

Untwisting twisted spectral triples

Magnus Goffeng*, Bram Mesland†, Adam Rennie‡*

* Department of Mathematical Sciences,
Chalmers University of Technology and University of Gothenburg,
Gothenburg, Sweden

†Max Planck Institut für Mathematik, Bonn, Germany

‡School of Mathematics and Applied Statistics, University of Wollongong
Wollongong, Australia

March 21, 2019

Abstract

We examine the index data associated to twisted spectral triples and higher order spectral triples. In particular, we show that a Lipschitz regular twisted spectral triple can always be “logarithmically dampened” through functional calculus, to obtain an ordinary (ie untwisted) spectral triple. The same procedure turns higher order spectral triples into spectral triples. We provide examples of highly regular twisted spectral triples with nontrivial index data for which Moscovici’s ansatz for a twisted local index formula is identically zero.

Contents

1	Kasparov modules and index theory	6
1.1	Twisted Kasparov modules	6
1.2	Invertible amplifications	8
1.3	The bounded transform for Lipschitz regular twisted Kasparov modules	11
1.4	Logarithmic dampening of Lipschitz regular twisted Kasparov modules	11
1.5	Higher order Kasparov modules	14
1.6	The bounded transform for higher order Kasparov modules	17
1.7	Logarithmic dampening of higher order Kasparov modules	20
2	Review of local index formulae	23
2.1	Smoothness and summability	23
2.2	The index cocycle	24
2.3	(Local) index theory for twisted spectral triples	25
3	Vanishing of the twisted local index formula	28
3.1	Set up and statement	28
3.2	The good	30
3.3	The bad	31
3.4	The ugly	33
3.4.1	The generators and relations picture	33

*email: goffeng@chalmers.se, brammesland@gmail.com, renniea@uow.edu.au

3.4.2	The shift space and groupoid picture	34
3.4.3	The construction of spectral triples for Cuntz-Krieger algebras	35
3.4.4	The counterexample	37

Introduction

The original definition of spectral triple [17, 18, 20] is motivated by the index theory of first order elliptic operators on manifolds. The general non-commutative and bivariant definition of unbounded Kasparov module was formalised in [3], and has been shown to capture a wide variety of geometric-dynamic examples. Despite this success, there are numerous examples which motivate more general definitions.

In this paper we consider index theoretic questions about twisted spectral triples and higher order spectral triples. Briefly, twisted spectral triples have bounded twisted commutators relative to an auxiliary algebra automorphism, while higher order spectral triples are analogues of elliptic operators of any order > 0 on a manifold.

Developing a consistent non-commutative geometry in a way that is compatible with index theory turns out to be hard, even for the most basic dynamical examples (see for instance [24, 27, 46]). Since index theory, and the underlying machinery of KK -theory, is our only tool for noncommutative algebraic topology, discussing geometry in a manner directly relatable to index theory is imperative.

An appropriate analogy is that the geometric meaning of differential forms in calculus can be directly related to topological structures via the close relationship between de Rham's differential topology and singular or Čech cohomology. While spectral triples (or more generally unbounded Kasparov modules) provide a geometric object, to obtain a connection to topology (via index theory) we need to know that the bounded transform

$$(\mathcal{A}, X_B, D) \mapsto (\mathcal{A}, X_B, F_D := D(1 + D^2)^{-1/2})$$

yields a Kasparov module and so a (preferably non-zero) KK -class. Under suitable conditions the bounded transform of both twisted and higher order variations of the notion of unbounded Kasparov module yields a Kasparov module.

Our first general result states that functional calculus using the C^1 -function

$$\text{sgnlog} : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \text{sgn}(x) \log(1 + |x|),$$

can be used to turn both twisted and higher order spectral triples into ordinary spectral triples without changing the K -homology class (for details on the twisted case, see Theorem 1 below). This logarithmic dampening has been used before in specific examples [7, 24, 27, 31, 46, 48], and in Section 1 of the present paper we formalise the procedure.

Thus, if a twisted or higher order spectral triple encodes nontrivial index theoretic data, then the same information can be recovered from an ordinary spectral triple, constructed through a well-defined procedure, albeit possibly less geometric than the original twisted spectral triple. Logarithmic dampening will for instance transform an elliptic differential operator to a pseudodifferential operator. For large portions of the paper, we work in the bivariant context, proving our general statements for unbounded Kasparov modules.

The motivation for introducing twisted spectral triples comes from situations where twisted commutators with a natural geometric differential operator are bounded whereas ordinary commutators are not [21]. This observation suggests that we seek computationally tractable representatives of K -homology classes using twisted commutators. Early successes of this philosophy concern conformal diffeomorphisms on manifolds [20, 49, 50], a class identified by Connes as tractable. Additionally, twists are well-known to improve ‘dimension drop’ problems in the cyclic homology of q -deformed algebras, [34, 35].

Another widely held hope is that while many C^* -algebras (such as those that arise in examples coming from dynamical systems) do not admit *finitely summable* ordinary spectral triples, by [16], there could be finitely summable twisted spectral triples. One reason for the interest in finite summability is the possibility of producing a computable cyclic cocycle formula for the index pairing.

An ansatz for computing such (twisted) index pairings was proposed by Moscovici [47], and shown to work for twists coming from “scaling automorphisms” [47, 49, 50]. Moscovici’s proposed formula was an analogue of the known local index formulae, adapted for regular twisted spectral triples, and computed in terms of residues of ζ -functions.

Our second main result concerns the existence of twisted spectral triples that pair nontrivially with K -theory, but for which all twisted higher residue cochains appearing in Moscovici’s ansatz vanish. The proof consists of examples where all ζ -functions coming from operators containing a twisted commutator as a factor are entire functions. This leaves little hope for a twisted higher residue cochain to compute the index pairing in general. For details see Theorem 2 (on page 5).

We produce several such examples. The first example arises by introducing a twist on the usual spectral triple on the circle. Next, we extend this twisted spectral triple on the circle to the crossed product by a group of Möbius transformations, thereby including examples of purely infinite C^* -algebras. The last class of examples comes from the action of the free group on the boundary of its Cayley graph. These are again purely infinite (and more involved), but the main idea in each of the examples is the same.

Our two main results indicate that the analytic aspect of index theory for finitely summable twisted spectral triples is quite involved. The proofs of these results indicate that the appropriate index theory is in fact closely related to the index theory for Li_1 -summable spectral triples (recently studied in [31]).

Main results

Let us discuss the main results in a bit more detail. It has been known since their introduction [21] that a twisted spectral triple $(\mathcal{A}, H, D, \sigma)$ defines a K -homology class only under some extra regularity assumption. Lipschitz regularity, namely that for all $a \in \mathcal{A}$ the twisted commutator

$$[|D|, a]_\sigma := |D|a - \sigma(a)|D|$$

extends to a bounded operator, is a sufficient condition¹. From the perspective of the local index formula, Lipschitz regularity is a necessary requirement and we adopt it here. We will

¹Other conditions are known: in [38], an elaborate set of conditions (adapted to Kasparov products) guaranteeing topological content is used.

in the first half of the paper work with *weakly* twisted spectral triples, meaning that the homomorphism σ need not preserve the algebra (for more details, see Remark 1.3 on page 7). The first of our main results untwists twisted spectral triples.

Theorem 1. *Let $(\mathcal{A}, H, D, \sigma)$ be a Lipschitz regular weakly twisted spectral triple such that D has compact resolvent. Define the self-adjoint operator*

$$D_{\log} := \text{sgnlog}(D) = D|D|^{-1} \log(1 + |D|).$$

Then $(\mathcal{A}, H, D_{\log})$ is a spectral triple defining the same class in K -homology as $(\mathcal{A}, H, D, \sigma)$. The weakly twisted spectral triple $(\mathcal{A}, H, D, \sigma)$ is finitely summable if and only if the spectral triple $(\mathcal{A}, H, D_{\log})$ is Li_1 -summable.

This result appears as Theorem 1.16 (see page 13), where it is proven in the larger generality of compact Lipschitz regular weakly twisted Kasparov modules. The summability statement is found in Proposition 1.17 (see page 14). In Theorem 1.37 (see page 22) we prove that the same logarithmic transform turns a higher order Kasparov module into an ordinary Kasparov module. For technical reasons, we restrict to the case of compact resolvent when considering twisted Kasparov modules but our results in the higher order case are proved in general.

The index theoretic content of Theorem 1 is as follows. If a K -homology class is represented by a weakly twisted spectral triple, then the K -homology class is also represented by an ordinary, i.e. untwisted, spectral triple, that can be constructed through a definite procedure.

We illustrate both higher order and twisted spectral triples by means of constructing various K -homologically nontrivial exotic spectral triples on the crossed product algebra arising from a non-isometric diffeomorphism on the circle. As a simple application, we show that the boundary map in the Pimsner-Voiculescu sequence can be computed at an unbounded level (under mild assumptions) using a combination of logarithmic dampening and higher order spectral triples. These results can be found in Subsection 1.6. This is an example of a setting in which twisted spectral triples do not provide a solution.

There is a left inverse (modulo bounded perturbations) to untwisting given by exponentiating an ordinary spectral triple satisfying further assumptions. Let (\mathcal{A}, H, D) be a spectral triple and write $F := D|D|^{-1}$. If for all $a \in \mathcal{A}$, we have

$$a \text{Dom}(e^{|D|}) \subseteq \text{Dom}(e^{|D|}), \text{ and } [F, a]H \subseteq \text{Dom}(e^{|D|}),$$

then $(\mathcal{A}, H, Fe^{|D|}, \sigma)$ is a weakly twisted spectral triple. The twist σ used in the exponentiation procedure is defined as $\sigma(a) := e^{|D|}ae^{-|D|}$. This way of twisting untwisted spectral triples preserves several pathological properties, and can be exploited to construct twisted spectral triples with exotic features.

In the second half of the paper we use the exponentiation process to construct twisted spectral triples that are regular, finitely summable, have finite discrete dimension spectrum, and pair nontrivially with K -theory. For the examples we construct, the cochain given by Moscovici's ansatz for a local index cocycle [47] vanishes identically, and therefore fails to compute the index pairing. These examples thus show that Moscovici's ansatz does not compute the index pairing in general. The details of the construction make clear that any index formula based on residues of traces is implausible for examples similar to ours.

The key to building such examples is the existence of ordinary Li_1 -summable spectral triples (\mathcal{A}, H, D) such that the commutators $[F, a]$ are finite rank or smoothing for all $a \in \mathcal{A}$. Then the conditions for exponentiation can be met, and the exponentiated triple $(\mathcal{A}, H, Fe^{|D|}, \sigma)$ is a finitely summable twisted spectral triple.

The fact that the commutators $[F, a]$ are smoothing or of finite rank is also used to show that the functionals appearing in Moscovici's ansatz (see Section 2 for detailed notation)

$$a_0, a_1, \dots, a_m \mapsto \text{Tr}(a_0[Fe^{|D|}, a_1]_\sigma^{(k_1)} \dots [Fe^{|D|}, a_m]_\sigma^{(k_m)} e^{-(2|k|+m+2z)|D|})$$

extend to entire functions of z . The cochain $(\phi_m)_{m \geq 1, \text{odd}}$ appearing in Moscovici's ansatz is a linear combination of residues of such expressions (for details, see Definition 2.12 on page 27).

The simplest example is given by the standard spectral triple $(C^\infty(S^1), L^2(S^1), -i\frac{d}{dx})$, which can be turned into a twisted spectral triple on the circle. Here we parametrise the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ by $x \mapsto e^{ix}$ and $-i\frac{d}{dx}$ is the ordinary Dirac operator on S^1 . An example on S^1 might seem too nice and forcing a twist on it as somewhat artificial. However, we show that the same method extends to crossed products by groups Γ of conformal diffeomorphisms

$$\left(C^\infty(S^1) \rtimes^{\text{alg}} \Gamma, L^2(S^1), -i\frac{d}{dx}, \sigma\right)$$

with appropriate twist σ defined from $-i\frac{d}{dx}$. For Fuchsian groups of the first kind the crossed product is purely infinite, and in this case finitely summable spectral triples do not exist.

Ponge and Wang studied the case of twisted spectral triples arising from groups of conformal diffeomorphisms in great detail [49, 50]. While they prove a local index formula, this is achieved by reducing to the case of trivial twist and using the local index formula for untwisted spectral triples. They noted in [49, Remark 4.1] that the question of whether Moscovici's ansatz computes the index for the standard conformal twist was open. Our example uses a twisting automorphism that is different from the conformal twist, leaving the question of Ponge-Wang unanswered. What our example does show is that Moscovici's ansatz does not compute the index pairing for all twisted spectral triples coming from conformal diffeomorphisms.

Our final set of examples are purely infinite Cuntz-Krieger algebras arising from the action of the free group \mathbb{F}_d on its boundary $\partial\mathbb{F}_d$, $d > 1$. The spectral triples we start with here were first studied in [27], where all the necessary index pairings and analytic behaviour were determined. It is nevertheless a lengthy calculation to prove the discrete dimension spectrum condition required. We omit the more onerous details in this example, since it uses the same idea as the other, simpler examples.

The above counterexamples can be summarised as our second main result.

Theorem 2. *There exist unital odd twisted spectral triples $(\mathcal{A}, H, Fe^{|D|}, \text{Ad}_{e^{|D|}})$ that are regular, finitely summable, have discrete dimension spectrum, and pair nontrivially with $K_1(\mathcal{A})$ such that the cochain $(\phi_m)_{m \geq 1, \text{odd}}$ provided by Moscovici's ansatz [47] for a twisted local index formula is identically zero.*

The precise statements for the three types of counterexample appear as Theorems 3.3, 3.6, 3.10. It should be noted that all three examples considered share the property that $[F, a]$ is smoothing. This property is (apparently) quite rare. The circle is the only connected closed

manifold admitting a metric for which the phase of the Dirac operator commutes with C^∞ -functions up to smoothing operators. Cuntz-Krieger algebras share several geometric features with the circle, and in fact, the circle and its crossed product by a Fuchsian group of the first kind are Cuntz-Krieger algebras (see [1]).

Acknowledgments

The authors would like to thank Heath Emerson and Bogdan Nica for several interesting discussions over the years that inspired the work in this paper, and Bob Yuncken for helpful comments. The authors acknowledge the support of the Erwin Schrödinger Institute where some of this work was conducted. A. R. and B. M. thank the Gothenburg Centre for Advanced Studies in Science and Technology for funding and the University of Gothenburg and Chalmers University of Technology for their hospitality in November 2017 when this project took shape. M. G. was supported by the Swedish Research Council Grant 2018-03509. B. M. thanks the Hausdorff Center for Mathematics and the Max Planck Institute for Mathematics in Bonn and the University of Wollongong for their hospitality and support.

1 Kasparov modules and index theory

In this section we recall the definition of two non-standard notions of unbounded Kasparov module, and hence of spectral triple. The first notion is that of twisted Kasparov module and is based on the behaviour of conformal diffeomorphisms on manifolds. The second notion is that of higher order Kasparov module and is inspired by elliptic pseudodifferential operators on manifolds of order ≥ 1 .

Both variations of Kasparov modules arise when one attempts to construct K -homologically nontrivial spectral triples for crossed products by non-measure preserving dynamics on metric measure spaces [4, 20, 27, 29, 46, 49, 50].

We first recall the fact that the bounded transform still produces bounded Kasparov modules in this generality. For twisted Kasparov modules we need to assume Lipschitz regularity, a mild condition used by several authors. Subsequently we prove that, using the logarithm function, these exotic Kasparov modules can be turned into ordinary Kasparov modules via the functional calculus, without changing the KK -class.

1.1 Twisted Kasparov modules

Let $\mathcal{A} \subset A$ be a dense $*$ -subalgebra of a unital C^* -algebra. Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be an algebra homomorphism that is compatible with the involution on A in the sense that $\sigma(a)^* = \sigma^{-1}(a^*)$. For a B -Hilbert C^* -module X , we let $\text{End}_B^*(X)$ denote the C^* -algebra of B -linear adjointable operators on X and $\mathbb{K}_B(X) \subseteq \text{End}_B^*(X)$ the C^* -subalgebra of B -compact operators.

Definition 1.1. A twisted Kasparov module $(\mathcal{A}, X, D, \sigma)$ is given by a $*$ -algebra \mathcal{A} with a regular automorphism σ , represented on a B -Hilbert C^* -module X via a $*$ -homomorphism

$$\pi : \mathcal{A} \rightarrow \text{End}_B^*(X).$$

The operator D is a densely defined regular self-adjoint operator

$$D : \text{Dom } D \subset X \rightarrow X$$

such that for all $a \in \mathcal{A}$ the following conditions are satisfied:

1. $\pi(a) \text{Dom } D \subset \text{Dom } D$ and the densely defined twisted commutator

$$[D, \pi(a)]_\sigma := D\pi(a) - \pi(\sigma(a))D,$$

is bounded (and so extends to a bounded adjointable operator on all of X by continuity);

2. the operator $\pi(a)(1 + D^2)^{-1/2}$ is a B -compact operator.

If we in addition have prescribed an operator $\gamma \in \text{End}_B^*(X)$ with $\gamma = \gamma^*$, $\gamma^2 = 1$, $D\gamma + \gamma D = 0$, and for all $a \in \mathcal{A}$ $\gamma\pi(a) = \pi(a)\gamma$, we call the spectral triple **even** or **graded**. Otherwise we say that it is **odd** or **ungraded**.

If \mathcal{A} is unital and $\pi(1) = 1$ we say that $(\mathcal{A}, X, D, \sigma)$ is **unital**. If D has compact resolvent (i.e. $(D \pm i)^{-1} \in \mathbb{K}_B(X)$) we say that $(\mathcal{A}, X, D, \sigma)$ is **compact**.

Remark 1.2. We will nearly always dispense with the representation π , treating \mathcal{A} as a subalgebra of $\text{End}_B^*(X)$. In general, π need not be faithful but for our purposes this issue does not play a role and will be disregarded.

Remark 1.3. It is not always the case that σ is an automorphism of \mathcal{A} . For instance, Kaaad [38] and Matassa-Yuncken [45], consider situations where σ fails to preserve \mathcal{A} . We consider the following situation. Let $\mathcal{A} \subset \text{End}_B^*(X)$ be a $*$ -subalgebra of bounded operators and σ a partially defined automorphism of $\text{End}_B^*(X)$ such that $\mathcal{A} \subseteq \text{Dom}(\sigma)$. Under these assumptions, we say that $(\mathcal{A}, X, D, \sigma)$ is a **weakly twisted Kasparov module** if the remaining conditions of Definition 1.1 are satisfied. The next result follows immediately from the bounded twisted commutator condition.

Proposition 1.4. *Let $(\mathcal{A}, X, D, \sigma)$ be a weakly twisted Kasparov module with \mathcal{A} contained in the domain $\cap_{k \in \mathbb{Z}} \text{Dom}(\sigma^k)$. We define \mathcal{A}_σ as the $*$ -algebra generated by $\cup_{k \in \mathbb{Z}} \sigma^k(\mathcal{A})$ and extend σ to an automorphism of \mathcal{A}_σ by multiplicativity. Assume that for all $a \in \mathcal{A}$,*

- $\sigma^k(a) \text{Dom}(D) \subseteq \text{Dom}(D)$ and $[D, \sigma^k(a)]_\sigma$ is bounded,
- $\sigma^k(a)(1 + D^2)^{-1/2}$ is B -compact for all $k \in \mathbb{Z}$.

Then $(\mathcal{A}_\sigma, X, D, \sigma)$ is a twisted Kasparov module.

We call $(\mathcal{A}_\sigma, X, D, \sigma)$ the **saturation** of the weakly twisted Kasparov module $(\mathcal{A}, X, D, \sigma)$. Whenever referring to the saturation of a weakly twisted Kasparov module $(\mathcal{A}, X, D, \sigma)$, we tacitly assume that $\mathcal{A} \subseteq \cap_{k \in \mathbb{Z}} \text{Dom}(\sigma^k)$ and that all the assumptions of Proposition 1.4 hold.

It is currently unclear if twisted Kasparov modules carry index theoretic information in all generality. They do if they satisfy the following mild condition ([20], see Proposition 1.13 below).

Definition 1.5 ([21]). A weakly twisted Kasparov module $(\mathcal{A}, X, D, \sigma)$ satisfies the twisted Lipschitz condition if

$$[|D|, a]_\sigma = |D|a - \sigma(a)|D|, \quad \text{is bounded for all } a \in \mathcal{A}.$$

In short, we say that $(\mathcal{A}, X, D, \sigma)$ is **Lipschitz regular**.

Example 1.6 (Non-isometric diffeomorphism on the circle). We describe the simplest case of a twisted spectral triple associated to a conformal diffeomorphism on a manifold (see [20, 47, 49, 50]). Denote by $S^1 \subset \mathbb{C}$ the unit circle and let $\gamma : S^1 \rightarrow S^1$ be a diffeomorphism, which generates an action of \mathbb{Z} on S^1 . A fact that we implicitly use in this example is that all diffeomorphisms of S^1 act conformally. Write $|\gamma'(z)|$ for the pointwise absolute value of the derivative $\frac{d\gamma}{dz}(z)$. Consider the unitary representation of \mathbb{Z} generated by the operator

$$\pi(\gamma) \in \mathcal{B}(L^2(S^1)), \quad \pi(\gamma)\phi(z) := |\gamma'(z)|^{-\frac{1}{2}}\phi(\gamma(z)). \quad (1.1)$$

Define $\sigma : C^\infty(S^1) \rtimes^{\text{alg}} \mathbb{Z} \rightarrow C^\infty(S^1) \rtimes^{\text{alg}} \mathbb{Z}$ by $\sigma(f\gamma^n)(z) = f|\gamma'|^n\gamma^n$. Then one easily checks that the data $(C^\infty(S^1) \rtimes^{\text{alg}} \mathbb{Z}, L^2(S^1), i\frac{d}{dx}, \sigma)$ defines a Lipschitz regular twisted spectral triple. Setting $\iota : C(S^1) \rightarrow C(S^1) \rtimes \mathbb{Z}$ to be the inclusion, it holds that

$$\iota^* \left[\left(C^\infty(S^1) \rtimes^{\text{alg}} \mathbb{Z}, L^2(S^1), i\frac{d}{dx}, \sigma \right) \right] = \left[\left(C^\infty(S^1), L^2(S^1), i\frac{d}{dx} \right) \right] \in K^1(C(S^1)).$$

This proves that

$$0 \neq \left[\left(C^\infty(S^1) \rtimes^{\text{alg}} \mathbb{Z}, L^2(S^1), i\frac{d}{dx}, \sigma \right) \right] \in K^1(C(S^1) \rtimes \mathbb{Z}).$$

By adding to $i\frac{d}{dx}$ the projection onto the constant functions in $L^2(S^1)$ we obtain an invertible self-adjoint operator D in $L^2(S^1)$ for which $(C^\infty(S^1) \rtimes^{\text{alg}} \mathbb{Z}, L^2(S^1), D, \sigma)$ is a twisted spectral triple representing the same K -homology class. Then applying [47, Proposition 3.3] we find that the difference

$$\sigma(a) - |D|a|D|^{-1} \in \mathcal{L}^{1,\infty}(L^2(S^1)),$$

and $(\sigma(a) - |D|a|D|^{-1})D$ extends to a bounded operator. Writing $\tilde{\sigma}(a) := |D|a|D|^{-1}$, we find that $(C^\infty(S^1) \rtimes^{\text{alg}} \mathbb{Z}, L^2(S^1), D, \tilde{\sigma})$ is a weakly twisted spectral triple, again representing the same K -homology class.

1.2 Invertible amplifications

In order to establish the link with KK -theory for Lipschitz regular twisted Kasparov modules, we first need a technical construction to rid possible problems related to invertibility of the operator D . The proof of the following lemma is analogous to that of [23, Lemma 3.6].

Lemma 1.7. *Let X be a Hilbert C^* -module over B and $D : \text{Dom } D \rightarrow X$ a self-adjoint regular operator with $(D \pm i)^{-1} \in \mathbb{K}_B(X)$. Set $\tilde{D} := D \oplus (-D)$ defined on $\text{Dom}(D) \oplus \text{Dom}(D) \subseteq X \oplus X$. Then there exists a self-adjoint compact operator $R \in \mathbb{K}_B(X \oplus X)$ such that*

1. $R(X \oplus X) \subseteq \text{Dom}(\tilde{D})$ and the operators $\tilde{D}R$ and $R\tilde{D}$ extend to B -compact operators on $X \oplus X$;

2. the operator $D_{\text{amp}} := \tilde{D} + R$ is invertible with $D_{\text{amp}}^{-1} \in \mathbb{K}_B(X \oplus X)$.

If X is $\mathbb{Z}/2$ -graded by γ and D is odd, we can take R odd for the grading $\gamma \oplus -\gamma$ on $X \oplus X$.

Proof. The operator

$$R := \begin{pmatrix} 0 & (1 + D^2)^{-1} \\ (1 + D^2)^{-1} & 0 \end{pmatrix},$$

satisfies all the necessary requirements. Indeed, D_{amp} is a bounded perturbation of \tilde{D} and thus has compact resolvent. Now $D_{\text{amp}}^2 = f(D) \oplus f(D)$, where $f(x) = x^2 + (1 + x^2)^{-2}$, which is a strictly positive function. Since $f > 1$ outside a compact interval around 0 and $f(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, it follows that $\frac{1}{f}$ is a C_0 -function, so D_{amp} has a compact inverse. \square

Definition 1.8. Let $(\mathcal{A}, X, D, \sigma)$ be a weakly twisted Kasparov module with $(D \pm i)^{-1} \in \mathbb{K}_B(X)$. An **invertible amplification** of $(\mathcal{A}, X, D, \sigma)$ is a quadruple $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ with D_{amp} as in Lemma 1.7. Here $X \oplus X$ is equipped with the A -action $a(x \oplus x') := ax \oplus ax'$ and the partially defined automorphism $\sigma_{\text{amp}}(T_1 \oplus T_2) := \sigma(T_1) \oplus T_2$ for $T_1 \in \text{Dom}(\sigma)$ and $T_2 \in \text{End}_B^*(X)$.

If $(\mathcal{A}, X, D, \sigma)$ is an even cycle with grading γ , we grade the module $X \oplus X$ in the cycle $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ by $\gamma \oplus -\gamma$, and tacitly assume that R from Lemma 1.7 is chosen to be an odd operator. Invertible amplifications will not play any conceptually important role in this paper, but rather a technical role as it allows us to reduce proofs to the case that D is invertible.

One of our main tools is the following integral formula for fractional powers of the invertible operator $1 + D^2$.

Lemma 1.9. *Let D be a regular self-adjoint operator on a Hilbert C^* -module X_B . For any $0 < r < 1$*

$$(1 + D^2)^{-r} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{-r} (1 + D^2 + \lambda)^{-1} d\lambda, \quad (1.2)$$

is a norm convergent integral. Moreover we have the estimates

$$\begin{aligned} \|(1 + D^2 + \lambda)^{-s}\|_{\text{End}_B^*(X)} &\leq (1 + \lambda)^{-s}, \\ \|D(1 + D^2 + \lambda)^{-\frac{1}{2}}\|_{\text{End}_B^*(X)} &\leq 1 \quad \text{and} \quad \|D^2(1 + D^2 + \lambda)^{-1}\|_{\text{End}_B^*(X)} \leq 1. \end{aligned}$$

The integral formula has been used in the Hilbert C^* -module context since the work of Baaj-Julg [3]. A detailed treatment can be found in [11, Appendix A, Remark 3]. The estimates can be found in [11, Appendix A, Remark 5].

For a closed operator $D : \text{Dom } D \rightarrow X$ we view $\text{Dom } D$ as a Hilbert C^* -module over B equipped with the graph inner product. The following Lemma is similar to results obtained in [11, Appendix A].

Lemma 1.10. *Let X be a B -Hilbert C^* -module, $D : \text{Dom } D \rightarrow X$ a self-adjoint regular operator and $R \in \mathbb{K}_B(X)$ is a self-adjoint operator such that*

$$R : X \rightarrow \text{Dom } D, \quad \text{and} \quad DR \in \mathbb{K}_B(X).$$

Then the bounded adjointable operator

$$(1 + D^2)^{-1/2} - (1 + (D + R)^2)^{-1/2}$$

on X defines a B -compact adjointable operator $X \rightarrow \text{Dom } D^2$. In particular $|D| - |D + R|$ has a bounded extension to X which is B -compact whenever $(D \pm i)^{-1} \in \mathbb{K}_B(X)$.

Proof. As DR is everywhere defined and compact, its densely defined adjoint RD extends to a compact operator as well. Moreover we have

$$(D + R)^2 = D^2 + RD + DR + R^2 : \text{Dom } D^2 \rightarrow X,$$

so $(D + R)^2$ is a compact perturbation of D^2 and $\text{Dom } D^2 = \text{Dom}(D + R)^2$. Using the integral formula (1.2), we write

$$\begin{aligned} D^2((1 + D^2)^{-1/2} - (1 + (D + R)^2)^{-1/2}) \\ = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} D^2(1 + D^2 + \lambda)^{-1} (DR + RD + R^2)(1 + (D + R)^2 + \lambda)^{-1} d\lambda, \end{aligned}$$

an identity valid on $\text{Dom } D$. Using that $DR + RD + R^2$ is compact, the integrand is compact and by Lemma 1.9 it is norm bounded by $\|DR + RD + R^2\| \lambda^{-1/2} (1 + \lambda)^{-1}$. Therefore the integral is norm convergent, and hence $(1 + D^2)^{-1/2} - (1 + (D + R)^2)^{-1/2}$ defines a compact operator $X \rightarrow \text{Dom } D^2$. Now consider $|D| - |D + R|$ and observe that the function $x \mapsto |x| - x^2(1 + x^2)^{-1/2}$ belongs to $C_0(\mathbb{R})$. It thus suffices to prove that $D^2(1 + D^2)^{-1/2} - (D + R)^2(1 + (D + R)^2)^{-1/2}$ is B -compact. This now follows from compactness of $DR + RD + R^2$, compactness of $(1 + D^2)^{-1/2} - (1 + (D + R)^2)^{-1/2}$ as an operator into $\text{Dom } D^2$ and boundedness of $D^2 : \text{Dom } D^2 \rightarrow X$. \square

Proposition 1.11. *Any compact weakly twisted Kasparov module $(\mathcal{A}, X, D, \sigma)$ admits an invertible amplification. Any invertible amplification $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ is a compact weakly twisted Kasparov module with D_{amp} invertible. Moreover $(\mathcal{A}, X, D, \sigma)$ is Lipschitz regular if and only if $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ is Lipschitz regular.*

Proof. While we assume that $(\mathcal{A}, X, D, \sigma)$ is compact, D is Fredholm and thus $(\mathcal{A}, X, D, \sigma)$ admits an invertible amplification using Lemma 1.7. It is a short algebraic verification to check that $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ is a weakly twisted Kasparov module. Since the inverse of D_{amp} is B -compact, its resolvent is B -compact and therefore $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ is compact. Recall the notation $\tilde{D} := D \oplus (-D)$ and note that $(\mathcal{A}, X, D, \sigma)$ is Lipschitz regular if and only if $(\mathcal{A}, X \oplus X, \tilde{D}, \sigma_{\text{amp}})$ is Lipschitz regular. The statement about Lipschitz regularity follows from that $|\tilde{D}| - |D_{\text{amp}}|$ is B -compact by Lemma 1.10. \square

Remark 1.12. It is clear from the construction that invertible amplifications commute with saturations in the following sense. Assume that $(\mathcal{A}, X, D, \sigma)$ is a unital weakly twisted Kasparov module satisfying all the assumptions of Proposition 1.4 (see page 7). Then $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ also satisfies all the assumptions of Proposition 1.4 and clearly

$$(\mathcal{A}_\sigma, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}}) = (\mathcal{A}_{\sigma_{\text{amp}}}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}}).$$

In this equation, the left hand side is the invertible amplification of the saturation, and the right hand side is the saturation of the invertible amplification.

1.3 The bounded transform for Lipschitz regular twisted Kasparov modules

We present a proof of the fact that the bounded transform of a Lipschitz regular weakly twisted unbounded Kasparov module is a well-defined bounded Kasparov module. Our proof closely follows the proof in the special case of twisted spectral triples due to Connes-Moscovici [21]. At the expense of a range of other assumptions, Kaaad in [38] proves a similar statement in the absence of the Lipschitz condition.

Recall that a *normalising function* χ is a function $\chi : \mathbb{R} \rightarrow \mathbb{R}$, continuous on $\text{Spec}(D)$, with the properties that $\chi(-x) = -\chi(x)$ and $\lim_{x \rightarrow \pm\infty} \chi(x) = \pm 1$.

Proposition 1.13 ([20]). *Let $(\mathcal{A}, X, D, \sigma)$ be a compact Lipschitz regular weakly twisted Kasparov module. Then for any choice of normalising function χ , the triple $(\mathcal{A}, X, \chi(D))$ is a bounded Kasparov module.*

Proof. Let $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ be an invertible amplification of $(\mathcal{A}, X, D, \sigma)$ as in Proposition 1.11. In the special case $\chi(x) = x(1+x^2)^{-1/2}$ we have that $\chi(D) \oplus (-\chi(D)) - \chi(D_{\text{amp}}) \in \mathbb{K}_B(X \oplus X)$ by Lemma 1.10. Up to compact perturbations, $\chi(D)$ and $\chi(D_{\text{amp}})$ are independent of the normalising function χ , so $\chi(D) \oplus (-\chi(D)) - \chi(D_{\text{amp}}) \in \mathbb{K}_B(X \oplus X)$ for any normalising function. It is clear that $(\mathcal{A}, X, \chi(D))$ is a bounded Kasparov module if and only if $(\mathcal{A}, X \oplus X, \chi(D) \oplus (-\chi(D)))$ is a bounded Kasparov module, which in turn is equivalent to $(\mathcal{A}, X \oplus X, \chi(D_{\text{amp}}))$ being a bounded Kasparov module by the preceding argument. Moreover, $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ is Lipschitz regular since $(\mathcal{A}, X, D, \sigma)$ is Lipschitz regular. We can in particular assume that $(\mathcal{A}, X, D, \sigma)$ satisfies that D is invertible.

Since $(\mathcal{A}, X, D, \sigma)$ is compact, the operator D has B -compact resolvent by definition, and it suffices to prove the proposition for the specific choice function $\chi(x) := \text{sgn}(x)$. We set $F := D|D|^{-1}$ and compute for $a \in \mathcal{A}$ that

$$\begin{aligned} [F, a] &= D|D|^{-1}a - aD|D|^{-1} = |D|^{-1}(Da - |D|aF) = |D|^{-1}([D, a]_{\sigma} + \sigma(a)D - |D|aF) \\ &= |D|^{-1}([D, a]_{\sigma} + (\sigma(a)|D| - |D|a)F) = |D|^{-1}([D, a]_{\sigma} - [|D|, a]_{\sigma}F). \end{aligned} \quad (1.3)$$

The operator $[D, a]_{\sigma} - [|D|, a]_{\sigma}F$ has a bounded extension by the bounded twisted commutator condition and the twisted Lipschitz condition. By B -compactness of $|D|^{-1}$, we conclude that $[F, a]$ is B -compact for all $a \in \mathcal{A}$. \square

We note that the operator $\chi(D)$ in Proposition 1.13 is independent of the twist σ . That is, if there exists an automorphism $\tilde{\sigma}$ of \mathcal{A} such that D and $|D|$ have bounded twisted commutators with \mathcal{A} , this suffices to prove that the bounded transform is a Fredholm module. This observation is of crucial importance in the sequel.

1.4 Logarithmic dampening of Lipschitz regular twisted Kasparov modules

In this subsection we associate to a Lipschitz regular weakly twisted unbounded Kasparov module an ordinary unbounded Kasparov module in the same KK -class. To this end we use the functional calculus and the logarithm function. In [48, Theorem 8] Pierrot observed that this is possible for crossed products by a conformal diffeomorphism. In [24, Theorem 9.14], [30, Section 10] and [31, Section 2.2.1 and 5.1] a related, but slightly broader set of examples is discussed. First, we require a series of lemmas.

Lemma 1.14. *Let Δ be a positive self-adjoint regular operator on a Hilbert C^* -module X with $\Delta^{-1} \in \text{End}_B^*(X)$. Let $a \in \text{End}_B^*(X)$ be a self-adjoint bounded operator on X such that $a \text{Dom } \Delta \subseteq \text{Dom } \Delta$. Then the operators $\Delta a \Delta^{-1}$, $\Delta^{-1} a \Delta$ and $[\log(\Delta), a]$ extend to bounded adjointable operators on X .*

Proof. The operator $\Delta a \Delta^{-1}$ is closed and defined on all of X , so by the closed graph theorem it is bounded. It has a densely defined adjoint $\Delta^{-1} a \Delta$, so $\Delta a \Delta^{-1}$ is bounded and adjointable. The adjoint of $\Delta a \Delta^{-1}$ is the closure of $\Delta^{-1} a \Delta$ which therefore extends to a bounded adjointable operator.

For each state $\varphi : B \rightarrow \mathbb{C}$, consider the Hilbert space $X_\varphi := X \otimes_\varphi H_\varphi$, with H_φ the associated GNS representation of B . Applying the Hadamard three lines theorem, for any $z \in \mathbb{C}$ with $\text{Re}(z) \in [-1, 1]$ the operator $\Delta^{-z} a \Delta^z$ has a bounded extension to X_φ and

$$\|\Delta^{-z} a \Delta^z\| \leq \|\Delta^{-1} a \Delta\|^{(1+\text{Re}(z))/2} \|\Delta a \Delta^{-1}\|^{(1-\text{Re}(z))/2}.$$

Moreover, the function $z \mapsto \Delta^{-z} a \Delta^z \in \mathcal{B}(X_\varphi)$ is holomorphic. We compute its derivative at 0 to be

$$\left. \frac{d}{dz} \right|_{z=0} \Delta^{-z} a \Delta^z = [\log(\Delta), a].$$

Holomorphicity of $z \mapsto \Delta^{-z} a \Delta^z \in \mathcal{B}(X_\varphi)$ allows us to deduce that $[\log(\Delta), a]$ extends to a bounded operator on X_φ . Since the state φ is arbitrary, all statements are valid in the C^* -module X by the local-global principle, [37]. \square

Lemma 1.15. *Let X be a B -Hilbert C^* -module, $D : \text{Dom } D \rightarrow X$ a self-adjoint regular operator and $R \in \mathbb{K}_B(X)$ a self-adjoint operator such that*

$$R : X \rightarrow \text{Dom } D, \quad \text{and} \quad DR \in \mathbb{K}_B(X).$$

Then $\log(1 + |D|) - \log(1 + |D + R|)$ has a bounded extension to X which is B -compact whenever $(D \pm i)^{-1} \in \mathbb{K}_B(X)$.

Proof. Since $x \mapsto 2 \log(1 + |x|) - \log(1 + x^2)$ is in $C_0(\mathbb{R})$, it suffices to prove that $\log(1 + D^2) - \log(1 + (D + R)^2)$ is B -compact. On the core $\text{Dom}(D^2)$ we use the strongly convergent integral expression

$$\log(1 + D^2) = \int_0^1 D^2 (1 + tD^2)^{-1} dt.$$

Since $\text{Dom } D^2 = \text{Dom}(D + R)^2$, we can write

$$\log(1 + D^2) - \log(1 + (D + R)^2) = \int_0^1 D^2 (1 + tD^2)^{-1} dt - \int_0^1 (D + R)^2 (1 + t(D + R)^2)^{-1} dt.$$

As $DR + RD + R^2 \in \mathbb{K}_B(X)$ and $\|(1 + t(D + R)^2)^{-1}\| \leq 1$, it suffices to prove compactness of

$$\begin{aligned} \int_0^1 D^2 ((1 + t(D + R)^2)^{-1} - (1 + tD^2)^{-1}) dt \\ = \int_0^1 t D^2 (1 + tD^2)^{-1} (DR + RD + R^2) (1 + t(D + R)^2)^{-1} dt. \end{aligned}$$

We have that $\|t D^2 (1 + tD^2)^{-1}\| \leq 1$ for all $t \in [0, 1]$, so the integrand is norm-continuous and compact on $[0, 1]$. We conclude that $\log(1 + D^2) - \log(1 + (D + R)^2)$ is B -compact. \square

Consider the continuous function $\text{sgnlog}(x) := x|x|^{-1}\log(1+|x|) \in C(\mathbb{R})$, whose derivative is the C_0 -function $x \mapsto (1+|x|)^{-1}$. For a self-adjoint regular operator D on a Hilbert C^* -module X , the self-adjoint regular operator $\text{sgnlog}(D)$ satisfies $\text{Dom } D \subset \text{Dom } \text{sgnlog}(D)$ and $\text{Dom } D$ is a core for $\text{sgnlog}(D)$.

Theorem 1.16. *Assume that $(\mathcal{A}, X, D, \sigma)$ is a compact Lipschitz regular weakly twisted unbounded Kasparov module. Then the logarithmic transform*

$$D_{\log} := \text{sgnlog}(D),$$

makes $(\mathcal{A}, X, D_{\log})$ into a compact unbounded Kasparov module which represents the same KK -class as $(\mathcal{A}, X, D, \sigma)$.

Proof. Consider an invertible amplification $(\mathcal{A}, X \oplus X, D_{\text{amp}}, \sigma_{\text{amp}})$ of $(\mathcal{A}, X, D, \sigma)$ as in Proposition 1.11. We note that $x \mapsto \text{sgnlog}(x) - x(1+x^2)^{-1/2}\log(1+|x|)$ is a C_0 -function, so $\text{sgnlog}(D) - D(1+D^2)^{-1/2}\log(1+|D|)$ and $\text{sgnlog}(D_{\text{amp}}) - D_{\text{amp}}(1+D_{\text{amp}}^2)^{-1/2}\log(1+|D_{\text{amp}}|)$ are B -compact. By Lemma 1.10 and Lemma 1.15 the operator

$$(\tilde{D}(1+\tilde{D}^2)^{-1/2}\log(1+|\tilde{D}|)) - D_{\text{amp}}(1+D_{\text{amp}}^2)^{-1/2}\log(1+|D_{\text{amp}}|)$$

is B -compact. We conclude that $\text{sgnlog}(\tilde{D}) = \text{sgnlog}(D) \oplus (-\text{sgnlog}(D))$ is a B -compact perturbation of $\text{sgnlog}(D_{\text{amp}})$ on $X \oplus X$. Since $[\text{sgnlog}(\tilde{D}), a] = [\text{sgnlog}(D), a] \oplus 0_X$, we can without loss of generality assume that D is invertible.

The only thing to prove is that D_{\log} has bounded commutators with \mathcal{A} . Consider

$$D_{\log,0} := |D|^{-1}\log(|D|)D = TD \quad \text{with} \quad T = |D|^{-1}\log(|D|).$$

Since $D_{\log} - D_{\log,0}$ is bounded, it suffices to prove that $D_{\log,0} = TD$ has bounded commutators with \mathcal{A} .

The operator $T[D, a]_{\sigma}$ is bounded and

$$[TD, a] = T[D, a]_{\sigma} + (T\sigma(a) - aT)D.$$

It therefore suffices to show that $(T\sigma(a) - aT)D$ is bounded. We look at

$$(T\sigma(a) - aT)D = ((|D|^{-1}\log|D|)\sigma(a) - a|D|^{-1}\log|D|)D. \quad (1.4)$$

Since

$$|D|^{-1}\sigma(a) - a|D|^{-1} = -|D|^{-1}[[D], a]_{\sigma}|D|^{-1},$$

it holds that

$$(|D|^{-1}\log|D|)\sigma(a) = (\log|D|)a|D|^{-1} - (\log|D|)|D|^{-1}[[D], a]_{\sigma}|D|^{-1}.$$

Thus up to a bounded operator (1.4) equals

$$[\log|D|, a]|D|^{-1}D = [\log|D|, a]F.$$

Now $a \text{Dom } |D| \subset \text{Dom } |D|$ and Lemma 1.14 applied to $\Delta = |D|$ gives that $[\log|D|, a]$ is bounded, so we are done. \square

Proposition 1.17. *Let $(\mathcal{A}, H, D, \sigma)$ be a compact Lipschitz regular weakly twisted spectral triple. For $s > 0$ we have $(1 + D^2)^{-s/2} \in \mathcal{L}^1$ if and only if $e^{-s|D_{\log}|} \in \mathcal{L}^1$. Hence the weakly twisted spectral triple $(\mathcal{A}, H, D, \sigma)$ is finitely summable if and only if the spectral triple $(\mathcal{A}, H, D_{\log})$ is Li_1 -summable.*

The proposition follows from the definitions, see [31], because

$$e^{-s|D_{\log}|} = e^{-s \log(1+|D|)} = (1 + |D|)^{-s}.$$

Example 1.18. Let us revisit the twisted spectral triple $(C^\infty(S^1) \rtimes^{\text{alg}} \mathbb{Z}, L^2(S^1), D, \sigma)$ of Example 1.6. The logarithmic dampening $(C^\infty(S^1) \rtimes^{\text{alg}} \mathbb{Z}, L^2(S^1), D_{\log})$ is an ordinary spectral triple by Theorem 1.16. We can replace the operator $D_{\log} := D|D|^{-1} \log(1 + |D|)$ by the operator ∂_{\log} , given by

$$\partial_{\log}(e^{2\pi i n t}) := (\text{sgn}(n) \log |n|) \cdot e^{2\pi i n t}, \quad \text{for } n \neq 0 \quad \text{and} \quad \partial_{\log}(1) = 0$$

as the operator $\partial_{\log} - D_{\log}$ is bounded.

Corollary 1.19. *Let (\mathcal{A}, X, D) be a triple containing the following information:*

- X is a countably generated Hilbert C^* -module over B ;
- D is a regular self-adjoint operator on X with B -compact inverse;
- \mathcal{A} is a $*$ -algebra represented on X such that $\mathcal{A} \text{Dom}(D) \subseteq \text{Dom}(D)$.

Let $F := D|D|^{-1}$ and assume that $[F, a] : X \rightarrow \text{Dom}(D)$ for all $a \in \mathcal{A}$. Then with

$$D_{\log} := \text{sgn} \log(D),$$

the collection $(\mathcal{A}, X, D_{\log})$ is an unbounded Kasparov module.

Proof. By Theorem 1.16, it suffices to prove that $(\mathcal{A}, X, D, \sigma)$ is a Lipschitz regular weakly twisted Kasparov module for $\sigma(a) := |D|a|D|^{-1}$. We have that $\mathcal{A} \subseteq \text{Dom}(\sigma)$ by Lemma 1.14 (see page 12) because \mathcal{A} preserves $\text{Dom}(D) = \text{Dom}(|D|)$. The closed graph theorem and boundedness of $[F, a]$ on X imply that if $[F, a]X \subseteq \text{Dom}(D)$ then $[F, a] : X \rightarrow \text{Dom}(D)$ is continuous. In particular, $|D|[F, a]$ is a bounded adjointable operator. Moreover, for any $a \in \mathcal{A}$, $[D, a]_{\sigma}$ is bounded because

$$[D, a]_{\sigma} = Da - \sigma(a)D = F|D|a - |D|aF = |D|[F, a].$$

Since D has B -compact inverse, the preceding argument shows that $(\mathcal{A}, X, D, \sigma)$ is a weakly twisted spectral triple. Clearly, $[|D|, a]_{\sigma} = |D|a - |D|a = 0$ is bounded so $(\mathcal{A}, X, D, \sigma)$ is a Lipschitz regular weakly twisted unbounded Kasparov module. \square

1.5 Higher order Kasparov modules

In this section we describe a weakening of the definition of Kasparov module, which as far as we know first appeared in the work of Wahl, [54]. Key observations can be found in [33, Lemma 51] and the notion reappeared in work by the first two listed authors on Cuntz-Krieger

algebras [27]. Related notions are anticipated in the literature, e.g. [11]. It allows for both higher order elliptic operators in classical settings, and provides a method for handling some of the difficulties that arise in dynamical examples. The main idea here is to relax the requirement that the commutators $[D, a]$ be bounded, by only asking for a weaker bound relative to D .

To introduce the concept we need to ensure that domain issues are appropriately addressed, and so need some preliminary definitions.

Definition 1.20. Let B be a C^* -algebra. Let X_B be a countably generated right B -Hilbert C^* -module, T a densely defined operator on X_B , D a densely defined self-adjoint regular operator on X_B and $\varepsilon > 0$. We say that T is ε -bounded with respect to D if the operators $T(1 + D^2)^{-\frac{1-\varepsilon}{2}}$ and $(1 + D^2)^{-\frac{1-\varepsilon}{2}}T$ are densely defined and norm-bounded.

Definition 1.21. Let B be a C^* -algebra. Let X_B be a countably generated right B -Hilbert C^* -module. An operator $a \in \text{End}_B^*(X)$ has ε -bounded commutators with the self-adjoint regular operator $D : \text{Dom } D \subset X_B \rightarrow X_B$ if

1. $a \text{Dom } D \subset \text{Dom } D$;
2. $[D, a]$ is ε -bounded with respect to D .

In short we say that $[D, a]$ is ε -bounded.

Example 1.22. Let us give a geometric example of ε -bounded commutators to explain the appearance of the parameter $\varepsilon > 0$, and the name ‘higher order spectral triple’. Let D be a self-adjoint elliptic pseudodifferential operator of order $m > 0$ acting on a vector bundle $E \rightarrow M$ on a closed manifold M . The Hilbert space is $H = L^2(M, E)$. The domain of D is the Sobolev space $W^{m,2}(M, E)$. If $a \in C^\infty(M)$, then it is well-known that $[D, a]$ is a pseudodifferential operator of order $m - 1$. Hence $(1 + D^2)^{\frac{1-m}{2m}}[D, a]$ and $[D, a](1 + D^2)^{\frac{1-m}{2m}}$ are pseudodifferential operators of order 0, thus bounded on $L^2(M, E)$.

We conclude that any $a \in C^\infty(M)$ has $1/m$ -bounded commutators with D . As such, one can consider the reciprocal ε^{-1} as an ‘order’ of the operator D appearing in an ε -bounded commutator.

Remark 1.23. Somewhat undermining the notion of order, it is for most purposes not actually necessary for the value of ε to be the same for all $a \in \mathcal{A}$. We will not carry this level of generality with us.

Wahl called these Kasparov modules ‘truly unbounded’. Goffeng-Mesland [27] dubbed them ε -unbounded Kasparov modules, with ε the analogue of the reciprocal of the order. Due to examples arising from higher order differential operators, we feel that the adjective ‘higher order’ is most appropriate to describe the notion.

Definition 1.24. Let \mathcal{A} be a $*$ -algebra, $m > 0$ and set $\varepsilon = \frac{1}{m}$. An odd order m Kasparov module is a triple (\mathcal{A}, X_B, D) where X_B is a countably generated B -Hilbert C^* -module with a $*$ -representation of \mathcal{A} and D is a self-adjoint operator such that

1. $a(1 + D^2)^{-\frac{1}{2}} \in \mathbb{K}_B(X)$ for all $a \in \mathcal{A}$ and
2. the image \mathcal{A} in $\text{End}_B^*(X)$ is contained in the space

$$\text{Lip}^\varepsilon(X, D) := \{a \in \text{End}_B^*(X) : a \cdot \text{Dom } D \subset \text{Dom } D, [D, a] \text{ is } \varepsilon\text{-bounded}\}.$$

If X_B is a $\mathbb{Z}/2\mathbb{Z}$ -graded B -Hilbert C^* -module, and (\mathcal{A}, X_B, D) is as above with D odd in the grading on X_B and A acting as even operators, we say that (\mathcal{A}, X_B, D) is an even order m spectral triple. Otherwise, we say that (\mathcal{A}, X_B, D) is odd.

As above (see Remark 1.2), we treat \mathcal{A} as a subalgebra of $\text{End}_B^*(X)$ despite it acting via a possibly non-faithful representation.

Remark 1.25. A higher order m -Kasparov module for $m > 0$ is a higher order m' -Kasparov module for $m' > 0$ whenever $m' \geq m$. If we can take $m = 1$ then we talk about an ordinary unbounded Kasparov module, and if $B = \mathbb{C}$ so that X is a Hilbert space we speak about higher order spectral triples.

Apart from higher order (elliptic) differential operators, higher order spectral triples arise in the construction of the unbounded Kasparov product. The context is typically that of a dynamical system on a metric measure space, for which the dynamics does not preserve the metric nor the measure. The phenomenon has been examined in detail for Cuntz-Krieger algebras [27, Section 5 and 6], Cuntz-Pimsner algebras of vector bundles [29, Section 4], group C^* -algebras and boundary crossed products of groups of Möbius transformations [46, Section 4] and Delone sets with finite local complexity [7, Section 5]. For later reference in this context, we include the following general construction.

Example 1.26. Let (\mathcal{A}, X, D) be an ordinary unbounded Kasparov module and $0 < s < 1$. If the bounded operators $[D, a]$ satisfy the mapping property

$$[D, a] : \text{Dom } D \rightarrow \text{Dom } |D|^s \quad \text{for all } a \in \mathcal{A}, \quad (1.5)$$

then the triple $(\mathcal{A}, X, D|D|^s)$ is a higher order Kasparov module of order $m = \frac{1+s}{s}$. This can be seen by writing $\text{Dom } D = (1 + |D|^{2+2s})^{-\frac{1-\varepsilon}{2}} X$ with $\varepsilon := \frac{s}{1+s}$. The operator

$$[D|D|^s, a](1 + |D|^{2+2s})^{-\frac{1-\varepsilon}{2}}, \quad (1.6)$$

is bounded because on $\text{Dom } D$ it holds that

$$[D|D|^s, a] = [|D|^s D, a] = |D|^s [D, a] + [|D|^s, a] D,$$

and $[|D|^s, a]$ is bounded for $0 < s < 1$ whereas $[D, a] : \text{Dom } D \rightarrow \text{Dom } |D|^s$ by assumption.

The notion of order carries some degree of arbitrariness due to Remark 1.25. In fact, if D was a self-adjoint first order elliptic operator on a closed manifold we would expect the order of $D|D|^s$ to be $1 + s$ rather than $\frac{1+s}{s}$. Under additional smoothness assumptions the order of $(\mathcal{A}, X, D|D|^s)$ improves. For simplicity, we assume that (\mathcal{A}, X, D) is an ordinary unbounded Kasparov module with D invertible. Let $s \in (-1, 1]$ and assume that

$$[|D|^s, a]|D|^{1-s} \quad \text{and} \quad |D|^s [D, a]|D|^{-s} \quad \text{are bounded for all } a \in \mathcal{A}. \quad (1.7)$$

These two conditions are indeed satisfied when D is a self-adjoint first order elliptic operator on a closed manifold and are often easy to check in practice. Set $\varepsilon = \frac{1}{1+s}$ so $\frac{1-\varepsilon}{2} = \frac{s}{2+2s}$. Under the conditions (1.7), we can continue the computation from Equation (1.6), obtaining

$$\begin{aligned} [D|D|^s, a](1 + |D|^{2+2s})^{-\frac{1-\varepsilon}{2}} &= (|D|^s [D, a] + [|D|^s, a] D) (1 + |D|^{2+2s})^{-\frac{s}{2+2s}} = \\ &= (|D|^s [D, a]|D|^{-s} + [|D|^s, a]|D|^{1-s} F) |D|^s (1 + |D|^{2+2s})^{-\frac{s}{2+2s}} \end{aligned}$$

Since $|D|^s (1 + |D|^{2+2s})^{-\frac{s}{2+2s}}$ is bounded, the conditions (1.7) ensure that $[D, a]$ is ε -bounded for $\varepsilon = \frac{1}{1+s}$. In particular, $(\mathcal{A}, X, D|D|^s)$ is of order $1 + s$.

1.6 The bounded transform for higher order Kasparov modules

We now come to the main result about higher order Kasparov cycles, concerning the bounded transform and the relation to KK -theory. The bounded transform,

$$F_D := D(1 + D^2)^{-1/2}, \quad (1.8)$$

of a higher order Kasparov module provides a Fredholm module.

Theorem 1.27 (cf. [3]). *The bounded transform $(A, X_B, D(1 + D^2)^{-\frac{1}{2}})$ of a higher order Kasparov module (A, X_B, D) is an (A, B) Kasparov module and hence defines a class in $KK_*(A, B)$. Moreover, for any choice of normalising function χ it holds that $a(D(1 + D^2)^{-\frac{1}{2}} - \chi(D)) \in \mathbb{K}_B(X)$ is B -compact for all $a \in A$.*

For $0 < \varepsilon < 1$, this theorem was first proved in [54, Discussion after Definition 2.4] and independently in [33, Lemma 51]. Several years later the result resurfaced in [27, Appendix], where several statements concerning the Kasparov product of higher order modules are discussed. The proof is based on the integral formula (1.2), and is essentially identical to the proof for ordinary unbounded Kasparov modules [3].

Example 1.28. Let $\gamma : S^1 \rightarrow S^1$ be a (non-isometric) diffeomorphism, generating an action of \mathbb{Z} on $C(S^1)$. Consider the $C(S^1)$ -Hilbert C^* -module $X := \ell^2(\mathbb{Z}) \otimes C(S^1)$, and write e_n for the standard basis of $\ell^2(\mathbb{Z})$. Represent $C(S^1)$ and γ on X via

$$\pi(f)(e_n \otimes \psi) := e_n \otimes (f \circ \gamma^{-n}) \cdot \psi, \quad \alpha(\gamma^k)(e_n \otimes \psi) := e_{n+k} \otimes \psi.$$

Then we can view $\alpha : \mathbb{Z} \rightarrow \text{End}_{C(S^1)}^*(X)$ as a unitary representation and it holds that

$$\alpha(\gamma^k)\pi(f)\alpha(\gamma^{-k}) = \pi(f \circ \gamma^k), \quad k \in \mathbb{Z}$$

and so (π, α) form a covariant representation as $C(S^1)$ -linear adjointable endomorphisms of $(C(S^1), \gamma)$ on X . We denote the corresponding representation by $\pi_X : C(S^1) \rtimes \mathbb{Z} \rightarrow \text{End}_{C(S^1)}(X)$. Define a self-adjoint regular operator N on X by $N(e_n \otimes \psi) := ne_n \otimes \psi$. Then

$$(1 + N^2)^{-1} \in \mathbb{K}_{C(S^1)}(X), \quad [N, \alpha(\gamma)] = \alpha(\gamma), \quad [N, \pi(f)] = 0,$$

and $(C^\infty(S^1) \rtimes_\gamma^{\text{alg}} \mathbb{Z}, X, N)$ is an unbounded Kasparov module. Its associated KK -class in $KK_1(C(S^1) \rtimes_\gamma \mathbb{Z}, C(S^1))$ coincides with that of the Pimsner-Voiculescu extension as its bounded transform defines the Toeplitz extension appearing in Cuntz' proof of the Pimsner-Voiculescu sequence [22].

Pick an $s \in (-1, 1]$. We use the convention that $0^s = 0$. It is easily verified that

$$[|N|^s, \alpha(\gamma)] = (|N|^s - |N - 1|^s)\alpha(\gamma).$$

It follows that

$$[|N|^s, \alpha(\gamma)]|N|^{1-s} = (|N|^s - |N - 1|^s)|N - 1|^{1-s}\alpha(\gamma),$$

which is bounded. Moreover,

$$|N|^s[N, \alpha(\gamma)]|N|^{-s} = |N|^s\alpha(\gamma)|N|^{-s} = |N|^s|N - 1|^{-s}\alpha(\gamma),$$

is also bounded. Therefore, $(C^\infty(S^1) \rtimes_\gamma^{\text{alg}} \mathbb{Z}, X, N)$ satisfies (1.7). The argument in Example 1.26 shows that

$$(C^\infty(S^1) \rtimes_\gamma^{\text{alg}} \mathbb{Z}, X, N_s), \quad N_s := N|N|^s, \quad s \in (-1, 1],$$

is an order $(1+s)$ -Kasparov module representing the same class as $(C^\infty(S^1) \rtimes_\gamma^{\text{alg}} \mathbb{Z}, X, N)$.

The modification N_s in Example 1.28 may seem artificial. The remainder of this section is devoted to the discussion of an example that shows that higher order Kasparov modules arise naturally in a dynamical context, in which the operator N_s plays an essential role. Theorem 1.27 above allows one to keep track of the index theoretic information in such cases. In particular, in Proposition 1.29 below, higher order spectral triples provide a solution to a problem where twisted spectral triples do not.

Consider the spectral triple $(C^\infty(S^1), L^2(S^1), \partial_{\log})$, with ∂_{\log} defined as in Example 1.18. Then the class

$$[(C^\infty(S^1), L^2(S^1), \partial_{\log})] \in K^1(C(S^1)),$$

generates $K^1(C(S^1))$. Let $H := X \otimes_{C(S^1)} L^2(S^1) \simeq \ell^2(\mathbb{Z}) \otimes L^2(S^1)$ and write $\pi := \pi_X \otimes 1$ for the representation of $C(S^1) \rtimes_\gamma \mathbb{Z}$ on H induced by the representation π_X on X from Example 1.28.

Define a self-adjoint operator \mathcal{D}_{\log} on H by $\mathcal{D}_{\log}(e_n \otimes \psi) := e_n \otimes \partial_{\log} \psi$, for $\psi \in \text{Dom } \partial_{\log}$ and write N_s for the self-adjoint operator $N_s \otimes 1$.

Proposition 1.29. *Let $\gamma : S^1 \rightarrow S^1$ be a (non-isometric) diffeomorphism. For any $0 < s \leq 1$*

$$\left(C^\infty(S^1) \rtimes_\gamma^{\text{alg}} \mathbb{Z}, H \oplus H, D := \begin{pmatrix} 0 & N_s + i\mathcal{D}_{\log} \\ N_s - i\mathcal{D}_{\log} & 0 \end{pmatrix} \right),$$

is a higher order spectral triple of order $\frac{1+s}{s}$. It represents the class

$$\partial \left[\left(C(S^1), L^2(S^1), i \frac{d}{dx} \right) \right] \in K^0(C(S^1) \rtimes_\gamma \mathbb{Z}),$$

where $\partial : K^1(C(S^1)) \rightarrow K^0(C(S^1) \rtimes_\gamma \mathbb{Z})$ is the boundary map in the Pimsner-Voiculescu exact sequence.

Proof. It is straightforward to check that D is essentially self-adjoint with compact resolvent, as $D^2 = \text{diag}(|N|^{2+2s} + \mathcal{D}_{\log}^2)$ on $C_c(\mathbb{Z}) \otimes^{\text{alg}} H^1(S^1) \otimes \mathbb{C}^2$. Moreover, the hypotheses of [27, Theorem A.7] are all trivially satisfied, so that if D defines a higher order spectral triple, it represents the product of the classes $[N_s]$ and $[\partial_{\log}]$.

It thus remains to verify ε -bounded commutators. For now we assume that $s \in (-1, 1]$ and consider an $\varepsilon \geq 0$. It holds that

$$[\mathcal{D}_{\log}, \alpha(\gamma)] = [N_s, \pi(f)] = 0, \tag{1.9}$$

and these are bounded operators. Since $\text{Dom } D = \text{Dom } N_s \cap \text{Dom } \mathcal{D}_{\log}$ the operator

$$(1 + N_s)^{\frac{1-\varepsilon}{2}} (1 + N_s^2 + \mathcal{D}_{\log}^2)^{-\frac{1-\varepsilon}{2}},$$

is bounded for $\varepsilon \in [0, 1]$. Thus for any T and $\varepsilon \in [0, 1]$, boundedness of $T(1 + N_s^2)^{-\frac{1-\varepsilon}{2}}$ implies boundedness of $T(1 + N_s^2 + \mathcal{D}_{\log}^2)^{-\frac{1-\varepsilon}{2}}$.

From Example 1.28, we know that

$$[N_s, \alpha(\gamma)](1 + N_s^2)^{-\frac{1-\varepsilon}{2}},$$

is a bounded operator for $0 \leq \varepsilon \leq \frac{1}{1+s}$. From this fact and Equation (1.9) we deduce that $[D, \alpha(\gamma)]$ is ε -bounded with respect to D for $0 < \varepsilon \leq \min\left(1, \frac{1}{1+s}\right)$.

Lastly we address $[\mathcal{D}_{\log}, \pi(f)](1 + N_s^2)^{-\frac{1-\varepsilon}{2}}$. Recall the notation $\pi(\gamma^n)$ from Example 1.6. It holds that

$$\left\| [\mathcal{D}_{\log}, \pi(f)](1 + |N|^{2+2s})^{-\frac{1-\varepsilon}{2}} \right\| = \sup_{n \in \mathbb{Z}} \left\| [\partial_{\log}, \pi(\gamma^n)\pi(f)\pi(\gamma^{-n})](1 + |n|^{2+2s})^{-\frac{1-\varepsilon}{2}} \right\|. \quad (1.10)$$

From

$$\begin{aligned} [\partial_{\log}, \pi(\gamma^n)\pi(f)\pi(\gamma^{-n})] &= [\partial_{\log}, \pi(\gamma^n)]\pi(f)\pi(\gamma^{-n}) + \pi(\gamma^n)[\partial_{\log}, \pi(f)]\pi(\gamma^{-n}) \\ &\quad + \pi(\gamma^n)\pi(f)[\partial_{\log}, \pi(\gamma^{-n})]. \end{aligned}$$

Since $\|[\partial_{\log}, \pi(\gamma^{-n})]\| \leq C|n|$ for $C := \|[\partial_{\log}, \pi(\gamma)]\|$, it follows that

$$\|[\partial_{\log}, \pi(\gamma^n)\pi(f)\pi(\gamma^{-n})]\| \leq 2C|n|\|\pi(f)\| + \|[\partial_{\log}, \pi(f)]\|.$$

The expression

$$|n|(1 + |n|^{2+2s})^{-\frac{1-\varepsilon}{2}},$$

is uniformly bounded in n if and only if $\varepsilon \leq \frac{s}{1+s}$. From this we deduce that the supremum in (1.10) is finite if and only if $\varepsilon \leq \frac{s}{1+s}$. By extension, Equation (1.9) allows us to deduce that $[D, \pi(f)]$ is ε -bounded with respect to D for $0 < \varepsilon \leq \frac{s}{1+s}$.

Summarizing, we need ε to satisfy $\varepsilon \in (0, 1]$, $\varepsilon \leq \frac{1}{1+s}$ and $\varepsilon \leq \frac{s}{1+s}$. It is therefore required that $0 < s \leq 1$. For $s \in [0, 1]$, $\frac{s}{1+s} \leq \frac{1}{1+s}$. These conditions can be condensed to

$$0 < s \leq 1 \quad \text{and} \quad 0 < \varepsilon \leq \frac{s}{1+s}.$$

We conclude that (1.28) is a higher order Kasparov module of order $\frac{1+s}{s}$ whenever $s \in (0, 1]$. \square

Remark 1.30. It should be noted that these estimates are not necessarily sharp. It can be verified that $[D, \alpha(\gamma)]$ is ε -bounded with respect to D if and only if $0 < \varepsilon \leq \min\left(1, \frac{1}{1+s}\right)$. It is however possible that a finer analysis of

$$\left\| [\partial_{\log}, \pi(\gamma^n)\pi(f)\pi(\gamma^{-n})] (1 + |n|^{2+2s} + \partial_{\log}^2)^{-\frac{1-\varepsilon}{2}} \right\|$$

would reveal that $[D, \pi(f)]$ is ε -bounded with respect to D for some range of $\varepsilon > \frac{s}{1+s}$.

Theorem 1.31. *Let (\mathcal{A}, H_0, D_0) be an odd spectral triple and set $F := \text{sgn}(D_0) = D_0|D_0|^{-1}$. Assume that γ is a $*$ -automorphism of \mathcal{A} that is implemented by a unitary U on H_0 such that*

$$[F, U] : H_0 \rightarrow \text{Dom}(D_0).$$

Set $H := X \otimes_{C(S^1)} H_0 = \ell^2(\mathbb{Z}) \otimes H_0$ and $\mathcal{D}_{\log} := 1_{\ell^2(\mathbb{Z})} \otimes \text{sgnlog}(D_0)$. For any $0 < s \leq 1$

$$\left(\mathcal{A} \rtimes_{\gamma}^{\text{alg}} \mathbb{Z}, H \oplus H, D := \begin{pmatrix} 0 & N_s + i\mathcal{D}_{\log} \\ N_s - i\mathcal{D}_{\log} & 0 \end{pmatrix} \right),$$

is a higher order spectral triple of order $\frac{1+s}{s}$. It represents the class

$$\partial[(\mathcal{A}, H_0, D_0)] \in K^0(\mathcal{A} \rtimes_{\gamma} \mathbb{Z}),$$

where $\partial : K^1(A) \rightarrow K^0(\mathcal{A} \rtimes_{\gamma} \mathbb{Z})$ is the boundary map in the Pimsner-Voiculescu exact sequence.

Proof. We note that the assumption on $[F, U]$ and Corollary 1.19 (see page 14) guarantee that we obtain a spectral triple $(\mathcal{A} \rtimes_{\gamma}^{\text{alg}} \mathbb{Z}, H_0, \text{sgnlog}(D_0))$ when letting $\mathcal{A} \rtimes_{\gamma}^{\text{alg}} \mathbb{Z}$ act on H_0 via U . The theorem is then proved ad verbatim to Proposition 1.29. \square

We note that we have used both the logarithmic dampening of ∂ and the higher order picture in order to achieve the last two results. If $[D_0, U]$ is bounded (ie γ is isometric in the case of Proposition 1.29), one can replace $N|N|^s$ by N and \mathcal{D}_{\log} by $\mathcal{D} = 1 \otimes D_0$. This results in an ordinary spectral triple, as was studied in detail in [4]. In general the operator \mathcal{D} gives rise to bounded twisted commutators and the operator N to bounded commutators, and this prevents their sum giving either a spectral triple or a twisted spectral triple.

1.7 Logarithmic dampening of higher order Kasparov modules

From the perspective of index theory, the homotopy class of the bounded transform F_D contains the relevant information. From a purely topological point of view, the unbounded representative of the KK -class of F_D is largely irrelevant and somewhat arbitrary. In this section we show that the logarithm function can be used to turn any higher order Kasparov module into an order 1 (i.e. ordinary) Kasparov module, *with B -compact commutators*. This logarithmic dampening supplies us with a class of unbounded Kasparov modules that are analytically very close to bounded Fredholm modules.

We first present a lemma concerning the integral representation of the logarithm function. Recall that given a C^* -subalgebra $A \subset \text{End}_B^*(E)$ an operator $K \in \text{End}_B^*(E)$ is **A -locally B -compact** if $aK, Ka \in \mathbb{K}_B(E)$.

Lemma 1.32. *Let K be positive and A -locally B -compact. Then $\log(1 + K)$ is A -locally B -compact.*

Proof. The operator norm convergent integral expression

$$\log(1 + K) = K \int_0^1 (1 + tK)^{-1} dt,$$

immediately gives the statement. \square

Lemma 1.33. *Let $0 \leq \alpha < \frac{1}{2}$, $a \text{Dom}(D) \subset \text{Dom}(D)$ and $[D, a](1 + D^2)^{-\alpha}$ be bounded. Then for all β with $\alpha < \beta < 1$, $D[(1 + D^2)^{-\beta}, a]$ is bounded.*

Proof. We use the integral formula (1.2).

$$\int_0^\infty D\lambda^{-\beta}(1+\lambda+D^2)^{-1}(D[D,a]+[D,a]D)(1+\lambda+D^2)^{-1}d\lambda$$

and we see that the two terms behave like $\lambda^{-\beta}(1+\lambda)^{-1+\alpha}$ and are therefore integrable at 0 and ∞ . \square

Lemma 1.34. *Let $0 \leq \alpha < \frac{1}{2}$, $a \text{ Dom}(D) \subset \text{Dom}(D)$ and $[D,a](1+D^2)^{-\alpha}$ be bounded. Then for $\beta > \alpha$ the operator $[(1+D^2)^{\frac{1}{2}-\beta}, a]$ is bounded.*

Proof. We prove that $(1+D^2)^{1/2-\beta}[(1+D^2)^{-\frac{1}{2}+\beta}, a](1+D^2)^{\frac{1}{2}-\beta}$ is bounded using the integral formula. Expansion gives us

$$\int_0^\infty \lambda^{-1/2+\beta}(1+D^2)^{\frac{1}{2}-\beta}(1+\lambda+D^2)^{-1}(D[D,a]+[D,a]D)(1+\lambda+D^2)^{-1}(1+D^2)^{\frac{1}{2}-\beta}d\lambda,$$

and both terms behave like $\lambda^{-\frac{1}{2}+\beta}(1+\lambda)^{-\beta}(1+\lambda)^{-\frac{1}{2}-\beta+\alpha}$, so are integrable at 0 and ∞ . \square

Lemma 1.35. *Let $0 \leq \alpha < \frac{1}{2}$, $a \text{ Dom}(D) \subset \text{Dom}(D)$ and $[D,a](1+D^2)^{-\alpha}$ be bounded. Then for $\beta > \alpha$ and $a, b \in \mathcal{A}$ the operator $D(1+D^2)^{-\beta}[(1+D^2)^{-\frac{1}{2}+\beta} \log(1+(1+D^2)^{\frac{1}{2}-\beta}), a]b$ is B -compact.*

Proof. Since $D(1+D^2)^{-\beta}$ behaves like $(1+D^2)^{\frac{1}{2}-\beta}$ we consider

$$(1+D^2)^{-\frac{1}{2}+\beta} \log(1+(1+D^2)^{\frac{1}{2}-\beta}) = \int_0^1 (1+t(1+D^2)^{\frac{1}{2}-\beta}) dt,$$

and expand the expression

$$(1+D^2)^{\frac{1}{2}-\beta}[(1+D^2)^{-\frac{1}{2}+\beta} \log(1+(1+D^2)^{\frac{1}{2}-\beta}), a]b \tag{1.11}$$

$$= \int_0^1 (1+D^2)^{\frac{1}{2}-\beta}(1+t(1+D^2))^{-\frac{1}{2}+\beta} t[(1+D^2)^{\frac{1}{2}-\beta}, a](1+t(1+D^2))^{-\frac{1}{2}+\beta} b dt. \tag{1.12}$$

By Lemma 1.34 $[(1+D^2)^{\frac{1}{2}-\beta}, a]$ is a bounded operator and the operators

$$t(1+D^2)^{\frac{1}{2}-\beta}(1+t(1+D^2))^{-\frac{1}{2}+\beta}, \quad (1+t(1+D^2))^{-\frac{1}{2}+\beta}b,$$

are uniformly bounded in t , and the latter operator is B -compact for $t \in (0, 1]$. Thus the left hand side of (1.11) is the integral of a uniformly bounded function with values in the B -compact operators, hence it is B -compact. \square

Definition 1.36. For a higher order Kasparov module (\mathcal{A}, X, D) and $0 < \beta < \frac{1}{2}$ we define its β -logarithmic transform $(\mathcal{A}, X, D_{\beta, \log})$ by the self-adjoint regular operator

$$D_{\beta, \log} := D(1+D^2)^{-\frac{1}{2}} \log(1+(1+D^2)^{\frac{1}{2}-\beta}).$$

Recall our notation $D_{\log} := \text{sgnlog}(D) = D|D|^{-1} \log(1+|D|)$. We now arrive at the main result of this section.

Theorem 1.37. *Let (\mathcal{A}, X, D) be a higher order Kasparov module. Then the logarithmic transform $(\mathcal{A}, X, D_{\log})$ is an ordinary unbounded Kasparov module representing the same KK -class. Moreover the commutators $[D_{\log}, ab]$ are compact for $a, b \in \mathcal{A}$.*

Proof. As the function

$$x \mapsto x(1+x^2)^{-\frac{1}{2}} \log(1+(1+x^2)^{\frac{1}{2}-\beta}) - (1-2\beta)\operatorname{sgn} \log(x),$$

belongs to $C_0(\mathbb{R})$, it follows that $D_{\beta, \log} - (1-2\beta)D_{\log}$ is an A -locally B -compact operator. It thus suffices to show that for some $\beta \in (0, 1/2)$, the β -logarithmic transform $(\mathcal{A}, X, D_{\beta, \log})$ is an ordinary unbounded Kasparov module with the commutators $[D_{\beta, \log}, ab]$ being compact for $a, b \in \mathcal{A}$. Since (\mathcal{A}, X, D) is a higher order Kasparov module, there is an $0 \leq \alpha < \frac{1}{2}$ such that for all $a \in \mathcal{A}$, $[D, a](1+D^2)^{-\alpha}$ and $(1+D^2)^{-\alpha}[D, a]$ are bounded. Take β with $\alpha < \beta < \frac{1}{2}$.

The operator $D_{\beta, \log}$ has A -locally B -compact resolvent by construction. We need to prove boundedness and B -compactness of the relevant commutators. To this we expand

$$[D_{\beta, \log}, a] = D(1+D^2)^{-\beta}[(1+D^2)^{-\frac{1}{2}+\beta} \log(1+(1+D^2)^{\frac{1}{2}-\beta}), a] \quad (1.13)$$

$$+ [D, a](1+D^2)^{-\beta}(1+D^2)^{-\frac{1}{2}+\beta} \log(1+(1+D^2)^{\frac{1}{2}-\beta}) \quad (1.14)$$

$$+ D[(1+D^2)^{-\beta}, a](1+D^2)^{-\frac{1}{2}+\beta} \log(1+(1+D^2)^{\frac{1}{2}-\beta}). \quad (1.15)$$

The first term is bounded by Lemma 1.35, the second term is bounded since $[D, a](1+D^2)^{-\alpha}$ is bounded and Lemma 1.33 gives boundedness of the third term. Moreover $[D, a]b$ is B -compact since for the summand (1.13), multiplication from the left by b makes the operator B -compact. For the terms (1.14) and (1.15), multiplication from the right by b makes them B -compact as

$$x \mapsto (1+x^2)^{-\frac{1}{2}+\beta} \log(1+(1+x^2)^{\frac{1}{2}-\beta}),$$

is a C_0 -function. Since \mathcal{A} is a $*$ -algebra and D is self-adjoint, we have that $[D, b^*]a^*$ is B -compact as well, and thus so are $a[D, b]$ and $[D, ab]$. \square

Corollary 1.38. *Every class in $KK_*(A, B)$ can be represented by an unbounded Kasparov module (\mathcal{A}, X, D) such that $\overline{AX} = X$ and for all $a \in \mathcal{A}$ the commutators $[D, a]$ are B -compact.*

Proof. It follows from [6, Proposition 18.3.6] and [41, Lemma 1.4]) that any class in $KK_*(A, B)$ can be represented by an unbounded Kasparov module (\mathcal{A}_0, X, D_0) with $\overline{AX} = X$. Then

$$\mathcal{A}_0^2 := \operatorname{span}_{\mathbb{C}} \{ab : a, b \in \mathcal{A}\},$$

is a dense $*$ -subalgebra of A . By Theorem 1.37 the triple

$$(\mathcal{A}, X, D) := (\mathcal{A}_0^2, X, (D_0)_{\log}),$$

is an unbounded Kasparov module for which all elements $a \in \mathcal{A}$ have B -compact commutators with D . \square

If one drops the requirement that $\overline{AX} = X$, then Corollary 1.38 is in fact implicitly proven in [3] (as noted by Kaad [39]), but that proof does not cover the stronger result presented here.

Corollary 1.39. *Let $\gamma : S^1 \rightarrow S^1$ be a (non-isometric) diffeomorphism. The logarithmic dampening of the higher order spectral triple (1.28) in Proposition 1.29 gives an ordinary spectral triple representing the class $\partial[(C(S^1), L^2(S^1), i \frac{d}{dx})] \in K^0(C(S^1) \rtimes_{\gamma} \mathbb{Z})$.*

2 Review of local index formulae

The first and most important application of spectral triples was to provide computationally tractable expressions for the index pairing. This programme was initiated by Connes and Moscovici to enable them to study the transverse fundamental class of foliations, crossed products and more generally triangular structures on manifolds, [20]. The outcome was the first expression and proofs of the local index formula.

Since then, refinements and extensions have been developed by Higson [36] and in [9, 10, 13, 14, 15, 52]. All the various statements of the local index formula rely on two basic assumptions: smoothness and finite summability. As summability requires reference to the Schatten ideals in $\mathcal{B}(H)$, it has so far not been developed for unbounded Kasparov modules. Moreover, summability in the non-unital case is more technical (see [8, 9]). In this section we therefore restrict our attention to unital (twisted) spectral triples. We remark that the discussion also extends to the semifinite setting (see more in [19, 5, 13, 14, 15, 9]), but we refrain from this level of generality.

2.1 Smoothness and summability

Here we discuss the standard definitions of smoothness and summability.

Definition 2.1. A unital spectral triple (\mathcal{A}, H, D) is regular if for all $k \geq 1$ and all $a \in \mathcal{A}$ the operators a and $[D, a]$ are in the domain of δ^k , where $\delta(T) = [|D|, T]$ is the partial derivation on $\mathcal{B}(H)$ defined by $|D|$.

Definition 2.2. A unital spectral triple (\mathcal{A}, H, D) is called finitely summable if there is a $s > 0$ such that $(i + D)^{-1} \in \mathcal{L}^s(H)$. If this is the case, we say that (\mathcal{A}, H, D) is s -summable. If (\mathcal{A}, H, D) is a QC^∞ finitely summable spectral triple, we call

$$p := \inf\{s \in \mathbb{R} : \text{Tr}((1 + D^2)^{-s/2}) < \infty\}$$

the **spectral dimension** of (\mathcal{A}, H, D) .

There are C^* -algebras that do not admit finitely summable spectral triples, even when they do admit finitely summable Fredholm modules (more on this later). We quote the following obstruction result.

Theorem 2.3 (Connes, [16]). *Let A be a unital C^* -algebra and (\mathcal{A}, H, D) a unital finitely summable spectral triple, with $\mathcal{A} \subset A$ dense. Then there exists a tracial state $\tau : A \rightarrow \mathbb{C}$ on A .*

This theorem was stated for ordinary spectral triples in [16], but the proof extends mutatis mutandis to the general compact higher order (and semi-finite) setting. More generally, if (\mathcal{A}, H, D) is a (semifinite higher order) spectral triple with $\text{Tr}(e^{-t|D|}) < \text{Tr}(\chi_{[0, \infty)}(D)) = \infty$ for all $t > 0$ then there is a tracial state on A (see [29, Theorem 3.22]). One can conclude that algebras with no non-zero trace, such as the Cuntz algebras, do not carry unital higher order (semi-finite) finitely summable spectral triples. In particular, the obstructions to finite summability of spectral triples remain when generalizing to higher order as well as to semi-finite spectral triples. The lack of finitely summable spectral triples does not preclude the existence of finitely summable Fredholm modules (see [25, 27]), and we exploit this later.

2.2 The index cocycle

For any finitely summable bounded Fredholm module one can construct an associated index cocycle called its Connes-Chern character [17, Chapter IV]. When the Fredholm module comes from a finitely summable regular spectral triple (\mathcal{A}, H, D) satisfying an additional meromorphicity assumption there is a different representative of the Connes-Chern character in the finitely supported (b, B) -bicomplex called the residue cocycle. Versions of this cocycle appear in the various local index formulae in non-commutative geometry, [20, 36, 13, 14].

Connes and Moscovici imposed the discrete dimension spectrum assumption to prove their original version of the local index formula, and we state this below. The proof of the local index formula in [13, 14] requires less restrictive hypotheses on the zeta functions, but we will focus here on discrete dimension spectrum which implies the conditions of [13, 14]. We will introduce some notation and definitions and then state the odd local index formula using [13].

Definition 2.4. Let (\mathcal{A}, H, D) be a regular spectral triple. The algebra $\mathcal{B}(\mathcal{A}) \subseteq \mathcal{B}(H)$ is the algebra of polynomials generated by $\delta^n(a)$ and $\delta^n([D, a])$ for $a \in \mathcal{A}$ and $n \geq 0$. A regular spectral triple (\mathcal{A}, H, D) has **discrete dimension spectrum** $\text{Sd} \subseteq \mathbb{C}$ if Sd is a discrete set and for all $b \in \mathcal{B}(\mathcal{A})$ the function $\zeta_b(z) := \text{Tr}(b(1 + D^2)^{-z})$ is defined and holomorphic for $\text{Re}(z)$ large, and analytically continues to $\mathbb{C} \setminus \text{Sd}$. We say the dimension spectrum is **simple** if this zeta function has poles of order at most one for all $b \in \mathcal{B}(\mathcal{A})$, **finite** if there is a $k \in \mathbb{N}$ such that the function has poles of order at most k for all $b \in \mathcal{B}(\mathcal{A})$ and **infinite**, if it is not finite.

Introduce multi-indices $\mathbb{k} = (k_1, \dots, k_m)$, $k_i = 0, 1, 2, \dots$, whose length m will always be clear from the context and let $|\mathbb{k}| = k_1 + \dots + k_m$. Define

$$\alpha(\mathbb{k}) = \frac{1}{k_1!k_2! \dots k_m! (k_1 + 1)(k_1 + k_2 + 2) \dots (|\mathbb{k}| + m)},$$

and the numbers $\tilde{\sigma}_{n,j}$ are defined by the equalities

$$\frac{\Gamma(n + z + 1/2)}{\Gamma(z + 1/2)} = \prod_{j=0}^{n-1} (z + j + 1/2) = \sum_{j=0}^n z^j \tilde{\sigma}_{n,j}.$$

If (\mathcal{A}, H, D) is a regular spectral triple and $T \in \mathcal{B}(H)$ then $T^{(n)}$ is the n^{th} iterated commutator with D^2 (whenever defined), that is, $T^{(n)} := [D^2, [D^2, [\dots, [D^2, T] \dots]]]$.

Now let (\mathcal{A}, H, D) be a spectral triple with finite dimension spectrum. For operators $b \in \mathcal{B}(H)$ of the form

$$b = a_0[D, a_1]^{(k_1)} \dots [D, a_m]^{(k_m)}(1 + D^2)^{-m/2 - |\mathbb{k}|}$$

we can define, for $j \in \mathbb{N}$, the functionals

$$\tau_j(b) := \text{res}_{z=0} z^j \zeta_b(z).$$

The hypothesis of finite dimension spectrum is clearly sufficient to define the residues. We adapt part of the statement of the odd local index formula from [13] to our situation.

Theorem 2.5 (Odd local index formula). *Let (\mathcal{A}, H, D) be an odd finitely summable regular spectral triple with spectral dimension $p \geq 1$ and discrete, finite dimension spectrum. Let P be the spectral projection of D corresponding to the interval $[0, \infty)$. Let $N = [p/2] + 1$ where $[\cdot]$ denotes the floor function (ie integer part), and let $u \in \mathcal{A}$ be unitary. Then the index pairing can be computed by means of the formula*

$$\langle [(\mathcal{A}, H, D)], [u] \rangle = \text{Index}(PuP) = \sum_m \phi_m(\text{Ch}_m(u))$$

where $\text{Ch}_m(u)$ are the components of the Chern character of u (see [13]), and for $a_0, \dots, a_m \in \mathcal{A}$

$$\begin{aligned} \phi_m(a_0, \dots, a_m) &= \sum_{|\mathbb{k}|=0}^{2N-1-m} (-1)^{|\mathbb{k}|} \alpha(k) \times \\ &\times \sum_{j=0}^{|\mathbb{k}|+(m-1)/2} \tilde{\sigma}_{(|\mathbb{k}|+(m-1)/2, j} \tau_j \left(a_0 [D, a_1]^{(k_1)} \dots [D, a_m]^{(k_m)} (1 + D^2)^{-|\mathbb{k}|-m/2} \right). \end{aligned}$$

The collection of functionals $(\phi_m)_{m=1, \text{odd}}^{2N-1}$ is a (b, B) -cocycle for \mathcal{A} .

2.3 (Local) index theory for twisted spectral triples

The obstruction to finite summability expressed in Theorem 2.3 and the dependency on finite summability in the local index formula of Theorem 2.5 call for a different approach in purely infinite situations. Theorem 2.3 does not rule out the existence of finitely summable twisted spectral triples in the absence of a trace.

Motivated by this issue, Moscovici gave an ansatz for a local index formula for twisted spectral triples in [47]. Moscovici's ansatz is a cyclic cochain in the (b, B) -bicomplex associated with a twisted spectral triple. Moscovici proved that his ansatz computes the index pairing with K -theory in a special case in [47]. In order to formulate Moscovici's ansatz, we need to adapt the notions of regularity, finite summability and dimension spectrum to the twisted setting. The notion of regularity we make use of is due to Matassa and Yuncken [45].

Definition 2.6. Let $(\mathcal{A}, H, D, \sigma)$ be a weakly twisted spectral triple.

- $(\mathcal{A}, H, D, \sigma)$ is said to be finitely summable if there is a $p > 0$ such that $(i+D)^{-1} \in \mathcal{L}^p(H)$. In this case, we say that $(\mathcal{A}, H, D, \sigma)$ is p -summable.
- $(\mathcal{A}, H, D, \sigma)$ is said to be regular if there is a $*$ -algebra $\mathcal{B} \subseteq \mathbb{B}(H)$ of bounded operators containing \mathcal{A} and $[D, \mathcal{A}]_\sigma$ to which σ extends to a linear isomorphism $\theta : \mathcal{B} \rightarrow \mathcal{B}$ such that \mathcal{B} is invariant under the derivation $\delta_\theta(b) := [|D|, b]_\theta = |D|b - \theta(b)|D|$.

Remark 2.7. If $(\mathcal{A}, H, D, \sigma)$ is regular, $|D|a - \sigma(a)|D| = \delta_\theta(a) \in \mathcal{B}$ for $a \in \mathcal{A}$. In particular, regular twisted spectral triples satisfy the twisted Lipschitz condition.

In fact, Matassa-Yuncken [45] showed that a regular twisted spectral triple admits a twisted pseudo-differential calculus (see [45, Definition 3.6]). We denote this twisted pseudo-differential calculus by $\Psi_{\sigma, D}^*(\mathcal{A})$. For full details on the twisted pseudo-differential calculus see [45], but let us point out that $\Psi_{\sigma, D}^*(\mathcal{A})$ contains \mathcal{A} , $[D, \mathcal{A}]_\sigma$, all powers of $|D|$ and is closed under δ_θ for a

suitable linear isomorphism θ of $\Psi_{\sigma,D}^*(\mathcal{A})$ extending σ . If we can take θ to be an automorphism we say that $(\mathcal{A}, H, D, \sigma)$ is *strongly regular*.

Before going into the index theory of twisted spectral triples, we record some basic results concerning regularity.

Proposition 2.8. *Let $(\mathcal{A}, H, D, \sigma)$ be a twisted spectral triple with D invertible and $\sigma(a) = |D|a|D|^{-1}$. Then $(\mathcal{A}, H, D, \sigma)$ is strongly regular and we can take $\Psi_{\sigma,D}^*(\mathcal{A})$ to be the filtered algebra generated by F , \mathcal{A} , $[D, \mathcal{A}]_\sigma$ and all complex powers of $|D|$ with $\theta = \sigma$, the filtering coming from the order of the power of $|D|$ and the one-parameter family of algebra automorphisms $(\Theta^z)_{z \in \mathbb{C}}$ is defined from conjugation by $|D|^z$.*

Proof. The definitions given in the statement imply that

$$\delta_\sigma(a) := [|D|, a]_\sigma = |D|a - |D|a|D|^{-1}|D| = |D|a - |D|a = 0.$$

In particular, the algebra generated by F , \mathcal{A} , $[D, \mathcal{A}]_\sigma$ and all powers of $|D|$ is invariant under δ_σ . It follows that the filtered algebra $\Psi_{\sigma,D}^*(\mathcal{A})$ satisfies the assumptions of [45, Definition 3.6] and forms a twisted pseudo-differential calculus. \square

Proposition 2.9. *Let $(\mathcal{A}, H, D, \sigma)$ be a regular weakly twisted spectral triple with D invertible, and set $\tilde{\sigma}(a) := |D|a|D|^{-1}$. Then $\sigma(a) - \tilde{\sigma}(a) \in \Psi_{\sigma,D}^{-1}(\mathcal{A})$. Moreover, $(\mathcal{A}, H, D, \tilde{\sigma})$ is a weakly twisted spectral triple and its saturation $(\mathcal{A}_{\tilde{\sigma}}, H, D, \tilde{\sigma})$ is a strongly regular twisted spectral triple.*

Proof. The first part of the result is a consequence of [45, Lemma 3.7]. The second part of the proposition is proven as follows. The property that $\sigma(a) - \tilde{\sigma}(a) \in \Psi_{\sigma,D}^{-1}(\mathcal{A})$ ensures that $(\mathcal{A}, H, D, \tilde{\sigma})$ is a weakly twisted spectral triple satisfying the conditions of Proposition 1.4 (see page 7) so its saturation $(\mathcal{A}_{\tilde{\sigma}}, H, D, \tilde{\sigma})$ is well defined. Strong regularity of its saturation follows from Proposition 2.8. \square

Remark 2.10. In the terminology of [45, Section 3], σ and $\tilde{\sigma}$ are equivalent twists. In particular, they can be equipped with related twisted pseudodifferential calculi.

Continuing towards Moscovici's ansatz, we define dimension spectrum for regular twisted spectral triples.

Definition 2.11. Let $(\mathcal{A}, H, D, \sigma)$ be a finitely summable regular spectral triple with twisted pseudo-differential calculus $\Psi_{\sigma,D}^*(\mathcal{A})$. We say that $(\mathcal{A}, H, D, \sigma)$ has discrete dimension spectrum if there is a discrete set $\text{Sd} \subseteq \mathbb{C}$ such that the ζ -functions

$$\zeta_T(z) := \text{Tr}(T|D|^{-2z}), \quad T \in \Psi_{\sigma,D}^*(\mathcal{A}),$$

defined for $\text{Re}(z) \gg 0$, have meromorphic extensions to \mathbb{C} holomorphic outside Sd . If there is an $N \in \mathbb{N}$ such that all poles of $\{\zeta_T : T \in \Psi_{\sigma,D}^*(\mathcal{A})\}$ are of order at most N in Sd , we say that $(\mathcal{A}, H, D, \sigma)$ has finite discrete dimension spectrum. If we can take $N = 1$, we say that $(\mathcal{A}, H, D, \sigma)$ has simple discrete dimension spectrum. For $T \in \Psi_{\sigma,D}^*(\mathcal{A})$ we write $T^{(1)} := [D^2, T]_{\sigma^2}$ and $T^{(k)} := [D^2, T^{(k-1)}]_{\sigma^2}$, and let $\tau_j(T) := \text{Res}_{z=0} z^j \text{Tr}(T|D|^{-2z})$

Moscovici introduced his ansatz for a twisted local index formula only for the case of simple dimension spectrum, but this is purely a matter of technical convenience. Adjusting Moscovici's constants as in the discussion of renormalisation in [20, Section II.3] brings them in line with the renormalised formula presented in Theorem 2.5.

We stress that **these constants do not affect our later arguments at all**. The crux of the issue for us is that, for certain examples, all multilinear functionals defined using residues of zeta functions are identically zero.

Definition 2.12. Let $(\mathcal{A}, H, D, \sigma)$ be a finitely summable regular twisted spectral triple with discrete dimension spectrum. For $m, j \in \mathbb{N}$ and $\mathbb{k} = (k_1, \dots, k_m) \in \mathbb{N}^m$, we define the m -cochain $\phi_{m,j,\mathbb{k}}$ on \mathcal{A} by setting

$$\begin{aligned} \phi_{m,j,\mathbb{k}}(a_0, a_1, \dots, a_q) \\ := \tau_j \left(\gamma a_0 [D, \sigma^{-2k_1-1}(a_1)]_\sigma^{(k_1)} \dots [D, \sigma^{-2(k_1+k_2+\dots+k_q)-1}(a_q)]_\sigma^{(k_q)} |D|^{-2|\mathbb{k}|-m} \right), \end{aligned}$$

for $a_0, a_1, \dots, a_m \in \mathcal{A}$. Here γ denotes the grading operator in the even case and $\gamma = F$ in the odd case. Adjusting the coefficients $c_{m,\mathbb{k}}$ from [47] to take care of the renormalisation procedure of [20], we define the m -cochain ϕ_m as

$$\phi_m(a_0, a_1, \dots, a_m) := \sum_{\mathbb{k} \in \mathbb{N}^m} (-1)^{|\mathbb{k}|} \alpha(\mathbb{k}) \sum_{j=0}^{|\mathbb{k}|+(m-1)/2} \tilde{\sigma}_{(|\mathbb{k}|+(m-1)/2,j)} \phi_{m,j,\mathbb{k}}(a_0, a_1, \dots, a_q).$$

The reader should note that due to the finite summability assumption, there are only finitely many non-zero terms in the sum defining ϕ_m and $\phi_m = 0$ for $m \gg 0$, sufficiently large. In [47], the cochain ϕ_m is defined by means of a slightly different expression, which makes use of the fact that Moscovici assumes simple dimension spectrum. Our definition extends Moscovici's in a way that is consistent with the derivation from the twisted JLO cocycle, taking into account the renormalisation procedure of [20].

For $j = 0, 1$ the parity of the regular and finitely summable twisted spectral triple $(\mathcal{A}, H, D, \sigma)$ with discrete dimension spectrum, we shall call the cochain $(\phi_{2m+j})_{m \in \mathbb{N}}$ in the (b, B) -bicomplex the *Moscovici ansatz* for an index cocycle. We note that the cochain $(\phi_{2m+j})_{m \in \mathbb{N}}$ is only an ansatz, and it is unclear how generally this ansatz actually provides a cocycle, let alone an index cocycle. In the next section (Section 3), we shall provide examples of regular finitely summable twisted spectral triples with finite discrete dimension spectrum for which Moscovici's ansatz can not represent an index cocycle.

Remark 2.13 (Computing index pairings using twisted spectral triples). The only to date known local index formula for twisted spectral triples is due to Moscovici who considered twisted spectral triples arising from scaling automorphisms. Given an ordinary spectral triple (\mathcal{A}, H, D) , a scaling automorphism is an automorphism of \mathcal{A} implemented by a unitary $U \in \mathcal{B}(H)$ such that there is a positive real number $\mu(U)$ for which $UDU^* = \mu(U)D$. If Γ is a group of scaling automorphisms, $(\mathcal{A} \rtimes \Gamma, H, D, \sigma)$, with $\sigma(aU) := \mu(U)aU$ is a twisted spectral triple.

In [47], it was proven that for twisted spectral triples associated to scaling automorphisms that

$$\langle [(\phi_{2m+j})_{m \in \mathbb{N}}], \text{Ch}(x) \rangle = \langle [(\mathcal{A}, H, D, \sigma)], x \rangle, \quad \forall x \in K_*(A). \quad (2.1)$$

Here the left hand side denotes the pairing of cyclic cohomology with the cyclic homology class given by the Chern character of the K -theory element x . The right hand side denotes the pairing of K -homology with K -theory. The counterexample alluded to in the preceding paragraph is found by providing a counterexample to the equality (2.1).

The local index theory was worked out in detail by Ponge and Wang [49, 50] for groups of conformal diffeomorphisms satisfying the scaling hypothesis. To do this they reduced to the case of trivial twist and employed the local index formula for ordinary spectral triples. This left open the question of whether Moscovici's ansatz computed the index. As noted in [49, 50], there are very few examples.

3 Vanishing of the twisted local index formula

One of the motivations to introduce twisted spectral triples is that twisting allows for the existence of Dirac operators with better spectral properties, as discussed in the Introduction and [21]. One sought after spectral property is finite summability, and one might hope for a local index formula for twisted spectral triples in the style of Connes-Moscovici's local index formula [20]. In [47], Moscovici provided an ansatz for a cyclic cocycle that generalized Connes-Moscovici's local index cocycle. Moscovici's ansatz reproduces the index character for twisted spectral triples associated to so-called scaling automorphisms.

In this subsection we discuss various examples showing that Moscovici's ansatz can not be extended to the case of a general finitely summable regular twisted spectral triple with finite discrete dimension spectrum. Our examples are highly regular odd twisted spectral triples pairing nontrivially with K -theory yet having all twisted higher residue cochains (appearing in Moscovici's ansatz) vanishing. Here "high regularity" means that all twisted commutators of the algebra elements with the Dirac operator in the twisted spectral triple are not just bounded but smoothing or even of finite rank – leaving little hope for a general index formula in terms of residues of ζ -functions.

3.1 Set up and statement

The construction of our counterexamples, and the proof of all their regularity properties follow a general pattern. The main idea is condensed in the following lemma. We will say that a Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$ is polynomially bounded if there are constants C and N such that $|f(x)| \leq C(1 + |x|)^N$ for all $x \in \mathbb{R}$.

Lemma 3.1. *Let (A, H, D, σ) be a finitely summable regular twisted spectral triple with D invertible and $\sigma(a) = |D|a|D|^{-1}$. Assume that*

- a) *For any two polynomially bounded Borel functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in A$ the twisted commutator $[D, a]_\sigma$ preserves $\text{Dom}(f_1(D))$ and the operator $f_1(D)[D, a]_\sigma f_2(D)$ extends to a trace class operator on H .*
- b) *There is a discrete subset $\mathcal{P} \subseteq \mathbb{C}$ such that for any $a \in A$, the ζ -functions $\zeta_a(z) := \text{Tr}(a|D|^{-2z})$ and $\zeta_{Fa}(z) := \text{Tr}(Fa|D|^{-2z})$ extend to meromorphic functions in \mathbb{C} holomorphic outside \mathcal{P} with poles in \mathcal{P} of uniformly bounded order.*

Then $(\mathcal{A}, H, D, \sigma)$ is a strongly regular finitely summable twisted spectral triple with finite discrete dimension spectrum. Moreover, for $m > 0$, $a_0, a_1, \dots, a_m \in \mathcal{A}$ and $\mathbb{k} \in \mathbb{N}^m$, the function

$$z \mapsto \text{Tr} \left(\gamma a_0 [D, \sigma^{-2k_1-1}(a_1)]_\sigma^{(k_1)} \dots [D, \sigma^{-2(k_1+k_2+\dots+k_m)-1}(a_m)]_\sigma^{(k_m)} |D|^{-2|\mathbb{k}|-m-2z} \right) \quad (3.1)$$

extends **holomorphically** to all of \mathbb{C} . In particular, $\phi_m = 0$ for $m > 0$.

Proof. Note that if $\sigma(a) = |D|a|D|^{-1}$, then

$$[D, a]_\sigma = Da - |D|a|D|^{-1}D = |D|(Fa - aF) = |D|[F, a]. \quad (3.2)$$

It follows from Proposition 2.8 that $(\mathcal{A}, H, D, \sigma)$ is strongly regular and that $\Psi_{\sigma, D}^*(\mathcal{A})$ is generated by F , \mathcal{A} , $[D, \mathcal{A}]_\sigma$ and all powers of $|D|$. We first show that $(\mathcal{A}, H, D, \sigma)$ has discrete dimension spectrum. It suffices to show that $\zeta_B(z)$ extends to a meromorphic function in \mathbb{C} holomorphic outside \mathcal{P} with possible poles in \mathcal{P} when B is a product of elements from $\{F\} \cup \mathcal{A} \cup [D, \mathcal{A}]_\sigma \cup \{|D|^k : k \in \mathbb{Z}\}$. It follows from Assumption a) that if B contains a factor from $[D, \mathcal{A}]_\sigma$ then $\zeta_B(z)$ extends holomorphically to \mathbb{C} .

If B does not contain a factor from $[D, \mathcal{A}]_\sigma$ then we can up to terms with factors in $[D, \mathcal{A}]_\sigma$ write $B = F^j |D|^z a |D|^w$ for some $j \in \mathbb{Z}$, $z, w \in \mathbb{C}$ since $\sigma(a) = |D|a|D|^{-1}$ and using (3.2). It follows from Assumption b) that $\zeta_B(z)$ extends to a meromorphic function in \mathbb{C} holomorphic outside \mathcal{P} with possible poles in \mathcal{P} .

It remains to show that for $m > 0$, $a_0, a_1, \dots, a_m \in \mathcal{A}$ and $\mathbb{k} \in \mathbb{N}^m$, the function in Equation (3.1) is holomorphic for all $z \in \mathbb{C}$. Indeed, since $m > 0$ we can for any $s \in \mathbb{R}$ write the expression

$$\gamma a_0 [D, \sigma^{-2k_1-1}(a_1)]_\sigma^{(k_1)} \dots [D, \sigma^{-2(k_1+k_2+\dots+k_m)-1}(a_m)]_\sigma^{(k_m)} |D|^{-2|\mathbb{k}|-m-2z},$$

as a sum of elements of the form

$$\gamma a_0 f_0(D) [D, b_1]_\sigma f_1(D) \dots f_{m-1}(D) [D, b_m]_\sigma f_m(D) |D|^{-s-2z},$$

for some elements $b_1, \dots, b_m \in \mathcal{A}$ and suitable polynomially bounded Borel functions $f_0, f_1, \dots, f_m : \mathbb{R} \rightarrow \mathbb{R}$. By Assumption b), there is a constant $s_0 \in \mathbb{R}$ independent of s such that the function

$$z \mapsto \text{Tr} \left(\gamma a_0 f_0(D) [D, b_1]_\sigma f_1(D) \dots f_{m-1}(D) [D, b_m]_\sigma f_m(D) |D|^{-s-2z} \right)$$

is holomorphic for $\text{Re}(z) > s_0 - s$. Therefore, the function in Equation (3.1) is holomorphic for $\text{Re}(z) > s_0 - s$, and since s is arbitrary the function in Equation (3.1) extends holomorphically to all of \mathbb{C} . We deduce that $\phi_{m, \mathbb{k}}(a_0, a_1, \dots, a_m) = 0$ for all $m > 0$, $a_0, a_1, \dots, a_m \in \mathcal{A}$ and $\mathbb{k} \in \mathbb{N}^m$ and so ϕ_m is zero. \square

The assumptions in Lemma 3.1 are very strong and may seem unrealistic. Nevertheless, we will construct examples of such twisted spectral triples below. The main ingredient comes from spectral triples with positive spectral dimension whose associated Fredholm modules are 0-dimensional. The equality (2.1) is disproved by the following theorem.

Theorem 3.2. *Let $(\mathcal{A}, H, D, \sigma)$ be an odd twisted spectral triple satisfying all the assumptions of Lemma 3.1 and such that for some $x \in K_1(\mathcal{A})$,*

$$\langle [(\mathcal{A}, H, D, \sigma)], x \rangle \neq 0.$$

Then $(\mathcal{A}, H, D, \sigma)$ is a strongly regular finitely summable twisted spectral triple with finite discrete dimension spectrum for which the equality (2.1) fails.

Proof. In the odd case we always have $m > 0$, and so the cochain provided by Moscovici's ansatz is zero by Lemma 3.1. Hence it can not compute the non-zero pairing. \square

So our task is to find a twisted spectral triple pairing nontrivially with K -theory and satisfying the assumptions of Lemma 3.1. We shall provide three examples where this phenomena occurs. The examples that matter most in this context are the ones that sidestep Connes' obstruction for finite summability. The construction relies on manipulating spectral triples (\mathcal{A}, H, D) for which commutators $[D|D|^{-1}, a]$ with the phase are smoothing or even finite rank.

3.2 The good

In this subsection and the subsequent two we construct counterexamples to the equality (2.1) consisting of finitely summable twisted spectral triples for purely infinite algebras. In the current section we present a simple, commutative counterexample, in which essentially all phenomena can be observed.

Consider the spectral triple for the circle

$$\left(C^\infty(S^1), L^2(S^1), D = -i \frac{d}{dx} + P_0 \right).$$

Here P_0 denotes the orthogonal projection onto the space of constant functions. It is well-known that commutators with the phase F of $D = -i \frac{d}{dx} + P_0$ are finite rank for polynomials in z and smoothing for C^∞ -functions. Consider the twist given by

$$\sigma(a) = |D|a|D|^{-1}.$$

In the first instance we only obtain a weakly twisted spectral triple but we can get a twisted spectral triple by taking the saturation \mathcal{A} as the $*$ -algebra generated by $\cup_k \sigma^k(C^\infty(S^1))$. We know that (the restriction to $C^\infty(S^1)$ of) this twisted spectral triple represents the class $[-id/dx]$ in K -homology, and so pairs nontrivially with $K_1(C(S^1))$. The twisted spectral triple $(\mathcal{A}, L^2(S^1), D, \sigma)$ is readily seen to be finitely summable. The fact that \mathcal{A} is contained in the classical order zero pseudodifferential operators on S^1 guarantees that it is regular and has simple discrete dimension spectrum.

Then $[D, a]_\sigma = |D|[F, a]$ is smoothing for $a \in C^\infty(S^1)$, and likewise if we consider $\sigma^k(a)$. A brief calculation shows that for $a_0, a_1 \in \mathcal{A}$ we have

$$\text{Tr}(a_0[D, \sigma^{-1}(a_1)]_\sigma |D|^{-(1+2z)}) = \text{Tr}(a_0[F, a_1] |D|^{-2z}),$$

and the right hand side is holomorphic for all $z \in \mathbb{C}$ when $a_1 \in C^\infty(S^1)$. Lemma 3.1 shows that similar comments apply for the other terms

$$\text{Tr}(a_0[D, \sigma^{-1}(a_1)]_\sigma^{(k_1)} |D|^{-(2k_1+1+2z)}),$$

appearing in the twisted local index formula. Hence the twisted local index formula can not compute the index pairing as all the residue functionals will vanish identically.

Theorem 3.3. *Let $(\mathcal{A}, L^2(S^1), D, \sigma)$ denote the regular finitely summable twisted spectral triple with simple discrete dimension spectrum obtained from saturating the weakly twisted spectral triple $(C^\infty(S^1), L^2(S^1), D, \sigma)$ with $D := -i\frac{d}{dx} + P_0$ and $\sigma(a) := |D|a|D|^{-1}$. Moscovici's ansatz does not compute the index pairing for $(\mathcal{A}, L^2(S^1), D, \sigma)$. More precisely, $(\mathcal{A}, L^2(S^1), D, \sigma)$ pairs nontrivially with $K_1(\mathcal{A})$ but the cochain (ϕ_m) provided by Moscovici's ansatz is the zero cochain. Hence the equality (2.1) does not hold.*

One could argue that this counterexample is artificial, and introduces a twist where none is needed. After all, $C^\infty(S^1)$ admits well-behaved, finitely summable spectral triples for which the Connes-Moscovici local index formula holds. In the next two sections we consider algebras for which no finitely summable spectral triples exist, and we construct finitely summable twisted spectral triples for which the twisted local index formula fails. While the details become more complicated, the essential story remains the same.

3.3 The bad

In this subsection we will construct finitely summable spectral triples on a class of C^* -algebras that in general do not admit finitely summable spectral triples.

Consider a discrete subgroup $\Gamma \subseteq SU(1, 1)$. The Lie group

$$SU(1, 1) := \left\{ \gamma = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}) : |a|^2 - |b|^2 = 1 \right\},$$

acts on the circle S^1 by Möbius transformations

$$\gamma(z) := \frac{az + b}{\bar{b}z + \bar{a}}.$$

We consider the algebra $\mathcal{A}_0 := C^\infty(S^1) \rtimes^{\text{alg}} \Gamma$. As in example 1.6, we can realize \mathcal{A}_0 as a $*$ -algebra of operators on $L^2(S^1)$ via the covariant representation

$$\pi(f)\phi(z) := f(z)\phi(z), f \in C^\infty(S^1), \quad \pi(\gamma)\phi(z) := |\gamma'(z)|^{-\frac{1}{2}}\phi(\gamma z), \gamma \in \Gamma, \quad (3.3)$$

of the C^* -dynamical system $(C(S^1), \Gamma)$, as in Equation (1.1). In the case that Γ is a nonelementary Fuchsian group of the first kind, the crossed product $C(S^1) \rtimes_r \Gamma$ is purely infinite (see [2, Proposition 3.1] and [44, Lemma 3.8]) and does not admit any finitely summable spectral triples.

We consider the self-adjoint elliptic first order pseudodifferential operator $D := -i\frac{d}{dx} + P_0$ on $L^2(S^1)$ as in Subsection 3.2. The classical order zero pseudodifferential operators on S^1 will be denoted by $\Psi_{\text{cl}}^0(S^1)$ and its algebraic crossed product with Γ by $\Psi_{\text{cl}}^0(S^1) \rtimes^{\text{alg}} \Gamma$. We define the regular automorphism σ of $\Psi_{\text{cl}}^0(S^1) \rtimes^{\text{alg}} \Gamma$ as $\sigma(a) := |D|a|D|^{-1}$. Since $\mathcal{A}_0 \subseteq \Psi_{\text{cl}}^0(S^1) \rtimes^{\text{alg}} \Gamma$, we have that $\mathcal{A}_0 \subseteq \bigcap_{k \in \mathbb{Z}} \text{Dom}(\sigma^k)$. In particular, we can define the saturation

$$\mathcal{A} := (\mathcal{A}_0)_\sigma \subseteq \Psi_{\text{cl}}^0(S^1) \rtimes^{\text{alg}} \Gamma.$$

We let $\Psi^{-\infty}(S^1)$ denote the $*$ -algebra of smoothing operators on $L^2(S^1)$, that is, operators with Schwartz kernel in $C^\infty(S^1 \times S^1)$.

Proposition 3.4. *The collection $(\mathcal{A}, L^2(S^1), D, \sigma)$ is a finitely summable regular twisted spectral triple such that for any $a \in \mathcal{A}$, the twisted commutator $[D, a]_\sigma \in \Psi^{-\infty}(S^1)$. Moreover, $\Psi_{\sigma, D}^*(\mathcal{A})$ can be taken to be generated by \mathcal{A} , $\Psi^{-\infty}(S^1)$ and all complex powers of $|D|$.*

Proof. We start by showing that $[D, a]_\sigma \in \Psi^{-\infty}(S^1)$ for all $a \in \mathcal{A}$. Note that for $a \in \mathcal{A}$,

$$[D, a]_\sigma = |D|[F, a].$$

It therefore suffices to show that $[F, a] \in \Psi^{-\infty}(S^1)$ for $a \in \mathcal{A}$. Since $[F, \sigma^k(a)] = \sigma^k([F, a])$, the Leibniz rule implies that it suffices to show that $[F, a] \in \Psi^{-\infty}(S^1)$ for $a \in \mathcal{A}_0$. We note that the (non-unitarised) action $\gamma\phi(z) := \phi(\gamma z)$ of $SU(1, 1)$ on $L^2(S^1)$ preserves

$$\text{im}(F + 1) = H^2(S^1) = \{f \in L^2(S^1) : f \text{ extends to a holomorphic function on the disc}\}.$$

Thus, since $z \mapsto |\gamma'(z)|^{-\frac{1}{2}} \in C^\infty(S^1)$, Equation (3.3) and the Leibniz rule imply that it suffices to show that $[F, a] \in \Psi^{-\infty}(S^1)$ for $a \in C^\infty(S^1)$, which we have already noted to be true.

Since D is an elliptic first order pseudodifferential operator, $(i \pm D)^{-1} \in \mathcal{L}^p(L^2(S^1))$ for all $p > 1$. We can conclude that $(\mathcal{A}, L^2(S^1), D, \sigma)$ is a finitely summable twisted spectral triple. It is regular and admits the prescribed twisted pseudodifferential calculus by Proposition 2.8 (see page 26). \square

Proposition 3.5. *The twisted spectral triple $(\mathcal{A}, L^2(S^1), D, \sigma)$ satisfies all the assumptions of Lemma 3.1 (see page 28). Moreover, $(\mathcal{A}, L^2(S^1), D, \sigma)$ has simple discrete dimension spectrum $\text{Sd} := 1 - \frac{1}{4}\mathbb{N}$.*

Proof. We begin by showing that for any two polynomially bounded Borel functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ we have that $[D, a]_\sigma$ preserves $\text{Dom}(f_1(D))$ and $f_1(D)[D, a]_\sigma f_2(D)$ extends to a trace class operator on $L^2(S^1)$. First, we note that since f_1 and f_2 are polynomially bounded, there is an $N \geq 0$ such that $|f_1(x)| \leq C_1(1 + |x|)^N$ and $|f_2(x)| \leq C_2(1 + |x|)^N$ for some $C_1, C_2 \geq 0$. In particular,

$$\text{Dom } |D|^N \subset \text{Dom } f_i(D), \quad i = 1, 2.$$

Since $[D, a]_\sigma \in \Psi^{-\infty}(S^1)$ it holds that $[D, a]_\sigma : L^2(S^1) \rightarrow \bigcap_{s \geq 0} \text{Dom } |D|^s$, and it follows that

$$(1 + |D|)^s f_1(D)[D, a]_\sigma f_2(D),$$

extends to a bounded operator for all $s \geq 0$. Since $(1 + |D|)^{-s}$ is a trace class operator for $s > 1$, it follows that

$$f_1(D)[D, a]_\sigma f_2(D) = (1 + |D|)^{-s} (1 + |D|)^s f_1(D)[D, a]_\sigma f_2(D),$$

is of trace class.

Next we show that for any $a \in \mathcal{A}$, the ζ -functions ζ_a and ζ_{Fa} extend meromorphically to functions on \mathbb{C} that are holomorphic outside $1 - \frac{1}{4}\mathbb{N}$ with at most simple poles in $1 - \frac{1}{4}\mathbb{N}$. It was shown in [47, Subsection 3.2] that for $A \in \Psi_{\text{cl}}^0(S^1) \rtimes^{\text{alg}} \Gamma$ the function $\zeta_A(z) := \text{Tr}(A|D|^{-2z})$ extends meromorphically to a function on \mathbb{C} that is holomorphic outside $1 - \frac{1}{4}\mathbb{N}$ with at most simple poles in $1 - \frac{1}{4}\mathbb{N}$. We can now conclude the desired result from noting that $\mathcal{A} + F\mathcal{A} \subseteq \Psi_{\text{cl}}^0(S^1) \rtimes^{\text{alg}} \Gamma$. \square

Consider the class $x \in K_1(\mathcal{A})$ of the unitary $u \in C^\infty(S^1) \subseteq \mathcal{A}$ defined by $u(z) := z$. By the index theorem for Toeplitz operators,

$$\langle [(\mathcal{A}, H, D, \sigma)], x \rangle = -1,$$

Using Theorem 3.2 we can now conclude the following theorem.

Theorem 3.6. *Let $(\mathcal{A}, L^2(S^1), D, \sigma)$ denote the regular finitely summable twisted spectral triple with simple discrete dimension spectrum obtained from saturating the weakly twisted spectral triple $(C^\infty(S^1) \rtimes^{\text{alg}} \Gamma, L^2(S^1), D, \sigma)$ with $D := -i\frac{d}{dx} + P_0$ and $\sigma(a) := |D|a|D|^{-1}$. Moscovici's ansatz does not compute the index pairing for $(\mathcal{A}, L^2(S^1), D, \sigma)$. More precisely, $(\mathcal{A}, L^2(S^1), D, \sigma)$ pairs nontrivially with $K_1(\mathcal{A})$ but the cochain (ϕ_m) provided by Moscovici's ansatz is the zero cochain. Hence the equality (2.1) does not hold.*

We note that the weakly twisted spectral triple $(C^\infty(S^1) \rtimes^{\text{alg}} \Gamma, L^2(S^1), D, \sigma)$ whose saturation appears in Theorem 3.6 is in fact the exponentiation (see page 4 of the Introduction) of the logarithmic dampening of a twisted spectral triple on $C^\infty(S^1) \rtimes^{\text{alg}} \Gamma$ constructed as in Example 1.6 (see page 8). This coincidence indicates that the study of index cocycles associated to twisted spectral triples is highly sensitive to the choice of twist.

3.4 The ugly

The algebras underlying our final family of counterexamples are saturations of crossed products arising from the free groups \mathbb{F}_d acting on the boundaries $\partial\mathbb{F}_d$ of their Cayley graphs. Here $d > 1$. The action of \mathbb{F} on $\partial\mathbb{F}_d$ is amenable, so there is no distinction between the full and reduced crossed products.

In order to obtain a twisted spectral triple for the purely infinite crossed product $C(\partial\mathbb{F}_d) \rtimes \mathbb{F}_d$, we utilise the fact that it is isomorphic to an explicit Cuntz-Krieger algebra. We then exploit recent advances in construction of spectral triples and K -homology classes for these algebras [27, 28, 29]. While these spectral triples are not finitely summable, commutators with the phase are again finite rank. As in the previous two examples, through exponentiation, we construct finitely summable regular weakly twisted spectral triples for $C(\partial\mathbb{F}_d) \rtimes \mathbb{F}_d$ whose saturation satisfies the hypotheses of Lemma 3.1.

3.4.1 The generators and relations picture

We begin our exposition with background on Cuntz-Krieger algebras and their spectral triples. Let $N \in \mathbb{N}_{>0}$ and $\mathbf{A} = (A_{ij})_{i,j=1}^N$ denote an $N \times N$ -matrix of 0's and 1's. For simplicity we assume that no row or column is 0. The associated Cuntz-Krieger algebra $\mathcal{O}_{\mathbf{A}}$ is defined as the universal unital C^* -algebra generated by N partial isometries S_1, S_2, \dots, S_N satisfying the relations

$$S_i^* S_j = \delta_{ij} \sum_{k=1}^N A_{jk} S_k S_k^* \quad \text{and} \quad \sum_{k=1}^N S_k S_k^* = 1.$$

Cuntz-Krieger algebras are nuclear. If \mathbf{A} is a primitive matrix, the C^* -algebra $\mathcal{O}_{\mathbf{A}}$ is simple and purely infinite. In particular, if \mathbf{A} is primitive, there are no traces on $\mathcal{O}_{\mathbf{A}}$ and a spectral triple

on $\mathcal{O}_{\mathbf{A}}$ is never finitely summable (by [16]). The K -theory and K -homology of Cuntz-Krieger algebras have been computed (see for instance [40, 51])

$$K_*(\mathcal{O}_{\mathbf{A}}) \cong \begin{cases} \text{coker}(1 - \mathbf{A}^T), & * = 0, \\ \ker(1 - \mathbf{A}^T), & * = 1, \end{cases}, \quad K^{*+1}(\mathcal{O}_{\mathbf{A}}) \cong \begin{cases} \text{coker}(1 - \mathbf{A}), & * = 0, \\ \ker(1 - \mathbf{A}), & * = 1. \end{cases}$$

Here we consider \mathbf{A} to be a matrix acting on \mathbb{Z}^N and $\text{coker}(1 - \mathbf{A}) = \mathbb{Z}^N / (1 - \mathbf{A})\mathbb{Z}^N$. By [40] that the index pairing $K_*(\mathcal{O}_{\mathbf{A}}) \times K^*(\mathcal{O}_{\mathbf{A}}) \rightarrow \mathbb{Z}$ (under the isomorphisms above) coincides with the pairing $\text{coker}(1 - \mathbf{A}^T) \times \ker(1 - \mathbf{A}) \rightarrow \mathbb{Z}$ induced from the inner product on \mathbb{Z}^N .

3.4.2 The shift space and groupoid picture

If $\mu \in \{1, \dots, N\}^k$ we call μ a word of length $|\mu| := k$ in the alphabet $\{1, \dots, N\}$. If the word $\mu = \mu_1 \cdots \mu_k$ satisfies that $A_{\mu_j, \mu_{j+1}} = 1$ for $j = 1, \dots, k-1$ we say that μ is **admissible**. We define the empty word \emptyset to be a word of length 0 and we define it to be admissible. We introduce the notation $\mathcal{V}_{\mathbf{A}, k}$ for the set of all admissible words of length k and $\mathcal{V}_{\mathbf{A}} := \cup_{k \in \mathbb{N}} \mathcal{V}_{\mathbf{A}, k}$. Similarly, we can consider infinite words $x = x_1 x_2 \cdots \in \{1, \dots, N\}^{\mathbb{N}_{>0}}$. The set of infinite admissible words is denoted by $\Omega_{\mathbf{A}}$ and is topologized as a compact Hausdorff space by its subspace topology $\Omega_{\mathbf{A}} \subseteq \{1, \dots, N\}^{\mathbb{N}_{>0}}$. The space $\Omega_{\mathbf{A}}$ is totally disconnected. If \mathbf{A} is a primitive matrix, $\Omega_{\mathbf{A}}$ contains no isolated point and $\Omega_{\mathbf{A}}$ is a Cantor space.

For $\mu = \mu_1 \cdots \mu_k \in \mathcal{V}_{\mathbf{A}}$ we define $S_{\mu} := S_{\mu_1} \cdots S_{\mu_k}$. The linear span \mathcal{A}_0 of the subset $\{S_{\mu} S_{\nu}^* : \mu, \nu \in \mathcal{V}_{\mathbf{A}}\}$ is a dense $*$ -subalgebra of $\mathcal{O}_{\mathbf{A}}$. The linear span of the subset $\{S_{\mu} S_{\mu}^* : \mu \in \mathcal{V}_{\mathbf{A}}\}$ is an abelian $*$ -algebra whose closure is a maximally abelian subalgebra of $\mathcal{O}_{\mathbf{A}}$ isomorphic to $C(\Omega_{\mathbf{A}})$ when identifying $S_{\mu} S_{\mu}^*$ with the characteristic function $\chi_{C_{\mu}} \in C(\Omega_{\mathbf{A}})$ of the cylinder set

$$C_{\mu} := \{x = x_1 x_2 \cdots \in \Omega_{\mathbf{A}} : x_1 x_2 \cdots x_k = \mu\}.$$

A more geometric approach to Cuntz-Krieger algebras stems from a description of $\mathcal{O}_{\mathbf{A}}$ as a groupoid C^* -algebra over $\Omega_{\mathbf{A}}$. The groupoid picture is useful for geometric constructions complementing the computational virtues of the description in terms of the generators S_1, \dots, S_N . The space $\Omega_{\mathbf{A}}$ carries a local homeomorphism

$$\sigma_{\mathbf{A}} : \Omega_{\mathbf{A}} \rightarrow \Omega_{\mathbf{A}}, \quad \sigma_{\mathbf{A}}(x_1 x_2 x_3 \cdots) := x_2 x_3 \cdots.$$

We define the groupoid $\mathcal{G}_{\mathbf{A}}$ over $\Omega_{\mathbf{A}}$ by

$$\mathcal{G}_{\mathbf{A}} := \{(x, n, y) \in \Omega_{\mathbf{A}} \times \mathbb{Z} \times \Omega_{\mathbf{A}} : \exists k \text{ such that } \sigma_{\mathbf{A}}^{n+k}(x) = \sigma_{\mathbf{A}}(y)\},$$

with domain map $d(x, n, y) := y$, range mapping $r(x, n, y) := x$, unit $e(x) = (x, 0, x)$ and composition $(x, n, y)(y, m, z) := (x, n+m, z)$. In the definition, it is implicit that $k, n+k \geq 0$. We also define the following integer valued functions on $\mathcal{G}_{\mathbf{A}}$:

$$c_{\mathbf{A}}(x, n, y) := n \quad \text{and} \quad \kappa_{\mathbf{A}}(x, n, y) := \min \{k \geq \max(0, -n) : \sigma_{\mathbf{A}}^{n+k}(x) = \sigma_{\mathbf{A}}^k(y)\}. \quad (3.4)$$

We topologize $\mathcal{G}_{\mathbf{A}}$ by declaring d and r to be local homeomorphisms and $c_{\mathbf{A}}$ and $\kappa_{\mathbf{A}}$ to be continuous. In this topology, $\mathcal{G}_{\mathbf{A}}$ is by definition étale. We define a clopen basis for the topology indexed by $\mu, \nu \in \mathcal{V}_{\mathbf{A}}$ as

$$C_{\mu, \nu} := \{(x, n, y) : n = |\mu| - |\nu|, x \in C_{\mu}, y \in C_{\nu}, \sigma_{\mathbf{A}}^{|\mu|}(x) = \sigma_{\mathbf{A}}^{|\nu|}(y)\}.$$

The mapping $\mathcal{O}_{\mathbf{A}} \rightarrow C^*(\mathcal{G}_{\mathbf{A}})$ defined by $S_{\mu} S_{\nu}^* \mapsto \chi_{C_{\mu, \nu}}$ is a $*$ -isomorphism. The image of the $*$ -algebra \mathcal{A}_0 generated by S_1, S_2, \dots, S_N coincides with the space of compactly supported locally constant functions on $\mathcal{G}_{\mathbf{A}}$ denoted by $C_c^{\infty}(\mathcal{G}_{\mathbf{A}})$.

3.4.3 The construction of spectral triples for Cuntz-Krieger algebras

Since $\mathcal{G}_{\mathbf{A}}$ is an étale groupoid over $\Omega_{\mathbf{A}}$, there is a conditional expectation $\Phi : \mathcal{O}_{\mathbf{A}} \rightarrow C(\Omega_{\mathbf{A}})$. Explicitly, it is defined from the property $\Phi(S_{\mu}S_{\nu}^*) := \delta_{\mu,\nu}S_{\mu}S_{\mu}^*$. We let $\Xi_{\mathbf{A}}$ denote the completion of $\mathcal{O}_{\mathbf{A}}$ as a right $C(\Omega_{\mathbf{A}})$ -Hilbert C^* -module in the inner product defined from Φ . It was shown in [29] that $\Xi_{\mathbf{A}}$ decomposes as a direct sum of the finitely generated projective $C(\Omega_{\mathbf{A}})$ -submodules $\Xi_{n,k} := C(c_{\mathbf{A}}^{-1}(n) \cap \kappa_{\mathbf{A}}^{-1}(k)) \subseteq \Xi_{\mathbf{A}}$. Moreover, we define a self-adjoint regular operator D_{ψ} densely on $\Xi_{\mathbf{A}}$ by for $f \in C_c(\mathcal{G}_{\mathbf{A}})$ setting

$$D_{\psi}f(x, n, y) := \psi(n, \kappa_{\mathbf{A}}(x, n, y))f(x, n, y), \quad \text{where} \quad \psi(n, k) := \begin{cases} n, & k = 0, \\ -|n| - k, & k > 0. \end{cases}$$

Note that if $k = 0$, then $n \geq 0$ is automatic from $c_{\mathbf{A}} + \kappa_{\mathbf{A}} \geq 0$. Equivalently, letting $p_{n,k}$ denote the orthogonal projection onto $\Xi_{n,k}$, D_{ψ} is the closure of the densely defined operator $\sum_{n,k} \psi(n, k)p_{n,k}$. Note that D_{ψ} is invertible on the orthogonal complement of $\Xi_{(0,0)}$ since $\psi(n, k) \neq 0$ for all n, k with equality if and only if $n = k = 0$.

It was proven in [27, Section 5] that $(\mathcal{A}_0, \Xi_{\mathbf{A}}, D_{\psi})$ is an unbounded $(\mathcal{O}_{\mathbf{A}}, C(\Omega_{\mathbf{A}}))$ -Kasparov module.

Recall that \mathcal{A}_0 is the $*$ -algebra generated by the generators S_1, \dots, S_N . We identify $\mathcal{A}_0 = C_c^{\infty}(\mathcal{G}_{\mathbf{A}})$ – the space of compactly supported locally constant functions. Let us define the relevant spectral triple on $\mathcal{O}_{\mathbf{A}}$. Pick a point $t \in \Omega_{\mathbf{A}}$ and define the discrete set $\mathcal{V}_t := d^{-1}(t)$. We identify

$$\mathcal{V}_t := \{(x, n) \in \Omega_{\mathbf{A}} \times \mathbb{Z} : \exists k \text{ such that } \sigma_{\mathbf{A}}^{n+k}(x) = \sigma_{\mathbf{A}}^k(t)\}. \quad (3.5)$$

We can consider \mathcal{V}_t as a subset of $\mathcal{V}_{\mathbf{A}} \times \mathbb{Z}$ via the embedding $(x, n) \mapsto (x_1 \cdots x_{n+\kappa_{\mathbf{A}}(x,n,t)}, n)$. If $n + \kappa_{\mathbf{A}}(x, n, t) = 0$, we interpret $x_1 \cdots x_{n+\kappa_{\mathbf{A}}(x,n,t)}$ as the empty word. We define the operator D_t on $\ell^2(\mathcal{V}_t)$ as

$$D_t \delta_{(x,n)} := \psi(n, \kappa_{\mathbf{A}}(x, n, t)) \delta_{(x,n)}.$$

It is proved in [27] that the collection $(\mathcal{A}_0, \ell^2(\mathcal{V}_t), D_t)$ is a spectral triple coinciding with the unbounded Kasparov product of the unbounded $(\mathcal{O}_{\mathbf{A}}, C(\Omega_{\mathbf{A}}))$ -Kasparov module $(\mathcal{A}_0, \Xi_{\mathbf{A}}, D, \psi)$ with the representation $\pi_t : C(\Omega_{\mathbf{A}}) \rightarrow \mathbb{C}$ given by point evaluation in t .

Proposition 3.7. *Let $\mathbf{A} = (A_{ij})_{i,j=1}^N$ be an $N \times N$ -matrix of 0's and 1's with no row or column being 0. Take $t \in \Omega_{\mathbf{A}}$. The odd spectral triple $(C_c^{\infty}(\mathcal{G}_{\mathbf{A}}), \ell^2(\mathcal{V}_t), D_t)$ constructed in Subsubsection 3.4.3 above satisfies the following properties:*

1. $(C_c^{\infty}(\mathcal{G}_{\mathbf{A}}), \ell^2(\mathcal{V}_t), D_t)$ is Li₁-summable, i.e. $\text{Tr}(e^{-s|D_t|})$ is finite for large enough $s \gg 0$;
2. The phase $F_t := 2\chi_{[0,\infty)}(D_t) - 1$ is such that for any two Borel functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in C_c^{\infty}(\mathcal{G}_{\mathbf{A}})$ the commutator $[F_t, a]$ preserves $\text{Dom}(f_1(D_t))$ and the operator

$$f_1(D_t)[F_t, a]f_2(D_t),$$

extends to a trace class operator on $\ell^2(\mathcal{V}_t)$;

3. Under the isomorphism $K^1(\mathcal{O}_{\mathbf{A}}) \cong \mathbb{Z}^N / (1 - \mathbf{A})\mathbb{Z}^N$ the class of $[(\mathcal{A}_0, \ell^2(\mathcal{V}_t), D_t)]$ is mapped to the class $\delta_j \bmod (1 - \mathbf{A})\mathbb{Z}^N$, where j is the first letter of t and δ_j denotes the j 'th basis vector in \mathbb{Z}^N .

Proof. We start by proving item 1): Li₁-summability. We compute that

$$\begin{aligned} \text{Tr}(e^{-s|D_t|}) &= \sum_{(x,n) \in \mathcal{V}_t} e^{-s|\psi(n, \kappa_{\mathbf{A}}(x,n,t))|} = \\ &= \sum_{n \in \mathbb{Z}} \sum_{k=\max(0,-n)}^{\infty} \#\{(x,n) \in \mathcal{V}_t : \kappa_{\mathbf{A}}(x,n,t) = k\} e^{-s|\psi(n,k)|}. \end{aligned}$$

If $(x,n) \in \mathcal{V}_t$ satisfies $\kappa_{\mathbf{A}}(x,n,t) = k$, then x is determined by the first $n+k$ letters of x and t . Therefore, we estimate

$$\#\{(x,n) \in \mathcal{V}_t : \kappa_{\mathbf{A}}(x,n,t) = k\} \leq N^{n+k}.$$

We can estimate

$$\text{Tr}(e^{-s|D_t|}) \leq \sum_{n \in \mathbb{Z}} \sum_{k=\max(0,-n)}^{\infty} N^{n+k} e^{-s(|n|+k+1)},$$

which is finite if $s > \log(N)$.

Next, we prove item 2). We identify $C_c(\mathcal{V}_t \times \mathcal{V}_t)$ with a space of finite rank operators on $\ell^2(\mathcal{V}_t)$ by $K\delta_{(x,n)} := \sum_{(x',n')} K((x',n'), (x,n))\delta_{(x',n')}$, for $K \in C_c(\mathcal{V}_t \times \mathcal{V}_t)$. It is clear that for any two Borel functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ and $K \in C_c(\mathcal{V}_t \times \mathcal{V}_t)$, the operator $f_1(D_t)Kf_2(D_t)$ again belongs to $C_c(\mathcal{V}_t \times \mathcal{V}_t)$, e.g. is of finite rank. By [27, Proof of Theorem 5.2.3], it holds that $[F_t, S_j] \in C_c(\mathcal{V}_t \times \mathcal{V}_t)$ for any j and therefore $[F_t, a] \in C_c(\mathcal{V}_t \times \mathcal{V}_t)$ for any $a \in \mathcal{A}_0$. Hence item 2) is true. Item 3) is proved in [27, Theorem 5.2.3]. \square

The remainder of this subsection will use the ingredients of the spectral triple in Proposition 3.7 to construct twisted spectral triples with the properties that we require.

Lemma 3.8. *Let $\mathbf{A} = (A_{ij})_{i,j=1}^N$ be an $N \times N$ -matrix of 0's and 1's with only nonzero rows and columns. Take $t \in \Omega_{\mathbf{A}}$ and consider the odd spectral triple $(\mathcal{A}_0, \ell^2(\mathcal{V}_t), D_t)$ from Proposition 3.7. For any $s \in \mathbb{C}$ and $a \in \mathcal{A}_0$, it holds that $\text{Dom}(e^{s|D_t|}) \subseteq \text{Dom}(e^{s|D_t|})$ and the operator $e^{s|D_t|} a e^{-s|D_t|}$ extends to a bounded operator on $\ell^2(\mathcal{V}_t)$.*

Proof. It suffices to consider $a = S_j$ for some j . A short computation shows that

$$S_j \delta_{(x,n)} = \begin{cases} \delta_{(jx, n+1)}, & \text{if } jx \text{ is admissible,} \\ 0, & \text{if } jx \text{ is not admissible.} \end{cases}$$

Therefore,

$$e^{s|D_t|} S_j \delta_{(x,n)} = \begin{cases} e^{s|\psi(n+1, \kappa_{\mathbf{A}}(jx, n+1, t))|} \delta_{(jx, n+1)}, & \text{if } jx \text{ is admissible,} \\ 0, & \text{if } jx \text{ is not admissible.} \end{cases}$$

Since

$$\left| |\psi(n+1, \kappa_{\mathbf{A}}(jx, n+1, t))| - |\psi(n, \kappa_{\mathbf{A}}(x, n, t))| \right| \leq 2,$$

we conclude that $S_j \text{Dom}(e^{s|D_t|}) \subseteq \text{Dom}(e^{s|D_t|})$ and $e^{s|D_t|} S_j e^{-s|D_t|}$ defines a bounded operator. \square

The ingredients to construct a twisted spectral triple are now all in place. For $t \in \Omega_{\mathbf{A}}$ and F_t as defined in Proposition 3.7 we define the self-adjoint operator

$$D_{\text{af},t} := F_t e^{|D_t|}. \quad (3.6)$$

The “af” stands for **actually finitely-summable**. Indeed $(i + D_{\text{af},t})^{-1} \in \mathcal{L}^p(\ell^2(\mathcal{V}_t))$ for any $p > \log(N)$ by the proof of Item 1) in Proposition 3.7 (see page 35). Using Lemma 3.8, for each $s \in \mathbb{C}$ we can define a homomorphism $\alpha_s : \mathcal{A}_0 \rightarrow \mathbb{B}(\ell^2(\mathcal{V}_t))$ as

$$\alpha_s(a) := |D_{\text{af},t}|^{is} a |D_{\text{af},t}|^{-is} = e^{is|D_t|} a e^{-is|D_t|}.$$

We write $\sigma := \alpha_{-i}$. Define \mathcal{A} as the saturation of \mathcal{A}_0 under σ , i.e. \mathcal{A} is the algebra generated by $\cup_{k \in \mathbb{Z}} \sigma^k(\mathcal{A}_0)$. Since $\sigma(a)^* = \sigma^{-1}(a^*)$, \mathcal{A} is a $*$ -algebra of bounded operators on $\ell^2(\mathcal{V}_t)$.

Proposition 3.9. *Let $\mathbf{A} = (A_{ij})_{i,j=1}^N$ be an $N \times N$ -matrix of 0's and 1's, with no row or column being 0, and take a $t \in \Omega_{\mathbf{A}}$. The collection $(\mathcal{A}, \ell^2(\mathcal{V}_t), D_{\text{af},t}, \sigma)$ is a strongly regular finitely summable twisted spectral triple. Moreover, for any two Borel functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ and $a \in \mathcal{A}$ the twisted commutator $[D_{\text{af},t}, a]_\sigma$ preserves $\text{Dom}(f_1(D_{\text{af},t}))$ and the operator $f_1(D_{\text{af},t})[D_{\text{af},t}, a]_\sigma f_2(D_{\text{af},t})$ extends to a trace class operator on H .*

Proof. That $(\mathcal{A}, \ell^2(\mathcal{V}_t), D_{\text{af},t}, \sigma)$ is a twisted spectral triple follows from noting that for $a \in \mathcal{A}$

$$[D_{\text{af},t}, a]_\sigma = D_{\text{af},t}[F_t, a], \quad (3.7)$$

which is bounded because for any $a \in \mathcal{A}_0$ and $k \in \mathbb{Z}$,

$$[F_t, \sigma^k(a)] = |D_{\text{af},t}|^k [F_t, a] |D_{\text{af},t}|^{-k},$$

which in turn belongs to $C_c(\mathcal{V}_t \times \mathcal{V}_t)$ because $[F_t, a] \in C_c(\mathcal{V}_t \times \mathcal{V}_t)$ by the proof of Proposition 3.7 (see page 35). By Proposition 2.8 (see page 26), $(\mathcal{A}, \ell^2(\mathcal{V}_t), D_{\text{af},t}, \sigma)$ is strongly regular. The twisted spectral triple $(\mathcal{A}, \ell^2(\mathcal{V}_t), D_{\text{af},t}, \sigma)$ is finitely summable since $(\mathcal{A}, \ell^2(\mathcal{V}_t), D_t)$ is Li_1 -summable. The last property stated in proposition follows from the identity (3.7) and Item 2) in Proposition 3.7. \square

Thus for any Cuntz-Krieger algebra we can construct a strongly regular twisted spectral triple satisfying condition a) of Lemma 3.1. To prove condition b) of Lemma 3.1 for some of these twisted spectral triples, we specialise to a particular family of Cuntz-Krieger algebras.

3.4.4 The counterexample

The case of interest to us is the Cuntz-Krieger algebra coinciding with the action of the free group on d generators acting on its Gromov boundary. We consider the $2d \times 2d$ -matrix

$$\mathbf{A} := \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & \vdots & \vdots \\ 1 & 1 & 1 & 0 & \cdots & \vdots & \vdots \\ 1 & 1 & 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & & \\ 1 & 1 & \cdots & & & 1 & 0 \\ 1 & 1 & 1 & \cdots & & 0 & 1 \end{pmatrix}. \quad (3.8)$$

In this case,

$$K_*(\mathcal{O}_{\mathbf{A}}) \cong K^{*+1}(\mathcal{O}_{\mathbf{A}}) \cong \begin{cases} \mathbb{Z}^d \oplus \mathbb{Z}/(d-1)\mathbb{Z}, & * = 0, \\ \mathbb{Z}^d, & * = 1. \end{cases}$$

For more details, see [27, Proposition 3.4.6 and 3.4.7] and references therein.

We identify the alphabet $\{1, 2, \dots, 2d\}$ with the alphabet $\{a_1, b_1, a_2, b_2, \dots, a_d, b_d\}$. We think of $\{a_1, b_1, a_2, b_2, \dots, a_d, b_d\}$ as a symmetric generating set for \mathbb{F}_d with $a_j = b_j^{-1}$. A word μ on the alphabet $\{a_1, b_1, a_2, b_2, \dots, a_d, b_d\}$ is admissible for \mathbf{A} if and only if μ thought of as a product of its letters in \mathbb{F}_d is a reduced word in the generating set $\{a_1, b_1, a_2, b_2, \dots, a_d, b_d\}$.

Therefore, identifying a finite admissible word μ with its product in \mathbb{F}_d induces a bijection of sets $\mathcal{V}_{\mathbf{A}} \rightarrow \mathbb{F}_d$. The space $\Omega_{\mathbf{A}}$ can be identified with the space $\partial\mathbb{F}_d$ of all infinite paths in \mathbb{F}_d and coincides with its Gromov boundary.

Let $\lambda_{\mu} \in C(\partial\mathbb{F}_d) \rtimes \mathbb{F}_d$ denote the unitary corresponding to the group element in \mathbb{F}_d defined by μ and $\chi_{C_{\mu}} \in C(\partial\mathbb{F}_d) \rtimes \mathbb{F}_d$ the characteristic function of the cylinder set C_{μ} . By [53, Section 2], the mapping defined from $S_{a_j} \mapsto \lambda_{a_j}(1 - \chi_{C_{b_j}})$ and $S_{b_j} \mapsto \lambda_{b_j}(1 - \chi_{C_{a_j}})$ gives an isomorphism $\mathcal{O}_{\mathbf{A}} \rightarrow C(\partial\mathbb{F}_d) \rtimes \mathbb{F}_d$.

We can describe the space $\ell^2(\mathcal{V}_t)$ and the spectral triple $(\mathcal{A}_0, \ell^2(\mathcal{V}_t), D_t)$ in terms of the free group in this case. We write $C^\infty(\partial\mathbb{F}_d)$ for the space of locally constant functions; it is generated by the cylinder functions $\{\chi_{C_{\mu}} : \mu \in \mathbb{F}_d\}$. First note that \mathcal{A}_0 coincides with the algebraic crossed product $C^\infty(\partial\mathbb{F}_d) \rtimes^{\text{alg}} \mathbb{F}_d$. For an element $\mu \in \mathbb{F}_d$ and $t \in \partial\mathbb{F}_d$ we define $\ell(\mu, t)$ as the number of cancellations occuring to put the product μt in reduced form. A short computation shows that the map

$$\varphi_t : \mathbb{F}_d \rightarrow \mathcal{V}_t, \quad \mu \mapsto (\mu t, |\mu| - 2\ell(\mu, t), t), \quad (3.9)$$

is a bijection of sets. Under φ_t and the identification $\mathcal{A}_0 = C^\infty(\partial\mathbb{F}_d) \rtimes^{\text{alg}} \mathbb{F}_d$, an element $a\lambda_{\gamma} \in \mathcal{A}_0$, where $a \in C^\infty(\partial\mathbb{F}_d)$ and $\gamma \in \mathbb{F}_d$, acts on $\ell^2(\mathbb{F}_d)$ as

$$a\lambda_{\gamma}\delta_{\mu} := a(\gamma\mu t)\delta_{\gamma\mu}.$$

Moreover,

$$\kappa_{\mathbf{A}}(\mu t, |\mu| - 2\ell(\mu, t), t) = \ell(\mu, t).$$

Therefore, $(\mathcal{A}_0, \ell^2(\mathcal{V}_t), D_t)$ is unitarily equivalent to $(C^\infty(\partial\mathbb{F}_d) \rtimes^{\text{alg}} \mathbb{F}_d, \ell^2(\mathbb{F}_d), D_{\mathbb{F}_d, t})$ where $D_{\mathbb{F}_d, t} := \varphi_t^{-1} D_t \varphi_t$ and we compute that for $\mu \in \mathbb{F}_d$ we have

$$D_{\mathbb{F}_d, t}\delta_{\mu} = \psi(|\mu| - 2\ell(\mu, t), \ell(\mu, t)) \delta_{\mu}.$$

In particular, $|D_{\mathbb{F}_d, t}\delta_{\mu}| = (||\mu| - 2\ell(\mu, t)| + \ell(\mu, t))\delta_{\mu}$.

Define the function

$$\Psi_t(\mu) := e^{||\mu| - 2\ell(\mu, t)| + \ell(\mu, t)}.$$

In this case, we can identify $D_{\text{af}, t}$ with the following operator on $\ell^2(\mathbb{F}_d)$:

$$D_{\text{af}, t}\delta_{\mu} = (2\chi_{\{0\}}(\ell(\mu, t)) - 1) \Psi_t(\mu)\delta_{\mu}. \quad (3.10)$$

The reader should note that the action of $C(\partial\mathbb{F}_d) \rtimes \mathbb{F}_d$ factors over the action of $C_b(\mathbb{F}_d) \rtimes \mathbb{F}_d$ on $\ell^2(\mathbb{F}_d)$ and the inclusion $i_t : C(\partial\mathbb{F}_d) \rtimes \mathbb{F}_d \hookrightarrow C_b(\mathbb{F}_d) \rtimes \mathbb{F}_d$ defined from the equivariant *-monomorphism $i_t : C(\partial\mathbb{F}_d) \rightarrow C_b(\mathbb{F}_d)$ given by $i_t(a)(\mu) := a(\mu t)$. It is clear from the definition that for $b \in C_b(\mathbb{F}_d)$ and $\gamma \in \mathbb{F}_d$,

$$\sigma(b\lambda_{\gamma})\delta_{\mu} = \Psi_t(\gamma\mu)\Psi_t(\mu)^{-1}b(\gamma\mu)\lambda_{\gamma}\delta_{\mu}.$$

We conclude that σ is a partially defined homomorphism $C_b(\mathbb{F}_d) \rtimes \mathbb{F}_d \rightarrow C_b(\mathbb{F}_d) \rtimes \mathbb{F}_d$. Moreover, the C^* -closure A of \mathcal{A} is an intermediate C^* -algebra

$$C(\partial\mathbb{F}_d) \rtimes \mathbb{F}_d = \mathcal{O}_{\mathbf{A}} \subseteq A \subseteq C_b(\mathbb{F}_d) \rtimes \mathbb{F}_d.$$

The proof that the twisted spectral triple $(C^\infty(\partial\mathbb{F}_d) \rtimes^{\text{alg}} \mathbb{F}_d, \ell^2(\mathbb{F}_d), D_{\text{af},t})$ constructed as in Proposition 3.9 satisfies condition b) of Lemma 3.1 is very long, but not particularly illuminating and so we omit them.

Indeed to check the detailed holomorphy statements, we resort to brute force calculation. We see no direct way of proving condition b) of Lemma 3.1 for a general Cuntz-Krieger algebra. It does however seem quite likely, to us, that condition b) of Lemma 3.1 holds for a more general class of Cuntz-Krieger algebras.

Theorem 3.10. *Take a fixed point $t \in \partial\mathbb{F}_d$. Let $(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)$ denote the regular finitely summable twisted spectral triple with finite discrete dimension spectrum obtained from saturating the weakly twisted spectral triple $(C^\infty(\partial\mathbb{F}_d) \rtimes^{\text{alg}} \mathbb{F}_d, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)$ with $D_{\text{af},t}$ as in Equation (3.10) and $\sigma(a) := |D_{\text{af},t}|a|D_{\text{af},t}|^{-1}$. Moscovici's ansatz does not compute the index pairing for $(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)$. More precisely, $(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)$ pairs nontrivially with $K_1(A)$ but the cochain (ϕ_m) provided by Moscovici's ansatz is the zero cochain. Hence the equality (2.1) does not hold.*

Proof. Set $A := \overline{\mathcal{A}} \subseteq \mathbb{B}(\ell^2(\mathbb{F}_d))$. We need to verify that $(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)$ satisfies the assumptions of Lemma 3.1 (see page 28) and that for some $x \in K_1(A)$,

$$\langle [(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)], x \rangle \neq 0.$$

The twisted spectral triple $(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)$ satisfies assumption a) of Lemma 3.1 by Proposition 3.9 (see page 37). As mentioned we omit the proof of assumption b) of Lemma 3.1, but note that it uses that t is a fixed point.

As for $\langle [(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)], x \rangle \neq 0$, we take $x := [\lambda_\gamma]$ where $\gamma \in \{a_1, b_1, \dots, a_d, b_d\}$ is one of the generators of \mathbb{F}_d . By the definition of the index pairing, $\langle [(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)], x \rangle$ coincides with the index pairing $\langle [(\mathcal{A}, \ell^2(\mathbb{F}_d), F_t)], x \rangle$ between the K -homology class $[(\mathcal{A}, \ell^2(\mathbb{F}_d), F_t)] \in K^1(A)$ and $x \in K_1(A)$. Again, going to the definition, the index pairing $\langle [(\mathcal{A}, \ell^2(\mathbb{F}_d), F_t)], x \rangle$ is by definition the index of the operator $P\lambda_\gamma P : P\ell^2(\mathbb{F}_d) \rightarrow P\ell^2(\mathbb{F}_d)$. Computing, we see that for $x = [\lambda_\gamma]$

$$\langle [(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)], x \rangle = \text{ind}(P\lambda_\gamma P : \ell^2(\mathcal{V}_{\mathbf{A},t_1}) \rightarrow \ell^2(\mathcal{V}_{\mathbf{A},t_1})).$$

Let us compute this index. We write $\mathbf{A} = (A_{i,j})_{i,j=1}^{2d}$ for the matrix in Equation (3.8) (i.e. $\mathcal{O}_{\mathbf{A}} \cong C(\partial\mathbb{F}_d) \rtimes \mathbb{F}_d$). Note that $\mathcal{V}_{\mathbf{A}} = \mathbb{F}_d$. For $\mu \in \mathcal{V}_{\mathbf{A},t_1}$ we have that

$$P\lambda_\gamma P\delta_\mu = \begin{cases} \delta_\gamma \mu, & |\mu| > 0, \\ A_{\gamma,t_1} \delta_\gamma, & |\mu| = 0. \end{cases}$$

In particular, $\dim \ker P\lambda_\gamma P = 1 - A_{\gamma,t_1}$. Therefore,

$$\text{ind}(P\lambda_\gamma P) = \dim \ker P\lambda_\gamma P - \dim \ker P\lambda_{\gamma^{-1}} P = A_{\gamma^{-1},t_1} - A_{\gamma,t_1} = \delta_{\gamma^{-1},t_1} - \delta_{\gamma,t_1}.$$

We see that for $\gamma = t_1^{\pm 1}$, $\langle [(\mathcal{A}, \ell^2(\mathbb{F}_d), D_{\text{af},t}, \sigma)], x \rangle \neq 0$. □

References

- [1] C. Anantharaman-Delaroche, *C^* -Algèbres de Cuntz-Krieger et groupes fuchsien*s. Operator Theory, Operator Algebras and Related Topics (Timișoara, 1996) 17–35. Theta Found., Bucharest, 1997.
- [2] C. Anantharaman-Delaroche, *Purely infinite C^* -algebras arising from dynamical systems*, Bull. Soc. Math. France, **125** (1997), 199–225.
- [3] S. Baaj, P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les C^* -modules hilbertiens*, C. R. Acad. Sci. Paris Sér. **I** **296** (21) (1983), 875–878.
- [4] J. Bellissard, M. Marcolli, K. Reihani, *Dynamical systems on spectral metric spaces*, arXiv:1008.4617
- [5] M. T. Benameur, T. Fack *Type II noncommutative geometry, I. Dixmier trace in von Neumann algebras*, Adv. Math. **199** (2006), 29–87.
- [6] B. Blackadar, *K -Theory for Operator Algebras*, 2nd ed., CUP, 1998.
- [7] C. Bourne and B. Mesland, *Index theory and topological phases of aperiodic lattices*, to appear in Ann. Henri Poincaré, DOI:10.1007/s00023-019-00764-9, arXiv:1807.03972.
- [8] A. Carey, V. Gayral, A. Rennie, F. Sukochev, *Integration for locally compact noncommutative spaces*, J. Funct. Anal., **263** (2012), 383–414.
- [9] A. Carey, V. Gayral, A. Rennie, F. Sukochev, *Index theory for locally compact noncommutative geometries*, Memoirs of the Amer. Math. Soc., **231** (2014).
- [10] A. Carey, S. Neshveyev, R. Nest, A. Rennie, *Twisted cyclic theory, equivariant KK -theory and KMS states*, J. Reine Angew. Math. **650** (2011), 161–191.
- [11] A. Carey, J. Phillips, *Unbounded Fredholm modules and spectral flow*, Canadian J. Math., **50**(4)(1998), 673–718.
- [12] A. Carey, J. Phillips, A. Rennie, *Spectral triples: examples and index theory*, in ‘Noncommutative Geometry and Physics: Renormalisation, Motives, Index Theory’, Ed. A. Carey, EMS (2011), 175–265.
- [13] A. Carey, J. Phillips, A. Rennie, F. Sukochev, *The local index formula in semifinite von Neumann algebras I: spectral flow*, Adv. Math. **202** (2006), 451–516.
- [14] A. Carey, J. Phillips, A. Rennie, F. Sukochev, *The local index formula in semifinite von Neumann algebras II: the even case*, Adv. Math. **202** (2006), 517–554.
- [15] A. Carey, J. Phillips, A. Rennie, F. Sukochev, *The Chern character of semifinite spectral triples*, J. Noncommut. Geom., **2** (2) (2008), 141–193.
- [16] A. Connes, *Compact metric spaces, Fredholm modules and hyperfiniteness*, Ergodic Th. Dynam. Sys. **9**(1989), 207–220.
- [17] A. Connes, *Noncommutative geometry*, Academic Press, San Diego, 1994.

- [18] A. Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. **34** (1995) 203–238.
- [19] A. Connes, J. Cuntz, *Quasi homomorphisms, cohomologie cyclique et positivité*, Commun. Math. Phys. **114** (1988), 515–526.
- [20] A. Connes, H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5** (1995) 174–243.
- [21] A. Connes, H. Moscovici, *Type III and spectral triples*, Traces in number theory, geometry and quantum fields, Aspects Math., Vieweg (2008), 57–71.
- [22] J. Cuntz, *K-theory and C^* -algebras*, Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), Lecture Notes in Math. **1046**, Springer, Berlin, 1984, 55–79.
- [23] R. J. Deeley, M. Goffeng, B. Mesland, *The bordism group of unbounded KK-cycles*, J. Topol. Anal. **10** (2) (2018), 355–400.
- [24] R. J. Deeley, M. Goffeng, B. Mesland, M.F. Whittaker, *Wieler solenoids, Cuntz-Pimsner algebras and K-theory*, Ergodic Th. Dynam. Sys. **38** (8) (2018), 2942–2988.
- [25] H. Emerson, B. Nica, *K-homological finiteness and hyperbolic groups*, J. Reine Angew. Math. **745** (2018), 189–229.
- [26] I. Forsyth, B. Mesland, A. Rennie, *Dense domains, symmetric operators and spectral triples*, New York J. Math., **20** (2014), 1001–1020.
- [27] M. Goffeng, B. Mesland, *Spectral triples and finite summability on Cuntz-Krieger algebras*, Doc. Math. **20** (2015), 89–170.
- [28] M. Goffeng, B. Mesland, *Spectral triples on O_N* , 2016 MATRIX Annals, 183–201, MATRIX Book Ser. **1**, Springer, Cham, 2018.
- [29] M. Goffeng, B. Mesland, A. Rennie, *Shift-tail equivalence and an unbounded representative of the Cuntz-Pimsner extension*, Ergodic Th. Dynam. Sys. **38** (4) (2018), 1389–1421.
- [30] M. Goffeng, A. Usachev, *Dixmier traces and residues on weak operator ideals*, arXiv:1710.08260.
- [31] M. Goffeng, A. Usachev, A. Rennie *Constructing KMS states from infinite-dimensional spectral triples*, arXiv:1811.06923.
- [32] J.M. Gracia-Bondia, J.C. Varilly, H. Figueroa, *Elements of noncommutative geometry*, Birkhauser, Boston, 2001.
- [33] M. Grensing, *Universal cycles and homological invariants of locally convex algebras*, J. Funct. Anal. **263** (8) (2012), 2170–2204.
- [34] T. Hadfield, U. Krähmer, *Twisted Homology of Quantum $SL(2)$* , K-theory **34** (4) (2005), 327–360.
- [35] T. Hadfield, U. Krähmer, *Twisted Homology of Quantum $SL(2)$ - Part II*, J. K-Theory, **6** (1) (2010), 69–98.

- [36] N. Higson *The local index formula in noncommutative geometry*, Contemporary Developments in Algebraic K -Theory, ICTP Lecture Notes, **15** (2003), 444–536.
- [37] J. Kaad, M. Lesch, *A local-global principle for regular operators on Hilbert modules*, J. Funct. Anal., **262** no. 10 (2012), 4540–4569.
- [38] J. Kaad, *The unbounded Kasparov product by a differentiable module*, arXiv:1509.09063.
- [39] J. Kaad, *On the unbounded picture of KK -theory*, arXiv:1901.05161.
- [40] J. Kaminker, I. Putnam, *K -theoretic duality of shifts of finite type*, Comm. Math. Phys. **187** (3) (1997), 509–522.
- [41] D. Kucerovsky, *A lifting theorem giving an isomorphisms of KK -products in bounded and unbounded KK -theory*, J. Operator Theory **44** (2000), 255–275.
- [42] A. Kumjian, D. Pask and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184**, (1998) 161–174.
- [43] E. C. Lance, *Hilbert C^* -Modules*, Cambridge University Press, Cambridge, 1995.
- [44] J. Lott, *Limit sets as examples in noncommutative geometry*, K -Theory **34** (2005), 283–326.
- [45] M. Matassa, R. Yuncken, *Regularity of twisted spectral triples and pseudodifferential calculi*, to appear in J. Noncomm. Geom. arXiv:1705.04178.
- [46] B. Mesland and M.H. Sengun, *Hecke operators in KK -theory and K -homology of Bianchi groups*, to appear in J. Noncomm. Geom., arXiv:1610.06808.
- [47] H. Moscovici, *Local index formula and twisted spectral triples*, Quanta of maths, 465–500, Clay Math. Proc. **11** (2010), Amer. Math. Soc., Providence, RI.
- [48] F. Pierrot, *Bimodules de Kasparov non bornés équivariants pour les groupoides topologiques localement compacts*, C. R. Math. Acad. Sci. Paris **342** (9) (2006), 661–663.
- [49] R. Ponge, H. Wang, *Noncommutative geometry and conformal geometry. I. Local index formula and conformal invariants*, J. Noncomm. Geom. **12** (4) (2018), 1573–1639.
- [50] R. Ponge, H. Wang, *Noncommutative geometry and conformal geometry II: Connes-Chern character and the local equivariant index theorem*, J. Noncomm. Geom. **10** (1) (2016), 303–374.
- [51] I. Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, **103** (2005), Amer. Math. Soc., Providence, RI.
- [52] A. Rennie, R. Senior, *The resolvent cocycle in twisted cyclic cohomology and a local index formula for the Podles sphere*, J. Noncommut. Geom., **8** (1) (2014), 1–43.
- [53] J. Spielberg, *Free-product groups, Cuntz-Krieger algebras, and covariant maps*, Internat. J. Math. **2** (4) (1991), 457–476.
- [54] C. Wahl, *On the noncommutative spectral flow*, J. Ramanujan Math. Soc. **22** (2), 2007, 135–187.