2 \rangle_{\mathbb{C}\ell(V^*CN)}([\phi]) := \psi_{V^*CN}(\rho^1([\phi]))\psi_{V^*CN}(\rho^2([\phi]))$$

and right action

$$(\rho \cdot a)([\phi]) := c_R(a([\phi]))\rho([\phi]),$$

where c_R is the right action of $\mathbb{C}\mathrm{liff}(V^*CN)$ on the Clifford bimodule E^1.

The isometric action of G on the Euclidean bundle VCN over CN gives rise to a unitary action of G on E^1, denoted for each $u \in G$ by $u_* : E^1_{[\phi]} \to E^1_{u\cdot[\phi]}$ for all $[\phi] \in CN_{s(u)}$, and hence determines an isomorphism $W_u^1 : \Gamma_0(CN_{s(u)}; E^1|_{CN_{s(u)}}) \to \Gamma_0(CN_{r(u)}; E^1|_{CN_{r(u)}})$ of Banach spaces given by the formula

$$(W_u^1\rho)([\phi]) := u_*\rho(u^{-1} \cdot [\phi])$$

for all $[\phi] \in CN_{r(u)}$. A routine calculation using Lemma 2.1 shows that

$$\langle W_u^1\rho^1, W_u^1\rho^2 \rangle_{\mathbb{C}\ell(V^*CN)} = \alpha_u^1(\langle \rho^1, \rho^2 \rangle_{\mathbb{C}\ell(V^*CN)}),$$

so (X_{E^1}, W^1) is a G-equivariant Hilbert $\mathbb{C}\ell(V^*CN)$-module.

Choose now a Euclidean metric for N. Such a choice is determined by a section $\sigma : M \to CN$ of $\pi_C : CN \to M$. For $[\phi_1], [\phi_2]$ in the same fibre CN_x, denote by $h([\phi_1], [\phi_2])$ the geodesic distance between $[\phi_1]$ and $[\phi_2]$ in the fibre, and then for any $[\phi_0] \in CN$ let $h^{[\phi_0]} : CN \to \mathbb{R}$ be the function

$$h^{[\phi_0]}([\phi]) := h([\phi_0], [\phi]).$$

In particular, for $x \in M$ and $[\phi] \in CN_x$, $h^{\sigma(x)}([\phi])$ gives the distance in the fibre between $[\phi]$ and the section σ. Consider now the vertical 1-form

$$Z_{[\phi]} := h^{\sigma(\pi_C([\phi]))}([\phi]) dh^{\sigma(\pi_C([\phi]))}_{[\phi]},$$

where d denotes the exterior derivative in the fibre. Define an operator B_1 on the dense submodule $X^c_{E^1} := \Gamma_c(CN; E^1)$ of X_{E^1} by the formula

$$(B_1\rho)([\phi]) := c_L(Z_{[\phi]})\rho([\phi]),$$

where c_L is the left representation of $\mathbb{C}\mathrm{liff}(V^*CN)$ on the Clifford bimodule E^1. Since c_L and c_R commute, B_1 commutes with the right action of $\mathbb{C}\ell(V^*CN)$. Finally, we let m be the representation of $C_0(M)$ on X_{E^1} by multiplication, that is

$$(m(f)\rho)([\phi]) := f(\pi_C([\phi]))\rho([\phi])$$

for all $f \in C_0(M)$ and $\rho \in X_{E^1}$. Equivariance of the map π_C tells us that m is an equivariant representation.

Proposition 3.3. *The triple* $(C_0(M), {}_mX_{E^1}, B_1)$ *is an unbounded* G*-equivariant Kasparov* $C_0(M)$*-*$\mathbb{C}\ell(V^*CN)$*-module, hence defines a class*

$$[B_1] \in KK^G(C_0(M), \mathbb{C}\ell(V^*CN)).$$

Proof. The first thing we need to prove is that B_1 is self-adjoint and regular. Observe first that B_1 is clearly symmetric. For each $[\phi] \in CN$, the localization $(X_{E^1})_{[\phi]}$ of X_{E^1} in the sense of [49] and [37] is just the finite dimensional Hilbert space

$$\mathcal{H}_{[\phi]} := \Lambda^*(V^*_{[\phi]}CN) \otimes \mathbb{C}$$

with the inner product coming from the Hermitian structure on $\Lambda^*(V^*_{[\phi]}CN) \otimes \mathbb{C}$, and the action of the localised operator $(B_1)_{[\phi]}$ on $\mathcal{H}_{[\phi]}$ is

$$(B_1)_{[\phi]}\eta := c_L(Z_{[\phi]})\eta.$$

Since $(B_1)_{[\phi]}$ is then self-adjoint on $\mathcal{H}_{[\phi]}$, it follows from [49, Théorème 1.18] that B_1 is self-adjoint and regular.

That $m(f)(1+B_1^2)^{-\frac{1}{2}}$ is a compact operator for all $f \in C_0(M)$ follows from the definition of Clifford multiplication. Indeed, one has $c_L(Z_{[\phi]})^2 = \|Z_{[\phi]}\|^2 = h^{\sigma(\pi_C([\phi]))}([\phi])^2$ since $dh^{\sigma(\pi_C([\phi]))}_{[\phi]}$ has norm 1 for all $[\phi]$ as the dual of the tangent to the unique unit speed geodesic joining $\sigma(\pi_C([\phi]))$ to $[\phi]$, and so for any $f \in C_0(M)$, one simply has

$$(m(f)(1 + B_1^2)^{-\frac{1}{2}}\rho)([\phi]) = \frac{f(\pi_C([\phi]))}{(1 + h^{\sigma(\pi_C([\phi]))}([\phi])^2)^{\frac{1}{2}}}\rho([\phi]).$$

Since f vanishes at infinity on the base M of $CN \to M$, and since $[\phi] \mapsto (1+h^{\sigma(\pi_C([\phi]))}([\phi])^2)^{-\frac{1}{2}}$ vanishes at infinity on the fibres of $CN \to M$, the function $[\phi] \mapsto f(\pi_C([\phi]))(1+h^{\sigma(\pi_C([\phi]))}([\phi])^2)^{-\frac{1}{2}}$ is an element of $C_0(CN)$, so that $m(f)(1 + B_1^2)^{-\frac{1}{2}}$ is indeed a compact operator on the $\mathbb{C}\ell(V^*CN)$-module X_{E^1}.

Concerning commutators, it is clear that B_1 commutes with the representation m of $C_0(M)$. Thus it only remains to prove that B_1 is appropriately equivariant. The idea of this is essentially the unbounded version of analogous results by Connes [19, Lemma 5.3] and Kasparov [41, Section 5.3], but the details are somewhat technical so we give them here. Fix $u \in G$ and $\rho \in \Gamma_c(CN_{r(u)}; E^1|_{CN_{r(u)}})$. We calculate

$$\begin{aligned}(B_1 - W_u^1 B_1 W_{u^{-1}}^1)\rho([\phi]) =& c_L(Z_{[\phi]})\rho([\phi]) - u_*(B_1 W_{u^{-1}}^1\rho)(u^{-1}\cdot[\phi])\\ =& c_L(Z_{[\phi]})\rho([\phi]) - u_*(c_L(Z_{u^{-1}\cdot[\phi]})(W_{u^{-1}}^1\rho)(u^{-1}\cdot[\phi]))\\ =& c_L(Z_{[\phi]})\rho([\phi]) - u_*(c_L(Z_{u^{-1}\cdot[\phi]})(u_*^{-1}\rho([\phi])))\\ =& c_L(Z_{[\phi]} - u_*Z_{u^{-1}\cdot[\phi]})\rho([\phi])\end{aligned}$$

where on the third line we have used the identity (4). Thus we must calculate a bound for the norm of the covector $Z_{[\phi]} - u_*Z_{u^{-1}\cdot[\phi]}$.

Denote $\sigma_r := \sigma(r(u))$ and $\sigma_s := \sigma(s(u))$. With this notation, we have

$$Z_{[\phi]} - u_*Z_{u^{-1}\cdot[\phi]} = h^{\sigma_r}([\phi])dh^{\sigma_r}_{[\phi]} - u_*h^{\sigma_s}(u^{-1}\cdot[\phi])dh^{\sigma_s}_{u^{-1}\cdot[\phi]}.$$

For any vector $\gamma \in V_{[\phi]}CN$ we have

$$(u_*dh^{\sigma_s}_{u^{-1}\cdot[\phi]})(\gamma) = dh^{\sigma_s}_{u^{-1}\cdot[\phi]}(u_*^{-1}\gamma) = d(h^{\sigma_s}\circ u^{-1})_{[\phi]}(\gamma),$$

giving $u_*dh^{\sigma_s}_{u^{-1}\cdot[\phi]} = d(h^{\sigma_s}\circ u^{-1})_{[\phi]}$, and since the action of G is isometric on the fibres we get

$$(h^{\sigma_s}\circ u^{-1})([\phi]) = h(\sigma_s, u^{-1}\cdot[\phi]) = h(u\cdot\sigma_s, [\phi]) = h^{u\cdot\sigma_s}([\phi]).$$

Thus

$$u_*dh^{\sigma_s}_{u^{-1}\cdot[\phi]} = dh^{u\cdot\sigma_s}_{[\phi]}.$$

We then see that

$$\begin{aligned}h^{\sigma_r}([\phi])dh^{\sigma_r}_{[\phi]} - u_*h^{\sigma_s}(u^{-1}\cdot[\phi])dh^{\sigma_s}_{u^{-1}\cdot[\phi]} =& h^{\sigma_r}([\phi])dh^{\sigma_r}_{[\phi]} - h^{u\cdot\sigma_s}([\phi])dh^{u\cdot\sigma_s}_{[\phi]}\\ =& \frac{1}{2}d\bigg((h^{\sigma_r})^2 - (h^{u\cdot\sigma_s})^2\bigg)_{[\phi]}\\ =& \frac{1}{2}d\bigg((h^{\sigma_r} - h^{u\cdot\sigma_s})(h^{\sigma_r} + h^{u\cdot\sigma_s})\bigg)_{[\phi]}.\end{aligned}$$

By the argument [41, Lemma 5.3], we have

$$\|dh^{\sigma_r}_{[\phi]} - dh^{u\cdot\sigma_s}_{[\phi]}\| \leq 2h(\sigma_r, u\cdot\sigma_s)(h^{\sigma_r}([\phi]) + h^{u\cdot\sigma_s}([\phi]))^{-1},$$

which we use to estimate

$$\begin{aligned}\|h^{\sigma_r}([\phi])dh^{\sigma_r}_{[\phi]} - u_*h^{\sigma_s}(u^{-1}\cdot[\phi])dh^{\sigma_s}_{u^{-1}\cdot[\phi]}\|^2 \leq& \frac{1}{4}\|(dh^{\sigma_r}_{[\phi]} - dh^{u\cdot\sigma_s}_{[\phi]})(h^{\sigma_r}([\phi]) + h^{u\cdot\sigma_s}([\phi]))\|^2\\ &+ \frac{1}{4}\|(h^{\sigma_r}([\phi]) - h^{u\cdot\sigma_s}([\phi]))(dh^{\sigma_r}_{[\phi]} + dh^{u\cdot\sigma_s}_{[\phi]})\|^2\\ \leq& h(\sigma_r, u\cdot\sigma_s)^2 + (h(\sigma_r,[\phi]) - h(u\cdot\sigma_s,[\phi]))^2\\ =& h(\sigma_r, u\cdot\sigma_s)^2 + h(\sigma_r,[\phi])^2 + h(u\cdot\sigma_s,[\phi])^2\\ &- 2h(\sigma_r,[\phi])h(u\cdot\sigma_s,[\phi])\\ \leq& 2h(\sigma_r, u\cdot\sigma_s)^2,\end{aligned}$$

where the last line is a consequence of the cosine inequality for spaces of non-positive sectional curvature [33, Corollary 13.2].

Thus for all $[\phi] \in CN_{r(u)}$, we have $\|Z_{[\phi]} - u_*Z_{u^{-1}\cdot[\phi]}\|^2 \leq 2h(\sigma(r(u)), u\cdot\sigma(s(u)))^2$ independently of $[\phi] \in CN_{r(u)}$, implying that $B_1 - W_u^1 B_1 W_{u^{-1}}^1$ extends to a bounded operator on

$(X_{E^1})_{r(u)}$. Moreover $u \mapsto h(\sigma(r(u)), u\cdot\sigma(s(u)))$ is continuous hence bounded on compact Hausdorff sets, so for any element U_i of the cover $\mathcal{U} = \{U_i\}_{i\in I}$ of G by Hausdorff open subsets, and for any $\varphi \in C_c(U_i)$ and $f \in C_0(M)$ we have that

$$\varphi \cdot m_i^r(r_i^*(f)) \cdot (r_i^* B_1 - (W^1)^i \circ s_i^* B_1 \circ ((W^1)^i)^{-1}) \in \mathcal{L}(r_i^* X_{E^1}).$$

It follows that $(C_0(M), {}_m X_{E^1}, B_1)$ is an unbounded equivariant Kasparov $C_0(M)$-$\mathbb{C}\ell(V^*CN)$-module. $\square$

3.2 The foliation of the Connes fibration

Before we can construct a second Kasparov module and the semifinite spectral triple associated to it, we need a closer study of the groupoid representation theory.

Let us come back to the frame bundle $\pi_F : F^+N \to M$. This bundle is *foliated* [39, Example 1.11] in the sense that it admits a foliation $\mathcal{F}_F$ of its total space F^+N, for which the differential of the projection π_F is an isomorphism of $T\mathcal{F}_F \subset TF^+N$ onto $T\mathcal{F} \subset TM$. We may then consider the normal bundle $N_F := TF^+N/T\mathcal{F}_F$

The choice of a connection on $\pi_F : F^+N \to M$ determines in the usual way a horizontal subbundle $HF^+N \subset TF^+N$ and a direct sum decomposition $TF^+N = VF^+N \oplus HF^+N$, where $VF^+N = \ker(d\pi_F)$ is the vertical subbundle. Now, $VF^+N \cap T\mathcal{F}_F$ is the zero section, and so we find that the normal bundle to the foliation $\mathcal{F}_F$ is

$$N_F = VF^+N \oplus (HF^+N/T\mathcal{F}_F). \tag{13}$$

The normal bundle N_F is again a G-equivariant bundle, and with respect to the splitting (13) we write

$$u_* = \begin{pmatrix} \tilde{a}(u) & \tilde{c}(u) \\ 0 & \tilde{d}(u) \end{pmatrix}$$

for the action of $u \in G$ on N_F. Note that the zero appearing in the bottom left corner is a consequence of the fact that by (8), G acts via diffeomorphisms between the fibres $GL^+(q,\mathbb{R})$ of $F^+N \to M$, and so preserves the bundle $VF^+N \to M$ of vectors tangent to the fibres.

Now we are not so interested in the frame bundle F^+N as the Connes fibration CN. Since the action of G on F^+N commutes with the right action of $SO(q,\mathbb{R})$, however, we find that we also obtain a foliation on the total space of $\pi_C : CN \to M$.

To be more specific, let $Q : F^+N \to CN$ be the quotient map. Then $T\mathcal{F}_C := dQ(T\mathcal{F}_F)$ is an integrable subbundle of TCN, which determines a foliation $\mathcal{F}_C$ of CN. Since $\pi_C \circ Q = \pi_F$, we see that $d\pi_C$ maps $T\mathcal{F}_C$ isomorphically onto $T\mathcal{F}$ making $\pi_C : CN \to M$ a foliated bundle. The normal bundle N_C of $\mathcal{F}_C$ also admits a splitting

$$N_C = VCN \oplus (HCN/T\mathcal{F}_C),$$

where HCN is the isomorphic image under dQ of the horizontal subbundle $HF^+N \subset TF^+N$. For convenience, we will denote $HCN/T\mathcal{F}_C$ by simply H. Thus,

$$N_C = VCN \oplus H.$$

Now, $d\pi_C$ maps the fibres of HCN isomorphically onto those of TM, and maps the fibres of $T\mathcal{F}_C$ isomorphically onto those of $T\mathcal{F}$. It follows that $d\pi_C$ induces an isomorphism of the fibres of $H = HCN/T\mathcal{F}_C$ onto those of $N = TM/T\mathcal{F}$. We can then equip H with a Euclidean metric in the following way, due to Connes [19, Page 38].

Proposition 3.4. *For $h_1, h_2 \in H_{[\phi]}$ and with $\cdot$ denoting the Euclidean inner product in $\mathbb{R}^q$, the formula*

$$m^H_{[\phi]}(h_1, h_2) := \phi^{-1}(d\pi_C(h_1)) \cdot \phi^{-1}(d\pi_C(h_2))$$

determines a well-defined Euclidean metric on the bundle $H \to CN$.

Proof. Suppose we were to choose a different representation $\phi' = \phi \circ A$ of $[\phi]$, where A is some matrix in $SO(q, \mathbb{R})$. Then by the invariance of the Euclidean inner product under special orthogonal transformations we have

$$\begin{aligned}(\phi')^{-1}(d\pi_C(h_1)) \cdot (\phi')^{-1}(d\pi_C(h_2)) &= (A^{-1}\phi^{-1}(d\pi_C(h_1))) \cdot (A^{-1}\phi^{-1}(d\pi_C(h_2))) \\ &= \phi^{-1}(d\pi_C(h_1)) \cdot \phi^{-1}(d\pi_C(h_2)),\end{aligned}$$

giving well-definedness. That we have defined a metric follows from the linearity of the maps ϕ and $d\pi_C$, and the fact that the Euclidean inner product is a metric on $\mathbb{R}^q$. $\square$

Remarkably, holonomy translations are orthogonal with respect to this Euclidean structure of H.

Proposition 3.5. *The normal bundle $N_C \to CN$ of the foliation $\mathcal{F}_C$ of CN is a G-equivariant vector bundle over the G-space CN. Moreover, with respect to the splitting $N_C = VCN \oplus H$, for $u \in G$ and $[\phi] \in CN_{s(u)}$ the holonomy action $u_* : (N_C)_{[\phi]} \to (N_C)_{u\cdot[\phi]}$ has the form*

$$u_* = \begin{pmatrix} a(u) & c(u) \\ 0 & d(u) \end{pmatrix}, \tag{14}$$

with $a(u) : V_{[\phi]}CN \to V_{u\cdot[\phi]}CN$ and $d(u) : H_{[\phi]} \to H_{u\cdot[\phi]}$ orthogonal and orientation-preserving.

Proof. The holonomy groupoid for the foliation $\mathcal{F}_C$ of CN is precisely the groupoid $CN \rtimes G$, under which the normal bundle $N_C \to CN$ is therefore equivariant. Thus $N_C \to CN$ is a G-equivariant vector bundle over the G-space CN.

Proposition 3.2 tells us that $a(u) : V_{[\phi]}CN \to V_{u\cdot[\phi]}CN$ is orthogonal and orientation-preserving, and that the vertical bundle is preserved under holonomy translation, which accounts for the 0 appearing in the bottom left corner of (14). Since $\pi_C : CN \to M$ is the anchor map for the G-space CN it is G-equivariant, implying that the identification $d\pi_C$ of fibres of H with those of N is also G-equivariant.

That $d(u) : H_{[\phi]} \to H_{u\cdot[\phi]}$ is orientation-preserving is then a consequence of the fact that it may be identified with the orientation-preserving action of u on the fibres of N. That $d(u)$ is orthogonal is a consequence of the following calculation for $h_1, h_2 \in H_{[\phi]}$:

$$\begin{aligned}m^H_{u\cdot[\phi]}(d(u)h_1, d(u)h_2) =& (u_* \circ \phi)^{-1}((d\pi_C \circ d(u))(h_1)) \cdot (u_* \circ \phi)^{-1}((d\pi_C \circ d(u))(h_1)) \\ =& (\phi^{-1} \circ u_*^{-1})((u_* \circ d\pi_C)(h_1)) \cdot (\phi^{-1} \circ u_*^{-1})((u_* \circ d\pi_C)(h_2)) \\ =& \phi^{-1}(d\pi_C(h_1)) \cdot \phi^{-1}(d\pi_C(h_2)) = m^H_{[\phi]}(h_1, h_2),\end{aligned}$$

where on the second line, we have used the equivariance of the anchor map $d\pi_C$ between H and N. $\square$

The triangular shape of the matrix in Proposition 3.5 is what is referred to as an *almost isometric* or *triangular structure* by Connes [19] and Connes-Moscovici [23] respectively.

The map $c(u) : H_{[\phi]} \to V_{u\cdot[\phi]}CN$, for $u \in G$ and $[\phi] \in CN_{s(u)}$, is where the interesting representation theory is encoded. Currently, however, the range of $c(u)$ is too high in dimension to be of much use, and these extra dimensions need to be "traced out". Observing that there is indeed a canonical trace $\mathrm{tr}_{F^+N} : VF^+N \to \mathbb{R}$ induced fibrewise by the usual matrix trace on $\mathfrak{gl}(q, \mathbb{R}) = M_q(\mathbb{R})$, we now check that we can apply this map to VCN also.

Lemma 3.6. *The map $\mathrm{tr}_{F^+N} : VF^+N \to \mathbb{R}$ descends to a well-defined map $\mathrm{tr}_{CN} : VCN \to \mathbb{R}$ for which $\mathrm{tr}_{CN} \circ a(u) = \mathrm{tr}_{CN}$ for all $u \in G$.*

Proof. For $A \in GL^+(q,\mathbb{R})$, we denote by $R_A : F^+N \to F^+N$ the map $\phi \mapsto \phi \cdot A$. By definition, the action of $A \in SO(q,\mathbb{R})$ on VF^+N is then given for $\phi \in F^+N$ and $v_\phi \in V_\phi F^+N$ by

$$v_\phi \cdot A := (dR_A)_\phi(v_\phi).$$

We compute

$$(dR_A)_\phi(v_\phi) = \frac{d}{dt}(\phi \cdot \exp(tv) \cdot A)\bigg|_{t=0} = \frac{d}{dt}((\phi \cdot A) \cdot (A^{-1}\exp(tv)A))\bigg|_{t=0} = (A^{-1}vA)_{\phi\cdot A},$$

from which we deduce that the action of $A \in SO(q,\mathbb{R})$ in the trivialisation $VF^+N = F^+N \times \mathfrak{gl}(q,\mathbb{R})$ is given by

$$(\phi, v) \cdot A = (\phi \cdot A, A^{-1}vA)$$

for all $\phi \in F^+N$, $v \in \mathfrak{gl}(q,\mathbb{R})$. Now, $\mathrm{tr}_{F^+N} : F^+N \times \mathfrak{gl}(q,\mathbb{R}) \to \mathbb{R}$ is by definition

$$\mathrm{tr}_{F^+N}(\phi, v) := \mathrm{tr}(v),$$

with tr denoting the usual matrix trace on $q\times q$ matrices, and with the range $\mathbb{R}$ of tr_{F^+N} carrying the trivial action of $SO(q,\mathbb{R})$. Then since the matrix trace is invariant under conjugation, we see that tr_{F^+N} is equivariant:

$$\mathrm{tr}_{F^+N}((\phi,v)\cdot A) = \mathrm{tr}(A^{-1}vA) = \mathrm{tr}(v) = \mathrm{tr}_{F^+N}(\phi,v)\cdot A,$$

and so descends to a well-defined map $\mathrm{tr}_{CN} : VCN \to \mathbb{R}$.

For the second assertion, note that since u commutes with the quotient map $Q : F^+N \to CN$ and since u_* acts as the identity on the fibres of $VF^+N = F^+N \times \mathbb{R}^{q^2}$ by (10), we have

$$\mathrm{tr}_{CN} \circ a(u) \circ dQ = \mathrm{tr}_{CN} \circ dQ \circ \mathrm{id} = \mathrm{tr}_{CN} \circ dQ.$$

Since dQ is surjective, we conclude that

$$\mathrm{tr}_{CN} \circ a(u) = \mathrm{tr}_{CN}$$

as claimed. □

Remark 3.7. Note that what makes Lemma 3.6 possible is the fact that the map $v \mapsto \mathrm{tr}(v)$ on $\mathfrak{gl}(q,\mathbb{R})$ is invariant under conjugation by invertible matrices. Thus in fact we could replace tr with any other invariant polynomial on $\mathfrak{gl}(q,\mathbb{R})$, parallelling the Chern-Weil construction of characteristic classes, and still obtain a well-defined (but no longer necessarily linear) map on the vertical tangent bundle of the Connes fibration. This observation is due to M. T. Benameur.

Let us put Lemma 3.6 to use in simplifying the groupoid representation theory. For $u \in G$ and $[\phi] \in CN_{s(u)}$, define

$$\delta(u) := \mathrm{tr}_{CN} \circ c(u) : H_{[\phi]} \to \mathbb{R}\,.$$

This $\delta(u)$ is linear, and so can be regarded as an element of $H^*_{[\phi]}$. We also define

$$\theta(u) := d(u^{-1})^t : H^*_{[\phi]} \to H^*_{u\cdot[\phi]},$$

the action on the covector bundle for H coming from the transpose of $d(u^{-1}) : H_{u\cdot[\phi]} \to H_{[\phi]}$. We have the following "$ax + b$ group"-type transformation laws.

Lemma 3.8. *For all $u, v \in G^{(2)}$, we have*

$$\theta(uv) = \theta(u)\theta(v), \quad \text{and} \quad \delta(uv) = \delta(v) + \theta(v^{-1})\delta(u).$$

Proof. These identities follow from the triangular structure of the matrices (14) and Lemma 3.6. Specifically, since G acts on N_C we have

$$\begin{pmatrix} a(uv) & c(uv) \\ 0 & d(uv) \end{pmatrix} = \begin{pmatrix} a(u) & c(u) \\ 0 & d(u) \end{pmatrix} \begin{pmatrix} a(v) & c(v) \\ 0 & d(v) \end{pmatrix} = \begin{pmatrix} a(u)a(v) & a(u)c(v) + c(u)d(v) \\ 0 & d(u)d(v) \end{pmatrix},$$

from which we immediately deduce that $d(uv) = d(u)d(v)$ and hence $\theta(uv) = \theta(u)\theta(v)$. We also calculate

$$\begin{aligned} \delta(uv) =& \operatorname{tr}_{CN} \circ c(uv) = \operatorname{tr}_{CN} \circ a(u) \circ c(v) + \operatorname{tr}_{CN} \circ c(u) \circ d(v) \\ =& \operatorname{tr}_{CN} \circ c(v) + \operatorname{tr}_{CN} \circ c(u) \circ d(v) = \delta(v) + \theta(v^{-1})\delta(u), \end{aligned}$$

using Lemma 3.6 for the third equality, giving the desired identities. □

3.3 The Vey Kasparov module

We now go about constructing a second Kasparov module, referred to in this paper as the Vey Kasparov module since it appears to be analogous to the Vey homomorphism considered in previous work [35, 26]. Our first job in constructing a second Kasparov module is to endow the total space H^* of the horizontal covector bundle $\pi_{H^*} : H^* \to CN$ with an action of G that encodes both θ and δ from Lemma 3.8.

Proposition 3.9. *For $u \in G$ and $\eta \in H^*|_{CN_{s(u)}}$, the formula*

$$u \cdot \eta := \theta(u)\eta + \delta(u^{-1})$$

determines the structure of a G-space on H^* with anchor map $\pi_C \circ \pi_{H^*} : H^* \to M$.

Proof. It is clear that $(\pi_C \circ \pi_{H^*})(u \cdot \eta) = r(u)$ for all $u \in G$ and $\eta \in H^*|_{CN_{s(u)}}$, and since by Lemma 3.8 θ is the identity on units and δ is zero on units we get $(\pi_C \circ \pi_{H^*})(\eta) \cdot \eta = \eta$ for all η. Thus it remains only to check that $(uv) \cdot \eta = u \cdot (v \cdot \eta)$ for all $(u, v) \in G^{(2)}$ and $\eta \in H^*|_{CN_{s(v)}}$. For this we simply have

$$(uv) \cdot \eta = \theta(uv)\eta + \delta(v^{-1}u^{-1}) = \theta(u)\big(\theta(v)\eta + \delta(v^{-1})\big) + \delta(u^{-1}) = u \cdot (v \cdot \eta),$$

with the second equality being a consequence of Lemma 3.8. □

We can now construct another dual Dirac class in much the same way as we did for the Connes fibration. Consider the bundle $VH^* := \ker(d\pi_{H^*})$ of vertical tangent vectors over the horizontal covector bundle $\pi_{H^*} : H^* \to CN$, and denote by $\pi_H : H \to CN$ the projection for the horizontal bundle. Since the fibres of H^* are vector spaces, we have $V_\eta H^*_{[\phi]} \cong H^*_{[\phi]}$ for all $[\phi] \in CN$ and $\eta \in H^*_{[\phi]}$. Thus the dual space $V^*_\eta H^*_{[\phi]}$ is a copy of $H_{[\phi]}$ and so we can write V^*H^* as the fibered product

$$V^*H^* \cong H^* \times_{\pi_{H^*}, \pi_H} H,$$

regarded as a vector bundle over H^* by using the projection onto the first factor. Since H is a G-equivariant Euclidean bundle over CN via the map d in Proposition 3.5, for all $u \in G$, $\eta \in H^*|_{CN_{s(u)}}$ and $h \in H|_{CN_{s(u)}}$, the formula

$$u_*(\eta, h) := (u \cdot \eta, d(u)h) = (\theta(u)\eta + \delta(u^{-1}), d(u)h)$$

defines on V^*H^* the structure of a G-equivariant Euclidean bundle over the G-space H^*. Then by functoriality $\mathbb{C}\mathrm{liff}(V^*H^*)$ is a G-equivariant bundle over H^*, and we denote the action of $u \in G$ on $k \in \mathbb{C}\mathrm{liff}(V^*H^*|_{H^*_{[\phi]}})$ by $k \mapsto u_\diamond k$ for all $[\phi] \in CN_{s(u)}$. Using these facts together with Proposition 3.9, the following result is clear.

Proposition 3.10. *The formula*

$$\alpha_u^2(\zeta)(\eta) := u_\diamond \zeta(u^{-1}\cdot\eta) = u_\diamond \zeta(\theta(u^{-1})\eta + \delta(u))$$

defined for $\zeta \in \mathbb{C}\ell(V^*H^*)$, $u \in G$ and $\eta \in H^*_{[\phi]}$ with $[\phi] \in CN_{r(u)}$, determines the structure of a G-algebra on $\mathbb{C}\ell(V^*H^*)$.

We now come to the definition of an appropriate Hilbert module. Let

$$E^2 := \Lambda^*(V^*H^*) \otimes \mathbb{C}$$

be the complexified exterior algebra bundle of V^*H^* over H^*, and define

$$X_{E^2} := \Gamma_0(H^*; E^2),$$

which is a Hilbert $\mathbb{C}\ell(V^*H^*)$-module whose structure as such is determined in the same way as for X_{E^1} using the identification of E^2 with $\mathbb{C}\mathrm{liff}(V^*H^*)$ as vector bundles.

By equivariance of V^*H^* over H^* and functoriality, for $u \in G$, $[\phi] \in CN_{s(u)}$ and $\eta \in H^*_{[\phi]}$ we obtain a unitary holonomy transport map $u_* : E^2_\eta \to E^2_{u\cdot\eta}$ and an isomorphism $W^2_u : \Gamma_0(H^*_{[\phi]}; E^2|_{H^*_{[\phi]}}) \to \Gamma_0(H^*_{u\cdot[\phi]}; E^2|_{H^*_{u\cdot[\phi]}})$ of Banach spaces defined by

$$(W^2_u\zeta)(\eta) := u_*\zeta(u^{-1}\cdot\eta) = u_*\zeta(\theta(u^{-1})\eta + \delta(u)).$$

Using Lemma 2.1, we observe that

$$\begin{aligned}\langle W^2_u\zeta_1, W^2_u\zeta_2\rangle_{\mathbb{C}\ell(V^*H^*)_{r(u)}}(\eta) &= u_\diamond\langle\zeta_1(\theta(u^{-1})\eta + \delta(u)), \zeta_2(\theta(u^{-1})\eta + \delta(u))\rangle \\ &= \alpha^2_u(\langle\zeta_1,\zeta_2\rangle_{\mathbb{C}\ell(V^*H^*)_{s(u)}})(\eta)\end{aligned}$$

for all $u \in G$, $[\phi] \in CN_{r(u)}$ and $\eta \in H^*_{[\phi]}$, so (X_{E^2}, W^2) is a G-Hilbert $\mathbb{C}\ell(V^*H^*)$-module.

We can define an unbounded operator B_2 on the dense submodule $X^c_{E^2} = \Gamma_c(H^*; E^2)$ of X_{E^2} by the formula

$$(B_2\zeta)(\eta) := c_L(\eta)\zeta(\eta),$$

where for $c_L(\eta)$ we regard $\eta \in H^*$ as a vertical covector in $V^*H^* = H^* \times_{\pi_{H^*},\pi_H} H$ using the Euclidean metric on H.

Finally, we take m^2 to be the representation of $C_0(CN)$ on X_{E^2} defined by

$$m^2(f)\zeta(\eta) := f(\pi_{H^*}(\eta))\zeta(\eta).$$

Using the fact that π_{H^*} is an equivariant map and that $\pi_{H^*}(\eta + \eta') = \pi_{H^*}(\eta) = [\phi]$ for all $[\phi] \in CN$ and $\eta, \eta' \in H^*_{[\phi]}$, a routine calculation shows that m^2 is an equivariant representation.

Proposition 3.11. *The triple $(C_0(CN), {}_{m^2}X_{E^2}, B_2)$ is an unbounded G-equivariant Kasparov $C_0(CN)$-$\mathbb{C}\ell(V^*H^*)$-module, defining a class*

$$[B_2] \in KK^G(C_0(CN), \mathbb{C}\ell(V^*H^*)).$$

Proof. The proof is essentially the same as the proof of Proposition 3.3. The only part that must be changed is checking the equivariance condition. For any $u \in G$, $[\phi] \in CN_{r(u)}$ and $\eta \in H^*_{[\phi]}$, we have

$$\begin{aligned}(W^2_u B_2 W^2_{u^{-1}})\zeta(\eta) =& u_*(B_2W^2_{u^{-1}}\zeta)\big(\theta(u^{-1})\eta + \delta(u)\big) \\ =& u_*\Big(c_L\big(\theta(u^{-1})\eta + \delta(u)\big)(W^2_{u^{-1}}\zeta)\big(\theta(u^{-1})\eta + \delta(u)\big)\Big) \\ =& u_*\Big(c_L\big(\theta(u^{-1})\eta + \delta(u)\big)\Big(u_*^{-1}\zeta\big(\theta(u)\big(\theta(u^{-1})\eta + \delta(u)\big) + \delta(u^{-1})\big)\Big)\Big) \\ =& u_*\Big(c_L\big(\theta(u^{-1})\eta + \delta(u)\big)\big(u_*^{-1}\zeta(\eta)\big)\Big) \\ =& c_L\big(\eta - \delta(u^{-1})\big)\zeta(\eta)\end{aligned}$$

where the last line follows from the identity $\theta(u)\delta(u) = -\delta(u^{-1})$ arising from Lemma 3.8, together with the identity (4). We then have

$$B_2 - W_u^2 B_2 W_{u^{-1}}^2 = c_L(\delta(u^{-1})),$$

which defines a bounded operator on $(X_{E^2})_{r(u)}$. The rest of the proof is then the same as in Proposition 3.3. $\square$

4 The index theorem

4.1 Some simplifications in codimension 1

There are important simplifications in the codimension 1 case. Observe that for a codimension 1, transversely orientable foliation $\mathcal{F}$ of M, the conormal bundle $N^* \to M$ is trivialised by a choice of orientation, which is given by a choice of a transverse volume form dx. Such a choice determines a dual section dx^* of $N \to M$ and hence a map $t : N \to \mathbb{R}$ defined by the equality $n = t(n)dx^*$ for $n \in N$. Thus

$$N = M \times \mathbb{R}.$$

The action of $u \in G$ on N will then be denoted by

$$u_*(s(u), n) := (r(u), \Delta(u)n), \tag{15}$$

with $\Delta : G \to \mathbb{R}_+^*$ a multiplicative homomorphism. Observe that under the correspondence $dx \mapsto dx^*$, this $\Delta(u)$ is precisely the Radon-Nikodym derivative of the transverse volume form dx with respect to the holonomy translation u. The principal $\mathbb{R}_+^*$-bundle F^+N of positively oriented frames for N, which coincides with the Connes fibration CN since $SO(1,\mathbb{R}) = 1$, is then also trivial under the map $\phi \mapsto (\pi_C(\phi), t \circ \phi)$:

$$CN = M \times \mathbb{R}_+^*.$$

The action of u on the fibres of CN, defined by (8) since $q = 1$, is induced by the same homomorphism $\Delta(u)$:

$$u \cdot (s(u), b) := (r(u), \Delta(u)b).$$

We will assume for ease of calculation that

$$CN = M \times \mathbb{R}$$

using the logarithm map on the fibres, so that the action of a groupoid element $u \in G$ on CN is now given by

$$u \cdot (s(u), c) = (r(u), c + \log \Delta(u)).$$

The horizontal and vertical bundles are both trivial line bundles, so

$$N_C = VCN \oplus H = CN \times (\mathbb{R} \oplus \mathbb{R}).$$

Here we regard the horizontal bundle $H = CN \times \mathbb{R}$ as a Euclidean bundle with metric m arising from CN defined as in Proposition 3.4 by

$$m^H_{(x,c)}(h_1, h_2) := (e^{-c}h_1) \cdot (e^{-c}h_2) = e^{-2c}h_1h_2.$$

We use the metric m^H to identify H with its dual H^*, by mapping $h \in H$ to the functional $m^H(h, \cdot)$. More precisely, we identify $h \in H_{(x,c)} = \mathbb{R}$ with $\eta_h := e^{-2c}h \in H^*_{(x,c)}$. We then find that the resulting metric on H^* is

$$m^{H^*}_{(x,c)}(\eta_h, \eta_{h'}) := m^H_{(x,c)}(h, h') = e^{-2c}hh' = e^{2c}\eta_h\eta_{h'}.$$

Under this identification, the map $\theta(u) : H^*_{(s(u),c)} \to H^*_{(r(u),c+\log\Delta(u))}$ is precisely $\eta \mapsto \Delta(u^{-1})\eta$.

With no need to trace over the vertical fibres in the codimension 1 case, we can then write the triangular structure of a holonomy transformation $u \in G$ as

$$u_* = \begin{pmatrix} 1 & \delta(u) \\ 0 & \Delta(u) \end{pmatrix}.$$

This action of u_* on $VCN \oplus H \subset TCN$ is the differential of the action of u on CN. It follows then that $\delta(u)$ is the derivative with respect to the transverse coordinate in M of the map $c \mapsto c + \log\Delta(u)$ on the fibres of CN. Since the normal bundle N over M has been trivialised, we can write this derivative as the scalar $\delta(u) = \partial \log\Delta(u)$, with ∂ denoting the derivative with respect to the transverse coordinate. Thus

$$u_* = \begin{pmatrix} 1 & \partial\log\Delta(u) \\ 0 & \Delta(u) \end{pmatrix}.$$

Let us now consider the Kasparov module $[B_2]$. The right-hand algebra in this case is $\mathbb{C}\ell(V^*H^*)$, and since for each $(x,c,\eta) \in H^*$ we can identify vertical tangent vectors in $V_{(x,c,\eta)}H^*$ with vectors in $H^*_{(x,c)}$, it follows that we can identify vertical covectors in $V^*_{(x,c,\eta)}H^*$ with linear functionals $H^*_{(x,c)} \to \mathbb{R}$. Observe then that there is a nonvanishing section κ of $V^*H^* \to H^*$ defined by

$$\kappa(x,c,\eta) := e^c\eta, \quad \text{for} \quad (x,c,\eta) \in H^*.$$

One has

$$\kappa(r(u), c+\log\Delta(u), \Delta(u^{-1})\eta) = e^{c+\log\Delta(u)}\Delta(u^{-1})\eta = e^c\eta = \kappa(s(u),c,\eta),$$

so κ is invariant under the action of G and therefore defines a trivialisation $V^*H^* \cong H^* \times \mathbb{R}$ for which the action of G is given by

$$u_*(s(u),c,\eta,s) = (r(u), c+\log\Delta(u), \Delta(u^{-1})\eta, s) \quad \text{for} \quad c \in CN, \quad s \in \mathbb{R}, \quad \eta \in H^*_{(s(u),c)}.$$

It follows that we can take $\mathbb{C}\ell(V^*H^*)$ to be $C_0(H^*)\otimes\mathrm{Cliff}(\mathbb{R})$, where G acts trivially on $\mathrm{Cliff}(\mathbb{R})$. That is, for all $f \otimes e \in C_0(H^*) \otimes \mathrm{Cliff}(\mathbb{R})$ we have

$$\alpha^2_u(f\otimes e)(r(u),c,\eta) = f(s(u), c - \log\Delta(u), \Delta(u)\eta + \partial\log\Delta(u)) \otimes e, \quad \eta \in H^*_{(r(u),c)}.$$

We define therefore an action α of G on $C_0(H^*)$ by

$$\alpha_u(f)(r(u),c,\eta) := f(s(u), c - \log\Delta(u), \Delta(u)\eta + \partial\log\Delta(u))$$

for $f \in C_0(H^*)$, so that $\alpha^2_u(f\otimes e) = \alpha_u(f)\otimes e$ for all $u \in G$ and $e \in \mathrm{Cliff}(\mathbb{R})$.

The same remarks carry over to the exterior bundle $\Lambda^*V^*H^*$, so that $\Gamma_0(H^*;\Lambda^*(V^*H^*)\otimes\mathbb{C})$ is just $C_0(H^*)\otimes\mathrm{Cliff}(\mathbb{R})$, on which the representation W^2 of G is defined by the same formula as α^2:

$$W^2_u(\rho\otimes e)(r(u),c,\eta) = \rho(s(u), c-\log\Delta(u), \Delta(u)\eta + \partial\log\Delta(u))\otimes e$$

for all $\rho\otimes e \in C_0(H^*)\otimes\mathrm{Cliff}(\mathbb{R})$. We thus define an action W of G on the Hilbert $C_0(H^*)$-module $C_0(H^*)$ by

$$W_u(\rho)(r(u),c,\eta) := \rho(s(u), c-\log\Delta(u), \Delta(u)\eta + \partial\log\Delta(u))$$

for all $\rho \in C_0(H^*)$, and we see that $W^2_u(\rho\otimes e) = W_u(\rho)\otimes e$ for all $u \in G$ and $e \in \mathrm{Cliff}(\mathbb{R})$.

Finally, the operator B_2 acts on $C_0(H^*)\otimes\mathrm{Cliff}(\mathbb{R})$ by

$$(B_2\rho\otimes e)(x,c,\eta) := e^c\eta\rho(x,c,\eta)\otimes c_L(e_1)e, \quad e \in \mathrm{Cliff}(\mathbb{R}), \ \eta \in H^*_{(x,c)},$$

where c_L is the left Clifford multiplication and e_1 is a fixed element of $\mathrm{Cliff}(\mathbb{R})$ with square 1. We can now proceed with the construction of a spectral triple from this data and the proof of the index theorem relating the spectral triple to the Godbillon-Vey invariant.

4.2 The spectral triple

Applying the descent map to the equivariant Kasparov module $(C_0(CN), {}_{m^2}X_{E^2}, B_2)$ of Proposition 3.11 in codimension 1 gives us by Proposition 2.11 a Kasparov module

$$(\Gamma_c(CN \rtimes G, \Omega^{\frac{1}{2}}), X_{E^2} \rtimes_r G, r^*B_2) \tag{16}$$

which defines a class in $KK(C_0(CN) \rtimes_r G, \mathbb{C}\ell(V^*H^*) \rtimes_r G)$. By the remarks of the previous section, we actually have

$$\mathbb{C}\ell(V^*H^*) \rtimes_r G = (C_0(H^*) \otimes \mathrm{Cliff}(\mathbb{R})) \rtimes_r G = (C_0(H^*) \rtimes_r G) \otimes \mathrm{Cliff}(\mathbb{R})$$

since G acts trivially on $\mathrm{Cliff}(\mathbb{R})$. Thus the module (16) can be replaced [20, Proposition 13, Appendix A, Chapter 4] by the odd Kasparov $C_0(CN) \rtimes_r G$-$C_0(H^*) \rtimes_r G$-module

$$(\Gamma_c(CN \rtimes G, \Omega^{\frac{1}{2}}), C_0(H^*) \rtimes_r G, \mathcal{B}) \tag{17}$$

where we define $\mathcal{B}$ on $\Gamma_c(H^* \rtimes G; \Omega^{\frac{1}{2}}) \subset C_0(H^*) \rtimes_r G$ by

$$(\mathcal{B}\,\rho)_u(x, c, \eta) := (\mathcal{B}_{r(u)}\,\rho_u)(x, c, \eta) := e^c \eta \rho_u(x, c, \eta), \quad \eta \in \mathbb{R}\,.$$

Here we are using density of $\Gamma_c(H^* \rtimes G; \Omega^{\frac{1}{2}})$ in $C_0(H^*) \rtimes_r G$ and density of $\Gamma_c(CN \rtimes G; \Omega^{\frac{1}{2}})$ in $C_0(CN) \rtimes_r G$ as in the final paragraph of Section 2.3.

The G-invariant transverse volume form on CN is $d\nu_{CN} = e^{-c}dxdc$, and we let τ_{CN} be the trace on $\Gamma_c(CN \rtimes G; \Omega^{\frac{1}{2}})$ defined by integration over CN with respect to $d\nu_{CN}$. The G-invariant transverse volume form on H^* is simply $d\nu_{H^*} = dxdcd\eta$, and we let τ_{H^*} be the trace on $\Gamma_c(H^* \rtimes G, \Omega^{\frac{1}{2}})$ induced by integration over H^* with respect to $d\nu_{H^*}$.

Putting the trace τ_{H^*} together with the odd Kasparov module (17), by Proposition 2.13 we obtain an odd semifinite spectral triple

$$(\mathcal{A}, \mathcal{H}, \mathcal{B})$$

relative to $(\mathcal{N}, \tau)$ where:

1. $\mathcal{A} = \Gamma_c(CN \rtimes G; \Omega^{\frac{1}{2}})$ acts by convolution operators on
2. $\mathcal{H}$, the Hilbert space completion of $\Gamma_c(H^* \rtimes G; \Omega^{\frac{1}{2}})$ in the inner product
$$(\rho_1|\rho_2) = \tau_{H^*}(\rho_1^* * \rho_2),$$
3. $\mathcal{B}$ is regarded as an operator on $\mathcal{H}$ with domain $\Gamma_c(H^* \rtimes G; \Omega^{\frac{1}{2}})$,
4. $\mathcal{N}$ is the weak closure of $\Gamma_c(H^* \rtimes G; \Omega^{\frac{1}{2}})$ in the bounded operators on $\mathcal{H}$ and,
5. τ is the normal extension of τ_{H^*} to $\mathcal{N}$.

We now apply the semifinite local index formula to $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ to prove the codimension 1 Godbillon-Vey index theorem.

4.3 The index theorem

We will apply the residue cocycle of [12, Definition 3.2] to prove the following theorem.

Theorem 4.1. *Let $(M, \mathcal{F})$ be a foliated manifold of codimension 1. The Chern character of the semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ given in Section 4.2 coincides up to a factor of $(2\pi i)^{\frac{1}{2}}$ with the Godbillon-Vey cyclic cocycle of Connes and Moscovici [22, Proposition 19].*

To apply the local index formula of [12] we need to check the summability and smoothness of the spectral triple.

Lemma 4.2. *The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ is smoothly summable of spectral dimension $p = 1$ and has isolated spectral dimension.*

Proof. We first check finite summability. For $s \in \mathbb{R}$, $a \in \Gamma_c(CN \rtimes G; \Omega^{\frac{1}{2}})$ and $\rho \in \mathcal{H}$, we calculate

$$\begin{aligned}
(a(1+\mathcal{B}^2)^{-\frac{s}{2}}\rho)_u(x,c,\eta) &= \int_{v\in G^{r(u)}} a_v(x,c)\big(W_v(1+\mathcal{B}^2_{s(v)})^{-\frac{s}{2}}\rho_{v^{-1}u}\big)(x,c,\eta) \\
&= \int_{v\in G^{r(u)}} a_v(x,c)\big(1+e^{2(c-\log\Delta(v))}(\Delta(v)\eta+\partial\log\Delta(v))^2\big)^{-\frac{s}{2}}\big(W_v\rho_{v^{-1}u}\big)(x,c,\eta) \\
&= \int_{v\in G^{r(u)}} a_v(x,c)(1+e^{2c}\Delta(v^{-1})^2(\Delta(v)\eta+\partial\log\Delta(v))^2)^{-\frac{s}{2}}(W_v\rho_{v^{-1}u})(x,c,\eta) \\
&= \int_{v\in G^{r(u)}} a_v(x,c)(1+e^{2c}(\eta-\partial\log\Delta(v^{-1}))^2)^{-\frac{s}{2}}(W_v\rho_{v^{-1}u})(x,c,\eta),
\end{aligned}$$

where on the last line we have used Lemma 3.8 in simplifying $\Delta(v^{-1})\partial\log\Delta(v) = -\partial\log\Delta(v^{-1})$. So $a(1+\mathcal{B}^2)^{-\frac{s}{2}}$ is the half-density on $H^* \rtimes G$ defined by

$$((x,c,\eta),u) \mapsto a_u(x,c)\big(1+e^{2c}(\eta-\partial\log\Delta(u^{-1}))^2\big)^{-\frac{s}{2}},$$

compactly supported in the u and (x,c) variables. Thus

$$\begin{aligned}
\tau_{H^*}(a(1+\mathcal{B}^2)^{-\frac{s}{2}}) &= \int_{M\times\mathbb{R}\times\mathbb{R}} a(x,c)\big(1+e^{2c}\eta^2\big)^{-\frac{s}{2}}dxdcd\eta \\
&= \int_{CN} a(x,c)d\nu_{CN}\int_{\mathbb{R}}\big(1+t^2\big)^{-\frac{s}{2}}dt,
\end{aligned}$$

where we have made the substitution $t = e^c\eta$. It is then clear that $\tau_{H^*}(a(1+\mathcal{B}^2)^{-\frac{s}{2}})$ is finite for all $s > 1$. For smoothness, we fix $a \in \Gamma_c(CN \rtimes G; \Omega^{\frac{1}{2}})$ and calculate

$$\begin{aligned}
([\mathcal{B}^2,a]\rho)_u(x,c,\eta) &= e^{2c}\eta^2\int_{v\in G^{r(u)}} a_v(x,c)(W_v\rho_{v^{-1}u})(x,c,\eta) \\
&\qquad - \int_{v\in G^{r(u)}} a_v(x,c)(W_v\,\mathcal{B}^2_{s(v)}\,\rho_{v^{-1}u})(x,c,\eta) \\
&= \int_{v\in G^{r(u)}} a_v(x,c)e^{2c}(\eta^2-\Delta(v^{-1})^2(\Delta(v)\eta+\partial\log\Delta(v))^2)(W_v\rho_{v^{-1}u})(x,c,\eta) \\
&= \int_{v\in G^{r(u)}} a_v(x,c)e^{2c}\big(2\eta\partial\log\Delta(v^{-1})-(\partial\log\Delta(v^{-1}))^2\big)(W_v\rho_{v^{-1}u})(x,c,\eta)
\end{aligned}$$

so that $[\mathcal{B}^2,a]$ is convolution by the half-density on $H^* \rtimes G$ defined by

$$((x,c,\eta),u) \mapsto a_u(x,c)e^{2c}\big(2\eta\partial\log\Delta(u^{-1})-(\partial\log\Delta(u^{-1}))^2\big).$$

We also calculate

$$\begin{aligned}
([\mathcal{B}^2,[\mathcal{B},a]]\rho)_u(x,c,\eta) =& e^{2c}\eta^2\big([\mathcal{B},a]\rho\big)_u(x,c,\eta) - \big([\mathcal{B},a]\,\mathcal{B}^2\,\rho\big)_u(x,c,\eta)\\
=& e^{2c}\eta^2\int_{v\in G^{r(u)}} a_v(x,c)e^c\partial\log\Delta(v^{-1})\big(W_v\rho_{v^{-1}u}\big)(x,c,\eta)\\
&- \int_{v\in G^{r(u)}} a_v(x,c)e^c\partial\log\Delta(v^{-1})\big(W_v\,\mathcal{B}^2_{s(v)}\,\rho_{v^{-1}u}\big)(x,c,\eta)\\
=& e^{2c}\eta^2\int_{v\in G^{r(u)}} a_v(x,c)e^c\partial\log\Delta(v^{-1})(W_v\rho_{v^{-1}u})(x,c,\eta)\\
&- \int_{v\in G^{r(u)}} a_v(x,c)e^{3c}\partial\log\Delta(v^{-1})\Delta(v^{-1})^2(\Delta(v)\eta+\partial\log\Delta(v))^2\\
&\times (W_v\rho_{v^{-1}u})(x,c,\eta)\\
=& \int_{v\in G^{r(u)}} a_v(x,c)e^{3c}\big(2\eta\partial\log\Delta(v^{-1}) - (\partial\log\Delta(v^{-1}))^2\big)\\
&\times \partial\log\Delta(v^{-1})(W_v\rho_{v^{-1}u})(x,c,\eta),
\end{aligned}$$

so that $[\mathcal{B}^2,[\mathcal{B},a]]$ is the half-density on $H^*\rtimes G$ defined by

$$((x,c,\eta),u)\mapsto a_u(x,c)e^{3c}\big(2\eta\partial\log\Delta(u^{-1}) - (\partial\log\Delta(u^{-1}))^2\big)\partial\log\Delta(u^{-1}).$$

More generally, setting $T^{(0)} := T$ and then inductively defining $T^{(k)} := [\mathcal{B}^2, T^{(k-1)}]$, we see that $[\mathcal{B},a]^{(k)}$ is the half-density on $H^*\rtimes G$ defined by

$$((x,c,\eta),u)\mapsto a_u(x,c)e^{(2k+1)c}\big(2\eta\partial\log\Delta(u^{-1}) - (\partial\log\Delta(u^{-1}))^2\big)^k\partial\log\Delta(u^{-1}).$$

Now these computations show that for $a\in\mathcal{A}$ and $k\in\mathbb{N}$, the operators $a^{(k)}$ and $[\mathcal{B},a]^{(k)}$ are half densities on $H^*\rtimes G$, with compact support in the $((x,c),u)\in CN\rtimes G$ variables equal to that of a, and growing like η^k in the fibre variable $\eta\in H^*_{(x,c)}$ for all $(x,c)\in CN$. Hence both $a^{(k)}(1+\mathcal{B}^2)^{-k/2}$ and $[\mathcal{B},a]^{(k)}(1+\mathcal{B}^2)^{-k/2}$ are bounded with compact support in the $CN\rtimes G$ directions. Hence for all $a\in\mathcal{A}$ the operator

$$(1+\mathcal{B}^2)^{-k/2-s/4}(a^{(k)})^*a^{(k)}(1+\mathcal{B}^2)^{-k/2-s/4}$$

is trace class whenever the real part of s is greater than 1, and similarly with a replaced by $[\mathcal{B},a]$. Thus $\mathcal{A}\cup[\mathcal{B},\mathcal{A}]\subset B_2^\infty(\mathcal{B},1)$ in the notation of [12]. Thus $\mathcal{A}^2$, the span of products from $\mathcal{A}$, satisfies $\mathcal{A}^2\cup[\mathcal{B},\mathcal{A}^2]\subset B_1^\infty(\mathcal{B},1)$, showing that the semifinite spectral triple over $\mathcal{A}^2$ is smoothly summable.

The last step to establish smooth summability is to observe that $\mathcal{A}$ has a (left) approximate unit for the inductive limit topology by [47, Proposition 6.8]. This ensures that any compactly supported section in $\mathcal{A}=\Gamma_c(CN\rtimes G;\Omega^{\frac{1}{2}})$ can be approximated by products while preserving summability.

Finally the computations also show that $(\mathcal{A},\mathcal{H},\mathcal{B})$ has isolated spectral dimension, as in [12, Definition 3.1], since for all multi-indices k of length $m\geq 0$ we have proved that

$$\tau_{H^*}(a_0[\mathcal{B},a_1]^{(k_1)}\cdots[\mathcal{B},a_m]^{(k_m)}(1+\mathcal{B}^2)^{-|k|-m/2-s})$$

has a meromorphic continuation in a neighbourhood of $s=0$. □

Finally we can prove the Theorem 4.1.

Proof of Theorem 4.1. Since the spectral dimension $p = 1$ and since the parity of the spectral triple is 1, the only nonzero term in the residue cocycle is ϕ_1 as defined in [12, Definition 3.2]. For any $a \in \Gamma_c(CN \rtimes G; \Omega^{\frac{1}{2}})$ we have

$$\begin{aligned}
\big([\mathcal{B}, a]\rho\big)_u(x, c, \eta) =& \mathcal{B}_{r(u)} \int_{v \in G^{r(u)}} a_v(x, c)(W_v \rho_{v^{-1}u})(x, c, \eta) \\
&- \int_{v \in G^{r(u)}} a_v(x, c)(W_v\, \mathcal{B}_{s(v)}\, \rho_{v^{-1}u})(x, c, \eta) \\
=& \int_{v \in G^{r(u)}} a_v(x, c)\big(\, \mathcal{B}_{r(v)} - W_v\, \mathcal{B}_{s(v)}\, W_{v^{-1}}\big)(W_v \rho_{v^{-1}u})(x, c, \eta) \\
=& \int_{v \in G^{r(u)}} a_v(x, c) e^c \partial \log \Delta(v^{-1})(W_v \rho_{v^{-1}u})(x, c, \eta) \\
=& (\delta_1(a)\rho)_u(x, c, \eta),
\end{aligned}$$

where δ_1 is the derivation of $\Gamma_c(CN \rtimes G; \Omega^{\frac{1}{2}})$ defined by

$$\delta_1(a)_u(x, c) := e^c \partial \log \Delta(u^{-1}) a_u(x, c).$$

The derivation δ_1 is precisely the same as that given in [22, Page 39]. Thus for $a_0, a_1 \in \Gamma_c(CN \rtimes G; \Omega^{\frac{1}{2}})$, we calculate

$$\begin{aligned}
\phi_1(a_0, a_1) =& 2(2\pi i)^{\frac{1}{2}} \operatorname{res}_{z=0} \tau_{H^*}\big(a_0[\mathcal{B}, a_1](1 + \mathcal{B}^2)^{-\frac{1}{2}-z}\big) \\
=& 2(2\pi i)^{\frac{1}{2}} \tau_{CN}(a_0 \delta_1(a_1)) \operatorname{res}_{z=0} \int_{\mathbb{R}} (1 + t^2)^{-\frac{1}{2}-z} dh \\
=& 2(2\pi i)^{\frac{1}{2}} \tau_{CN}(a_0 \delta_1(a_1)) \operatorname{res}_{z=0} \frac{\Gamma(1/2)\Gamma(z)}{2\Gamma(1/2 + z)} \\
=& (2\pi i)^{\frac{1}{2}} \tau_{CN}(a_0 \delta_1(a_1)).
\end{aligned}$$

This is, up to the factor $(2\pi i)^{\frac{1}{2}}$, the Godbillon-Vey cyclic cocycle from [22, Proposition 19]. □

5 Concluding remarks

It is tempting to view the higher codimension version of the codimension 1 Kasparov module and spectral triple as analogous data representing the Godbillon-Vey invariant in higher codimension. Sadly, despite the naturality of the constructions presented here, it is far from clear that such an interpretation is warranted. Without an identification of the Chern character of these spectral triples with the Godbillon-Vey class, they must remain an interesting construction.

One final remark on the constructions presented here: they all pass to real algebras and real KK-theory. All our constructions are Real [40] for the obvious variations of complex conjugation, in part because of our systematic use of the exterior algebra rather than the spinor bundle. This means that we can at all stages retain contact with homology of manifolds with real coefficients.

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