CHERN NUMBERS, LOCALISATION AND THE BULK-EDGE CORRESPONDENCE FOR CONTINUOUS MODELS OF TOPOLOGICAL PHASES

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ABSTRACT. In order to study continuous models of disordered topological phases, we construct an unbounded Kasparov module and a semifinite spectral triple for the crossed product of a separable C^*-algebra by a twisted $\mathbb{R}^d$-action. The spectral triple allows us to employ the non-unital local index formula to obtain the higher Chern numbers in the continuous setting with complex observable algebra. In addition, the pairing can be extended to a larger algebra closely related to dynamical localisation, as in the tight-binding approximation. The Kasparov module allows us to exploit the Wiener–Hopf extension and the Kasparov product to obtain a bulk-boundary correspondence for continuous models of disordered topological phases.

CONTENTS

1. Introduction

This paper examines the noncommutative index theory of twisted crossed products of a separable C^*-algebra B by $\mathbb{R}^d$. Our motivation for studying such algebras comes from its application to continuous models of disordered quantum systems, where the algebra of observables can be described by the twisted crossed product $C(\Omega) \rtimes_\theta \mathbb{R}^d$ [8, 9]. Numerous results in condensed matter physics which can be proved in the tight-binding approximation have not been addressed for continuum models. Here we study higher Chern numbers, the bulk-edge correspondence and stability of phases in the strongly disordered/dynamically localised regime for continuum models. Because of the anti-linear symmetries that appear in topological insulator systems, we will consider both complex and real C^*-algebras and crossed products.

The key to our approach is the construction of a Kasparov module and a semifinite spectral triple modelling the geometry of the noncommutative disordered Brillouin zone. The spectral triple satisfies the strongest summability conditions of [22], allowing us to employ the local index formula for complex algebras.

The local index formula yields the higher Chern numbers directly, in complete analogy with the formula for the higher Chern numbers in the tight-binding approximation [69, 70, 71, 72].

In Section 5 we extend the formulae for the higher Chern numbers to a larger Sobolev algebra that is constructed using the non-commutative L^p-spaces and closely related to regions of dynamical localisation.

Kellendonk and Richard [48] use the Wiener–Hopf extension to model the relationship between bulk and edge observables,

$$(1) \quad 0 \to \mathcal{K} \otimes (B \rtimes_\theta \mathbb{R}^{d-1}) \to \left(C_0(\mathbb{R} \cup \{+\infty\}) \otimes (B \rtimes_\theta \mathbb{R}^{d-1}) \right) \rtimes \mathbb{R} \to (B \rtimes_\theta \mathbb{R}^{d-1}) \rtimes \mathbb{R} \to 0.$$

We prove, in Section 6, that our Kasparov module for a twisted $\mathbb{R}^d$-action factorises (up to a sign) into the product of a Kasparov module for a twisted $\mathbb{R}^{d-1}$-action with the extension class from Equation (1) linking the bulk and edge algebras. This factorisation implies a bulk-edge correspondence for the semifinite index pairing as well as more general pairings of our Kasparov module with real or complex K-theory classes.

We return to our initial motivation in Section 7 and include a case-study of how our theory applies to disordered quantum systems and their topological properties. The example of disordered magnetic Schrödinger operators on $L^2(\mathbb{R}^d)$ also allows us to consider the connection of our Sobolev algebra to the localised states studied in [1, 34, 35]. We compare our results and those in [1], where we show that if the Fermi energy lies in a region of dynamical localisation, then our bulk indices and Chern number formulas are still well-defined and stable under homotopies within a mobility gap. In case of complex topological insulators, we are also able to extend our results on the bulk-edge correspondence and show that non-trivial bulk invariants imply delocalised edge states on the boundary, analogous to the discrete case in [71, Section 6.6].

Finally, Appendix A gives a brief summary of non-unital index theory and the tools from Kasparov theory we require.

2. Kasparov modules for twisted crossed products by $\mathbb{R}^d$

In this section we construct a Kasparov module for twisted crossed products $B \rtimes_\theta \mathbb{R}^d$ where B is a real or complex separable C^*-algebra; see Appendix A for the definition of an unbounded Kasparov module. This Kasparov module is closely related to the Connes–Thom class in Kasparov theory when the crossed product is untwisted. The inverse class was studied in [3, 2] for a different class of twisted crossed products.

2.1. **Preliminaries on twisted dynamical systems.** Let B be a C^*-algebra with $(B, \mathbb{R}^d, \alpha, \theta)$ a twisted dynamical system [64]. We consider the $*$-algebra $C_c(\mathbb{R}^d, B)$ with operations,

$$(f_1 * f_2)(x) = \int_{\mathbb{R}^d} \alpha_{-x}(\theta(y, x - y)) \alpha_{y-x}(f_1(y)) f_2(x - y) \, dy, \qquad f^*(x) = \alpha_{-x}(f(-x)^*).$$

The unitary-valued function $\theta : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{U}(\mathcal{M}(B))$ encodes the twist and takes values in the unitaries of the multiplier algebra of B. The twist θ is required to satisfy the cocycle identities

(2) $\quad \theta(x,y)\theta(x+y,z) = \alpha_x(\theta(y,z))\theta(x,y+z), \qquad \theta(x,0) = \theta(0,x) = 1 \quad$ for all $x,\, y,\, z \in \mathbb{R}^d$,

and the following relationship with the action:

(3) $\qquad\qquad\qquad \alpha_x \circ \alpha_y(b) = \theta(x,y)\,\alpha_{x+y}(b)\,\theta(x,y)^*, \quad x, y \in \mathbb{R}^d,\ b \in B.$

We denote the crossed product completion $B \rtimes_\theta \mathbb{R}^d$ by A.

We do not consider crossed products with arbitrary twists θ, but restrict to the case that $\theta(x,-x) = 1$ for all $x \in \mathbb{R}^d$. This simplifies many of our arguments and still encompasses the examples of interest (e.g. a disordered quantum system with continuously changing magnetic field).

If for every $x, y \in \mathbb{R}^d$, $\theta(x,y)$ is constant in B (e.g. θ comes from a magnetic field with constant strength), then the twist reduces to a map $\theta : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{U}(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Thus for complex algebras θ is a group cocycle $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{T}$ and therefore is related to the Moore cohomology group $H^2(\mathbb{R}^d, \mathbb{T})$, which is constructed from Borel multipliers of $\mathbb{R}^d$. In the real case, we are interested in $H^2(\mathbb{R}^d, \{\pm 1\})$. In such situations we note the following.

Proposition 2.1 ([81], Lemma 8.3). *If $\theta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{T}$ (resp. $\theta : \mathbb{R}^d \times \mathbb{R}^d \to \{\pm 1\}$) is a Borel multiplier and its class $[\theta] \in H^2(\mathbb{R}^d, \mathbb{T})$ (resp. $H^2(\mathbb{R}^d, \{\pm 1\})$) is non-torsion, then θ is cohomologous to $\widetilde{\theta}$ with $\widetilde{\theta}(x,-x) = 1$ for all $x \in \mathbb{R}^d$.*

Proof. By the cocycle property of θ, we first note that

$$\theta(x,-x)\theta(0,x) = \theta(x,0)\theta(-x,x)$$

so $\theta(x,-x) = \theta(-x,x)$. Next, provided $[\theta]$ is non-torsion, we can define $\lambda(x) = [\theta(x,-x)]^{1/2}$ where we take the square root with argument in $[0,\pi)$ (if the cocycle is $\{\pm 1\}$-valued, then we take the argument to be 0). Then we have that

$$\partial\lambda(x,-x) = \lambda(0)\lambda(x)^{-1}\lambda(-x)^{-1} = \theta(x,-x)^{-1}.$$

Lastly, we define $\widetilde{\theta} = \theta\partial\lambda$ which by construction is such that $\widetilde{\theta}(x,-x) = 1$. $\qquad\square$

For the case that $B = C(\Omega)$ for some compact and second countable space Ω with twisted action, the assumption $\theta(x,-x) = 1$ means that there is an explicit isomorphism

$$C(\Omega) \rtimes_\theta \mathbb{R}^d \cong \big(C(\Omega) \rtimes_\theta \mathbb{R}^{d-1}\big) \rtimes \mathbb{R}$$

where the crossed product by $\mathbb{R}$ is untwisted, see [48]. This decomposition allows us to relate the twisted crossed product $C(\Omega) \rtimes_\theta \mathbb{R}^d$ to the Wiener–Hopf extension

$$0 \to (C(\Omega) \rtimes_\theta \mathbb{R}^{d-1}) \otimes \mathcal{K}[L^2(\mathbb{R})] \to C_0(\mathbb{R} \cup \{+\infty\}, C(\Omega) \rtimes_\theta \mathbb{R}^{d-1}) \rtimes \mathbb{R} \to C(\Omega) \rtimes_\theta \mathbb{R}^d \to 0.$$

Such an extension plays a crucial role in the bulk-edge correspondence for disordered topological phases with a boundary in Section 6.

For more general twisted actions, we first use [64, Theorem 4.1] to decompose

$$B \rtimes_\theta \mathbb{R}^d \cong \big(B \rtimes_{\theta_e} \mathbb{R}^{d-1}\big) \rtimes_\sigma \mathbb{R}$$

with θ_e the restriction of θ to $\mathbb{R}^{d-1} \times \{0\}$. Then, letting $C = B \rtimes_{\theta_e} \mathbb{R}^{d-1}$ and using the Packer–Raeburn stabilisation trick [64, Section 3],

$$K_*(C \rtimes_\sigma \mathbb{R}) \cong K_*\big((C \otimes \mathcal{K}) \rtimes \mathbb{R}\big) \cong K_{*-1}(C \otimes \mathcal{K}) \cong K_{*-1}(C) \cong K_*(C \rtimes \mathbb{R}).$$

Therefore the Packer–Raeburn stabilisation isomorphism gives us an invertible element in $KK(C \rtimes_\sigma \mathbb{R}, C \rtimes \mathbb{R})$ which allows us to relate the twisted crossed product $C \rtimes_\sigma \mathbb{R}$ to the Wiener–Hopf extension

$$0 \to C \otimes \mathcal{K} \to (C_0(\mathbb{R} \cup \{+\infty\}) \otimes C) \rtimes \mathbb{R} \to C \rtimes \mathbb{R} \to 0$$

and corresponding class in $KK^1(C \rtimes \mathbb{R}, C)$. Hence, from the perspective of Kasparov theory, we can assume that our twisted action $A = B \rtimes_\theta \mathbb{R}^d$ is such that $A \cong (B \rtimes_\theta \mathbb{R}^{d-1}) \rtimes \mathbb{R}$ without losing

any index-theoretic information. This unwinding of the crossed product will be important for boundary maps under the Wiener–Hopf extension and the bulk-edge correspondence in Section 6.

Example 2.2 (Magnetic twists, [10, 56, 59]). Let $B = C(\Omega)$ with Ω the compact space of disorder configurations with a (twisted) action by $\mathbb{R}^d$ of magnetic translations. Consider a magnetic field in $\mathbb{R}^d$ with components $\{B_{jk}^\omega\}_{j,k=1}^d$ that continuously depend on $\omega \in \Omega$. We then regard the cocycle θ as a function of ω, where

$$\theta(x,y)(\omega) = \exp\left(-i\Gamma^{B^\omega}\langle 0, x, x+y\rangle\right)$$

with $\Gamma^{B^\omega}\langle 0, x, x+y\rangle$ the flux of the magnetic field through the triangle defined by the points 0, x and $x+y$. We see that in this case $\theta(x,-x) = 1$ for all $\omega \in \Omega$ as required. The algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ models continuous and disordered quantum systems with a (not necessarily constant) magnetic field.

Let us extend this example slightly by considering the case when $B = C(\Omega) \rtimes_\phi \mathbb{R}^k$ for $1 \leq k < d$. Following [64, Theorem 4.1], there is a decomposition

$$C(\Omega) \rtimes_\theta \mathbb{R}^d \cong \left(C(\Omega) \rtimes_\phi \mathbb{R}^k\right) \rtimes_\sigma \mathbb{R}^{d-k},$$

where, because the subgroup and quotient of $\mathbb{R}^d$ we consider is easy, the action and twist of $\mathbb{R}^k$ and $\mathbb{R}^{d-k}$ is simply the restriction of the action and twist of $\mathbb{R}^d$ to $\mathbb{R}^k \times \{0\}^d$ and $\{0\}^k \times \mathbb{R}^{d-k}$ respectively. Hence we retain that both $\theta(x,-x)(\omega) = 1$ and $\sigma(z,-z)(\omega) = 1$ for $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^{d-k}$. Such a decomposition of twisted crossed products has applications to so-called weak topological insulators, where we may use this decomposition to extract $(d-k)$-dimensional invariants from d-dimensional systems. We will not emphasise this application here, though the interested reader can consult [72, Section 7, 8] for results in the discrete setting.

We also remark that magnetic twists for real algebras and real crossed products are less interesting as we require $\theta(x,y)$ to be an orthogonal operator in $\mathcal{M}(C(\Omega, \mathbb{R}))$. This puts large constraints on the type of magnetic field we can consider and will often mean that the magnetic field vanishes. We will return to crossed products twisted by a magnetic field in Section 7.

We will now restrict to twisted dynamical systems $(B, \mathbb{R}^d, \alpha, \theta)$ such that $\theta(x,-x) = 1$ for all $x \in \mathbb{R}^d$.

2.2. **An unbounded Kasparov module.** We consider the Hilbert C^*-module $L^2(\mathbb{R}^d, B) \cong L^2(\mathbb{R}^d) \otimes B$ with right action by right-multiplication and inner-product

$$(f_1 \mid f_2)_B = \int_{\mathbb{R}^d} f_1(x)^* f_2(x) \, \mathrm{d}x.$$

Lemma 2.3. *If the twist θ is such that $\theta(x,-x) = 1$ for all $x \in \mathbb{R}^d$, then the Hilbert module $L^2(\mathbb{R}^d, B)$ is isometrically isomorphic to the C^*-module E_B given by the completion of $C_c(\mathbb{R}^d, B)$ with respect to the inner product $(f_1 \mid f_2)_B = (f_1^* * f_2)(0)$.*

Proof. The inner-product on E_B takes the form

$$(f_1 \mid f_2)_B = \int_{\mathbb{R}^d} \theta(y,-y)\alpha_y(f_1^*(y))f_2(-y) \, \mathrm{d}y = \int_{\mathbb{R}^d} \theta(-y,y)f_1(y)^* f_2(y) \, \mathrm{d}y.$$

If $\theta(y,-y) = 1$ then the inner products coincide and the right-action of B by right-multiplication is compatible with the inner product on E_B. Hence the two spaces are isomorphic as C^*-modules. $\qquad\square$

Proposition 2.4. *Let $C_c(\mathbb{R}^d, B)$ act on E_B by left convolution multiplication. Then this action extends to an adjointable representation of $A = B \rtimes_\theta \mathbb{R}^d$.*

Proof. The action is adjointable on a dense subspace as

$$(f_1 * f_2 \mid f_3)_B = (f_2^* * f_1^* * f_3)(0) = (f_2 \mid f_1^* * f_3)_B, \quad f_1, f_2, f_3 \in C_c(\mathbb{R}^d, B)$$

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Furthermore, the action is bounded since

$$(f_1 * f_2 \mid f_1 * f_2)_B = (f_2^* * f_1^* * f_1 * f_2)(0) \leq \|f_1^* * f_1\|(f_2 \mid f_2)_B, \quad f_1, f_2 \in C_c(\mathbb{R}^d, B),$$

and so it extends to an adjointable action on the whole space by the completion $B \rtimes_\theta \mathbb{R}^d$. $\square$

Using the identification of E_B with $L^2(\mathbb{R}^d, B)$, we can define an adjointable action of $C_c(\mathbb{R}^d, B)$ (which extends to an action of $B \rtimes_\theta \mathbb{R}^d$) on $L^2(\mathbb{R}^d, B)$ by

$$(\pi(f)\psi)(x) = \int_{\mathbb{R}^d} \alpha_{-x}(\theta(y, x - y))\alpha_{y-x}(f(y))\psi(x - y) \, \mathrm{d}y$$

$$= \int_{\mathbb{R}^d} \alpha_{-x}(\theta(x - u, u))\alpha_{-u}(f(x - u))\psi(u) \, \mathrm{d}u$$

(4)
$$= \int_{\mathbb{R}^d} \theta(-x, x - u)\alpha_{-u}(f(x - u))\psi(u) \, \mathrm{d}u,$$

where we have made the substitution $u = x - y$ and used the identity from Equation (2),

$$\alpha_{-x}(\theta(x - u, u))\theta(-x, x) = \theta(-x, x - u)\theta(-u, u),$$

which together with the assumption $\theta(x, -x) = 1$ implies that $\alpha_{-x}(\theta(x - u, u)) = \theta(-x, x - u)$.

Remark 2.5. The two presentations of the right-B C^*-module are useful in different contexts. The module E_B allows us to easily define a left-action of A, while $\mathrm{End}_B^0(L^2(\mathbb{R}^d, B)) \cong \mathcal{K}[L^2(\mathbb{R}^d)] \otimes B$, and so the presentation $L^2(\mathbb{R}^d, B)$ is useful for more analytic arguments.

The algebra $C_c(\mathbb{R}^d, B)$ comes with the derivations $(\partial_j f)(x) = x_j f(x)$ (where x_j is the j-th component of $x \in \mathbb{R}^d$) and we observe that $\partial_j(C_c(\mathbb{R}^d, B)) \subset C_c(\mathbb{R}^d, B)$. A brief computation relates the derivations $\{\partial_j\}_{j=1}^d$ to the unbounded position operators $\{X_j\}_{j=1}^d$ on $L^2(\mathbb{R}^d, B)$, where for $f \in C_c(\mathbb{R}^d, B)$,

(5)
$$\pi(\partial_j f) = [X_j, \pi(f)].$$

To construct the unbounded operator for our Kasparov module, we use the $\mathbb{Z}_2$-graded exterior algebra $\bigwedge^* \mathbb{R}^d$ and Clifford representations on this space. We first establish our notation and conventions for the Clifford algebras $C\ell_{p,q}$, namely

$$C\ell_{p,q} = \mathrm{span}_\mathbb{R} \left\{ \gamma^1, \ldots, \gamma^p, \rho^1, \ldots, \rho^q \mid (\gamma^i)^2 = 1, (\gamma^i)^* = \gamma^i, (\rho^i)^2 = -1, (\rho^i)^* = -\rho^i \right\},$$

and such that all the various γ^j, ρ^k are odd and mutually anti-commute. The exterior algebra $\bigwedge^* \mathbb{R}^d$ has representations of $C\ell_{0,d}$ and $C\ell_{d,0}$ with generators

$$\rho^j(\omega) = e_j \wedge \omega - \iota(e_j)\omega, \qquad\qquad \gamma^j(\omega) = e_j \wedge \omega + \iota(e_j)\omega,$$

where $\{e_j\}_{j=1}^d$ is the standard basis of $\mathbb{R}^d$ and $\iota(\nu)\omega$ is the contraction of ω along ν. One readily checks that ρ^j and γ^j mutually anti-commute and generate representations of $C\ell_{0,d}$ and $C\ell_{d,0}$ respectively.

Proposition 2.6. *The triple*

$$\lambda_d = \left(C_c(\mathbb{R}^d, B)\hat{\otimes}C\ell_{0,d}, \, L^2(\mathbb{R}^d, B)_B\hat{\otimes}\bigwedge^* \mathbb{R}^d, \, X = \sum_{j=1}^d X_j\hat{\otimes}\gamma^j \right)$$

is a real or complex unbounded Kasparov module.

Proof. The first thing to observe is that X is self-adjoint and regular. This can be proved directly, or by using the local-global principle [40, 67] and the fact that (up to Clifford variables) we have a multiplication operator. For $f \in C_c(\mathbb{R}^d, B)$, Equation (5) says that

$$[X, \pi(f)] = \sum_{j=1}^d [X_j, \pi(f)]\hat{\otimes}\gamma^j = \sum_{j=1}^d \pi(\partial_j f)\hat{\otimes}\gamma^j,$$

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and $\pi(\partial_j f) \in \text{End}_B(L^2(\mathbb{R}^d, B))$ for all $j \in \{1, \dots, d\}$. For $\rho^k \in C\ell_{0,d}$ we know that $\rho^k \gamma^j = -\gamma^j \rho^k$ for $j \in \{1, \dots, d\}$, so $C\ell_{0,d}$ graded commutes with X. Thus (graded) commutators of X with elements of $C_c(\mathbb{R}^d, B) \hat{\otimes} C\ell_{0,d}$ are defined on $\text{Dom}(X)$ and extend to adjointable operators. Therefore all that we need to show is that $\pi(f)(1+X^2)^{-1/2}$ is compact in $L^2(\mathbb{R}^d, B)$ for $f \in C_c(\mathbb{R}^d, B)$. Using Equation (4), we note that $\pi(f)(1+X^2)^{-1/2}$ has the B-valued integral kernel

(6)
$$k_f(x, y) = \theta(-x, x-y)\alpha_{-y}(f(x-y))(1+|y|^2)^{-1/2} \hat{\otimes} \text{Id}_{\bigwedge^* \mathbb{R}^d}.$$

The continuity of f and θ shows that $k_f \in C_0(\mathbb{R}^d \times \mathbb{R}^d) \otimes B$. This now allows us to find a sequence in $C_c(\mathbb{R}^d) \otimes C_c(\mathbb{R}^d) \otimes B^2$ such that

$$k_f = \lim_{n \to \infty} \sum_{j=1}^{N_n} f_{n,j} \otimes g_{n,j} \otimes b_{n,j} c_{n,j}.$$

Then computing shows that the sum of rank one operators (see the Appendix)

$$\pi(f)(1+X^2)^{-1/2} = \lim_{n \to \infty} \sum_{j=1}^{N_n} \Theta_{f_{n,j} \otimes b_{n,j}, \overline{g_{n,j}} \otimes c_{n,j}^*}$$

converges in the operator norm topology of operators on $L^2(\mathbb{R}^d, B)$. Hence $\pi(f)(1+X^2)^{-1/2}$ is the norm limit of compact operators, and thus is compact. $\square$

We have used the orientation of $\mathbb{R}^d$ to construct the Kasparov module using the operator $X = \sum_{j=1}^d X_j \hat{\otimes} \gamma^j$ and left-Clifford multiplication on $\bigwedge^* \mathbb{R}^d$, [58, Section 4]. The exterior algebra construction has the benefit that the differences between the real and complex cases are minimal and the Kasparov modules we construct behave well under Kasparov products (see Section 6).

3. Traces and a semifinite spectral triple

If the algebra B has a faithful, semifinite and norm lower-semicontinous tracial weight, τ_B, that is invariant under the twisted $\mathbb{R}^d$-action, there is a general method by which we can obtain a semifinite spectral triple, [55, 65, 42, 20]. Again, a summary of the relevant definitions and results are contained in the Appendix.

The existence of such a trace on B is satisfied in the physically interesting case of $B = C(\Omega, M_N(\mathbb{C}))$ (or $M_N(\mathbb{R})$), where the disorder space of configurations Ω (typically compact) is equipped with a probability measure $\mathbf{P}$ such that $\text{supp}(\mathbf{P}) = \Omega$. In examples from aperiodic media, the measure $\mathbf{P}$ is often invariant and ergodic under the $\mathbb{R}^d$-action by translations, though many of our results only require that τ_B is invariant under the group action.

In our examples, the semifinite spectral triple we obtain is also smoothly summable in the sense of Definition A.11, which allows us to employ the local index formula, Theorem A.14 and A.15 [22, Theorem 3.33]. In turn, the local index formula gives us the higher Chern numbers and an approach to understanding localisation.

Given the C^*-module $L^2(\mathbb{R}^d, B)$ and trace τ_B, we complete $C_c(\mathbb{R}^d) \otimes \text{Dom}(\tau_B)$ in the norm coming from the inner-product

$$\langle \lambda_1 \otimes b_1, \lambda_2 \otimes b_2 \rangle = \tau_B((\lambda_1 \otimes b_1 \mid \lambda_2 \otimes b_2)_B) = \langle \lambda_1, \lambda_2 \rangle_{L^2(\mathbb{R}^d)} \tau_B(b_1^* b_2),$$

which defines the Hilbert space $L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B)$ where $L^2(B, \tau_B)$ is the GNS space.

Lemma 3.1. *The algebra $A = B \rtimes_\theta \mathbb{R}^d$ acts on $L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B)$.*

Proof. This follows from the identification $L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B) \cong L^2(\mathbb{R}^d, B) \otimes_B L^2(B, \tau_B)$ and Proposition 2.4. $\square$

Proposition 3.2 ([55], Theorem 1.1). *Given $T \in \text{End}_B(L^2(\mathbb{R}^d, B))$ with $T \geq 0$, define*

$$\text{Tr}_\tau(T) = \sup_I \sum_{\xi \in I} \tau_B[(\xi \mid T\xi)_B],$$

where the supremum is taken over all finite subsets $I \subset L^2(\mathbb{R}^d, B)$ with $\sum_{\xi \in I} \Theta_{\xi, \xi} \leq 1$.

1) Then Tr_τ is a semifinite norm lower-semicontinuous trace on the compact endomorphisms $\mathrm{End}_B^0(L^2(\mathbb{R}^d, B))$ with the property $\mathrm{Tr}_\tau(\Theta_{\xi_1, \xi_2}) = \tau_B((\xi_2 \mid \xi_1)_B)$.

2) Let $\mathcal{N}$ be the von Neumann algebra $\mathrm{End}_B^{00}(L^2(\mathbb{R}^d, B))'' \subset \mathcal{B}(L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B))$. Then the trace Tr_τ extends to a faithful semifinite trace on the positive cone $\mathcal{N}_+$.

The semifinite trace Tr_τ on $\mathcal{N}$ gives a semifinite trace $\mathrm{Tr}_\tau \hat{\otimes} \mathrm{Tr}_{\bigwedge^* \mathbb{R}^d}$ on $\mathcal{N} \hat{\otimes} \mathrm{End}(\bigwedge^* \mathbb{R}^d)$. To simplify our notation, we will often suppress the finite-trace and finite-dimensional von Neumann algebra $\mathrm{End}(\bigwedge^* \mathbb{R}^d)$.

Lemma 3.3. *If $f \in C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B)^{1/2})$, then $\pi(f)(1 + X^2)^{-s/4}$ is Hilbert-Schmidt with respect to Tr_τ for $s > d$.*

Proof. The operator $\pi(f)(1 + X^2)^{-s/4}$ has the integral kernel
$$k_f(x, y) = \theta(-x, x - y)\alpha_{-y}(f(x - y))(1 + |y|^2)^{-s/4} \hat{\otimes} \mathrm{Id}_{\bigwedge^* \mathbb{R}^d}.$$

Ignoring the factor $\mathrm{Id}_{\bigwedge^* \mathbb{R}^d}$, the kernel of $(\pi(f)(1 + X^2)^{-s/4})^* = (1 + X^2)^{-s/4}\pi(f^*)$ is then
$$\begin{aligned}
\widetilde{k}_{f^*}(x, y) &= (1 + |x|^2)^{-s/4}\theta(-x, x - y)\alpha_{-y}(f^*(x - y)) \\
&= (1 + |x|^2)^{-s/4}\theta(-x, x - y)\alpha_{-y} \circ \alpha_{y-x}(f(y - x)^*) \\
&= (1 + |x|^2)^{-s/4}\theta(-x, x - y)\theta(-y, y - x)\alpha_{-x}(f(y - x)^*)\theta(-y, y - x)^* \\
&= (1 + |x|^2)^{-s/4}\alpha_{-x}(f(y - x)^*)\theta(-y, y - x)^*,
\end{aligned}$$
where we used the definition of f^*, Equation (3) on the twisting of α, and the cocycle identity
$$\theta(-x, x - y)\theta(-y, y - x) = \alpha_{-x}(\theta(x - y, y - x))\theta(-x, 0) = \alpha_{-x}(1)1 = 1$$
under the added assumption $\theta(u, -u) = 1$.

Because τ_B is a faithful, semifinite and norm lower-semicontinuous tracial weight on B, the trace-class operators $\mathcal{L}^1(\mathcal{N}, \mathrm{Tr}_\tau)$ contains $\mathcal{L}^1(L^2(\mathbb{R}^d)) \otimes \mathrm{Dom}(\tau_B)$ (algebraic tensor product), and the trace restricted to this set is $\mathrm{Tr}_{L^2(\mathbb{R}^d)} \otimes \tau_B$. Ignoring the trace over $\bigwedge^* \mathbb{R}^d$, we compute directly
$$\begin{aligned}
\mathrm{Tr}_\tau\left((1 + X^2)^{-s/4}\pi(f^*f)(1 + X^2)^{-s/4}\right) &= \int_{\mathbb{R}^{2d}} \tau_B\left(\widetilde{k}_{f^*}(x, y)k_f(y, x)\right)\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^{2d}} \tau_B\Big((1 + |x|^2)^{-s/4}\alpha_{-x}(f(y - x)^*)\theta(-y, y - x)^* \\
&\qquad\qquad \times \theta(-y, y - x)\alpha_{-x}(f(y - x))(1 + |x|^2)^{-s/4}\Big)\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^{2d}} \tau_B\Big(\alpha_{-x}(f(y - x)^*)\alpha_{-x}(f(y - x))\Big)(1 + |x|^2)^{-s/2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}^{2d}} \tau_B\Big(f(y - x)^*f(y - x)\Big)(1 + |x|^2)^{-s/2}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$
where we have used the invariance of τ_B under the $\mathbb{R}^d$-action. Next we make the substitution $u = y - x$, $v = x$ and use the compact support of f on u to estimate, for $s > d$,
$$\left\|\pi(f)(1 + X^2)^{-s/4}\right\|_2^2 = \int_{\mathbb{R}^{2d}} \tau_B(f(u)^*f(u))(1 + |v|^2)^{-s/2}\,\mathrm{d}u\,\mathrm{d}v = C_s\int_{\mathbb{R}^d} \tau_B(f(u)^*f(u))\,\mathrm{d}u < \infty.$$

The trace over $\bigwedge^* \mathbb{R}^d$ does not change the argument, only adding a factor of 2^d, and so we are done. $\square$

In the language of semifinite spectral triples (summarised in the Appendix), the Lemma says that $C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B)^{1/2})$ is contained in $\mathcal{B}_2(X, d)$, the 'square integrable' operators. In fact $C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B)^{1/2})$ is contained in $\mathcal{B}_2^\infty(X, d)$, the 'smooth square integrable' operators.

Lemma 3.4. *For $f \in C_c(\mathbb{R}^d, B)$, let $\delta(\pi(f)) = [|X|, \pi(f)]$, defined initially on $\mathrm{Dom}(X)$. Then for all $m = 1, 2, 3, \ldots$, and all $f \in C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B)^{1/2})$, the operator $\delta^m(\pi(f))(1 + X^2)^{-s/4}$ is Hilbert-Schmidt with respect to Tr_τ.*

Proof. The proof is much like that of the previous lemma. We just note that the operator $\delta^m(\pi(f))(1 + X^2)^{-s/4}$ has B-valued integral kernel

$$k_{f,m}(x, y) = \theta(-x, x - y)(|x| - |y|)^m \alpha_{-y}(f(x - y))(1 + |y|^2)^{-s/4} \hat{\otimes} \mathrm{Id}_{\bigwedge^* \mathbb{R}^d}.$$

Then just as in Lemma 3.3, the kernel of $(\delta^m(\pi(f))(1+X^2)^{-s/4})^* = (1+X^2)^{-s/4}\delta^m(\pi(f^*))(-1)^m$ is then

$$\widetilde{k}_{f^*,m}(x, y) = (1 + |x|^2)^{-s/4}(|x| - |y|)^m (-1)^m \alpha_{-x}(f(y - x)^*)\theta(-y, y - x)^* \hat{\otimes} \mathrm{Id}_{\bigwedge^* \mathbb{R}^d}.$$

Then we compute as before,

$$(-1)^m \mathrm{Tr}_\tau \left((1 + X^2)^{-s/4} \delta^m(\pi(f^*)) \delta^m(\pi(f))(1 + X^2)^{-s/4} \right)$$

$$= \int_{\mathbb{R}^{2d}} \tau_B \left(f(y - x)^* f(y - x) \right) (|x| - |y|)^{2m} (-1)^m (1 + |x|^2)^{-s/2} \, \mathrm{d}x \, \mathrm{d}y.$$

Now taking absolute values, using $(|x| - |y|)^{2m} \leq |x - y|^{2m}$, and changing variables as in the last lemma we find that for $s > d$,

$$\left\| \delta^m(\pi(f))(1 + X^2)^{-s/4} \right\|_2^2 \leq \int_{\mathbb{R}^{2d}} \tau_B \left(f(u)^* f(u) \right) |u|^{2m} (1 + |v|^2)^{-s/2} \, \mathrm{d}u \, \mathrm{d}v$$

$$= C_s \int_{\mathbb{R}^d} \tau_B \left(f(u)^* f(u) \right) |u|^{2m} \, \mathrm{d}u < \infty. \qquad \square$$

The next theorem is the main result of this section. The analogous result in the tight-binding approximation can be proved much more simply. While the proof here is quite short, it relies on quite substantial machinery, which we summarise in the Appendix. The result justifies the use of this extra machinery, because once we have shown that our spectral triple satisfies the additional requirement of smooth summability, we can employ the local index formula, at least in the case of complex C^*-algebras. Ultimately the local index formula yields the higher Chern numbers and the extension of the index pairing to the localised regime.

Theorem 3.5. *Let $\mathcal{A} = C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B))$. Then*

$$\left(\mathcal{A} \hat{\otimes} C\ell_{0,d}, \ L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B) \hat{\otimes} \bigwedge{}^* \mathbb{R}^d, \ X = \sum_{j=1}^d X_j \otimes 1 \hat{\otimes} \gamma^j, \ (\mathcal{N}, \mathrm{Tr}_\tau) \right)$$

is a ($\mathbb{Z}_2$-graded) smoothly summable semifinite spectral triple with spectral dimension d.

Proof. The boundedness of commutators $[X, \pi(f)]$ is the same as in the Kasparov module case and the self-adjointness of X is clear. By Lemma 3.3, $\pi(f)(1+X^2)^{-s/4}$ is Tr_τ-Hilbert-Schmidt for $s > d$ and therefore compact in $(\mathcal{N}, \mathrm{Tr}_\tau)$ [32]. As $s \to 2$, $\pi(f)(1 + X^2)^{-s/4} \to \pi(f)(1 + X^2)^{-1/2}$ in operator norm, whence $\pi(f)(1 + X^2)^{-1/2}$ is a norm-limit of compact operators and so is compact. In all these statements, and below, we write π instead of $\pi \hat{\otimes} 1_{\bigwedge^* \mathbb{R}^d}$.

Using the notation from Section A.2, our spectral triple will be smoothly summable if we can show that $\pi(\mathcal{A}) \cup [X, \pi(\mathcal{A})] \subset \mathcal{B}_1^\infty(X, d)$. Since δ is a derivation, for any m and any $f, g \in C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B)^{1/2})$

$$\delta^m(\pi(f)\pi(g)) = \sum_{k=0}^m \binom{m}{k} \delta^k(\pi(f))\delta^{m-k}(\pi(g))$$

and this is an element of $\mathcal{B}_1(X, d)$ by Lemma 3.4. Hence

$$C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B)^{1/2})^2 = C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B)) \subset \mathcal{B}_1^\infty(X, d).$$

Proposition A.12 then implies that the spectral triple is finitely summable with spectral dimension d.

Next we consider $\delta^m([X, \pi(fg)])$ and note that

$$[X, \pi(fg)] = \sum_{j=1}^{d} [X_j, \pi(fg)] \,\hat{\otimes}\, \gamma^j = \sum_{j=1}^{d} \partial_j(fg) \,\hat{\otimes}\, \gamma^j$$

by Equation (5). Because $\partial_j(fg) \in \mathcal{A}$ and $|X|$ commutes with the γ^j, $[X, \pi(\mathcal{A})] \subset \mathcal{B}_1^\infty(X, d)$ by the same argument as $\pi(\mathcal{A})$ and we are done. $\qquad\square$

Because we have a smoothly-summable spectral triple, we may complete $\mathcal{A}$ in the δ-φ topology (see Equation (24) in the appendix) to obtain an algebra $\mathcal{A}_{\delta,\varphi}$ that is Fréchet and stable under the holomorphic functional calculus [22, Proposition 2.20], so that $K_*(\mathcal{A}_{\delta,\varphi}) \cong K_*(A)$. Thus any K-theory class for $B \rtimes_\theta \mathbb{R}^d$ has a representative in a matrix algebra over $\mathcal{A}_{\delta,\varphi}$. In addition, the spectral triple

$$\left(\mathcal{A}_{\delta,\varphi} \hat{\otimes} C\ell_{0,d}, \; L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B) \hat{\otimes} {\textstyle\bigwedge}^* \mathbb{R}^d, \; X, \; (\mathcal{N}, \mathrm{Tr}_\tau) \right)$$

is smoothly summable with spectral dimension d [22, Proposition 2.20], and so our analytic formulae extend to pairings with projections or unitaries over $\mathcal{A}_{\delta,\varphi}$.

4. Continuous Chern numbers for complex systems

Now that we have a semifinite spectral triple satisfying the regularity properties required for the local index formula, we restrict to complex algebras and Hilbert spaces to consider the semifinite index pairing with K-theory classes in $K_*(B \rtimes_\theta \mathbb{R}^d)$. The limitations of this approach for real algebras will be discussed below.

Our main aim is to obtain the higher Chern numbers of continuum systems. Various tight-binding versions of these results were obtained in [69, 70, 71, 72].

To better align our notation with the other literature on the topic, we consider the unbounded trace $\mathcal{T}$ on $B \rtimes_\theta \mathbb{R}^d$ by the formula $\mathcal{T}(f) = \tau_B(f(0))$ for $f \in C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B))$. We note that $\mathcal{T}(f) = \mathrm{Tr}_\tau(f)$ for $f \in C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B))$ by an argument analogous to the proof of Lemma 2.3.

The first observation we make is that the semifinite local index formula is currently only valid for ungraded and complex algebras (acting on possibly graded spaces),[1] while our semifinite spectral triple is defined over a graded algebra $\mathcal{A} \hat{\otimes} C\ell_{0,d}$.

For complex algebras we can work with the semifinite spectral triple coming from the spin^c structure on $\mathbb{R}^d$. This is also what is used in [69, 70, 71, 72]. Namely, we let $\nu = 2^{\lceil (d-1)/2 \rceil}$. Then the triple

$$(7) \qquad \left(\mathcal{A} = C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B)), \; L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B) \hat{\otimes} \mathbb{C}^\nu, \; X = \sum_{j=1}^{d} X_j \otimes 1 \hat{\otimes} \gamma^j \right)$$

is a complex and smoothly summable semifinite spectral triple of spectral dimension d and relative to $(\mathcal{N} \hat{\otimes} \mathrm{End}(\mathbb{C}^\nu), \mathrm{Tr}_\tau \hat{\otimes} \mathrm{Tr}_{\mathbb{C}^\nu})$. The spectral triple is odd (ungraded) if d is odd and is even for d even with grading operator $\gamma = (-i)^{d/2} \gamma^1 \cdots \gamma^d$.

For d even, the semifinite spectral triple from Equation (7) is easily related to our original semifinite spectral triple from Theorem 3.5 by the external product with the Morita equivalence bimodule $\left(\mathbb{C}\ell_d, \mathbb{C}_{\mathbb{C}}^{2^{d/2}}, 0 \right)$, which gives an invertible class in $KK(\mathbb{C}\ell_d, \mathbb{C})$. For d odd, we first turn our ungraded triple into a graded triple over $\mathcal{A} \hat{\otimes} \mathbb{C}\ell_1$, then the Morita equivalence between $\mathbb{C}\ell_{d-1}$ and $\mathbb{C}$ recovers the original spectral triple. In both even and odd cases we do not lose any information.

Proposition 4.1. *Let $f \in C_c(\mathbb{R}^d, \mathrm{Dom}(\tau_B))$. If τ_B is invariant under the action of $\mathbb{R}^d$, then the complex function*

$$s \mapsto \zeta_f(s) = \mathrm{Tr}_\tau(\pi(f)(1 + |X|^2)^{-s/2})$$

---

[1]The proofs of the local index formula given in [22, 24] can naturally be recast for graded algebras, but the validity of the result needs to be checked. For real (graded) algebras this will be necessary.

is holomorphic for $\Re(s) > d$ with at worst a simple pole at $s = d$ with residue

$$\operatorname*{res}_{s=d} \operatorname{Tr}_\tau(\pi(f)(1+|X|^2)^{-s/2}) = \operatorname{Vol}_{d-1}(S^{d-1})\,\mathcal{T}(f).$$

Proof. Because $\pi(f)(1+|X|^2)^{-s/2}$ is trace-class for $\Re(s) > d$, we can compute directly using that $\operatorname{Tr}_\tau = \operatorname{Tr}_{L^2(\mathbb{R}^d)} \otimes \tau_B$ (on the algebraic tensor product of $\mathcal{L}^1(L^2(\mathbb{R}^d))$ and $\operatorname{Dom}(\tau) \subset B$). Using the formula in Equation (6) for the integral kernel, we find that for $\Re(s) > d$,

$$\begin{aligned}
\operatorname{Tr}_\tau\left(\pi(f)(1+|X|^2)^{-s/2}\right) &= \int_{\mathbb{R}^d} \tau_B\big(k_f(x,x)\big)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \tau_B\Big(\theta(-x,0)\alpha_{-x}(f(0))\Big)(1+|x|^2)^{-s/2}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \tau_B\Big(\alpha_{-x}(f(0))\Big)(1+|x|^2)^{-s/2}\,\mathrm{d}x \\
&= \tau_B(f(0)) \int_{\mathbb{R}^d} (1+|x|^2)^{-s/2}\,\mathrm{d}x,
\end{aligned}$$

where we have used the invariance of the $\mathbb{R}^d$-action on the fourth line. Using polar coordinates we can compute explicitly for $\Re(s) > d$,

$$\operatorname{Tr}_\tau\left(\pi(f)(1+|X|^2)^{-s/2}\right) = \tau_B(f(0))\operatorname{Vol}_{d-1}(S^{d-1}) \int_0^\infty (1+r^2)^{-s/2}r^{d-1}\,\mathrm{d}r$$

(8)
$$= \mathcal{T}(f)\operatorname{Vol}_{d-1}(S^{d-1})\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{s-d}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)}.$$

The right hand side of Equation (8) has an analytic continuation to the complex plane that is holomorphic for $\Re(s) > d$ and with a simple pole at $\Re(s) = d$. Taking the residue yields

$$\operatorname*{res}_{s=d} \operatorname{Tr}_\tau(\pi(f)(1+|X|^2)^{-s/2}) = \mathcal{T}(f)\operatorname{Vol}_{d-1}(S^{d-1})$$

as required. $\square$

In the case of complex algebras and Kasparov modules, the semifinite spectral triple from Equation (7) and tracial weight τ_B give a well-defined map

$$K_*(B \rtimes_\theta \mathbb{R}^d) \times KK^*(B \rtimes_\theta \mathbb{R}^d, B) \to KK(\mathbb{C}, B) \xrightarrow{(\tau_B)_*} \mathbb{R}.$$

The semifinite local index formula [22, Theorem 3.33] gives us computable expressions for this K-theoretic composition, which we now present.

4.1. **Odd formula.** Because the spectral triple of Equation (7) is smoothly summable with spectral dimension d, the odd local index formula gives that

$$\langle [u], [(\mathcal{A}, \mathcal{H}, X_{\mathrm{odd}})]\rangle = \frac{-1}{\sqrt{2\pi i}} \operatorname*{res}_{r=(1-d)/2} \sum_{m=1,\mathrm{odd}}^{d} \phi_m^r(\mathrm{Ch}^m(u)),$$

where u is a unitary in $M_q(\mathcal{A}^\sim)$ and

$$\mathrm{Ch}^{2n+1}(u) = (-1)^n n!\, u^* \otimes u \otimes u^* \otimes \cdots \otimes u \quad (2n+2 \text{ entries}).$$

The functional ϕ_m^r is the resolvent cocycle from Definition A.13. Using notation from Section A.1.1, we can write the pairing $\langle [u], [(\mathcal{A}, \mathcal{H}, X_{\mathrm{odd}})]\rangle$ as a semifinite Fredholm index,

$$\langle [u], [(\mathcal{A}, \mathcal{H}, X_{\mathrm{odd}})]\rangle = \operatorname{Index}_{\tau \otimes \operatorname{Tr}_{\mathbb{C}^{2q}}}\left((P \otimes 1_{2q})\hat{u}(P \otimes 1_{2q})\right), \qquad \hat{u} = \begin{pmatrix} u & 0 \\ 0 & 1_u \end{pmatrix},$$

with $1_u = \pi^n(u)$ for $\pi^n : M_q(\mathcal{A}) \to M_q(\mathbb{C})$ the quotient map from the unitisation and $P = \frac{1}{2}(1+F_X)$ given as in Proposition A.4. We also write $\operatorname{Index}_\tau$ to refer to the semifinite Fredholm index with respect to Tr_τ.

Theorem 4.2 (Odd index formula). *Let d be odd and u a complex unitary in $M_q(\mathcal{A}^\sim)$ where $\mathcal{A}^\sim$ is the minimal unitisation of $\mathcal{A}$. If the trace τ_B on B is invariant under the action of $\mathbb{R}^d$, then the semifinite index pairing with the semifinite spectral triple from Equation (7) with d odd is given by the formula*

$$\mathrm{Index}_{\tau \otimes \mathrm{Tr}_{\mathbb{C}^{2q}}}((P_X \otimes 1_{2q})\hat{u}(P_X \otimes 1_{2q})) = C_d \sum_{\sigma \in S_d} (-1)^\sigma \, (\mathrm{Tr}_{\mathbb{C}^q} \otimes \mathcal{T}) \left(\prod_{i=1}^d u^* \partial_{\sigma(i)} u \right),$$

where $C_{2n+1} = \frac{2(2\pi i)^n n!}{(2n+1)!}$, $\mathrm{Tr}_{\mathbb{C}^q}$ is the matrix trace on $\mathbb{C}^q$ and S_d is the permutation group on d letters.

We give the proof in the case $q = 1$, since we can extend to matrices by the standard extension of spectral triples over $\mathcal{A}$ to $M_q(\mathcal{A})$. Except in cases where we need specific results about the spinor trace $\mathrm{Tr}_{\mathbb{C}^\nu}$, we will write the trace $\mathrm{Tr}_\tau \hat{\otimes} \mathrm{Tr}_{\mathbb{C}^\nu}$ as just Tr_τ.

To compute the index pairing we make the following important observation.

Lemma 4.3 ([11], §11.1). *The only term in the sum $\sum_{m=1,\mathrm{odd}}^d \phi_m^r(\mathrm{Ch}^m(u))$ that contributes to the index pairing is the term with $m = d$.*

Proof. We first note that the spinor trace of the product of $d = 2n + 1$ Clifford generators is given by

$$(9) \qquad\qquad \mathrm{Tr}_{\mathbb{C}^\nu}(\gamma^1 \cdots \gamma^d) = (-1)^{n+1} i^{-n} 2^n$$

and will vanish on any product of k Clifford generators with $0 < k < d$. The resolvent cocycle involves the spinor trace of terms

$$a_0 R_s(\lambda)[X, a_1] R_s(\lambda) \cdots [X, a_m] R_s(\lambda), \qquad R_s(\lambda) = (\lambda - (1 + s^2 + X^2))^{-1},$$

for $a_0, \ldots, a_m \in \pi(\mathcal{A})$. We note that $[X, a_l] = \sum_{j=1}^d \partial_j(a_l) \hat{\otimes} \gamma^j$ and $R_s(\lambda)$ is diagonal in the spinor representation. Hence the product $a_0 R_s(\lambda)[X, a_1] R_s(\lambda) \cdots [X, a_m] R_s(\lambda)$ will be in the span of m Clifford generators for $0 < m < d$ acting on $L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B) \hat{\otimes} \mathbb{C}^\nu$. Furthermore, our trace estimates ensure that each spinor component

$$\int_\ell \lambda^{-d/2-r} a_0 (\lambda - (1 + s^2 + X^2))^{-1} \partial_{j_1}(a_1) \cdots \partial_{j_m}(a_m)(\lambda - (1 + s^2 + X^2))^{-1} \, d\lambda$$

is trace-class for $a_0, \ldots, a_m \in \mathcal{A}$ and $\Re(r)$ sufficiently large. Hence for $0 < m < d$, the spinor trace will vanish for $\Re(r) > 0$ and so $\phi_m^r(\mathrm{Ch}^m(u))$ does not contribute to the index pairing for $0 < m < d$. $\square$

Proof of Theorem 4.2. Lemma 4.3 simplifies the index computation substantially, where the index is now given by the expression

$$\langle [u], [(\mathcal{A}, \mathcal{H}, X_{\mathrm{odd}})] \rangle = \frac{-1}{\sqrt{2\pi i}} \operatorname*{res}_{r=(1-d)/2} \phi_d^r(\mathrm{Ch}^d(u)).$$

Here the resolvent cocycle is formed using the trace $\mathrm{Tr}_{\mathbb{C}^q} \otimes \mathrm{Tr}_\tau \otimes \mathrm{Tr}_{\mathbb{C}^\nu}$ where $\mathrm{Tr}_{\mathbb{C}^\nu}$ is the trace on the spinor representation. We simplify the formulae below by taking $q = 1$ and suppressing the spinor trace. Therefore we need to compute the residue at $r = (1 - d)/2$ of

$$\frac{(-1)^{n+1} n! \eta_d}{(2\pi i)^{3/2}} \int_0^\infty s^d \, \mathrm{Tr}_\tau \left(\int_\ell \lambda^{-d/2-r} u^* R_s(\lambda)[X, u] R_s(\lambda)[X, u^*] \cdots [X, u] R_s(\lambda) \, d\lambda \right) ds,$$

where $d = 2n + 1$ and

$$\eta_d = -\sqrt{2i} \, 2^{d+1} \frac{\Gamma(d/2 + 1)}{\Gamma(d + 1)}.$$

To compute this residue we move all terms $R_s(\lambda)$ to the right, which can be done up to a function holomorphic at $r = (1 - d)/2$ by an argument similar to the proof of Proposition 4.1. This allows

us to take the Cauchy integral. We then observe that $\underbrace{[X,u][X,u^*]\cdots[X,u]}_{d \text{ terms}} \in \pi(\mathcal{A})\,\hat{\otimes}\,1_{\mathbb{C}^\nu}$, so Proposition 4.1 implies that the zeta function

$$\mathrm{Tr}_\tau\Big(u^*[X,u][X,u^*]\cdots[X,u](1+X^2)^{-z/2}\Big)$$

has at worst a simple pole at $\Re(z)=d$. Therefore we can explicitly compute for $d=2n+1$,

$$\frac{-1}{\sqrt{2\pi i}}\operatorname*{res}_{r=(1-d)/2}\phi_d^r(\mathrm{Ch}^d(u)) = (-1)^{n+1}\frac{n!}{d!}\frac{\Gamma(d/2)}{\sqrt{\pi}}\operatorname*{res}_{z=d}\mathrm{Tr}_\tau\Big(u^*[X,u][X,u^*]\cdots[X,u](1+X^2)^{-z/2}\Big)$$

and so our index pairing can be written as

$$\mathrm{Index}_\tau(P\hat{u}P) = (-1)^{n+1}\frac{n!\Gamma(d/2)}{d!\sqrt{\pi}}\operatorname*{res}_{z=d}\mathrm{Tr}_\tau\Big(u^*[X,u][X,u^*]\cdots[X,u](1+X^2)^{-z/2}\Big).$$

We make use of the identity $[X,u^*]=-u^*[X,u]u^*$, which allows us to rewrite

$$u^*\underbrace{[X,u][X,u^*]\cdots[X,u]}_{d=2n+1 \text{ terms}} = (-1)^n u^*[X,u]u^*[X,u]u^*\cdots u^*[X,u] = (-1)^n\,(u^*[X,u])^d.$$

Recall that $[X,u]=\sum_{j=1}^d[X_j,u]\hat{\otimes}\gamma^j = \sum_{j=1}^d\partial_j(u)\hat{\otimes}\gamma^j$ so we have the relation $u^*[X,u]=\sum_{j=1}^d u^*\partial_j(u)\hat{\otimes}\gamma^j$. Taking the d-th power

$$(u^*[X,u])^d = \sum_{J=(j_1,\ldots,j_d)} u^*\partial_{j_1}(u)\cdots u^*\partial_{j_d}(u)\,\hat{\otimes}\,\gamma^{j_1}\cdots\gamma^{j_d}$$

where the sum is extended over all multi-indices J. Note that every term in the sum is a multiple of the identity on $\mathbb{C}^\nu$ and so has a non-zero spinor trace. Writing this product in terms of permutations,

$$(-1)^n\,(u^*[X,u])^d = (-1)^n\sum_{\sigma\in S_d}(-1)^\sigma\prod_{j=1}^d(u^*\partial_{\sigma(j)}(u)\hat{\otimes}\gamma^j),$$

where S_d is the permutation group of d letters. Combining these results yields

$$\mathrm{Index}_\tau(P\hat{u}P) = (-1)^{n+1}\frac{n!\Gamma(d/2)}{d!\sqrt{\pi}}\operatorname*{res}_{z=d}\mathrm{Tr}_\tau\Big(u^*[X,u][X,u^*]\cdots[X,u](1+X^2)^{-z/2}\Big)$$

$$= -\frac{n!\Gamma(d/2)}{d!\sqrt{\pi}}\operatorname*{res}_{z=d}\mathrm{Tr}_\tau\Bigg(\bigg(\sum_{\sigma\in S_d}(-1)^\sigma\prod_{j=1}^d(u^*\partial_{\sigma(j)}(u)\hat{\otimes}\gamma^j)\bigg)(1+X^2)^{-z/2}\Bigg).$$

Recalling the spinor degrees of freedom, we can apply Equation (9) and Proposition 4.1 to reduce the formula to

$$\mathrm{Index}_\tau(P\hat{u}P) = (-1)^n\frac{n!\Gamma(d/2)\mathrm{Vol}_{d-1}(S^{d-1})2^n}{i^n d!\sqrt{\pi}}\sum_{\sigma\in S_d}(-1)^\sigma\,\mathcal{T}\bigg(\prod_{j=1}^d u^*\partial_{\sigma(j)}(u)\bigg).$$

Finally we use the equation $\mathrm{Vol}_{d-1}(S^{d-1})=\frac{d\pi^{d/2}}{\Gamma(d/2+1)}$ to simplify our formula to

$$\mathrm{Index}_\tau(P\hat{u}P) = C_d\sum_{\sigma\in S_d}(-1)^\sigma\,\mathcal{T}\bigg(\prod_{j=1}^d u^*\partial_{\sigma(j)}(u)\bigg)$$

with $C_{2n+1}=\frac{2(-2\pi)^n n!}{i^n(2n+1)!}=\frac{2(2\pi i)^n n!}{(2n+1)!}$. $\qquad\square$

We see our result as analogous to the higher dimensional Chern numbers of discrete crossed products considered in [70, 71, 72] for $C(\Omega)\rtimes_\theta\mathbb{Z}^d$ for d odd. For $d=1$ and an untwisted crossed product, $B\rtimes\mathbb{R}$, we recover the results studied in [57, 66, 20].

4.2. Even formula.

We now consider the case of even dimensions and recall the even local index formula,

$$\langle [p] - [1_p], [(\mathcal{A}, \mathcal{H}, X_{\text{even}})] \rangle = \operatorname*{res}_{r=(1-d)/2} \sum_{m=0,\text{even}}^{d} \phi_m^r(\text{Ch}^m(p) - \text{Ch}^m(1_p)),$$

$$\text{Ch}^{2n}(p) = (-1)^n \frac{(2n)!}{2(n!)}(2p-1) \otimes p^{\otimes 2n}, \qquad \text{Ch}^0(p) = p,$$

where ϕ_m^r is the resolvent cocycle of Definition A.13 and $1_p = \pi^q(p)$ for $\pi^q : M_q(\mathcal{A}^\sim) \to M_q(\mathbb{C})$ the quotient map. We will again use Proposition A.4 to write the pairing as a semifinite Fredholm index.

Theorem 4.4 (Even index formula). *Let p be a projection in $M_q(\mathcal{A}^\sim)$ with d even. If the trace τ_B on B is invariant under the action of $\mathbb{R}^d$, then the semifinite index pairing can be expressed by the formula*

$$\text{Index}_{\tau \otimes \text{Tr}_{\mathbb{C}^{2q}}}(\hat{p}(F_X \otimes 1_{2q})_+\hat{p}) = \frac{(-2\pi i)^{d/2}}{(d/2)!} \sum_{\sigma \in S_d} (-1)^\sigma \, (\text{Tr}_{\mathbb{C}^q} \otimes \mathcal{T})\left(p \prod_{j=1}^{d} \partial_{\sigma(j)} p \right),$$

where S_d is the permutation group of d letters.

Like the setting with d odd, our computation can be simplified with some preliminary results. We again focus on the case $q = 1$.

Lemma 4.5. *The index pairing reduces to the computation* $\operatorname*{res}_{r=(1-d)/2} \phi_d^r(\text{Ch}^d(p))$.

Proof. We first note that for $m > 0$, $\phi_m^r(\text{Ch}(1_p)) = 0$ as these terms involve the commutators $[X, 1_p] = 0$. The proof used in Lemma 4.3 also holds here to show that $\phi_m^r(\text{Ch}^m(p))$ does not contribute to the index pairing for $0 < m < d$. The $m = 0$ term is of the form

$$\phi_0^r(p - 1_p) = 2 \int_0^\infty \text{Tr}_\tau\left(\gamma(p - 1_p)(1 + s^2 + X^2)^{-d/2-r} \right) ds,$$

Because there is a symmetry of the operator $(p - 1_p)(1 + s^2 + X^2)^{-d/2-r}$ between the ± 1 eigenspaces of the grading operator $\gamma = (-i)^{d/2}\gamma^1\gamma^2 \cdots \gamma^d$, the graded trace will vanish provided $\Re(r)$ is sufficiently large. Therefore $\phi_0^r(p - 1_p)$ analytically continues as a function holomorphic in a neighbourhood of $r = (1 - d)/2$, hence the residue will vanish. $\square$

Proof of Theorem 4.4. Lemma 4.5 implies our index computation is reduced to

$$\langle [p] - [1_p], [(\mathcal{A}, \mathcal{H}, X_{\text{even}})] \rangle = \operatorname*{res}_{r=(1-d)/2} \phi_d^r(\text{Ch}^d(p)),$$

which is a residue at $r = (1 - d)/2$ of the term

$$\frac{(-1)^{d/2}d!\eta_d}{(d/2)!2\pi i} \int_0^\infty s^d \, \text{Tr}_\tau\left(\gamma \int_\ell \lambda^{-d/2-r}(2p-1)R_s(\lambda)[X,p]R_s(\lambda) \cdots [X,p]R_s(\lambda) \, d\lambda \right) ds$$

with $\eta_d = 2^{d+1}\frac{\Gamma(d/2+1)}{\Gamma(d+1)}$. Like the case of d odd, we can move the resolvent terms to the right up to a holomorphic error in order to take the Cauchy integral. Proposition 4.1 also implies that the semifinite trace $\text{Tr}_\tau\left(\gamma(2p-1)([X,p])^d(1+X^2)^{-s/2}\right)$ has at worst a simple pole at $s = d$. Computing the residue explicitly using the formula of Definition A.13, we find

$$\operatorname*{res}_{r=(1-d)/2} \phi_d^r(\text{Ch}^d(p)) = \frac{(-1)^{d/2}}{2((d/2)!)}((d/2) - 1)! \operatorname*{res}_{z=d} \text{Tr}_\tau\left(\gamma(2p-1)([X,p])^d(1+X^2)^{-z/2} \right),$$

or

$$\text{Index}_\tau(\hat{p}(F_X)_+\hat{p}) = (-1)^{d/2}\frac{1}{d} \operatorname*{res}_{z=d} \text{Tr}_\tau\left(\gamma(2p-1)([X,p])^d(1+X^2)^{-z/2} \right).$$

Next we compute

$$[X,p]^d = \sum_{J=(j_1,\ldots,j_d)} [X_{j_1},p] \cdots [X_{j_d},p] \,\hat{\otimes}\, \gamma^{j_1} \cdots \gamma^{j_d} = i^{d/2} \sum_{\sigma \in S_d} (-1)^\sigma [X_{\sigma(1)},p] \cdots [X_{\sigma(d)},p] \,\hat{\otimes}\, \gamma$$

as $\gamma = (-i)^{d/2} \gamma^1 \cdots \gamma^d$. Since $[X,p] \in \mathcal{B}_1^\infty(X,d)$ we can cycle the final term $[X_{\sigma(d)},p]$ in this product to the front when we apply the trace, to find that

$$\mathrm{Tr}_\tau \left(\gamma \sum_{\sigma \in S_d} (-1)^\sigma \left([X_{\sigma(1)},p] \cdots [X_{\sigma(d)},p] \otimes \gamma \right) (1+X^2)^{-z/2} \right)$$

$$= \mathrm{Tr}_\tau \left(\sum_{\sigma \in S_d} (-1)^\sigma \left([X_{\sigma(1)},p] \cdots [X_{\sigma(d)},p](1+|X|^2)^{-z/2} \right) \hat{\otimes} \, \mathrm{Id}_{\mathbb{C}^\nu} \right)$$

$$= \mathrm{Tr}_\tau \left(\sum_{\sigma \in S_d} (-1)^\sigma \left([X_{\sigma(d)},p][X_{\sigma(1)},p] \cdots [X_{\sigma(d-1)},p](1+|X|^2)^{-z/2} \right) \hat{\otimes} \, \mathrm{Id}_{\mathbb{C}^\nu} \right).$$

Since the cyclic permutation exchanging the first and last term is odd, we see that this sum runs over the same set of permutations twice, once with a plus sign and once with a minus sign. Hence for the real part of z greater than d we have

$$\mathrm{Tr}_\tau \left(\gamma([X,p])^d (1+X^2)^{-z/2} \right) = 0,$$

and so we need only compute the remaining term with 'integrand' $2p([X,p])^d$. As above

$$p([X,p])^d = p \sum_{\sigma \in S_d} (-1)^\sigma \prod_{j=1}^d \partial_{\sigma(j)}(p) \,\hat{\otimes}\, \gamma^j.$$

Therefore, using the relation $\mathrm{Tr}_{\mathbb{C}^\nu}(\gamma \gamma^1 \cdots \gamma^d) = i^{d/2} 2^{d/2-1}$ and Proposition 4.1,

$$\mathrm{Index}_\tau(\hat{p}(F_X)_+ \hat{p}) = (-1)^{d/2} \frac{1}{d} \operatorname*{res}_{z=d} \mathrm{Tr}_\tau \left(\gamma \, 2p([X,p])^d (1+X^2)^{-z/2} \right)$$

$$= \frac{(-2i)^{d/2} \mathrm{Vol}_{d-1}(S^{d-1})}{d} \, \mathcal{T} \left(p \sum_{\sigma \in S_d} (-1)^\sigma \prod_{j=1}^d \partial_{\sigma(j)}(p) \right)$$

We use the equation $\mathrm{Vol}_{d-1}(S^{d-1}) = \frac{d\pi^{d/2}}{(d/2)!}$ for d even to simplify

$$(10) \qquad \mathrm{Index}_\tau(\hat{p}(F_X)_+ \hat{p}) = \frac{(-2\pi i)^{d/2}}{(d/2)!} \sum_{\sigma \in S_d} (-1)^\sigma \mathcal{T} \left(p \prod_{j=1}^d \partial_{\sigma(j)}(p) \right). \qquad \square$$

We remark that Equation (10) appears in the case $B = C(\Omega)$ and $d = 2$ in [84, 62]. To relate Equation (10) to the results in [69, 71, 72], we note that we have used the derivation $\partial_j(a) = [X_j, a]$, whereas Prodan et al. use $\tilde{\partial}(a) = \pm i[X_j, a]$. Applying our argument with $\tilde{\partial}$ as our algebraic derivation will bring in an extra factor of $i^d = (-1)^{d/2}$ and, hence, we have that

$$\mathrm{Index}_\tau(\hat{p}(F_X)_+ \hat{p}) = \frac{(2\pi i)^{d/2}}{(d/2)!} \sum_{\sigma \in S_d} (-1)^\sigma \mathcal{T} \left(p \prod_{j=1}^d \tilde{\partial}_{\sigma(j)}(p) \right).$$

We compare this expression to [69, Equation (4)] and see that, in the case of $B = C(\Omega)$ with invariant probability measure, we have reproduced the expression for the higher-dimensional even Chern numbers in the continuous (non-unital) setting. Of course, Theorem 4.2 and 4.4 are valid for a wider range of examples by taking B to be a more general C^*-algebra.

5. Extending the index pairing

In this section we exploit the 'flatness' of the (possibly noncommutative) Euclidean spaces which comprise our observable algebra. Of course there is also the disorder space Ω, or 'base algebra' B more generally, but our operator X does not see this data. As a consequence of the flatness, all but one term of the local index formula is identically zero, and this allows us to extend the index pairing to a larger algebra.

Let $\mathcal{M} = \pi_{\mathrm{GNS}}(B)''$ denote the weak closure of B under the GNS representation $B \to \mathcal{B}[L^2(B, \tau_B)]$. The action α on B extends to $\mathcal{M}$ and we can consider the von Neumann crossed product $\mathcal{M} \rtimes_\theta \mathbb{R}^d$. We note the following equivalent presentations,

$$\mathcal{M} \rtimes_\theta \mathbb{R}^d \cong (B \rtimes_\theta \mathbb{R}^d)'' \cong \mathrm{End}_B^{00}(L^2(\mathbb{R}^d, B))''$$

and so $\mathcal{M} \rtimes_\theta \mathbb{R}^d$ is the same as the semifinite von Neumann algebra $\mathcal{N}$ considered in the previous section. The algebra $\mathcal{M} \rtimes_\theta \mathbb{R}^d$ is the measurable functions $f : \mathbb{R}^d \to \mathcal{M}$ with respect to the trace τ_B and subject to a convolution product and covariance relation. We denote by $\tilde{\pi}$ the representation of $B \rtimes_\theta \mathbb{R}^d$ on $L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B)$ extended to $\mathcal{M} \rtimes_\theta \mathbb{R}^d$.

The operator $X = \sum_{j=1}^d X_j \hat{\otimes} \gamma^j$ is affiliated to $\mathcal{N} \hat{\otimes} \mathrm{End}(\bigwedge^* \mathbb{R}^d)$ and is measurable with respect to the trace $\mathrm{Tr}_\tau \hat{\otimes} \mathrm{Tr}_{\bigwedge^* \mathbb{R}^d}$. Therefore we can consider refined notions of summability in this larger algebra using the Fréchet $*$-algebras $\mathcal{B}_2(X, d)$ and $\mathcal{B}_1(X, d)$ introduced in Appendix A.2.

Proposition 5.1. *Suppose that $g_1, g_2 \in \mathcal{M} \rtimes_\theta \mathbb{R}^d$ are such that $g_j(x) \in \mathrm{Dom}(\tau_B)^{1/2}$ for almost all x and satisfy the bound*

$$(11) \qquad \int_{\mathbb{R}^d} (1 + |x|^2)^n \tau_B(|g_j(x)|^2) \, \mathrm{d}x < \infty, \qquad j = 1, 2, \ n \in \mathbb{N}_+.$$

Then $g_1 g_2 \in \mathcal{B}_1^n(X, d)$. In particular, any projection in $\mathcal{M} \rtimes_\theta \mathbb{R}^d$ that satisfies Equation (11) is in $\mathcal{B}_1^n(X, d)$.

The proof of Lemma 3.3 also shows that the ambiguity of the notation $|g(x)|$ (as convolution or pointwise product absolute value) disappears.

Proof. Recall from the appendix, the norms φ_s on $\mathcal{B}_2^n(X, d)^2$. A short calculation shows that in our case

$$\varphi_{d+1/m}(|\delta^k(g)|^2) = C_{d+1/m} \int_{\mathbb{R}^d} 2|x|^{2k} \tau_B(|g(x)|^2) \, \mathrm{d}x + \|\delta^k(g)\|^2,$$

where we have used the cyclicity of τ_B, $\tau_B(b^*b) = \tau_B(bb^*)$. We use this equality to estimate in $\mathcal{B}_2^n(X, d)^2 \subset \mathcal{B}_1^n(X, d)$,

$$\mathcal{P}_{n,l}(g_1 g_2) \leq \sum_{k=0}^l \mathcal{Q}_n(\delta^k(g_1)) \mathcal{Q}_n(\delta^{l-k}(g_2))$$

$$\leq \sum_{k=0}^l \left(\left(\|\delta^k(g_1)\|^2 + 2 \int_{\mathbb{R}^d} |x|^{2k} \tau(|g_1(x)|^2) \, \mathrm{d}x \right) \left(\|\delta^{l-k}(g_2)\|^2 + 2 \int_{\mathbb{R}^d} |z|^{2(l-k)} \tau(|g_2(z)|^2 \, \mathrm{d}z) \right) \right)$$

$$\leq \sum_{k=0}^l C_k \int_{\mathbb{R}^d} (1 + 3|x|^{2k}) \tau(|g_1(x)|^2) \, \mathrm{d}x \times \int_{\mathbb{R}^d} (1 + 3|z|^{2(l-k)}) \tau(|g_2(z)|^2) \, \mathrm{d}z$$

$$\leq \max_j C \int_{\mathbb{R}^d} (1 + |x|^2)^l \tau(|g_j(x)|^2) \, \mathrm{d}x$$

The third inequality uses the fact that the L^1-norm dominates the crossed product norm. Hence the seminorms $P_{n,l}$ are finite for $l \leq n$. $\square$

To extend our index pairing to a larger algebra, we use the Sobolev spaces and Sobolev algebra considered in [69, 71] for the discrete setting.

Definition 5.2. The Sobolev spaces $\mathcal{W}_{r,p}$ are defined as the Banach spaces obtained as the completion of $C_c(\mathbb{R}^d, B)$ in the norms

$$\|f\|_{r,p} = \sum_{|\alpha| \leq r} \mathrm{Tr}_\tau \left(|\partial^\alpha f|^p \right)^{1/p}, \qquad r \in \mathbb{N}, \ p \in [1, \infty),$$

where we use multi-index notation, $\alpha \in \mathbb{N}^d$, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$.

The Sobolev spaces are not closed under multiplication, but if we consider the Hölder inequality of noncommutative L^p-spaces (cf. [32, Theorem 4.2]),

$$\|a_1 \cdots a_k\|_{r,p} \leq \|a_1\|_{r,p_1} \cdots \|a_k\|_{r,p_k}, \qquad \frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{p},$$

then we obtain an algebra structure if we take the intersection of such spaces.

Definition 5.3. The Sobolev algebra $\mathcal{A}_{\mathrm{Sob}}$ is defined as the algebraic span of products $\mathcal{A}_{\mathrm{Sob}} = \mathrm{span}\{ab : a, b \in \widetilde{\mathcal{A}}_{\mathrm{Sob}}\}$ with $\widetilde{\mathcal{A}}_{\mathrm{Sob}}$ the intersection

$$\widetilde{\mathcal{A}}_{\mathrm{Sob}} = \Big(\bigcap_{r \in \mathbb{N}, \, p \in \mathbb{N}_+} \mathcal{W}_{r,p} \Big) \cap \mathcal{M} \rtimes_\theta \mathbb{R}^d.$$

Our Sobolev algebra $\mathcal{A}_{\mathrm{Sob}}$ is defined using the span of products rather than the algebra $\widetilde{\mathcal{A}}_{\mathrm{Sob}}$ for largely technical reasons that appear in the non-unital setting. For applications to topological phases, this extra detail is not an issue as the K-theoretic phase of interest is constructed out of the Fermi projection $P_\mu = P_\mu^2$ or $1 - 2P_\mu$. Another advantage of taking the algebra of products is the next result.

Lemma 5.4. If $a \in \mathcal{A}_{\mathrm{Sob}}$, then $a \in \mathcal{B}_1^n(X, d)$ for any $n \in \mathbb{N}_+$.

Proof. Because $\mathcal{B}_2^n(X, d)^2 \subset \mathcal{B}_1^n(X, d)$, the result follows if we can show that $b \in \mathcal{B}_2^n(X, d)$ for $b \in \widetilde{\mathcal{A}}_{\mathrm{Sob}}$. Using the condition from Equation (11), the result is immediate as $b \in \mathcal{W}_{r,p}$ for r arbitrarily large. $\qquad \square$

By adapting arguments developed for $\mathcal{B}_1^n(X, d)$, Lemma 5.4 can then be used to obtain the following.

Lemma 5.5 (c.f. [22], Proposition 1.18)**.** *For any $n \in \mathbb{N}$, the algebra $M_n(\mathcal{A}_{\mathrm{Sob}})$ is stable under the holomorphic functional calculus.*

Proof. This is basically the same as [22, Proposition 1.18], except that we replace the norms on $\mathcal{B}_1^n(X, d)$ and $\mathcal{B}_2^n(X, d)$ with the Sobolev norms $\| \cdot \|_{r,n}$ and $\| \cdot \|_{r,2n}$ respectively. $\qquad \square$

Lemma 5.6. *If $a \in \mathcal{B}_1^{\lfloor d/2 \rfloor + 1}(X, d)$, then $a(1 + X^2)^{-d/2 - r} \in \mathcal{L}^1(\mathcal{M} \rtimes_\theta \mathbb{R}^d, \mathrm{Tr}_\tau)$ for all $r > 0$.*

Proof. We start by writing

$$a(1 + X^2)^{-d/2 - r} = (1 + X^2)^{d/4 + r/2} \Big((1 + X^2)^{-d/4 - r/2} a (1 + X^2)^{-d/4 - r/2} \Big) (1 + X^2)^{-d/4 - r/2}.$$

Now by [22, Lemma 1.13, Proposition 1.14] we can write $a = \sum_{j=1}^4 b_j c_j$ with $b_j, c_j \in \mathcal{B}_2(X, d)$. Since $a \in \mathcal{B}_1^{\lfloor d/2 \rfloor + 1}(X, d)$, we can then show that each $b_j, c_j \in \mathcal{B}_2^{\lfloor d/2 \rfloor + 1}(X, d)$. For notational simplicity we write $a = bc$ with $b, c \in \mathcal{B}_2^{\lfloor d/2 \rfloor + 1}(X, d)$. Then we know that for all $r > 0$,

$$(1 + X^2)^{-d/4 - r/2} a (1 + X^2)^{-d/4 - r/2} \in \mathcal{L}^1(\mathcal{M} \rtimes_\theta \mathbb{R}^d, \mathrm{Tr}_\tau).$$

Thus we have reduced the problem to considering the behaviour of the one-parameter group $T \mapsto \sigma^z(T) = (1 + X^2)^{z/2} T (1 + X^2)^{-z/2}$ as in [22, Section 1.4]. In particular, for sufficiently smooth elements $b, c \in \mathcal{B}_2(X, d)$, we wish to show that

$$(1 + X^2)^{z/2 - d/2 - r} bc (1 + X^2)^{-z/2 - d/2 - r}$$

is trace class. Since $(1 + X^2)^{1/2}(1 + |X|)^{-1}$ is a bounded invertible element in $L^\infty(|X|)$, we can simplify the computations by removing the square roots. It also suffices to consider $0 < r < 1/2$,

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and so we let m be the greatest integer less than or equal to $d/2$. Iterating the identity $(1 + |X|)T(1 + |X|)^{-1} = T + \delta(T)(1 + |X|)^{-1}$ for $T \in \mathcal{B}_1^{\lfloor d/2 \rfloor}(X, d)$ we have

$$(1 + |X|)^{d/2+r}T(1 + |X|)^{-d/2-r} = (1 + |X|)^{d/2+r-m} \sum_{j=0}^{m} \delta^j(T)(1 + |X|)^{-d/2-r+m-j},$$

and so we will be done if we can show that for $T_1, T_2 \in \mathcal{B}_2^1(X, d)$ and $0 \leq \alpha < 1$

$$(1 + |X|)^\alpha T_1 T_2 (1 + |X|)^{-\alpha} - T_1 T_2 \in \mathcal{B}_1(X, d).$$

For this we use the integral formula for fractional powers, [23, p701], and write

$$(1 + |X|)^\alpha T_1 T_2 (1 + |X|)^{-\alpha} = (1 + |X|)^\alpha \, T_1 T_2 \, \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha}(1 + \lambda + |X|)^{-1} \, d\lambda.$$

Taking commutators yields

$$(1 + |X|)^\alpha T_1 T_2 (1 + |X|)^{-\alpha} = T_1 T_2 + (1 + |X|)^\alpha \, \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha}[T_1 T_2, (1 + \lambda + |X|)^{-1}] \, d\lambda$$

$$= T_1 T_2 - (1 + |X|)^\alpha \, \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha}(1 + \lambda + |X|)^{-1}(\delta(T_1)T_2 + T_1\delta(T_2))(1 + \lambda + |X|)^{-1} \, d\lambda.$$

Using $(1 + |X|)^\alpha(1 + \lambda + |X|)^{-1} \leq 1$ and $\|(1 + \lambda + |X|)^{-1}\| \leq \frac{1}{1+\lambda}$ we find that we can estimate the n-th seminorm $\mathcal{P}_n$ on $\mathcal{B}_1(X, d)$ by the n-th seminorm $\mathcal{Q}_n$ on $\mathcal{B}_2(X, d)$ via

$$\mathcal{P}_n\Big((1 + |X|)^\alpha \, T_1 T_2 \, (1 + |X|)^{-\alpha} - T_1 T_2\Big)$$

$$\leq \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} \frac{1}{1+\lambda}\Big(\mathcal{Q}_n(\delta(T_1))\mathcal{Q}_n(T_2) + \mathcal{Q}_n(T_1)\mathcal{Q}_n(\delta(T_2))\Big) \, d\lambda$$

and this is finite for every $\alpha > 0$. In particular for $T_1, T_2 \in \mathcal{B}_2^1(X, d)$, for all $r > 0$ and $\alpha > 0$, the operator

$$(1 + |X|)^{\alpha-d-r} \, T_1 T_2 \, (1 + |X|)^{-\alpha-d-r}$$

is trace class.

Lastly, we note that in the above proof, at no point do we need to apply δ to either of the factors b, c more than $\lfloor d/2 \rfloor + 1$ times. $\qquad \square$

Proposition 5.7. *The tuple* $\Big(\mathcal{A}_{\mathrm{Sob}}\hat{\otimes}C\ell_{0,d}, L^2(\mathbb{R}^d) \otimes L^2(B, \tau_B)\hat{\otimes}\bigwedge^* \mathbb{R}^d, \sum_j X_j\hat{\otimes}\gamma^j\Big)$ *is a finitely summable semifinite spectral triple with spectral dimension* d.

Proof. The operators $[X, \tilde{\pi}(g)]$ are bounded by the regularity of elements in the Sobolev spaces and algebra. For finite summability we apply Lemmas 5.4 and 5.6. $\qquad \square$

A difficulty that we encounter with extending the index pairing is that, as defined, there is no guarantee that the algebra $\mathcal{A}_{\mathrm{Sob}}$ is separable, and typically it will not be. For index pairings the lack of separability is not a problem: given a projection or unitary over $\mathcal{A}_{\mathrm{Sob}}$, we can restrict to the separable algebra generated by this projection or unitary as in [11], and so formulae for the pairing are valid. What is in question is homotopy invariance of the pairing for homotopies continuous in the topology of $\mathcal{A}_{\mathrm{Sob}}$.

Proposition 5.7 is valid for both real and complex Sobolev algebras. We now restrict to complex pairings and the extension of the Chern number formulas derived in Section 4.

Lemma 5.8. *The multi-linear functional*

$$\phi(a_0, a_1, \ldots, a_d) = \mathrm{res}_{s=d} \, \mathrm{Tr}_\tau\Big(a_0\partial_1(a_1)\cdots\partial_d(a_d)(1 + X^2)^{-s/2}\Big), \qquad a_0, \ldots, a_d \in \mathcal{A}_{\mathrm{Sob}}$$

is well-defined and continuous with respect to the topology on $\mathcal{A}_{\mathrm{Sob}}$. *The functional* ϕ *defines a Hochschild cocycle on* $\mathcal{A}_{\mathrm{Sob}}$. *Furthermore,*

$$\phi(a_0, a_1, \ldots, a_d) = C_d \sum_{\sigma \in S_d} (-1)^\sigma \mathcal{T}\Big(a_0\partial_{\sigma(1)}a_1\cdots\partial_{\sigma(d)}a_d\Big).$$

Proof. The functional is well-defined and continuous by the Hölder inequality of the Sobolev spaces (or Lemma 5.6). That we obtain a Hochschild cocycle follows from [24, Lemmas 3, 13]. Finally, the last equality follows by analogous algebraic arguments as was done in Section 4 and the observation that Proposition 4.1 can also be applied to elements in $\mathcal{A}_{\mathrm{Sob}}$. □

Theorem 5.9. *The index formulas given in Theorems 4.2 and 4.4 extend to any projection or unitary in $M_q(\mathcal{A}_{\mathrm{Sob}}^{\sim})$ (complex algebras).*

Proof. By Lemma 5.8, we know that the formula for the index is well-defined and so we just need to identify the formula with the index pairing.

By [22, Proposition 2.14] our Sobolev spectral triple determines a semifinite Fredholm module with operator $X(1 + X^2)^{-1/2}$ and is $(d + 1)$-summable over $\mathcal{A}_{\mathrm{Sob}}$. Hence we obtain a Chern character for $\left(\mathcal{A}_{\mathrm{Sob}}, \mathcal{H}, X(1 + X^2)^{-1/2}\right)$ which we immediately see is the continuous extension of the Chern character for our smoothly summable spectral triple to the algebra $\mathcal{A}_{\mathrm{Sob}}$.

The Chern character of the smoothly summable spectral triple coincides with its Hochschild class by Lemmas 4.3 and 4.5. The residue formula for the Hochschild class extends to define a continuous multi-linear functional on $\mathcal{A}_{\mathrm{Sob}}$, by Lemma 5.8. Since the Chern character for $\mathcal{A}_{\mathrm{Sob}}$ is the continuous extension of the smooth Chern character, the formulae for the index pairings extend to $\mathcal{A}_{\mathrm{Sob}}$.

Because $\mathcal{A}_{\mathrm{Sob}}$ is a local Banach $*$-algebra in the sense of [13, Definition 3.1.1], homotopic projections $p_1 \sim_h p_2$ are unitarily equivalent [13, Proposition 4.3.3]. Because cyclic cohomology is invariant for inner automorphisms and we remain inside the minimal unitisation of $\mathcal{A}_{\mathrm{Sob}}$, the cyclic pairing and therefore the index for p_1 and p_2 is constant. The result also holds for the odd pairing as the connected components in $\mathcal{U}(\mathcal{A}_{\mathrm{Sob}}^{\sim})$ coincide with path-connected components in $\mathcal{U}(\mathcal{A}_{\mathrm{Sob}}^{\sim})$. □

Let us now consider the range and stability of the Sobolev pairing. For any separable sub-algebra $\mathcal{C}$ of $\mathcal{A}_{\mathrm{Sob}}$ (e.g. the algebra generated by a projection or unitary, similar to what is considered in [11]), we define a new Kasparov module. The semifinite spectral triple above has Hilbert space $\mathcal{H} = L^2(E, \tau_B)$. Here $E_B \cong L^2(\mathbb{R}^d, B)$ is the B-module of the Kasparov A-B-module λ_d from Proposition 2.6, and $\tau_B : B \to \mathbb{C}$ the trace. The inner product on E_B remains well-defined for elements of $\mathcal{C} \cdot E$, and we complete $(\mathcal{C} \cdot E \mid \mathcal{C} \cdot E) \subset \pi_{GNS}(B)''$ in norm to obtain an algebra D. In turn we complete $\mathcal{C} \cdot E$ in the resulting Hilbert module norm, and we obtain a Kasparov C-D-module. So we obtain well-defined pairings

$$K_*(C) \times KK^*(C, D) \to K_0(D).$$

We can then use Proposition A.4 in the appendix to conclude that the semifinite index represents the composition

$$K_*(C) \times KK^*(C, D) \to K_0(D) \xrightarrow{\tau_B} \mathbb{R}.$$

Because D can be taken to be separable, we therefore have that the range of the semifinite index is countably generated (though not necessarily discrete).

5.1. Pairings with ergodic measures.
We consider the case of $B = C(\Omega)$ with Ω a compact topological space with a faithful measure $\mathbf{P}$ that is invariant and ergodic under a twisted $\mathbb{R}^d$-action.

The Sobolev algebra is then a subalgebra of the von Neumann crossed product $L^\infty(\Omega) \rtimes_\theta \mathbb{R}^d$ and naturally acts on $L^2(\mathbb{R}^d \times \Omega, \mathrm{d}x \times \mathrm{d}\mathbf{P})$. We can still consider the pointwise representations $\widetilde{\pi}_\omega$ of $L^\infty(\Omega) \rtimes_\theta \mathbb{R}^d$ (and $\mathcal{A}_{\mathrm{Sob}}$) on $L^2(\mathbb{R}^d)$. To do this we use the fact that the evaluations $f \mapsto f(\omega)$ are norm continuous on $C(\Omega) \rtimes_\theta \mathbb{R}^d$ and so have normal extensions to $L^\infty(\Omega) \rtimes_\theta \mathbb{R}^d$.

Thus we can construct the triple

$$\widetilde{\lambda}_d(\omega) = \left(\mathcal{A}_{\mathrm{Sob}}, \widetilde{\pi}_\omega L^2(\mathbb{R}^d)\hat{\otimes}\mathbb{C}^\nu, X\right),$$

which has the properties that $[X, \widetilde{\pi}_\omega(f)]$ is $\mathbf{P}$-almost surely bounded and $\widetilde{\pi}_\omega(f)(1 + X^2)^{-s/2}$ is $\mathbf{P}$-almost surely trace-class for $s > d$. However, the zero-measure set where the commutator may be unbounded and the trace-class condition might fail depends on the element $f \in \mathcal{A}_{\mathrm{Sob}}$.

Lemma 5.10. *Let $p \in M_q(\mathcal{A}_{\mathrm{Sob}}^{\sim})$ be a projection and $u \in M_q(\mathcal{A}_{\mathrm{Sob}}^{\sim})$ a unitary. Then there is a well-defined $\mathbb{Z}$-valued pairing*

$$\langle p, \widetilde{\lambda}_d(\omega) \rangle = \mathrm{Index}\left(\widetilde{\pi}_\omega(\hat{p})(F_X \otimes 1_{2q})_+ \widetilde{\pi}_\omega(\hat{p}) \right), \quad F_X = X(1 + X^2)^{-1/2},$$

$$\langle u, \widetilde{\lambda}_d(\omega) \rangle = \mathrm{Index}\left(P_{2q} \widetilde{\pi}_\omega(\hat{u}) P_{2q} - (1 - P_{2q}) \right), \quad P_{2q} = \frac{1}{2}(1 + F_X) \otimes 1_{2q}.$$

The pairing is $\mathbf{P}$-almost surely constant in Ω and, as such, extends to a pairing with the family $\{\widetilde{\lambda}_d(\omega)\}_{\omega \in \Omega}$.

Proof. Because the measure on Ω is ergodic, it suffices to check that the pairing is $\mathbf{P}$-almost surely constant on an orbit. To show this constancy, we remark that if $\omega' = T_{-a}\omega$, then using the corresponding covariance relation, F_X is unitarily equivalent to F_{X+a}, the bounded transform of $\sum_j (X_j + a_j) \hat{\otimes} \gamma^j$, via the unitary U_a implementing T_{-a}. Since $U_a[X, U_a^*]$ is bounded, we have a bounded perturbation of the unbounded operator X. This implies that for $f \in \mathcal{A}_{\mathrm{Sob}}$, the difference $\widetilde{\pi}_\omega(f)(F_X - F_{X+a})$ is $\mathbf{P}$-almost surely compact. Applying the unitary transformation to the index,

$$\mathrm{Index}\left(\widetilde{\pi}_\omega(\hat{p})(F_X \otimes 1_{2q})_+ \widetilde{\pi}_\omega(\hat{p}) \right) = \mathrm{Index}\left(\widetilde{\pi}_{T_{-a}\omega}(\hat{p})(F_X \otimes 1_{2q})_+ \widetilde{\pi}_{T_{-a}\omega}(\hat{p}) + K \right)$$
$$= \mathrm{Index}\left(\widetilde{\pi}_{T_{-a}\omega}(\hat{p})(F_X \otimes 1_{2q})_+ \widetilde{\pi}_{T_{-a}\omega}(\hat{p}) \right),$$

where K is compact. The odd pairing follows an analogous argument. $\square$

Theorem 5.11. *If the trace τ on $C(\Omega)$ comes from a faithful measure $\mathbf{P}$ that is invariant and ergodic under the twisted $\mathbb{R}^d$-action, then the index formulas given in Theorems 4.2 and 4.4 extend to $\mathcal{A}_{\mathrm{Sob}}$ and are $\mathbf{P}$-almost surely integer valued. Furthermore, the index is invariant under continuous deformations in the Sobolev topology.*

Proof. The cyclic formula is well-defined and represents the semifinite Chern character of the semifinite spectral triple from Proposition 5.7 by Theorem 5.9. We just need to connect this result to our new analytic indices. Because the measure is ergodic, for any $f \in \mathcal{A}_{\mathrm{Sob}}$, Birkhoff's theorem [63, Section 4] gives that $\mathrm{Tr}_\tau(f) = \mathrm{Tr}_{\mathrm{Vol}}(\widetilde{\pi}_\omega(f))$ with $\mathrm{Tr}_{\mathrm{Vol}}$ the trace per unit volume,

$$\mathrm{Tr}_{\mathrm{Vol}}(\widetilde{\pi}_\omega(f)) = \lim_{\Lambda \nearrow \mathbb{R}^d} \frac{1}{\mathrm{Vol}(\Lambda)} \mathrm{Tr}\left(P_\Lambda \widetilde{\pi}_\omega(f) \right), \qquad P_\Lambda : L^2(\mathbb{R}^d) \to L^2(\Lambda).$$

In particular, by Proposition 4.1 and the regularity properties of $\mathcal{A}_{\mathrm{Sob}}$ we have that

$$\mathrm{res}_{s=d} \mathrm{Tr}_\tau(\widetilde{\pi}(f)(1 + |X|^2)^{-s/2}) = \mathrm{Vol}_{d-1}(S^{d-1}) \mathrm{Tr}_{\mathrm{Vol}}(\widetilde{\pi}_\omega(f)), \qquad f \in \mathcal{A}_{\mathrm{Sob}}$$

$\mathbf{P}$-almost surely (see also [15, Lemma 3.3.7]). This implies that the semifinite Chern character of the semifinite spectral triple over $\mathcal{A}_{\mathrm{Sob}}$ is $\mathbf{P}$-almost surely the same as the Chern character of $\widetilde{\lambda}_d(\omega)$, which can be computed by the $\mathbb{Z}$-valued analytic pairings. Hence in the ergodic measure setting, our Sobolev index formula is $\mathbf{P}$-almost surely $\mathbb{Z}$-valued and constant in Ω.

Next, we consider a continuous deformation in $\mathcal{A}_{\mathrm{Sob}}$. Because the Hochschild cocycle is continuous in the Sobolev topology, the cyclic expression for the index will change continuously as we make this deformation. However, the equality of the cyclic formula with the Fredholm index for *any* projection or unitary in $\mathcal{A}_{\mathrm{Sob}}^{\sim}$ ensures that the cyclic pairing is always $\mathbb{Z}$-valued. Therefore if the cyclic pairing were to change during this deformation, then there must be a discontinuity, which is not possible. Thus the index formula is constant throughout the deformation. $\square$

Remarks 5.12.   (1) If we take a homotopy of projections in $\mathcal{A}_{\mathrm{Sob}}$, then the corresponding path $\widetilde{\pi}_\omega(p_t)(F_X)_+ \widetilde{\pi}_\omega(p_t)$ from $\widetilde{\pi}_\omega(p_0)(F_X)_+ \widetilde{\pi}_\omega(p_0)$ is $\mathbf{P}$-almost surely constant mod compacts, since

$$\widetilde{\pi}_\omega(p_t)(F_X)_+ \widetilde{\pi}_\omega(p_t) - \widetilde{\pi}_\omega(p_0)(F_X)_+ \widetilde{\pi}_\omega(p_0)$$
$$= \widetilde{\pi}_\omega(p_t - p_0)(F_X)_+ \widetilde{\pi}_\omega(p_t) + \widetilde{\pi}_\omega(p_0)(F_X)_+ \widetilde{\pi}_\omega(p_t - p_0)$$

is trace norm continuous. Therefore, the index pairing is constant on homotopy classes of projections in $\mathcal{A}_{\text{Sob}}$. Let us emphasise, however, that the homotopy $\widetilde{\pi}_\omega(p_t)(F_X)_+\widetilde{\pi}_\omega(p_t)$ in $\mathcal{A}_{\text{Sob}}$ is *not* a continuous path in the operator norm topology. Indeed, the topology of $\mathcal{A}_{\text{Sob}}$ is fundamentally different from the operator norm, which is required if we wish to meaningfully extend our index pairing. Hence, while we can make sense of index pairings in $\mathcal{A}_{\text{Sob}}$, it is not easy to relate the local Banach algebra $\mathcal{A}_{\text{Sob}}$ to the C^*-algebra we first considered.

(2) While taking the intersection over all Sobolev spaces appear to be quite restrictive, we will see in Section 7.4 that dynamically localised observables often have, on average over the configuration space Ω, *exponentially decaying* integral kernels. As such, our index theory over $\mathcal{A}_{\text{Sob}}$ be be applied in this situation. We will also show that (under extra restrictions), deformations within a region of dynamical localisation are continuous in the Sobolev topology.

6. The bulk-edge correspondence

The bulk-edge correspondence is a key property of topological states of matter, where non-trivial topological properties in the bulk (interior) of a physical system give rise to edge behaviour, e.g. the existence of stable edge states and edge conductivity. Driving the bulk-edge correspondence for the C^*-algebraic approach to condensed matter physics is a short exact sequence linking bulk and edge observable algebras [49, 51, 48, 38].

The short exact sequence encodes the boundary map in K-theory or K-homology (or their extension KK-theory), where one shows that non-trivial bulk K-theory classes or Fredholm modules are mapped to non-trivial edge invariants under the boundary map. Because the topological phases of interest arise as index pairings, our K-theoretic result on boundary maps immediately implies the bulk-edge correspondence for index pairings and invariants.

6.1. **The Wiener–Hopf extension.** By considering crossed product algebras by $\mathbb{R}$, there is a natural short exact sequence, namely the Wiener–Hopf extension (see for example [77]). Recalling the discussion in Section 2.1, we can decompose the twisted crossed product $B \rtimes_\theta \mathbb{R}^d$ as an iterated crossed product of a twisted crossed product by $\mathbb{R}^{d-1}$ and an untwisted $\mathbb{R}$-crossed product, $(B \rtimes_\theta \mathbb{R}^{d-1}) \rtimes \mathbb{R}$. In the case $B = C(\Omega)$ with $\theta(x,-x) = 1$ this can be done via an explicit isomorphism [48]. For general B, the decomposition is equivalent to our original twisted crossed product at the level of KK-theory. We let $A_e = B \rtimes_\theta \mathbb{R}^{d-1}$, the observables on the edge of a system with boundary, and $A_b = A_e \rtimes \mathbb{R}$ the algebra on a boundaryless system.

Following [48, 51] our bulk-edge short exact sequence is

$$(12) \qquad 0 \to \mathcal{K} \otimes A_e \to (C_0(\mathbb{R} \cup \{+\infty\}) \otimes A_e) \rtimes \mathbb{R} \to A_e \rtimes \mathbb{R} \to 0$$

where the $\mathbb{R}$-action on $C_0(\mathbb{R} \cup \{+\infty\}) \otimes A_e$ is by translation on $C_0(\mathbb{R} \cup \{+\infty\})$ (with fixed point at $+\infty$) and by the automorphism on A_e such that $A_b = A_e \rtimes \mathbb{R}$. In order to compute boundary maps in KK-theory, we first represent Equation (12) as an unbounded Kasparov module by the isomorphism $KKO(A\hat{\otimes}C\ell_{0,1}, C) \cong \text{Ext}^{-1}(A, C)$ for separable C^*-algebras A and C [43, §7].

Proposition 6.1. *The unbounded crossed-product Kasparov module*

$$(13) \qquad \left(C_c(\mathbb{R}, A_e)\hat{\otimes}C\ell_{0,1}, \; L^2(\mathbb{R}, A_e)_{A_e}\hat{\otimes}\bigwedge\nolimits^* \mathbb{R}, \; X_{\text{ext}}\hat{\otimes}\gamma_{\text{ext}} \right),$$

represents the class of the extension of Equation (12) in $KKO(A_b\hat{\otimes}C\ell_{0,1}, A_e)$. Here γ_{ext} is the generator of $C\ell_{1,0}$ and X_{ext} is the multiplication operator by the independent variable in $\mathbb{R}$.

Proof. Our Kasparov module is precisely the unbounded Kasparov module λ_d we have already considered in Proposition 2.6 for $d = 1$. Our task, therefore, is to show that this unbounded module represents the Wiener–Hopf extension in Equation (12).

Associated to the graded Kasparov module from the Equation (13) is the ungraded (odd) module $\left(C_c(\mathbb{R}, A_e), L^2(\mathbb{R}, A_e), X_{\text{ext}} \right)$, from which we can construct an extension. First we use

Connes' trick [27] to double our unbounded Kasparov module to the tuple

$$\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, L^2(\mathbb{R}, A_e) \oplus L^2(\mathbb{R}, A_e), X_m = \begin{pmatrix} X_{\text{ext}} & m \\ m & -X_{\text{ext}} \end{pmatrix} \right), \quad m > 0,$$

which does not change the class in $KKO^1(A_b, A_e)$ and has the advantage that X_m has a spectral gap around 0 (see also [22, Section 2.7] for another method). Next we let $P = \chi_{[0,\infty)}(X_m)$, which up to a locally compact pertubation is exactly the projection $\Pi \oplus \Pi$, with $\Pi : L^2(\mathbb{R}, A_e) \to L^2(\mathbb{R}_+, A_e)$ the projection onto the half-space Hilbert module. Therefore given the module $\left(C_c(\mathbb{R}, A_e), L^2(\mathbb{R}, A_e), X_{\text{ext}} \right)$ we can associate the extension

$$0 \to \mathcal{K}[L^2(\mathbb{R}_+, A_e)] \to C^*(PA_bP, \mathcal{K}[L^2(\mathbb{R}_+, A_e)]) \to A_b \to 0$$

with positive semisplitting by P. Hence the Kasparov module gives rise to the Busby invariant

$$\phi : A_b \to \mathcal{Q}(A_e), \qquad \phi(a) = p(PaP),$$

with $p : \mathcal{M}(A_e \otimes \mathcal{K}) \to \mathcal{Q}(A_e \otimes \mathcal{K})$ the corona projection. Next we consider the Wiener–Hopf extension

$$0 \to \mathcal{K} \otimes A_e \to (C_0(\mathbb{R} \cup \{+\infty\}) \otimes A_e) \rtimes \mathbb{R} \to A_b \to 0.$$

We take a function $g \in C_0(\mathbb{R} \cup \{+\infty\})$ that is 0 for $x \leq 0$, smoothly goes to 1 for $0 \leq x \leq m/2$ and is 1 for all $x > m/2$. Then the map $f \mapsto gf$ for $f \in C_c(\mathbb{R}, A_e)$ gives rise to a map $A_e \rtimes \mathbb{R} \to (C_0(\mathbb{R} \cup \{+\infty\}) \otimes A_e) \rtimes \mathbb{R}$ and the Busby invariant $\tilde{\phi}(f) = p((gf)(+\infty))$, where

$$\tilde{\phi}(f) = p((gf)(+\infty)) \in \mathcal{Q}(C_0(\mathbb{R}) \rtimes \mathbb{R} \otimes A_e) \cong \mathcal{Q}(\mathcal{K} \otimes A_e).$$

The maps $a \mapsto PaP$ and $f \mapsto gf$ differ by a compact operator on $L^2(\mathbb{R}_+, A_e)$ and, so we have that $\phi = \tilde{\phi}$ and the extensions are equivalent. $\qquad \square$

Remark 6.2 (The Thom class). We note that the unbounded Kasparov module coming from an (untwisted) $\mathbb{R}$-action and representing the Wiener–Hopf extension is the inverse of the class in KK-theory implementing the Connes–Thom isomorphism. This statement follows immediately from the work of Rieffel [77], who showed that the boundary map from the Wiener–Hopf extension implements the inverse of the Connes–Thom isomorphism. An explicit representative of the inverse to the class from Proposition 6.1 is constructed in [2, 3].

6.2. **The edge Kasparov module and the product.** Given the edge algebra $A_e = B \rtimes_\theta \mathbb{R}^{d-1}$ with $d \geq 2$, we can construct an unbounded Kasparov module

$$\lambda_{d-1} = \left(C_c(\mathbb{R}^{d-1}, B) \hat{\otimes} C\ell_{0,d-1}, L^2(\mathbb{R}^{d-1}, B) \hat{\otimes} \bigwedge{}^* \mathbb{R}^{d-1}, \sum_{j=1}^{d-1} X_j \hat{\otimes} \gamma^j \right)$$

by Proposition 2.6. The internal Kasparov product of the extension class from Proposition 6.1 with λ_{d-1} defines a map $KKO^1(A_b, A_e) \times KKO^{d-1}(A_e, B) \to KKO^d(A_b, B)$. Our central result of this section is that the product at the unbounded level produces, up to a permutation of Clifford generators, the 'bulk' Kasparov module λ_d. The result is a continuous analogue of [16, 17, 18], which studied crossed products by $\mathbb{Z}^d$.

Theorem 6.3. *The Kasparov product $[\text{ext}] \hat{\otimes}_{A_e} [\lambda_{d-1}]$ is represented by the unbounded Kasparov module,*

$$\left(C_c(\mathbb{R}^d, B) \hat{\otimes} C\ell_{0,d}, L^2(\mathbb{R}^d, B) \hat{\otimes} \bigwedge{}^* \mathbb{R}^d, X_d \hat{\otimes} \gamma^1 + \sum_{j=1}^{d-1} X_j \hat{\otimes} \gamma^{j+1} \right).$$

Furthermore $[\text{ext}] \hat{\otimes}_{A_e} [\lambda_{d-1}] = (-1)^{d-1} [\lambda_d]$, where $-[x]$ represents the inverse class in the KK-group.

Proof. We will focus on the real setting as the case of complex algebras and spaces follows the same argument. We denote by $A_e = B \rtimes_\theta \mathbb{R}^{d-1}$ and $A_b = B \rtimes_\theta \mathbb{R}^d \cong A_e \rtimes \mathbb{R}$. We are taking the internal product of a $A_b \hat{\otimes} C\ell_{0,1}$-$A_e$ module with an $A_e \hat{\otimes} C\ell_{0,d-1}$-$B$ module. To take this

product, we first take the external product of the $A_b\hat{\otimes}C\ell_{0,1}$-$A_e$ module with the identity class in $KKO(C\ell_{0,d-1}, C\ell_{0,d-1})$. This class can be represented by the Kasparov module

$$\Big(C\ell_{0,d-1}, (C\ell_{0,d-1})_{C\ell_{0,d-1}}, 0\Big)$$

with right and left actions given by right and left multiplication. The external product gives the $A_b\hat{\otimes}C\ell_{0,d}$-$A_e\hat{\otimes}C\ell_{0,d-1}$ module

$$\left(C_c(\mathbb{R}, A_e)\hat{\otimes}C\ell_{0,1}\hat{\otimes}C\ell_{0,d-1}, \left(L^2(\mathbb{R}, A_e)\hat{\otimes}\bigwedge{}^* \mathbb{R}\hat{\otimes}C\ell_{0,d-1}\right)_{A_e\hat{\otimes}C\ell_{0,d-1}}, X_{\text{ext}}\hat{\otimes}\gamma_{\text{ext}}\hat{\otimes}1\right).$$

We now take the internal product of this module with the edge module λ_{d-1}. We start with the C^*-modules, where

$$\left(L^2(\mathbb{R}, A_e)\hat{\otimes}_{\mathbb{R}}\bigwedge{}^*\mathbb{R}\,\hat{\otimes}_{\mathbb{R}}\,C\ell_{0,d-1}\right)\hat{\otimes}_{A_e\hat{\otimes}C\ell_{0,d-1}}\left(L^2(\mathbb{R}^{d-1}, B)\hat{\otimes}_{\mathbb{R}}\bigwedge{}^*\mathbb{R}^{d-1}\right)$$

$$\cong \left(L^2(\mathbb{R}, A_e)\otimes_{A_e} L^2(\mathbb{R}^{d-1}, B)\right)\hat{\otimes}_{\mathbb{R}}\bigwedge{}^*\mathbb{R}\,\hat{\otimes}_{\mathbb{R}}\left(C\ell_{0,d-1}\cdot\bigwedge{}^*\mathbb{R}^{d-1}\right)$$

$$\cong \left(L^2(\mathbb{R}, A_e)\otimes_{A_e} L^2(\mathbb{R}^{d-1}, B)\right)\hat{\otimes}_{\mathbb{R}}\bigwedge{}^*\mathbb{R}\,\hat{\otimes}_{\mathbb{R}}\bigwedge{}^*\mathbb{R}^{d-1}$$

as the action of $C\ell_{0,d-1}$ on $\bigwedge^*\mathbb{R}^{d-1}$ by left-multiplication is nondegenerate.

Next we define $1\otimes_\nabla X_j$ on the dense submodule $C_c(\mathbb{R}^{d-1}, B)\otimes_{\mathcal{A}_e} L^2(\mathbb{R}^{d-1}, B)$ for all $j\in\{1,\ldots,d-1\}$ and $\mathcal{A}_e = C_c(\mathbb{R}^{d-1}, B)$. We consider the connection $\nabla_j : \mathcal{A}_e \to \mathcal{A}_e\otimes_{\mathcal{A}_e^\sim}\Omega^1(\mathcal{A}_e^\sim)$ defined from the derivation $\partial_j a_e = [X_j, a_e]$. From this connection we construct the unbounded operator

$$(14) \qquad (1\otimes_\nabla X_j)(\psi_1\otimes\psi_2) = \psi_1\otimes X_j\psi_2 + \nabla_j(\psi_1)\psi_2$$

for $\psi_1\in C_c(\mathbb{R}^{d-1}, B)$ and $\psi_2\in\text{Dom}(X_j)\subset L^2(\mathbb{R}^{d-1}, B)$. We refer the reader to [60, 41, 61] for more details on connections and the construction of operators like $1\hat{\otimes}_\nabla X_j$. Then

$$(15) \quad \left(C_c(\mathbb{R}, \mathcal{A}_e)\hat{\otimes}C\ell_{0,1}\hat{\otimes}C\ell_{0,d-1}, \left(L^2(\mathbb{R}, A_e)\otimes_{A_e} L^2(\mathbb{R}^{d-1}, B)\right)\hat{\otimes}_{\mathbb{R}}\bigwedge{}^*\mathbb{R}\,\hat{\otimes}_{\mathbb{R}}\bigwedge{}^*\mathbb{R}^{d-1},\right.$$

$$\left. X_{\text{ext}}\otimes 1\hat{\otimes}\gamma_{\text{ext}}\hat{\otimes}1 + \sum_{j=1}^{d-1}(1\otimes_\nabla X_j)\hat{\otimes}1\hat{\otimes}\gamma^j\right)$$

is a candidate for the unbounded product module, where the Clifford actions take the form

$$\rho_{\text{ext}}\hat{\otimes}1(\omega_1\hat{\otimes}\omega_2) = (e_1\wedge\omega_1 - \iota(e_1)\omega_1)\hat{\otimes}\omega_2$$

$$1\hat{\otimes}\rho^j(\omega_1\hat{\otimes}\omega_2) = (-1)^{|\omega_1|}\omega_1\hat{\otimes}(e_j\wedge\omega_2 - \iota(e_j)\omega_2),$$

for $j\in\{1,\ldots,d-1\}$ and $|\omega_1|$ is the degree of the form ω_1. Similarly $\gamma_{\text{ext}}\hat{\otimes}1$ and $1\hat{\otimes}\gamma^j$. Arguments very similar to the proof of Proposition 2.6 show that Equation (15) is a real or complex Kasparov module depending on what setting we are in. A simple check of Kucerovsky's criterion [54, Theorem 13], as in [17, 18], shows that the unbounded Kasparov module of Equation (15) is an unbounded representative of the class $[\text{ext}]\hat{\otimes}_{A_e}[\lambda_{d-1}]$.

Our next task is to relate the module (15) to λ_d. We first identify $\bigwedge^*\mathbb{R}\,\hat{\otimes}_{\mathbb{R}}\bigwedge^*\mathbb{R}^{d-1}\cong\bigwedge^*\mathbb{R}^d$ and use the graded isomorphism $C\ell_{p,q}\hat{\otimes}C\ell_{r,s}\cong C\ell_{p+r,q+s}$ from [43, §2.16] on the left and right Clifford generators by the mapping

$$\rho_{\text{ext}}\hat{\otimes}1\mapsto\rho^1, \qquad\qquad\qquad 1\hat{\otimes}\rho^j\mapsto\rho^{j+1},$$

$$\gamma_{\text{ext}}\hat{\otimes}1\mapsto\gamma^1, \qquad\qquad\qquad 1\hat{\otimes}\gamma^j\mapsto\gamma^{j+1}.$$

Applying this isomorphism gives the unbounded Kasparov module representing the product,

$$\left(C_c(\mathbb{R}, \mathcal{A}_e)\hat{\otimes}C\ell_{0,d}, (L^2(\mathbb{R}, A_e)\otimes_{A_e} L^2(\mathbb{R}^{d-1}, B))\hat{\otimes}\bigwedge{}^*\mathbb{R}^d, X_{\text{ext}}\otimes 1\hat{\otimes}\gamma^1 + \sum_{j=1}^{d-1}(1\otimes_\nabla X_j)\hat{\otimes}\gamma^{j+1}\right),$$

with $C\ell_{0,d}$-action generated by $\rho^j(\omega) = e_j \wedge \omega - \iota(e_j)\omega$ and $C\ell_{d,0}$-action generated by $\gamma^j(\omega) = e_j \wedge \omega + \iota(e_j)\omega$ for $\omega \in \bigwedge^* \mathbb{R}^d$ and $\{e_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$.

Next we define a unitary map $L^2(\mathbb{R}, A_e) \otimes_{A_e} L^2(\mathbb{R}^{d-1}, B) \to L^2(\mathbb{R}^d, B)$. To write this map, it is advantageous to use the isomorphisms from Lemma 2.3,

$$L^2(\mathbb{R}, A_e) \cong \overline{C_c(\mathbb{R}, A_e)}, \qquad L^2(\mathbb{R}^{d-1}, B) \cong \overline{C_c(\mathbb{R}^{d-1}, B)} \qquad L^2(\mathbb{R}^d, B) \cong \overline{C_c(\mathbb{R}^d, B)},$$

with inner-products $(f_1 \mid f_2) = (f_1^* * f_2)(0)$ and the left-action is the extension of left multiplication. We work with the dense submodule

$$C_c(\mathbb{R}, \mathcal{A}_e) \cong C_c(\mathbb{R}, C_c(\mathbb{R}^{d-1}, B)) \cong C_c(\mathbb{R}) \otimes C_c(\mathbb{R}^{d-1}, B),$$

which allows us to write down the map

$$\varrho : C_c(\mathbb{R}) \otimes C_c(\mathbb{R}^{d-1}, B) \otimes_{C_c(\mathbb{R}^{d-1}, B)} C_c(\mathbb{R}^{d-1}, B) \to C_c(\mathbb{R}^d, B),$$

$$\varrho(f_1 \otimes f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3) = f_1 \otimes f_2 * f_3 \in C_c(\mathbb{R}) \otimes C_c(\mathbb{R}^{d-1}, B) \cong C_c(\mathbb{R}^d, B).$$

As the action of left-multiplication is non-degenerate and uniformly bounded, this map extends to a unitary map when we take the closure in the module norm. It is easy to check that

$$\varrho\big((X_{\mathrm{ext}} \otimes 1 \otimes 1)(f_1 \otimes f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3)\big) = (X_d f_1) \otimes f_2 * f_3 = X_d \, \varrho(f_1 \otimes f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3).$$

For the left-action, we see that for $g_1 \otimes g_2 \in C_c(\mathbb{R}) \otimes C_c(\mathbb{R}^{d-1}, B)$,

$$\begin{aligned}
\varrho\Big(\pi(g_1 \otimes g_2)(f_1 \otimes f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3)\Big) &= \varrho\Big((g_1 * f_1) \otimes (g_2 * f_2) \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3\Big) \\
&= (g_1 * f_1) \otimes (g_2 * f_2) * f_3 \\
&= (g_1 * f_1) \otimes g_2 * (f_2 * f_3) \\
&= (g_1 \otimes g_2)(f_1 \otimes f_2 * f_3) \\
&= \pi(g_1 \otimes g_2)\varrho(f_1 \otimes f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3).
\end{aligned}$$

Again, as the left-action is uniformly bounded, the result extends to show that the left action is compatible with the unitary map on the whole module. Finally we note that on the dense submodule, the operator $1 \otimes_\nabla X_j$ from Equation (14) has the form

$$(1 \otimes_\nabla X_j)(f_1 \otimes f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3) = f_1 \otimes X_j f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3 + f_1 \otimes f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} X_j f_3.$$

Therefore we compute

$$\begin{aligned}
\varrho\Big((1 \otimes_\nabla X_j)(f_1 \otimes f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3)\Big) &= f_1 \otimes ((X_j f_2) * f_3 + f_2 * (X_j f_3)) \\
&= f_1 \otimes X_j \, (f_2 * f_3) \\
&= X_j \, \varrho(f_1 \otimes f_2 \otimes_{C_c(\mathbb{R}^{d-1}, B)} f_3)
\end{aligned}$$

as the operator X_j is a derivation on $C_c(\mathbb{R}^{d-1}, B)$. Taking closures, we have that $1 \otimes_\nabla X_j \mapsto X_j$ under ϱ. To summarise, the unbounded Kasparov module representing the product is unitarily equivalent to

$$(16) \qquad \left(C_c(\mathbb{R}^d, B) \hat{\otimes} C\ell_{0,d}, \; L^2(\mathbb{R}^d, B) \hat{\otimes} \bigwedge^* \mathbb{R}^d, \; X_d \hat{\otimes} \gamma^1 + \sum_{j=1}^{d-1} X_j \hat{\otimes} \gamma^{j+1} \right)$$

with left and right Clifford actions as before. The only difference between our product module and the bulk module λ_d from Proposition 2.6 is the labelling of the Clifford basis. The map $\eta(\gamma^j) = \gamma^{\sigma(j)}$ and $\eta(\rho^j) = \rho^{\sigma(j)}$ for $\sigma(j) = (j-1) \bmod d$ is an isomorphism of Clifford algebras that may reverse the orientation of the algebra. Taking the canonical orientation $\omega_{C\ell_{0,d}} = \rho^1 \cdots \rho^d$ of $C\ell_{0,d}$,

$$\eta(\omega_{C\ell_{0,d}}) = \rho^d \rho^1 \cdots \rho^{d-1} = (-1)^{d-1} \rho^1 \cdots \rho^d = (-1)^{d-1} \omega_{C\ell_{0,d}},$$

similarly γ^j and $C\ell_{d,0}$. Using [43, §5: Theorem 3], such a map on Clifford algebras will send the KK-class of the Kasparov module of Equation (16) to its inverse if $\eta(\omega) = -\omega$ or will leave

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the class invariant if $\eta(\omega) = \omega$. Hence, at the level of KK-classes, $[\text{ext}]\hat{\otimes}_{A_e}[\lambda_{d-1}] = (-1)^{d-1}[\lambda_d]$ as required. $\square$

6.3. **Pairings and the bulk-edge correspondence.**

Given a real or complex K-theory element $[x] \in KO_j(B \rtimes_\theta \mathbb{R}^d)$ (we will consider the real setting as the complex case is simpler), the unbounded Kasparov module λ_d from Propostion 2.6 gives a map

$$KO_j(B \rtimes_\theta \mathbb{R}^d) \times KKO^d(B \rtimes_\theta \mathbb{R}^d, B) \to KO_{j-d}(B)$$

via the internal product. Theorem 6.3 implies that we may decompose this bulk pairing as the product $(-1)^{d-1}[x]\hat{\otimes}_{A_b}([\text{ext}]\hat{\otimes}_{A_e}[\lambda_{d-1}])$. The associativity of the Kasparov product ensures that this product can be expressed as $(-1)^{d-1}([x]\hat{\otimes}_{A_b}[\text{ext}])\hat{\otimes}_{A_e}[\lambda_{d-1}]$, which is now a pairing over the edge algebra

$$KO_{j-1}(B \rtimes_\theta \mathbb{R}^{d-1}) \times KKO^{d-1}(B \rtimes_\theta \mathbb{R}^{d-1}, B) \to KO_{j-d}(B).$$

The equality of these bulk and boundary pairings is the bulk-edge correspondence.

If $d = 1$, then $A_b = B \rtimes \mathbb{R}$, $A_e = B$ and $[x]\hat{\otimes}_{A_b}[\lambda_1] = \partial[x] \in KO_{j-1}(B)$ as λ_1 represents the extension class of the Wiener–Hopf extension. As the boundary map of the Wiener–Hopf extension is an isomorphism in K-theory, the bulk pairing will be non-trivial only if the boundary K-theory class is non-trivial.

The bulk-edge correspondence is usually associated to topological states of matter though we note that there may be other applications of such a result. We may iterate the bulk-edge correspondence to say that pairings of continuous crossed products $B \rtimes_\theta \mathbb{R}^d$ can be expressed as (up to a sign) a pairing of a crossed product of any order $B \rtimes \mathbb{R}^k$ (though this will place restrictions on the allowed twists). At the level of KK-classes, this result follows from the Connes–Thom isomorphism, but our explicit formulae allow us to derive concrete formulas for index pairings in terms of the physical operators.

Theorem 6.3 immediately implies that there is also a bulk-edge correspondence of the semifinite index pairing. This remark is particularly useful for complex algebras as the bulk and edge invariants may be computed by the local formulas from Theorem 4.2 and 4.4. Hence we obtain an equality of cyclic pairings for twisted dynamical systems $(B, \mathbb{R}^d, \alpha, \theta)$ with a faithful and invariant tracial weight τ_B on B. Furthermore, we know that the range of these cyclic pairings is countably generated and often discrete. Hence the bulk-edge result extends to a wide range of potential examples and index pairings.

7. APPLICATIONS TO DISORDERED QUANTUM SYSTEMS AND TOPOLOGICAL PHASES

Here we link our mathematical framework back to continuous models of free-fermionic quantum mechanical systems and their topological invariants. Our aim is to show how the framework for modelling such systems developed in [62, 8, 9] naturally fits into our constructions and results from the previous sections. We also make some comments on localisation and the stability of the index pairings in the strong disorder regime.

7.1. **Review: Disordered Hamiltonians and twisted crossed products.**

We model a particle in $\mathbb{R}^d$ subject to a uniform magnetic field perpendicular to the sample. There is a choice of magnetic potential A, where $B = \mathrm{d}A + A \wedge A$ is the magnetic field. We take $A = (A_1, \ldots, A_d)$ such that $A_j \in L^2_{\text{loc.}}(\mathbb{R}^d)$ and differentiable with

$$\frac{\partial}{\partial x_j}A_k - \frac{\partial}{\partial x_k}A_j = B_{j,k} = \text{const.}$$

for all $j, k \in \{1, \ldots, d\}$. The case of a magnetic field continuously depending on x is also possible (cf. Example 2.2), though we will consider constant field strength for simplicity. We direct the reader to [10, 56, 59] for more detailed study on magnetic fields and twisted crossed products.

The Schrödinger operator is given by

$$H_0 = \frac{1}{2m^*} \sum_{j=1}^{d} \left(-i\hbar \frac{\partial}{\partial x_j} - eA_j \right)^2,$$

where m^* is the effective mass of the particle. We choose units such that $m^* = \frac{\hbar}{2}$ and introduce the operators $K_j = -i\frac{\partial}{\partial x_j} - \frac{e}{\hbar}A_j$ for $j \in \{1, \ldots, d\}$. We choose the symmetric gauge and define $A_j = -\frac{1}{2}\sum_{k=1}^{d} B_{j,k}x_k$ for $j = 1, \ldots, d$, where $B_{j,k}$ is antisymmetric and real. We introduce the parameter $\theta_{j,k}$ so that we can rewrite

$$K_j = -i\frac{\partial}{\partial x_j} - \sum_{k=1}^{d} \theta_{j,k}X_k \quad \text{and} \quad \sum_{j=1}^{d} K_j^2 = \frac{2m^*}{\hbar^2}H_0 = H_0.$$

Example 7.1 (Quantum Hall Hamiltonian). In the case where $d = 2$, our Hamitonian is given in the symmetric gauge as

$$H_0 = \left(-i\frac{\partial}{\partial x_1} + \theta X_2 \right)^2 + \left(-i\frac{\partial}{\partial x_2} - \theta X_1 \right)^2,$$

where $\theta \in \mathbb{R}$ represents the magnetic flux through a unit cell. We recognise this Hamiltonian as the 2-dimensional Landau Hamiltonian used to model the quantum Hall effect.

The presence of the magnetic field means that H_0 does not commute with ordinary translation operators S_a, where $(S_a\psi)(x) = \psi(x - a)$ for $\psi \in L^2(\mathbb{R}^d)$ and $x, a \in \mathbb{R}^d$. However, we may define the so-called *magnetic translations* U_a such that in the symmetric gauge $(U_a\psi)(x) = e^{-i\theta(x \wedge a)}(x - a)$ for $\psi \in L^2(\mathbb{R}^d)$, where $\theta(x \wedge a) = \sum_{j,k=1}^{d} \theta_{j,k}x_j a_k$. We note that $\theta(x \wedge x) = 0$ and $\theta(x \wedge y) = -\theta(y \wedge x)$. One checks that $[U_a, K_j] = 0$ on $\text{Dom}(K_j)$ for any $a \in \mathbb{R}^d$ and $j \in \{1, \ldots, d\}$ (see [85] for more details on magnetic translations for general gauge choices).

We wish to consider a system with disorder or impurities. Following Bellissard and co-authors [8, 62], such effects can be encoded by the hull. We let $H = H_0 + V$ with V a potential coming from an essentially bounded, real-valued and measurable function on $\mathbb{R}^d$. Consider the set

$$\Omega = \overline{\{U_a V U_{-a} : a \in \mathbb{R}^d\}},$$

with compact closure in the weak operator topology. Clearly Ω is endowed with a twisted action of $\mathbb{R}^d$ by magnetic translations, denoted by $\{T_a\}_{a \in \mathbb{R}^d}$. By considering the possible translates of the potential V, we obtain a family of Hamiltonians $\{H_\omega\}_{\omega \in \Omega}$ representing a disordered or aperiodic system.

Proposition 7.2 ([8], §2.4. Also see [15], Corollary 3.2.11)**.** *Denote by V_ω the bounded function representing the point $\omega \in \Omega$. Then there is a Borel function v on Ω such that $V_\omega(x) = v(T_{-x}\omega)$ for almost all $x \in \mathbb{R}^d$ and all $\omega \in \Omega$. If in addition V is uniformly continuous and bounded, v is continuous.*

The proposition shows that if our disordered potential V is uniformly continuous and bounded, we can associate a continuous function $v \in C(\Omega)$, a unital C^*-algebra. Furthermore, $C(\Omega)$ comes with a twisted action of $\mathbb{R}^d$ via the action on Ω. Hence we may consider the twisted crossed product $C(\Omega) \rtimes_\theta \mathbb{R}^d$. We start with the dense subalgebra $\mathcal{A} = C_c(\mathbb{R}^d, C(\Omega))$ and denote functions $f \in C_c(\mathbb{R}^d, C(\Omega))$ by $f(x; \omega)$. We use the symmetric gauge which determines the twist θ and gives rise to the $*$-algebra structure,

$$(f_1 * f_2)(x; \omega) = \int_{\mathbb{R}^d} e^{i\theta(x \wedge y)} f_1(y; \omega) f_2(x - y; T_{-y}\omega) \, dy, \qquad f^*(x; \omega) = \overline{f(-x; T_{-x}\omega)}.$$

For a fixed $\omega \in \Omega$, we can represent $\mathcal{A}$ on $L^2(\mathbb{R}^d)$ by the map π_ω, where

$$[\pi_\omega(f)\psi](x) = \int_{\mathbb{R}^d} e^{-i\theta(x \wedge y)} f(y - x; T_{-x}\omega) \psi(y) \, dy$$

for all $\psi \in L^2(\mathbb{R}^d)$. A computation shows that π_ω is a $*$-algebra homomorphism. Furthermore, the representation satisfies the covariance condition

$$(17) \qquad\qquad U_a \pi_\omega(f) U_{-a} = \pi_{T_a \omega}(f)$$

for all $\omega \in \Omega$ and $f \in C_c(\mathbb{R}^d, C(\Omega))$. Hence we obtain a family of representations of the resulting crossed product completion $A = C(\Omega) \rtimes_\theta \mathbb{R}^d$. A key result of [8] is the following.

Theorem 7.3 ([8], Theorem 6). *Take $H = \sum_j K_j^2 + V$ acting on $L^2(\mathbb{R}^d)$ with hull Ω. For each z in the resolvent set of H and $x \in \mathbb{R}^d$ there is an element $R(z; x) \in A$ such that for all $\omega \in \Omega$, $\pi_\omega[R(z; x)] = (z - H_{T_{-x}\omega})^{-1}$.*

Therefore we can take the algebra of observables of a disordered magnetic Hamiltonian to be $C(\Omega) \rtimes_\theta \mathbb{R}^d$. Hence we may consider topological properties of the physical system by studying the topology of the crossed product algebra.

7.1.1. *Measures and traces.* We now suppose that the disorder space of configurations Ω has a probability measure $\mathbf{P}$ that is invariant under the $\mathbb{R}^d$-action and $\mathrm{supp}(\mathbf{P}) = \Omega$. The measure $\mathbf{P}$ induces a trace $\tau_{\mathbf{P}}$ on $C(\Omega)$ by integration that is semifinite, norm lower-semicontinuous (by Fatou's lemma applied to $\mathbf{P}$) and faithful by the support assumption on $\mathbf{P}$.

We now extend $\tau_{\mathbf{P}}$ to an unbounded trace on the crossed product $C(\Omega) \rtimes_\theta \mathbb{R}^d$. For $f \in C_c(\mathbb{R}^d, C(\Omega))$ and $f \geq 0$, we define

$$\mathcal{T}(f) = \tau_{\mathbf{P}}[f(0)] = \int_\Omega f(0; \omega) \, \mathrm{d}\mathbf{P}(\omega).$$

Using [55], and Propositions 2.4, 2.6, we deduce that $\mathcal{T}$ satisfies our technical hypotheses.

Lemma 7.4. *The functional $\mathcal{T}$ is a faithful semifinite norm lower-semicontinous trace with $\mathcal{A} \subset \mathrm{Dom}(\mathcal{T})$. Furthermore, $\mathcal{T}$ extends to a semifinite trace on $L^\infty(A, \mathcal{T}) = (C(\Omega) \rtimes_\theta \mathbb{R}^d)'' \subset \mathcal{B}[L^2(\mathbb{R}^d) \otimes L^2(\Omega, \mathbf{P})]$.*

Proposition 7.5 ([9], Proposition 1). *Let $f \in \mathcal{A}_+$. If $\mathbf{P}$ is an ergodic measure (that is, the only functions $\xi \in L^2(\Omega, \mathbf{P})$ such that $\xi(T_a \omega) = \xi(\omega)$ for all $a \in \mathbb{R}^d$ are constant functions), then for almost all $\omega \in \Omega$,*

$$\mathcal{T}(f) = \mathrm{Tr}_{\mathrm{Vol}}[\pi_\omega(f)],$$

with $\mathrm{Tr}_{\mathrm{Vol}}$ the trace per unit volume on $L^2(\mathbb{R}^d)$.

Proof. Given $g \in \mathcal{A}$, we know that

$$[\pi_\omega(g)\psi](x) = \int_{\mathbb{R}^d} e^{-i\theta(x \wedge y)} g(y - x; T_{-x}\omega)\psi(y) \, \mathrm{d}y$$

so $\pi_\omega(g)$ is an integral operator with kernel $k_\omega(x, y) = e^{-i\theta(x \wedge y)} g(y - x; T_{-x}\omega)$. We let Λ_j be a sequence of increasing sets that converge in the appropriate sense to $\mathbb{R}^d$, e.g. $\Lambda_j = [-j, j]^d$. Because Λ_j is bounded and $k_\omega(x, y)$ is continuous, $\pi_\omega(g)$ is Hilbert-Schmidt on $L^2(\Lambda_j)$ by [73, Theorem VI.23] for any $g \in C_c(\mathbb{R}^d, C(\Omega))$. Therefore we can say that the product $\pi_\omega(g^*g)$ is trace-class by [73, Theorem VI.22, part (h)] for $g \in C_c(\mathbb{R}^d, C(\Omega))$. We can take the trace Tr_{Λ_j} by integrating along the diagonal [82, Theorem 3.9]. Computing the trace of $f = g^*g$,

$$\mathrm{Tr}_{\Lambda_j}[\pi_\omega(f)] = \int_{\Lambda_j} k_\omega(x, x) \, \mathrm{d}x = \int_{\Lambda_j} e^{-i\theta(x \wedge x)} f(x - x; T_{-x}\omega) \, \mathrm{d}x = \int_{\Lambda_j} f(0; T_{-x}\omega) \, \mathrm{d}x.$$

As the action of $\mathbb{R}^d$ by T on Ω is $\mathbf{P}$-measure preserving, a continuous version of Birkhoff's Ergodic Theorem in higher dimensions [63, Section 4] gives that

$$\mathrm{Tr}_{\mathrm{Vol}}[\pi_\omega(f)] = \lim_{j \to \infty} \frac{1}{|\Lambda_j|} \mathrm{Tr}_{\Lambda_j}[\pi_\omega(f)] = \lim_{j \to \infty} \frac{1}{|\Lambda_j|} \int_{\Lambda_j} f(0; T_{-x}\omega) \, \mathrm{d}x = \int_\Omega f(0; \omega) \, \mathrm{d}\mathbf{P}(\omega) = \mathcal{T}(f)$$

for almost all ω. $\qquad\square$

7.2. Invariants of topological systems. Because the hull Ω is compact and Hausdorff, $C(\Omega)$ is a separable C^*-algebra and our general theory applies to the example. We note that a disordered family of Hamiltonians $\{H_\omega\}_{\omega\in\Omega}$ affiliated to $C(\Omega) \rtimes_\theta \mathbb{R}^d$ can be considered as a single element H affiliated to $C(\Omega) \rtimes_\theta \mathbb{R}^d$. Often we will work with H considered as a family of Hamiltonians, though occasionally it will be useful to consider a particular configuration H_ω acting on $L^2(\mathbb{R}^d)$ for $\omega \in \Omega$.

We may also be interested in the case when the Hamiltonian H satisfies a CT-type symmetry. That is, H is time reversal and/or particle-hole and/or chiral symmetric. If we are interested in anti-linear symmetries on H, then this introduces a Real structure on the crossed product algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ via complex conjugation. As such we are interested in the topological invariants of the real subalgebra of $C(\Omega) \rtimes_\theta \mathbb{R}^d$ that is compatible with the symmetries under consideration. Such an algebra is still a real crossed product $C(\Omega) \rtimes_\theta \mathbb{R}^d$, but the anti-linear symmetries present may put limitations on the type of magnetic fields and twists θ that are possible.

Proposition 7.6 ([33, 83, 46, 53, 17]). *If the disordered family of Hamiltonians $\{H_\omega\}_{\omega\in\Omega}$ satisfies a CT-type symmetry and retains a spectral gap for all $\omega \in \Omega$, then we can associate a class in $KO_n(C(\Omega) \rtimes_\theta \mathbb{R}^d)$ or $K_n(C(\Omega) \rtimes_\theta \mathbb{R}^d)$, where n is determined by the symmetry.*

We can pair the K-theory class from Proposition 7.6 with our complex or real unbounded Kasparov module

$$(18) \qquad \left(\mathcal{A}\hat{\otimes}C\ell_{0,d},\ L^2(\mathbb{R}^d, C(\Omega))_{C(\Omega)}\hat{\otimes}\bigwedge\nolimits^* \mathbb{R}^d,\ \sum_{j=1}^d X_j\hat{\otimes}\gamma^j \right).$$

Furthermore, the invariant trace $\tau_{\mathbf{P}}$ on $C(\Omega)$ allows us to construct the semifinite spectral triple

$$\left(\mathcal{A}\hat{\otimes}C\ell_{0,d},\ L^2(\mathbb{R}^d) \otimes L^2(\Omega, \mathbf{P})\hat{\otimes}\bigwedge\nolimits^* \mathbb{R}^d,\ \sum_{j=1}^d X_j\hat{\otimes}\gamma^j \right)$$

relative to the trace $\mathcal{T}$ and von Neumann algebra $L^\infty(A, \mathcal{T})$. This spectral triple is smoothly summable with spectral dimension d.

Remark 7.7. Let us briefly comment on the possible K-theoretic phases encoded by the group $KO_n(C(\Omega) \rtimes_\theta \mathbb{R}^d)$ or $K_n(C(\Omega) \rtimes_\theta \mathbb{R}^d)$ from Proposition 7.6. We note that the (twisted) d-fold Connes–Thom isomorphism can be implemented by the Kasparov product with the Kasparov module from Equation (18). Hence we have the explicit isomorphism $KO_n(C(\Omega) \rtimes_\theta \mathbb{R}^d) \cong KO_{n-d}(C(\Omega))$ (similarly complex).

Let us consider the computation of $KO_{n-d}(C(\Omega))$ in a few simple cases. If Ω is contractible, then $KO_{n-d}(C(\Omega)) \cong KO_{n-d}(\mathbb{R})$. If Ω a totally disconnected space, then by the continuity of the K-functor $KO_{n-d}(C(\Omega)) \cong C(\Omega, KO_{n-d}(\mathbb{R}))$. Outside of these examples, the computation of the K-theory of the configuration space is much more involved. For continuous hulls Ω that come from certain tilings, Savinien and Bellissard construct a spectral sequence that converges to $K_{n-d}(C(\Omega))$ and whose page-2 is isomorphic to the integer Čech cohomology of Ω [78]. Hence the computation of $KO_n(C(\Omega) \rtimes_\theta \mathbb{R}^d)$ (or complex) in general is a highly non-trivial problem which we will not consider here.

7.2.1. *Complex invariants.* For a disordered Hamiltonian $H = \{H_\omega\}_{\omega\in\Omega}$ without additional symmetries, the K-theory class of interest is the Fermi projection $P_\mu = \chi_{(-\infty,\mu]}(H)$. If $P_\mu \in \mathcal{A}_{\text{Sob}}$, the Sobolev algebra from Section 5, then the index pairing with P_μ is well-defined. We will show in Section 7.4 that $P_\mu \in \mathcal{A}_{\text{Sob}}$ when the Fermi energy μ is in a region of dymanical localisation (using results from [1]). If μ is in a gap in the spectrum, then $P_\mu \in \mathcal{A}$, the smooth subalgebra of $C(\Omega) \rtimes_\theta \mathbb{R}^d$.

If the Hamiltonian is chiral symmetric, then H is invertible and there is a self-adjoint complex unitary R_c such that $R_c H R_c^* = -H$. Diagonalising R_c if necessary, this implies that H can be

written as an off-diagonal matrix. We can use the chiral symmetry to define the so-called Fermi unitary U_μ, where

$$1 - 2P_\mu = \mathrm{sgn}(H) = \begin{pmatrix} 0 & U_\mu^* \\ U_\mu & 0 \end{pmatrix}.$$

If H is invertible, then $U_\mu = \frac{1}{2}(1 - R_c)(1 - 2P_\mu)\frac{1}{2}(1 + R_c) \in \mathcal{A}^\sim$ is a 'smooth' unitary provided $R_c \in \mathcal{A}^\sim$. We can also consider a more general setting where $P_\mu \in \mathcal{A}_{\mathrm{Sob}}$ and $R_c \in \mathcal{A}_{\mathrm{Sob}}^\sim$; then $U_\mu = \frac{1}{2}(1 - R_c)(1 - 2P_\mu)\frac{1}{2}(1 + R_c) \in \mathcal{A}_{\mathrm{Sob}}^\sim$. In either setting we we obtain a K-theory class $[U_\mu] \in K_1(\mathcal{A})$ or $[U_\mu] \in K_1(\mathcal{A}_{\mathrm{Sob}})$. Other assumptions are also possible to ensure that U_μ is a well-defined unitary in $\mathcal{A}_{\mathrm{Sob}}^\sim$.

If $P_\mu \in M_q(\mathcal{A}_{\mathrm{Sob}})$ and $U_\mu \in M_q(\mathcal{A}_{\mathrm{Sob}}^\sim)$, then our semifinite pairing gives the cyclic expressions

$$\langle [U_\mu], [X] \rangle = C_d \sum_{\sigma \in S_d} (-1)^\sigma \, (\mathrm{Tr}_{\mathbb{C}^q} \otimes \mathcal{T}) \Big(\prod_{i=1}^{d} U_\mu^* \partial_{\sigma(i)} U_\mu \Big), \quad d \text{ odd},$$

$$\langle [P_\mu], [X] \rangle = \frac{(-2\pi i)^{d/2}}{(d/2)!} \sum_{\sigma \in S_d} (-1)^\sigma \, (\mathrm{Tr}_{\mathbb{C}^q} \otimes \mathcal{T}) \Big(P_\mu \prod_{i=1}^{d} \partial_{\sigma(i)} P_\mu \Big), \quad d \text{ even}$$

where, we recall, $C_{2n+1} = \frac{2(2\pi i)^n n!}{(2n+1)!}$ and $(\partial_j f)(x; \omega) = x_j f(x; \omega)$ with the property that $\pi_\omega(\partial_j f) = [X_j, \pi_\omega(f)]$ for all $\omega \in \Omega$ and $f \in C_c(\mathbb{R}^d, C(\Omega))$ (and then extended to $\mathcal{A}_{\mathrm{Sob}}$).

We require that the probability measure $\mathbf{P}$ on Ω is invariant under the action of magnetic translations. If we also assume that the $\mathbf{P}$ is ergodic under the twisted $\mathbb{R}^d$-action, then by Proposition 7.5 we can write

$$\langle [U_\mu], [X] \rangle = C_d \sum_{\sigma \in S_d} (-1)^\sigma \, (\mathrm{Tr}_{\mathbb{C}^q} \otimes \mathrm{Tr}_{\mathrm{Vol}}) \Big(\prod_{j=1}^{d} \pi_\omega(U_\mu)^* [X_{\sigma(j)}, \pi_\omega(U_\mu)] \Big), \quad d \text{ odd},$$

$$\langle [P_\mu], [X] \rangle = \frac{(-2\pi i)^{d/2}}{(d/2)!} \sum_{\sigma \in S_d} (-1)^\sigma \, (\mathrm{Tr}_{\mathbb{C}^q} \otimes \mathrm{Tr}_{\mathrm{Vol}}) \Big(\pi_\omega(P_\mu) \prod_{j=1}^{d} [X_{\sigma(j)}, \pi_\omega(P_\mu)] \Big), \quad d \text{ even}$$

for almost all $\omega \in \Omega$. By Theorem 5.11, the pairings for ergodic measures are almost surely integer valued and constant in ω. Furthermore, the formulas for the pairing now involve a specific configuration H_ω acting on the concrete Hilbert space $L^2(\mathbb{R}^d)$ with physical trace $\mathrm{Tr}_{\mathrm{Vol}}$. Therefore our cyclic formulas can be linked to physical phenomema more easily. It is shown in [71, Chapter 5, 7] that, for $A = C(\Omega) \rtimes_\theta \mathbb{Z}^d$, the Chern number formulas can be linked to transport coefficients of the linear conductivity tensor of solid state systems. The link between lower order coefficients of the conductivity tensor and our cyclic pairings in the continuous case has been studied in [29].

7.2.2. *Real invariants.* For non-torsion invariants, we can pair the semifinite spectral triple with a class in $KO_d(A)$ to obtain a number in $(\tau_{\mathbf{P}})_*[KO_0(C(\Omega))]$. If the class in $KO_0(C(\Omega))$ can be represented by a matrix-valued function $g \in M_q(C(\Omega))$, then the induced trace is just point-wise integration

$$\int_\Omega \mathrm{Tr}_{\mathbb{C}^q}(g(\omega)) \, d\mathbf{P}(\omega).$$

Hence the non-torsion invariants that arise are averaged over the disorder. Similarly, we can apply the induced trace to non-torsion elements in $KO_4(C(\Omega)) \cong KO_0(\mathbb{H} \otimes C(\Omega))$ provided the measure on Ω is sufficiently compatible with the quaternionic structure that comes from classes in KO_4.

For products that give rise to torsion invariants in $KO_0(C(\Omega))$, the semifinite pairing will be trivial as it involves taking the trace, a functional that will vanish on elements of finite order. Furthermore, the semifinite pairing is not well-defined for products that take value in $KO_{n-d}(C(\Omega))$ for $n-d$ not 0 or 4 (mod 8). In such circumstances, we instead take the Kasparov

product of the K-theory class of the Hamiltonian with the Kasparov module from Equation (18),

$$KO_n(A) \times KKO^d(A, C(\Omega)) \to KKO(C\ell_{n,d}, C(\Omega)) \cong KO_{n-d}(C(\Omega)).$$

The class in $KO_{n-d}(C(\Omega))$ is encoded via a Clifford index, analogous to the approach of [4]. Using the unbounded representative of the Kasparov product, $(C\ell_{n,d}, E_{C(\Omega)}, \tilde{X})$, this Clifford index is given by the equivalence class of $\mathrm{Ker}(\tilde{X})$ as a graded $C\ell_{n,d}$-module (when this makes sense). Analytic formulas for this index can be written down in concrete examples of Hamiltonians, see [17, Section 4.1] for example.

If the class in $KO_k(C(\Omega))$ with $k \neq 0, 4$ is non-torsion, we can also obtain 'disorder averaged' invariants by first noting that

$$KO_k(C(\Omega)) \cong KKO(\mathbb{R}, C(\Omega)\hat{\otimes}C\ell_{0,k}) \cong KKO(C\ell_{k,0}, C(\Omega))$$

and then applying the graded trace on $C(\Omega)\hat{\otimes}C\ell_{0,k}$. If we wish to avoid graded traces, then we can use Bott periodicity in KKO-theory to relate $KKO(\mathbb{R}, C(\Omega)\hat{\otimes}C\ell_{0,k}) \cong KKO(\mathbb{R}, C(\Omega) \otimes C_0(\mathbb{R}^k))$, which comes via the product with the Bott class

$$\left(C\ell_{0,k}, \, C_0(\mathbb{R}^k)_{C_0(\mathbb{R}^k)}\hat{\otimes}\bigwedge\nolimits^* \mathbb{R}^k, \, \sum_{j=1}^k x_j\hat{\otimes}\gamma^j \right),$$

see [43, §5]. The trace on $C(\Omega) \otimes C_0(\mathbb{R}^k)$ is easier to understand, but using this trace requires that we have to take another product. A concrete expression for this pairing would depend on the specific symmetries of the Hamiltonian that feed into the construction of the K-theory class in Proposition 7.6; see [37] for similar examples.

Let us return to the case of strong phases defined via the Kasparov product and Clifford index, which encompasses the most general case. We assume for the time being that the Hamiltonian has a spectral gap and so we can work in the smooth algebra $\mathcal{A} \subset C(\Omega) \rtimes_\theta \mathbb{R}^d$ (the case of the Sobolev algebra and the link to dynamical localisation will be considered in Section 7.6). For a fixed $\omega \in \Omega$, the equivalence class of the Clifford module $\mathrm{Ker}(\tilde{X}_\omega)$ can be can be written as an analytic formula using the index map on skew-adjoint Fredholm operators considered in [5]. Suppose that T is an odd Fredholm operator on a graded Hilbert space $\mathcal{H} \cong \mathcal{H}_+ \oplus \mathcal{H}_-$ and T anti-commutes (graded-commutes) with a representation of $C\ell_{r,s}$ on $\mathcal{H}$. Then $\mathrm{Ker}(T)$ has the structure of a graded left $C\ell_{r,s}$-module to which we can be associate the analytic index

$$(19) \quad \mathrm{Index}_{r,s}(T) = \begin{cases} \dim_{\mathbb{R}} \mathrm{Ker}(T_+) - \dim_{\mathbb{R}} \mathrm{Ker}(T_+^*), & r - s = 0 \bmod 8, \\ \dim_{\mathbb{R}} \mathrm{Ker}(T_+) \bmod 2, & r - s = 1 \bmod 8, \\ \dim_{\mathbb{C}} \mathrm{Ker}(T_+) \bmod 2, & r - s = 2 \bmod 8, \quad T_\pm : \mathcal{H}_\pm \to \mathcal{H}_\mp. \\ \dim_{\mathbb{H}} \mathrm{Ker}(T_+) - \dim_{\mathbb{H}} \mathrm{Ker}(T_+^*), & r - s = 4 \bmod 8, \\ 0, & \text{otherwise,} \end{cases}$$

In particular, suppose the Kasparov module $(C\ell_{n,d}, \mathcal{H}, \tilde{X}_\omega)$ represents the product of $[H] \in KO_n(\mathcal{A})$ (from Proposition 7.6) with the class of the spectral triple $[\lambda_d(\omega)] \in KO^d(\mathcal{A})$ that comes from the evaluation map. Then the Clifford module valued index $[\mathrm{Ker}(\tilde{X}_\omega)]$ will be non-trivial (where the class is zero if $\mathrm{Ker}(\tilde{X}_\omega)$ comes from the restriction of a left $C\ell_{n+1,d}$-module) if and only if $\mathrm{Index}_{n,d}\big((\tilde{X}_\omega)_+\big)$ is non-zero.

The above arguments can be extended to indices taking value in $KO_{n-d}(C(\Omega))$, see [79, Section 2.2]. Let us also remark that we can also define real indices with values in $KO_{n-d}(C(\Omega))$ using the local-global principle, provided that the $\mathbb{R}^d$ action is ergodic. The most useful way to view this index is as the class of a right $C(\Omega)$-module (so sections of a real vector bundle over Ω) with a left $C\ell_{n,d}$ action.

7.2.3. *Example: The disordered Kane–Mele model.* We consider the famous Kane–Mele model for two-dimensional topological insulators with fermionic time-reversal symmetry. The Hamiltonian of interest is

$$H^\omega_{KM} = \begin{pmatrix} h_\omega & g^* \\ g & \mathcal{C}h_\omega\mathcal{C} \end{pmatrix},$$

where h_ω is a self-adjoint operator acting on $L^2(\mathbb{R}^2, \mathbb{C}^n)$, g is playing the role of the Rashba coupling in the continuous setting and $\mathcal{C}$ is (point-wise) complex conjugation on $L^2(\mathbb{R}^2, \mathbb{C}^n)$. The time reversal involution we consider is given by $R_T = \begin{pmatrix} 0 & -\mathcal{C} \\ \mathcal{C} & 0 \end{pmatrix}$. We see that for H^ω_{KM} to be time-reversal symmetric, then we require $g^* = -\mathcal{C}g\mathcal{C}$. In fact for K-theoretic purposes, it would also be sufficient for g to be sufficiently bounded by h_ω that we can homotopy (in the resolvent topology) to a Hamiltonian with $g = 0$.

We assume that the disorder on h_ω is such that $\{h_\omega\}_{\omega\in\Omega}$ is affiliated to $C(\Omega, M_n(\mathbb{C})) \rtimes \mathbb{R}^2$ (e.g. Ω is the hull of a non-periodic but absolutely continuous potential V) and so H_{KM} is affiliated to $C(\Omega, M_{2n}(\mathbb{C})) \rtimes \mathbb{R}^2$. We restrict to the time-reversal invariant and real subalgebra $A_{KM} = C(\Omega, M_{2n}(\mathbb{R})) \rtimes \mathbb{R}^2$.

Let us assume for the time being that H^ω_{KM} has a gap for all $\omega \in \Omega$. Then the Fermi projection $P_\mu \in C(\Omega, M_{2n}(\mathbb{R})) \rtimes \mathbb{R}^2$. If we incorporate the extra structure $R_T P_\mu R_T^* = P_\mu$ and $R_T^2 = -1$, we obtain a projection compatible with a quaternionic structure and so a class $[P_\mu] \in KO_4(A_{KM})$. Following [17] this class is represented by the Kasparov module

$$\left(C\ell_{4,0}, \, P_\mu(A_{KM})^{\oplus 2}_{A_{KM}}, \, 0 \right),$$

where the $C\ell_{4,0}$-action comes from the equivalence between quaternionic spaces and $C\ell_{4,0}$-modules [33, Appendix B]. We pair this class with our unbounded Kasparov module

$$\lambda_2 = \left(C_c(\mathbb{R}^2, C(\Omega, M_{2n}))\hat{\otimes}C\ell_{0,2}, \, L^2(\mathbb{R}^2, C(\Omega, M_{2n}))_{C(\Omega,M_{2n})}\hat{\otimes}\bigwedge\nolimits^*\mathbb{R}^2, \, X = \sum_{j=1}^2 X_j\hat{\otimes}\gamma^j \right),$$

where, if we complexify this module, then we have that

$$(R_T \otimes 1_{\bigwedge^*\mathbb{C}^d})X(R_T \otimes 1_{\bigwedge^*\mathbb{C}^d})^* = \sum_{j=1}^2 R_T(X_j \otimes 1_{\mathbb{C}^{2n}})R_T^*\hat{\otimes}\gamma^j = \sum_{j=1}^2 \mathcal{C}X_j\mathcal{C} \otimes 1_{\mathbb{C}^{2n}}\hat{\otimes}\gamma^j = X.$$

The product of the two Kasparov modules gives the following,

$$\left(C\ell_{4,2}, \, P_\mu(L^2(\mathbb{R}^2, C(\Omega, M_{2n}))_{C(\Omega,M_{2n})})^{\oplus 2}\hat{\otimes}\bigwedge\nolimits^*\mathbb{R}^2, \, \sum_{j=1}^2 P_\mu(X_j \otimes 1)P_\mu\hat{\otimes}\gamma^j \right).$$

The topological information of interest is obtained in the class of $\mathrm{Ker}(P_\mu X P_\mu)$ considered as a Clifford module over $C\ell_{4,2}$.

By composing our pairing with the evaluation map at $\omega \in \Omega$, we can use the indices for skew-adoint Fredholm operators considered in [5] to obtain analytic formulas for the map

$$KO_4(A_{KM}) \times KKO^2(A_{KM}, C(\Omega, M_{2n})) \to KO_2(C(\Omega, M_{2n})) \xrightarrow{\mathrm{ev}_\omega} KO_2(\mathbb{R}) \cong \mathbb{Z}_2.$$

Taking this composition, we have that

$$\langle[P_\mu], [\lambda_2]\rangle(\omega) = \dim_\mathbb{C} \mathrm{Ker}\left(\pi_\omega(P_\mu)(X_1 \otimes 1\hat{\otimes}\gamma^1 + X_2 \otimes 1\hat{\otimes}\gamma_2)_+\pi_\omega(P_\mu) \right) \bmod 2$$
$$= \dim_\mathbb{C} \mathrm{Ker}\left(\pi_\omega(P_\mu)(X_1 \otimes 1 + iX_2 \otimes 1)\pi_\omega(P_\mu) \right) \bmod 2,$$

where in the last line we have chosen particular Clifford generators. We recognise this index as analogous to the indices considered in [80, 45].

It is shown in [45] that for the Kane–Mele model (without disorder), the defined analytic index agrees with the Kane–Mele invariant and is non-trivial. This result is proved in the discrete setting, but can be linked to our framework via the Bloch–Floquet transform for periodic potentials, see [74, Chapter XIII.16]. We also remark that the operator $\pi_\omega(P_\mu)(X_1 \otimes 1 + iX_2 \otimes$

1)$\pi_\omega(P_\mu)$ continues to be Fredholm if $P_\mu \in \mathcal{A}_{\text{Sob}}$. Hence, our $\mathbb{Z}_2$-valued index is still well-defined as a pairing over the Sobolev algebra. Then applying results from Section 7.4 and 7.6, our formula is still well-defined and constant in regions of dynamical localisation.

We note that if the Rashba coupling is zero, we have that $[\sigma_z, H_{KM}^\omega] = 0$ for all $\omega \in \Omega$. Therefore there is a decomposition of P_μ into $P_\mu^\pm$ corresponding to the $+1$ and -1 eigenspaces of the spin operator σ_z. From this point we can directly adapt results from [80, 47] to simplify the computation of the mod 2 index. Namely, provided that $P_\mu^\pm \in \mathcal{A}_{\text{Sob}}$, then using that the time-reversal involution R_T is such that $R_T\pi_\omega(P_\mu^\pm)R_T^* = \pi_\omega(P_\mu^\mp)$, we compute with F the phase of $X_1 \otimes 1 + iX_2 \otimes 1$ and such that $R_T F R_T^* = F^*$,

$$\dim_{\mathbb{C}} \text{Ker}(\pi_\omega(P_\mu)F\pi_\omega(P_\mu)) = \dim_{\mathbb{C}} \text{Ker}(\pi_\omega(P_\mu^+)F\pi_\omega(P_\mu^+)) + \dim_{\mathbb{C}} \text{Ker}(\pi_\omega(P_\mu^-)F\pi_\omega(P_\mu^-))$$

$$= \text{Index}(\pi_\omega(P_\mu^+)F\pi_\omega(P_\mu^+)) + \dim_{\mathbb{C}} \text{Ker}(\pi_\omega(P_\mu^+)F^*\pi_\omega(P_\mu^+))$$

$$+ \dim_{\mathbb{C}} \text{Ker}(\pi_\omega(P_\mu^-)F\pi_\omega(P_\mu^-))$$

$$= \text{Index}(\pi_\omega(P_\mu^+)F\pi_\omega(P_\mu^+)) + \dim_{\mathbb{C}} \text{Ker}(R_T\pi_\omega(P_\mu^+)F^*\pi_\omega(P_\mu^+)R_T^*)$$

$$+ \dim_{\mathbb{C}} \text{Ker}(\pi_\omega(P_\mu^-)F\pi_\omega(P_\mu^-))$$

$$= \text{Index}(\pi_\omega(P_\mu^+)F\pi_\omega(P_\mu^+)) + 2\dim_{\mathbb{C}} \text{Ker}(\pi_\omega(P_\mu^-)F\pi_\omega(P_\mu^-))$$

and so $\dim_{\mathbb{C}} \text{Ker}(P_\mu F P_\mu) \bmod 2 = \text{Index}(\pi_\omega(P_\mu^+)F\pi_\omega(P_\mu^+)) \bmod 2$ (the same formula is also true for $\pi_\omega(P_\mu^-)$). Because we have assumed $P_\mu^\pm \in \mathcal{A}_{\text{Sob}}$, we can use the cyclic formula for the index pairing to conclude that,

$$\langle [P_\mu], [\lambda_2] \rangle(\omega) = -2\pi i \, \mathcal{T}\big(\pi_\omega(P_\mu^\pm)\big[[X_1, \pi_\omega(P_\mu^\pm)], [X_2, \pi_\omega(P_\mu^\pm)]\big]\big) \bmod 2.$$

Similar results with less restrictive assumptions can be found in [47, Section 6].

We now consider the relation of our pairing to (real) Poincaré duality when there is no disorder. If there is no disorder and the potential is periodic, then using the Bloch–Floquet transform, the relevant observable algebra is (up to stabilisation) the real C^*-algebra $C(\{\text{pt}\}, \mathbb{R}) \rtimes \mathbb{Z}^2 \cong C^*(\mathbb{Z}^2) \cong C(i\mathbb{T}^2)$ with

$$C(i\mathbb{T}^2) = \left\{ f \in C(\mathbb{T}^2, \mathbb{C}) : \overline{f(k)} = f(-k) \right\}, \qquad KO_*(C(i\mathbb{T}^2)) \cong KR^{-*}(\mathbb{T}^2, \zeta)$$

with ζ the involution $k \mapsto -k$. The Kasparov module λ_2 is now just a real spectral triple for $C(i\mathbb{T}^2) \otimes Cl_{0,2}$ and can be extended to a spectral triple Λ_2 for $C(i\mathbb{T}^2) \otimes C(i\mathbb{T}^2)\hat{\otimes}Cl_{0,2}$ (via the diagonal map $\Delta : \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$). Hence we have a representative of Kasparov's fundamental class [44] and obtain a graded group isomorphism

$$\cdot \otimes_{C(i\mathbb{T}^2)} [\Lambda_2] : \ KO_*(C(i\mathbb{T}^2)) \to KO^{*-2}(C(i\mathbb{T}^2)).$$

Thus for every non-zero element $[x] \in KO_*(C(i\mathbb{T}^2))$ we have $[x] \otimes_{C(i\mathbb{T}^2)} [\Lambda_2] \neq 0$. Now pairing the K-homology class $[x] \otimes_{C(i\mathbb{T}^2)} [\Lambda_2]$ with $[1] \in KO_0(C(i\mathbb{T}^2))$ gives

$$[1] \otimes_{C(i\mathbb{T}^2)} ([x] \otimes_{C(i\mathbb{T}^2)} [\Lambda_2]) = [x] \otimes_{C(i\mathbb{T}^2)} [\lambda_2].$$

Using the Atiyah–Bott–Shapiro framework, the pairing on the right hand side is also computable as follows. One finds a representative $(Cl_{0,2}, \mathcal{H}, T)$ of the pairing and then regards the kernel of the operator T in $KO^{*-2}(C(i\mathbb{T}^2))$ as a graded left $Cl_{0,2}$-module [4]. Actually, given the structure of the product module, we actually start with $Cl_{4,2}$-modules, and have simply removed a $Cl_{2,2}$ module using the Morita equivalence $Cl_{2,2} \sim \mathbb{R}$. As noted above, this index lies in a group isomorphic to $\mathbb{Z}_2$, and is computable as a complex index computed mod 2.

It remains to see that we surject onto $KO_2(\mathbb{R})$. Recall that $KO_4(C(i\mathbb{T}^2)) \cong KR^{-4}(\mathbb{T}^2, \zeta) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Here $\mathbb{Z}$ is the quaternionic rank of the Bloch bundle $\{P_\mu(k)\}_{k \in \mathbb{T}^d}$ and $\mathbb{Z}_2$ is the Kane–Mele invariant. Since the Bloch bundle is quaternionic, it has even complex dimension, and thus the mod 2 kernel dimension is zero.

For the torsion generator, suppose that $\dim_{\mathbb{C}} \text{Ker}\big(\pi_\omega(P_\mu)(X_1 \otimes 1 + iX_2 \otimes 1)\pi_\omega(P_\mu)\big) \bmod 2 = 0$. If this index is trivial, then our $Cl_{4,2}$ module is the restriction of a $Cl_{5,2}$-module. Using that $Cl_{4,2} \cong Cl_{4,0}\hat{\otimes}Cl_{0,2}$, which is our decomposition of the index pairing, then the restriction of a

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$C\ell_{5,2}$-module implies that the original class $[P_\mu] \in KKO(C\ell_{4,0}, C(i\mathbb{T}^2))$ is the restriction of a $C\ell_{5,0}$ module (where we now refer to a Clifford module over a finitely generated and projective $C(i\mathbb{T}^2)$-module). We remark that the $C\ell_{5,2}$-module cannot come from a $Cl_{1,2}$ structure on λ_2 as $\mathrm{End}_\mathbb{R}(\bigwedge^* \mathbb{R}^2) \cong C\ell_{0,2}\hat{\otimes}C\ell_{2,0}$ and the entirety of $\mathrm{End}_\mathbb{R}(\bigwedge^* \mathbb{R}^2)$ is used in the construction of λ_2. If $[P_\mu]$ is the restriction of a $C\ell_{5,0}$-module, this implies that the class $[P_\mu] \in KO_4(C(i\mathbb{T}^2))$ is zero. The contrapositive of this argument then implies that a non-trivial $\mathbb{Z}_2$ component of the class $[P_\mu]$ will then give a non-trivial index pairing with λ_2. That is, the torsion part of $KO_4(A_{KM})$ is detected by λ_2, at least in the absence of disorder.

7.3. **The bulk-edge correspondence.** Kellendonk and Richard consider disordered systems with boundary using the short exact sequence coming from the Wiener–Hopf extension,

$$0 \to (C(\Omega) \rtimes_\theta \mathbb{R}^{d-1})\otimes\mathcal{K} \to \Big(C_0(\mathbb{R} \cup \{+\infty\}) \otimes C(\Omega) \rtimes_\theta \mathbb{R}^{d-1}\Big) \rtimes \mathbb{R} \to (C(\Omega) \rtimes_\theta \mathbb{R}^{d-1}) \rtimes \mathbb{R} \to 0,$$

see [48]. It is proved in the case $d = 2$ in [50] that a disordered Hamiltonian H_ω acting on $L^2(\mathbb{R}^{d-1} \times (-\infty, s])$ is affiliated to the algebra $\big(C_0(\mathbb{R} \cup \{+\infty\}) \otimes C(\Omega) \rtimes_\theta \mathbb{R}^{d-1}\big) \rtimes \mathbb{R}$ for any $s \in \mathbb{R}$. Hence we can think of the Wiener–Hopf algebra as representing the half-infinite system with boundary.

Recall the unbounded Kasparov module $[\lambda_d] \in KKO^d(C(\Omega) \rtimes_\theta \mathbb{R}^d, C(\Omega))$ that is used to derive the noncommutative Chern numbers and disordered Clifford indices from Section 7.2. Analogous to the discrete setting in [18], factorisation of this Kasparov module via the Wiener–Hopf extension (Theorem 6.3) means that our analytic indices can be written as pairings over the bulk or edge algebra. Up to the sign $(-1)^{d-1}$, the bulk and edge pairings coincide. In particular, non-trivial topological effects are present on the boundary if and only if non-trivial effects are present in the bulk.

For complex invariants, our Chern number formulas apply for both bulk and edge invariants. For dimensions 1, 2 and 3, work by [51, 71] explicitly links our edge pairings, $\langle\partial[U_\mu], [X_{d-1}]\rangle$ and $\langle\partial[P_\mu], [X_{d-1}]\rangle$, to the edge states, edge conductance or surface quantum Hall-like effect of a disordered Hamiltonian acting on a system with boundary (here ∂ is the boundary map in complex K-theory of the Wiener–Hopf extension).

For real invariants, while we have an explicit equivalence of the analytic bulk and edge pairings, the link to the physical system is much harder to interpret. This is particularly true for torsion invariants which often cannot be detected by any local formula (see for example the discussion in [6, p148]).

7.4. **Localisation of complex bulk invariants.** As a final step in our study of continuous models of disordered quantum systems, we connect the Sobolev index pairings considered in Section 5 to dynamical localised observables.

Recall $L^\infty(A, \mathcal{T})$, the $*$-algebra of weakly measurable families $\Omega \ni \omega \mapsto B_\omega \in \mathcal{B}[L^2(\mathbb{R}^2)]$ which satisfy the covariance condition $U_a B_\omega U_{-a} = B_{T_a\omega}$ for $a \in \mathbb{R}^d$. The algebra $L^\infty(A, \mathcal{T})$ is a von Neumann algebra under the norm $\|B\|_\infty = \mathbf{P}\text{-ess}\sup_\omega \|B_\omega\|_{\mathcal{B}[L^2(\mathbb{R}^d)]}$. We can also interpret this algebra as the von Neumann crossed product $L^\infty(\Omega, \mathbf{P}) \rtimes_\theta \mathbb{R}^d$ with elements as bounded measurable functions $g : \mathbb{R}^d \times \Omega \to \mathbb{C}$ (or $\mathbb{R}$). As before, self-adjoint elements H affiliated to $L^\infty(\Omega, \mathbf{P}) \rtimes_\theta \mathbb{R}^d$ represent a family $\{H_\omega\}_{\omega\in\Omega}$ acting on $L^2(\mathbb{R}^d) \otimes L^2(\Omega, \mathbf{P})$ with the realisations H_ω acting on $L^2(\mathbb{R}^d)$ for each $\omega \in \Omega$.

The study of localisation of observables in continuous models is considerably more complicated than its discrete counterpart. We will simply quote a result from [1] and apply it to our models of interest. Similar results can also be found in [34, 35]. We first note some notation. In the following theorem, we let $\chi_I(A)$ be the spectral projection of an operator A associated to the interval $I \subset \mathbb{R}$. The kinds of random potentials treated in [1] are very general, but not completely so. The characterisation of the class of potentials requires a fixed radius r as described in [1, Section 1.7]). Given this r, we denote by χ_x the characteristic function of a ball B_x^r centred at x of radius r.

Theorem 7.8 ([1], Theorem 1.1). *Let $\{H_\omega\}_{\omega \in \Omega}$ be a family of random magnetic Schrödinger operators satisfying regularity assumptions outlined in [1, Section 1.7]. Let Ξ be an open subset of $\mathbb{R}^d$, and Λ_n an increasing sequence of bounded open subsets of Ξ with $\cup \Lambda_n = \Xi$. Suppose that for some $0 < s < 1$ and an open bounded interval J there are constants $C < \infty$ and $m > 0$ such that*

$$
(20) \qquad \int_J \mathbb{E}\left(\left\| \chi_x (H_\omega^{(\Lambda_n)} - E)^{-1} \chi_y \right\|^s \right) \mathrm{d}E \leq C A e^{-m \operatorname{dist}_{\Lambda_n}(x,y)}
$$

for all $n \in \mathbb{N}$, $x, y \in \Lambda_n$. Then for every $v < 1/(2s)$ there exists $C_v < \infty$ such that, for all $x, y \in \Xi$,

$$
(21) \qquad \mathbb{E}\left(\sup_{g: |g| \leq 1} \left\| \chi_x g(H_\omega^{(\Xi)}) \chi_J(H_\omega^{(\Xi)}) \chi_y \right\| \right) \leq C_v e^{-vm \operatorname{dist}_\Xi(x,y)},
$$

where the supremum is taken over all Borel measurable functions g which satisfy $|g| \leq 1$ pointwise.

Applying Equation (21) to the operators $g_t(H_\omega) = e^{-itH_\omega}$ yields

$$
\mathbb{E}\left(\sup_t \left\| \chi_x e^{-itH_\omega^{(\Xi)}} \chi_J(H_\omega^{(\Xi)}) \chi_y \right\| \right) \leq C_v e^{-vm \operatorname{dist}_\Xi(x,y)},
$$

a strong version of dynamical localisation.

The Hamiltonians we consider for applications to topological phases will always be bounded from below and affiliated to $L^\infty(\Omega, \mathbf{P}) \rtimes_\theta \mathbb{R}^d$. Though we will occasionally require stricter assumptions.

Corollary 7.9. *Let H be affiliated to $L^\infty(\Omega) \rtimes_\theta \mathbb{R}^d$ and representing a family $\{H_\omega\}_{\omega \in \Omega}$ of disordered Hamiltonians. If $\{H_\omega\}_{\omega \in \Omega}$ satisfies the hypothesis of Theorem 7.8, then $\chi_{(-\infty, E]}(H) \in \mathcal{A}_{\mathrm{Sob}}$ for any E in the localised region $J \subset \mathbb{R}$. In particular, if the Fermi energy μ is in a mobility gap, then the Fermi projection $P_\mu \in \mathcal{A}_{\mathrm{Sob}}$.*

Proof. Applying the theorem, the operator P_μ has, on average over Ω, an exponentially decaying integral kernel (in particular, see [1, Equation (1.10)]). We note that $P_\mu = \tilde{\pi}(p_\mu)$ with $p_\mu \in L^\infty(\Omega, \mathbf{P}) \rtimes_\theta \mathbb{R}^d$. Therefore we check the Sobolev condition where by Theorem 7.8 there are strictly positive constants C_1 and C_2 such that

$$
\|p_\mu\|_{r,1} \leq C_r \int_{\mathbb{R}^d} \int_\Omega (1 + |x|)^r p_\mu(x; \omega) \, \mathrm{d}\mathbf{P}(\omega) \, \mathrm{d}x \leq C_1 C_r \int_{\mathbb{R}^d} (1 + |x|^2)^r e^{-C_2 |x|} \, \mathrm{d}x < \infty,
$$

and so is finite for any $r \in \mathbb{N}$. Next we use that p_μ is a projection so $p_\mu \in \mathcal{W}_{0,p}$ for any $p \in [1, \infty)$. The localisation bound can then again be used to give that $p_\mu \in \mathcal{W}_{r,p}$ for any r, p. $\square$

A key property of a dynamically-localised region of the spectrum $J \subset \sigma(H_\omega)$ (also called a mobility gap) is that the pure point spectrum of H_ω in J is $\mathbf{P}$-almost surely dense in J [1]. Considering the element H affiliated to $\mathcal{A}_{\mathrm{Sob}}$ represented by the family $\{H_\omega\}_{\omega \in \Omega}$, then with probability 1 the pure point spectrum of H in J is also dense in J. Therefore if $\mu \in J$, then $\mathbf{P}$-almost surely μ is a limit point of eigenvalues and $\mu \in \sigma_{\mathrm{ess}}(H)$.

A key success of the noncommutative geometry approach to the quantum Hall effect is the proof that the Hall conductance is constant within a mobility gap, proved in the discrete case in [9, Section 5]. The continuous analogue of this result is quite involved. We have not been able to resolve this question fully, but present a result for a random family of Hamiltonians $H = \{H_\omega\}_{\omega \in \Omega}$ bounded from below and affiliated to the C^*-algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ (*not* the von Neumann closure $L^\infty(\Omega, \mathbf{P}) \rtimes_\theta \mathbb{R}^d$).

Proposition 7.10 ([68], Proposition 3.31). *Let $H = \{H_\omega\}_{\omega \in \Omega}$ be a self-adjoint element that is bounded from below and affiliated to the C^*-algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ with mobility gap J. Then $G(h) \in \mathcal{A}_{\mathrm{Sob}}$ for every Borel function G with support in J.*

The proof in [68, Proposition 3.31] is for a different setting with a slightly different localisation bound, but we observe that the key argument continues to hold here. The main difference is the replacement of a sum with an integral in [68, Equation (3.63), (3.69)] and we use the bound from Equation (20) rather than [68, Equation (3.55)].

A key motivation for considering $\mathcal{A}_{\mathrm{Sob}}$ was to find a topology such that the Hochschild cocycle is still continuous, but deformations in a fixed mobility gap are also continuous. Because $\mathcal{A}_{\mathrm{Sob}}$ is defined using a tracial norm (which is weaker than the operator norm) but with strong regularity under the algebraic derivations, it is able to manage these two roles. The following result demonstrates this property.

Proposition 7.11. *Let $H = \{H_\omega\}_{\omega \in \Omega}$ be a random family of Hamiltonians affiliated to $C(\Omega) \rtimes_\theta \mathbb{R}^d$ with mobility gap J and bounded from below. Then The map $J \ni E \mapsto \chi_{(-\infty,E]}(H) \in \mathcal{A}_{\mathrm{Sob}}$ is continuous.*

Proof. We use the notation $P_E = \chi_{(-\infty,E]}(H)$ and $P_{[E,E']} = \chi_{[E,E']}(H)$. By Proposition 7.10, the spectral projections $P_{[a_n,b_n]} \in \mathcal{A}_{\mathrm{Sob}}$ for any $a_n, b_n \in J$. In particular, by the Borel functional calculus (see [73, Theorem VIII.5] for example) if $a_n \to a \in J$ and $b_n \to b \in J$, then $P_{[a_n,b_n]} \to P_{[a,b]}$ in the strong operator topology with $P_{[a,b]} \in \mathcal{A}_{\mathrm{Sob}}$.

Next we note that if p_n is a sequence of trace-class projections with $p_n \to p$ strongly with p trace-class, then $\mathrm{Tr}(p_n) \to \mathrm{Tr}(p)$. Because the Sobolev algebra contains trace-class elements (under the dual trace), we can say that $\mathrm{Tr}_\tau(P_{[a_n,b_n]}) \to \mathrm{Tr}_\tau(P_{[a_n,b_n]})$ and in particular $P_{[a_n,b_n]} \to P_{[a,b]}$ in $\mathcal{W}_{0,1}$ (the Sobolev space). Assuming that $P_{[a_n,b_n]} \preceq P_{[a,b]}$ (e.g. $[a_n,b_n] \subset [a,b]$ for all n), then we also have that $P_{[a_n,b_n]} \to P_{[a,b]}$ in $\mathcal{W}_{0,p}$. Because $P_{[a_n,b_n]}, P_{[a,b]} \in \mathcal{W}_{r,p}$ for any r, p and all n, the localisation bound and the convergence of $P_{[a_n,b_n]}$ in the trace norm ensures that $P_{[a_n,b_n]} \to P_{[a,b]}$ in $\mathcal{W}_{r,p}$ for all r, p. That is, deformations of the spectral projection of h within a fixed mobility gap J are continuous in the Sobolev topology. Hence, for $E, E' \in J$ with $E < E'$, then $\|P_{E'} - P_E\|_{\mathrm{Sob}} = \|P_{[E,E']}\|_{\mathrm{Sob}}$ can be controlled by $|E' - E|$. $\square$

By considering sufficiently regular symmetry operators, we can extend the above result to more general K-theoretic phases.

Proposition 7.12. *Let H be a Hamiltonian satisfying the assumptions of Proposition 7.11. Suppose further that H is CT symmetric with symmetry operators in $\mathcal{A}_{\widetilde{\mathrm{Sob}}}$ and symmetries uniform in the mobility gap. Then the K-theoretic phase of H (in the Sobolev topology) is constant in J.*

Proof. Consider a deformation that respects the symmetries of H and is uniform in the mobility gap. If H has no symmetries, then by Proposition 7.11, the deformation H_t is a homotopy of projections in $\mathcal{A}_{\mathrm{Sob}}$ and the phase is therefore constant. If H is CT-symmetric, then the K-theoretic phase is determied by the symmetry operators and P_μ or $\mathrm{sgn}(H)$ (with $0 \in J$). Because we can write $\mathrm{sgn}(H_t) = 1 - 2P_\mu(t)$ and the symmetries hold throughout J, we can again use Proposition 7.11 for symmetric deformations. $\square$

Let us summarise our results for complex pairings.

Corollary 7.13. *Let $H = \{H_\omega\}_{\omega \in \Omega}$ be a random family of Hamiltonians satisfying the assumptions of Proposition 7.11 and fix an ergodic measure $\mathbf{P}$ on Ω. Then $\mathbf{P}$-almost surely, the complex topological pairings from Section 7.2 are well defined, $\mathbb{Z}$-valued, constant in Ω and constant in a region of dynamical localisation.*

Proof. The pairing extends to $\mathcal{A}_{\mathrm{Sob}}$ by Theorem 5.9. If the measure is ergodic, then the cyclic valued is $\mathbb{Z}$-valued by Theorem 5.11. If we make a deformation within a mobility gap, this is continuous in the Sobolev topology by Proposition 7.11 and so the pairing is constant over this deformation by Theorem 5.11. $\square$

Hence we are able to obtain analogous results to those in [62, 9, 69, 70].

7.5. Delocalisation of complex edge states.

Our argument follows [71, Section 6.6]. Let us briefly review our basic setup as well as some additional assumptions we will require. We have the short-exact sequence

$$0 \to C(\Omega) \rtimes_\theta \mathbb{R}^{d-1} \otimes \mathcal{K} \to \mathcal{E} \to C(\Omega) \rtimes_\theta \mathbb{R}^d \to 0,$$

where H_s a disordered magnetic Schrödinger acting on $L^2(\mathbb{R}^{d-1} \times (-\infty, s])$ with Dirichlet boundary conditions and affiliated to $\mathcal{E} = \big(C_0(\mathbb{R} \cup \{+\infty\}) \otimes C(\Omega) \rtimes_\theta \mathbb{R}^{d-1}\big) \rtimes \mathbb{R}$ for every $s \in \mathbb{R}$. If H_s is chiral symmetric, we assume that the chiral involution R_c is sufficiently local so that R_c is in the minimal unitisation of (matrices of) $C(\Omega) \rtimes_\theta \mathbb{R}^d$, $\mathcal{E}$ and $C(\Omega) \rtimes_\theta \mathbb{R}^{d-1}$. Often $R_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and so this criterion is trivially satisfied here.

Lemma 7.14 (See [71], Section 4.3, or [39], Chapter 4)**.** *If the Fermi level is in a gap in the spectrum, then*

$$\partial[P_\mu] = [\exp(2\pi i f_{\exp}(H_s))] \in K_1(C(\Omega) \rtimes_\theta \mathbb{R}^{d-1} \otimes \mathcal{K})$$

with $f_{\exp}(H_s)$ a smooth non-decreasing function that is 0 below the spectral gap and 1 above the spectral gap.

If H is chiral symmetric with chiral unitary $U_\mu = \frac{1}{2}(1 - R_c)(1 - 2P_\mu)\frac{1}{2}(1 + R_c)$, then

$$\partial[U_\mu] = \big[e^{-i\frac{\pi}{2}f_{\mathrm{ind}}(H_s)}\frac{1}{2}(1 + R_c)e^{-i\frac{\pi}{2}f_{\mathrm{ind}}(H_s)}\big] - \big[\frac{1}{2}(1 + R_c)\big] \in K_0(C(\Omega) \rtimes_\theta \mathbb{R}^{d-1} \otimes \mathcal{K}).$$

with f_{ind} an odd and smooth non-decreasing function that is -1 below the spectral gap and $+1$ above the spectral gap.

We wish to consider our Chern number formulas for the edge pairings with $\partial[P_\mu]$ and $\partial[U_\mu]$ for a particular disorder space. Namely we take $\Omega \times [-L, L]$ for L large but finite. This space still has an $\mathbb{R}^{d-1}$-action and given the invariant measure $\mathbf{P}$ on Ω, we can extend $\mathbf{P}$ to $\tilde{\mathbf{P}}$, the product of $\mathbf{P}$ and (normalised) integration. The new measure $\tilde{\mathbf{P}}$ is still invariant under the $\mathbb{R}^{d-1}$-action on Ω and so defines an unbounded trace $\tilde{\mathcal{T}}$ on $C(\Omega \times [-L, L]) \rtimes_\theta \mathbb{R}^{d-1}$. As our general theorems only require an invariant tracial weight, our key Chern number results still apply for $\Omega \times [-L, L]$ and $\tilde{\mathbf{P}}$.

Theorem 7.15. *Suppose that the Fermi level μ is in a spectral gap $J \subset \mathbb{R}$ of the bulk Hamiltonian H, and adopt the notation of Theorem 7.8.*

(1) *Suppose that d is even and $\mathrm{Ch}_d(P_\mu)$ is non-zero. Let H_s be the Hamiltonian for the system with boundary. The localisation bound with respect to $\Omega \times [-L, L]$ of H_s,*

$$\int_J \mathbb{E}_{\tilde{\mathbf{P}}}\left(\left\|\chi_x\big((H_\omega)_s^{(\Lambda_n)} - E\big)^{-1}\chi_y\right\|^s\right) \mathrm{d}E \leq CAe^{-m\,\mathrm{dist}_{\Lambda_n}(x,y)},$$

cannot hold for large but finite L.

(2) *If d is odd and $\mathrm{Ch}_d(U_\mu)$ is non-zero, then a localisation bound of H_s cannot hold for large but finite L.*

The above result says that if our bulk-invariants are non-trivial, then the boundary spectrum of H_s cannot become localised by the addition of an arbitrarily thick surface layer.

Proof. Using the measure $\tilde{\mathbf{P}}$, Lemma 7.14 and the bulk-edge correspondence, we have the equality for d even

$$\mathrm{Ch}_d(P_\mu) = -C_{d-1} \sum_{\sigma \in S_{d-1}} (-1)^\sigma (\tilde{\mathcal{T}} \otimes \mathrm{Tr}_{L^2(\mathbb{R})}) \Big(\prod_{j=1}^{d-1} (e^{2\pi i f_{\exp}(H_s)})^* \partial_{\sigma(j)} e^{2\pi i f_{\exp}(H_s)} \Big),$$

where we also take the trace over $L^2(\mathbb{R})$ as the image of the boundary map is represented by elements in $C(\Omega \times [-L, L]) \rtimes_\theta \mathbb{R}^{d-1} \otimes \mathcal{K}[L^2(\mathbb{R})]$. If the localisation bound were to hold for H_s, we could take a homotopy in $\mathcal{A}_{\mathrm{Sob}}$ from $f_{\exp}(H_s)$ to $1 - \chi_{(-\infty,\mu]}(H_s)$ without changing the right-hand side by Theorem 5.9. However, the real part of $e^{2\pi i(1-\chi_{(-\infty,\mu]}(H_s))}$ is precisely $\chi_{(-\infty,\mu]}(H_s)$

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and so the Fermi projection lifts to another projection, which implies that the image of P_μ under the boundary map will be trivial. Hence our boundary pairing must be trivial, which contradicts that $\mathrm{Ch}_d(P_\mu)$ is non-zero.

Similarly for d odd and non-trivial bulk pairing, if the localisation bound holds, then we can homotopy $f_{\mathrm{ind}}(H_s)$ to $\mathrm{sgn}(H_s)$ in the topology of $\mathcal{A}_{\mathrm{Sob}}$. Then, because H_s is chiral symmetric, $\mathrm{sgn}(H_s)$ anti-commutes with R_c and so $e^{-i\frac{\pi}{2}\mathrm{sgn}(H_s)}\frac{1}{2}(1+R_c)e^{-i\frac{\pi}{2}\mathrm{sgn}(H_s)} = \frac{1}{2}(1+R_c)$ Hence $\partial[U_\mu]$ will be trivial and the edge pairing will vanish, again contradicting the assumption on the bulk pairing. $\qquad\square$

We remark that the hypotheses of Theorem 7.15 still require a spectral gap of the bulk Hamiltonian H affiliated to $C(\Omega) \rtimes_\theta \mathbb{R}^d$. This is because we used the boundary map in K-theory associated to the Wiener–Hopf extension, and we do not have an analogous extension for the Sobolev algebra $\mathcal{A}_{\mathrm{Sob}}$. A weakening of this assumption as in [31, 36] for the discrete case, while desirable, is beyond the scope of this manuscript.

7.6. **Real pairings and localisation.** In general our real pairings are Clifford indices representing classes in $KO_{j-d}(C(\Omega))$ and cannot be detected by the semifinite index unless $j = d$ or $j - d = 4$ with the measure on Ω compatible with the quaternionic structure. Instead we fix an ergodic measure on Ω and consider the analytic pairing with the family $\{\widetilde{\lambda}_d(\omega)\}_{\omega\in\Omega}$ considered in Section 5.1 with

$$\widetilde{\lambda}_d(\omega) = \left(\mathcal{A}_{\mathrm{Sob}}\hat{\otimes}C\ell_{0,d}, \,_{\tilde{\pi}_\omega}L^2(\mathbb{R}^d)\hat{\otimes}\bigwedge\nolimits^*\mathbb{R}^d, \, X = \sum_{j=1}^d X_j\hat{\otimes}\gamma^j\right),$$

where we recall that for any $f \in \mathcal{A}_{\mathrm{Sob}}$, $\widetilde{\pi}_\omega(f)(1+X^2)^{-1/2} \in \mathcal{L}^{(d,\infty)}[L^2(\mathbb{R}^d,\mathbb{R}^m)]$ **P**-almost surely (though this set of measure one depends on f). Our aim is to extend the analytic $\mathbb{Z}$ or $\mathbb{Z}_2$-valued skew-adjoint Fredholm indices from Equation (19) to the Sobolev algebra and topological phases in strong disorder. We make this link by using the K-theoretic phase $[H] \in KO_n(\mathcal{A}_{\mathrm{Sob}})$ (or a separable subalgebra) to endow the kernel of $\widetilde{\pi}_\omega(f)(1 + X^2)^{-1/2}$ with a left $C\ell_{n,d}$ Clifford module structure (**P**-almost surely).

Lemma 7.16. *Suppose that the Fermi energy lies in a mobility gap and the measure* **P** *on* Ω *is invariant and ergodic under the* $\mathbb{R}^d$-action. *Let* $(C\ell_{n,0}, p\mathcal{A}_{\mathrm{Sob}\,\mathcal{A}_{\mathrm{Sob}}}^{\oplus N})$ *be a finitely-generated and projective* $\mathcal{A}_{\mathrm{Sob}}$ *module with a left* $C\ell_{n,0}$-action *that represents the* K-theory class $[H] \in KO_n(\mathcal{A}_{\mathrm{Sob}})$ *from Proposition 7.6 (see [17, Section 3.3] for a construction of this class). Let* $p_\omega F_X p_\omega = \widetilde{\pi}_\omega(p)(X(1+X^2)^{-1/2} \otimes 1_N)\widetilde{\pi}_\omega(p)$ *with* X *the unbounded operator from* $\widetilde{\lambda}_d(\omega)$. *Then the skew-adjoint Fredholm index* $\mathrm{Index}_{n,d}\big((p_\omega F_X p_\omega)_+\big)$ *from Equation (19) is well-defined,* **P**-*almost surely constant, and yields a* $\mathbb{Z}$ *or* $\mathbb{Z}_2$-valued pairing of $[H]$ *with* $\{\widetilde{\lambda}_d(\omega)\}_{\omega\in\Omega}$.

Proof. As in Theorem 5.11, the ergodic assumption means that we only have to check constancy of the Clifford indices on an orbit in Ω. Using the covariance relation $F_X = X(1 + X^2)^{-1/2}$ is unitarily equivalent to F_{X+a} (modulo compacts) via the unitary U_a implementing translation on the product module $p\mathcal{H}$ (where, to be precise $\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{R}^N)\hat{\otimes}\bigwedge^*\mathbb{R}^d$). By the stability of the index pairings, the unitary transformation shows, along with the covariance described in Equation (17), that defined index is **P**-almost surely constant. $\qquad\square$

We have defined $\mathbb{Z}$ or $\mathbb{Z}_2$-valued analytic indices whenever the Fermi projector lies in the Sobolev algebra and we have an ergodic action. The torsion-valued indices can not be computed in a straightforward way using cyclic theory; see [47] for further progress on these questions.

Our last step is to consider the stability of the real Sobolev index pairings under perturbations within a mobility gap.

Proposition 7.17. *Let* $H = \{H_\omega\}_{\omega\in\Omega}$ *be a random family of Hamiltonians that is bounded from below and affiliated to* $C(\Omega) \rtimes_\theta \mathbb{R}^d$ *with a mobility gap* J *(in the sense of Theorem 7.8). Suppose* H *has a time reversal and/or particle-hole symmetry via an involution* $R_T, R_P \in \mathcal{A}_{\mathrm{Sob}}^\sim$. *If the Fermi energy is* μ *is in the mobility gap and the measure* **P** *on* Ω *is ergodic, then the* $\mathbb{Z}$

*and $\mathbb{Z}_2$-valued strong phases from Lemma 7.16 are well-defined and **P**-almost surely constant in Ω. Furthermore, these indices are invariant under deformations of the Hamiltonian that do not change the symmetries of the Hamiltonian and are uniform in the mobility gap.*

Proof. For $\mu \in J$, we know that $P_\mu \in \mathcal{A}_{\mathrm{Sob}}$ and, provided R_T or $R_P \in \widetilde{\mathcal{A}_{\mathrm{Sob}}}$, we can represent the K-theoretic phase of the Hamiltonian via the finitely generated and projective module $(C\ell_{n,0}, p\mathcal{A}_{\mathrm{Sob}}^{\oplus N})$ (following the construction in [17, Section 3.3]).[2] Then applying Lemma 7.16 we have that the pairing is well-defined and **P**-almost surely constant in Ω.

Consider now a deformation within the mobility gap J that respects the symmetries of H. By the additional assumptions on H and Proposition 7.12, this deformation is continuous in $\mathcal{A}_{\mathrm{Sob}}$. Hence, we consider the path $F_t^\omega = \widetilde{\pi}_\omega(p_t)(F_X \otimes 1_N)\widetilde{\pi}_\omega(p_t)$ that is continuous in $\mathcal{A}_{\mathrm{Sob}}$ (but will *not* be continuous in the operator norm). Then the difference $F_t^\omega - F_0^\omega \in \widetilde{\pi}_\omega(\mathcal{W}_{r,p})$ is **P**-almost surely compact and the defined analytic indices will not change along this path. □

Our argument for delocalisation of boundary states for complex topological phases in Section 7.5 relies on an explicit computation of the boundary map in K-theory of the symmetry class of the Hamiltonian. While we have constructed an explicit representative of the extension class in Proposition 6.1, a representative of $[H]\hat{\otimes}[\mathrm{ext}]$ in KO-theory expressed in terms of the Hamiltonian with boundary H_s is, in general, a much harder task than the complex case. Explicit formulas for boundary maps in real K-theory are known, see [14], but the passage from a disordered Schrödinger operator to a unitary with symmetries that is required for the Boersema–Loring picture is non-trivial. Hence we leave the delocalisation of edge states for non-trivial bulk pairings anti-linear symmetries as an open problem.

8. Concluding Remarks

We finish by making some brief comments on our results, their possible extensions and current limitations.

Throughout the paper, whenever the semifinite local index theorem is employed, we are restricted to complex algebras and spectral triples only. This is a large limitation as there are many materials of interest which have invariants arising from a non-torsion real pairing (see [37] for example). A local and cyclic expression for real pairings of the form

$$KO_d(B \rtimes_\theta \mathbb{R}^d) \times KKO^d(B \rtimes_\theta \mathbb{R}^d, B) \to KO_0(B) \xrightarrow{(\tau_B)_*} \mathbb{R}$$

should be possible, though the details of the proof of the even local index formula need to be carefully checked to see if they extend. Pairings that take value in $KO_j(B)$ for $j \neq 0$ could also be studied in this way, though this would require using the (graded) trace on the Clifford algebra or working with suspensions.

The lack of a local formula for torsion invariants, while unsurprising, means that an explicit link between torsion-valued pairings and physical phenomena is a key challenge for mathematical physicists interested in topological insulators. See [47] for recent progress.

In order to study the stability of phases under perturbations within a fixed mobility gap, we had to increase the hypothesis on our Hamiltonian so that it is affiliated to the C^*-algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ and not the von Neumann closure $L^\infty(\Omega, \mathbf{P}) \rtimes_\theta \mathbb{R}^d$. While a complete resolution of this somewhat technical problem may be quite difficult, perhaps analogous results to ours can be obtained for specific model Hamiltonians of interest to condensed matter physicists and with weaker affiliation hypotheses.

We have only considered the extension of bulk phases to strong disorder. The question of edge indices and the bulk-boundary correspondence in a mobility gap as in [31, 36] requires extra study. One immediate problem is whether there is an analogous version of the Wiener–Hopf extension for Sobolev algebra $\mathcal{A}_{\mathrm{Sob}}$. The description of K-theory using local Banach algebras as in [13] may offer further pathways to consider this problem.

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[2]Note that p need not be the Fermi projection, but may be a different projection constructed from P_μ.

Appendix A. Summary of non-unital index theory

In what follows we will assume that the algebras we deal with are separable. A useful exposition of KK-theory and its applications can be found in [13, 72] and [19] for the unbounded setting.

Given a real or complex C^*-module E_B over a $\mathbb{Z}_2$-graded C^*-algebra B, we denote by $\mathrm{End}_B(E)$ the algebra of adjointable endomorphisms of E subject to the B-valued inner-product $(\cdot \mid \cdot)_B$. The finite rank endomorphisms $\mathrm{End}_B^{00}(E)$ is the algebraic span of the operators Θ_{e_1,e_2} for $e_1, e_2 \in E$ such that

$$\Theta_{e_1,e_2}(e_3) = e_1 \cdot (e_2 \mid e_3)_B$$

with $e \cdot b$ the (possibly graded) right-action of B on E_B. The algebra of compact endomorphisms $\mathrm{End}_B^0(E)$ is the C^*-closure of $\mathrm{End}_B^{00}(E)$.

Definition A.1. Let A and B be real $\mathbb{Z}_2$-graded C^*-algebras. A real unbounded Kasparov module $(\mathcal{A}, {}_\pi E_B, D)$ is a $\mathbb{Z}_2$-graded real C^*-module E_B, a graded representation of A on E_B, $\pi : A \to \mathrm{End}_B(E)$, and an unbounded self-adjoint, regular and odd operator D such that for all $a \in \mathcal{A} \subset A$, a dense $*$-subalgebra,

$$[D, \pi(a)]_\pm \in \mathrm{End}_B(E), \qquad \pi(a)(1 + D^2)^{-1/2} \in \mathrm{End}_B^0(E).$$

For a complex Kasparov module, one simply replaces all spaces and algebras with complex ones. Where unambiguous, we will omit the representation π and write unbounded Kasparov modules as $(\mathcal{A}, E_B, D)$. The results of Baaj and Julg [7] continue to hold for real Kasparov modules, so given an unbounded module $(\mathcal{A}, E_B, D)$ we apply the bounded transformation to obtain the real Kasparov module $(A, E_B, D(1 + D^2)^{-1/2})$.

A.1. **Semifinite theory.** An unbounded A-$\mathbb{C}$ or A-$\mathbb{R}$ Kasparov module is precisely a complex or real spectral triple as defined by Connes. Complex spectral triples satisfying additional regularity properties have the advantage that the local index formula by Connes and Moscovici [28] gives computable expressions for the index pairing with K-theory, a special case of the Kasparov product

$$K_j(A) \times KK^j(A, \mathbb{C}) \to K_0(\mathbb{C}) \cong \mathbb{Z}.$$

We can extend this general framework by working with semifinite spectral triples.

Let τ be a fixed faithful, normal, semifinite trace on a von Neumann algebra $\mathcal{N}$. We let $\mathcal{K}_\mathcal{N}$ be the τ-compact operators in $\mathcal{N}$ (that is, the norm closed ideal generated by the projections $P \in \mathcal{N}$ with $\tau(P) < \infty$).

Definition A.2. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ relative to $(\mathcal{N}, \tau)$ is given by a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, a graded $*$-algebra $\mathcal{A} \subset \mathcal{N}$ with (graded) representation on $\mathcal{H}$ and a densely defined odd unbounded self-adjoint operator D affiliated to $\mathcal{N}$ such that

    (1) $[D, a]_\pm$ is well-defined on $\mathrm{Dom}(D)$ and extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$,

    (2) $a(1 + D^2)^{-1/2} \in \mathcal{K}_\mathcal{N}$ for all $a \in \mathcal{A}$.

For complex algebras and spaces, we can also remove the gradings, in which case the semifinite spectral triple is called odd (otherwise even).

If we take $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and $\tau = \text{Tr}$, then we recover the usual definition of a spectral triple.

Theorem A.3 ([42, 22]). *Let $(\mathcal{A}, \mathcal{H}, D)$ be a complex semifinite spectral triple associated to $(\mathcal{N}, \tau)$ with A the C^*-completion of $\mathcal{A}$. Then $(\mathcal{A}, \mathcal{H}, D)$ determines a class in $KK(A, C)$ with C a subalgebra of $\mathcal{K}_\mathcal{N}$. If $\mathcal{A}$ is separable, we can take C to be separable. If $(\mathcal{A}, \mathcal{H}, D)$ is odd, then the triple determines a class in $KK^1(A, C)$.*

A.1.1. *The semifinite index pairing.* Semifinite spectral triples $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}$ separable and ungraded can be paired with K-theory elements by the following composition

(22) $$K_j(A) \times KK^j(A, C) \to K_0(C) \xrightarrow{\tau_*} \mathbb{R},$$

with the class in $KK^j(A, C)$ coming from Theorem A.3. The image of the semifinite index pairing is a countably generated subset of $\mathbb{R}$ and, as such, can potentially detect finer invariants than the usual pairing of K-theory with K-homology. We call the map from Equation (22) the *semifinite index pairing* of a K-theory class with a semifinite spectral triple and is denoted $\langle [e], [(\mathcal{A}, \mathcal{H}, D)] \rangle$ for $[e] \in K_j(A)$.

We can also describe the semifinite index pairing analytically using the semifinite Fredholm index. Given a semifinite von Neumann algebra $(\mathcal{N}, \tau)$, an operator $T \in \mathcal{N}$ that is invertible modulo $\mathcal{K}_\mathcal{N}$ has semifinite Fredholm index

$$\text{Index}_\tau(T) = \tau(P_{\text{Ker}(T)}) - \tau(P_{\text{Ker}(T^*)}).$$

We can use the semifinite index to write down an analytic formula for the semifinite pairing. Let $\mathcal{A}^\sim = \mathcal{A} \oplus \mathbb{C}$ be the minimal unitisation of $\mathcal{A}$. Given $b \in M_n(\mathcal{A}^\sim)$ we let

$$\hat{b} = \begin{pmatrix} b & 0 \\ 0 & 1_b \end{pmatrix},$$

where $1_b = \pi^n(b)$ and $\pi^n : M_n(\mathcal{A}^\sim) \to M_n(\mathbb{C})$ is the quotient map coming from the unitisation.

Proposition A.4 ([22], Proposition 2.13). *Let $(\mathcal{A}, \mathcal{H}, D)$ be a complex semifinite spectral triple relative to $(\mathcal{N}, \tau)$ with $\mathcal{A}$ separable and D invertible. Let e be a projector in $M_n(\mathcal{A}^\sim)$, which represents $[e] \in K_0(\mathcal{A})$ and u a unitary in $M_n(\mathcal{A}^\sim)$ representing $[u] \in K_1(\mathcal{A})$. In the even case, define $T_\pm = \frac{1}{2}(1 \mp \gamma)T\frac{1}{2}(1 \pm \gamma)$ with γ the grading on $\mathcal{H}$. Then with $F = D|D|^{-1}$ and $P = (1 + F)/2$, the semifinite index pairing is represented by*

$$\langle [e] - [1_e], (\mathcal{A}, \mathcal{H}, D) \rangle = \text{Index}_{\tau \otimes \text{Tr}_{\mathbb{C}^{2n}}}(\hat{e}(F \otimes 1_{2n})_+ \hat{e}), \qquad \textit{even case,}$$

$$\langle [u], (\mathcal{A}, \mathcal{H}, D) \rangle = \text{Index}_{\tau \otimes \text{Tr}_{\mathbb{C}^{2n}}}((P \otimes 1_{2n})\hat{u}(P \otimes 1_{2n})), \qquad \textit{odd case.}$$

If D is not invertible, we take $m > 0$ and define the double spectral triple $(\mathcal{A}, \mathcal{H} \oplus \mathcal{H}, D_m)$ relative to $(M_2(\mathcal{N}), \tau \otimes \text{Tr}_{\mathbb{C}^2})$, where the operator D_m and the action of $\mathcal{A}$ is given by

$$D_m = \begin{pmatrix} D & m \\ m & -D \end{pmatrix}, \qquad \qquad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

for all $a \in \mathcal{A}$. If $(\mathcal{A}, \mathcal{H}, D)$ is graded by γ, then the double is graded by $\hat{\gamma} = \gamma \oplus (-\gamma)$. Doubling the spectral triple does not change the K-homology class and ensures that the unbounded operator D_m is invertible [27].

Remark A.5 (Semifinite spectral triples and torsion invariants). The pairing of Equation (22) is valid in both the complex and real setting. The pairing is, however, unhelpful in the case of torsion invariants as if $[x] \in K_0(C)$ has finite order, then $\tau_*([x]) = 0$. In particular, torsion invariants are common in real K-theory and play an important role in, for example, characterising the topological phase of a free-fermionic system [52]. In order to access torsion invariants we do not take the induced trace and consider the more general product

(23) $$K_j(A) \times KK^d(A, C) \to K_{j-d}(C)$$

for real or complex algebras. Hence we work with Kasparov modules directly. Unbounded Kasparov theory is a useful method for computing internal Kasparov products, as one must when the product represents a torsion class.

The image of the K-theoretic pairing from Equation (23) can be interpreted as a Clifford module, a finitely generated subspace of a countably generated C^*-module E_C with graded Clifford action. We can associate an analytic index to elements in $KO_{j-d}(C)$ or $K_{j-d}(C)$ via an analogue of Atiyah–Bott–Shapiro theory of Clifford modules, see [4, 17].

A.1.2. *Kasparov modules to semifinite spectral triples.* We can associate a Kasparov module to a semifinite spectral triple by Theorem A.3. One may ask if the converse is true. Given an unbounded Kasparov A-B module with B containing a faithful semifinite norm lower-semicontinuous trace (or tracial weight), we can often construct semifinite spectral triples using the dual trace construction (see Section 3 or [65] for a simple example).

The dual-trace method of constructing semifinite spectral triples has the advantage that the algebra B is often more closely related to the problem under consideration than the algebra C from Theorem A.3. In particular, the semifinite index pairing from Equation (22) can be rewritten with B in the place of C.

Given a sufficiently regular (complex) semifinite spectral triple from an unbounded Kasparov module, we may use the semifinite local index formula to compute the K-theoretic semifinite index pairing. As the local index formula is a cyclic expression involving traces and derivations, semifinite spectral triples and index theory can be employed in order to more easily compute pairings of K-theory classes with unbounded Kasparov modules as in Equation (22).

A.2. **Summability of non-unital spectral triples.** Spectral triples often contain more than just K-homological data. Hence we introduce extra structure on spectral triples that have the interpretation of a differential structure and measure theory. If the algebra is non-unital and non-local in the sense of [75], then we require the noncommutative measure theory developed in [21, 22]. Our brief exposition follows [30, Section 2]. In order to discuss smoothness and summability for non-unital spectral triples, we need to introduce an analogue of L^p-spaces for operators and weights over a semifinite von Neumann algebra $(\mathcal{N}, \tau)$.

Definition A.6. Let D be a densely defined self-adjoint operator affiliated to $\mathcal{N}$. Then for each $p \geq 1$ and $s > p$ we define a weight φ_s on $\mathcal{N}$ by

$$\varphi_s(T) = \tau\left((1 + D^2)^{-s/4} T (1 + D^2)^{-s/4}\right)$$

for T a positive element in $\mathcal{N}$. We define the subspace $\mathcal{B}_2(D, p)$ of $\mathcal{N}$ by

$$\mathcal{B}_2(D, p) = \bigcap_{s>p} \left(\mathrm{Dom}(\varphi_s)^{1/2} \bigcap (\mathrm{Dom}(\varphi_s)^{1/2})^*\right).$$

Take $T \in \mathcal{B}_2(D, p)$. The norms

$$\mathcal{Q}_n(T) = \left(\|T\|^2 + \varphi_{p+1/n}(|T|^2) + \varphi_{p+1/n}(|T^*|^2)\right)^{1/2}$$

for $n = 1, 2, \dots$ take finite values on $\mathcal{B}_2(D, p)$ and provide a topology on $\mathcal{B}_2(D, p)$ stronger than the norm topology. The space $\mathcal{B}_2(D, p)$ is a Fréchet algebra [22, Proposition 1.6] and can be interpreted as the bounded square integrable operators.

To introduce the bounded integrable operators, first take the span of products, $\mathcal{B}_2(D, p)^2$, and define the norms

$$\mathcal{P}_n(T) = \inf\left\{\sum_{i=1}^{k} \mathcal{Q}_n(T_{1,i}) \mathcal{Q}_n(T_{2,i}) \,:\, T = \sum_{i=1}^{k} T_{1,i} T_{2,i}, \ T_{1,i}, T_{2,i} \in \mathcal{B}_2(D, p)\right\},$$

where the sums are finite and the infimum is over all possible such representations of T. It is shown in [22, p12–13] that $\mathcal{P}_n$ are norms on $\mathcal{B}_2(D, p)^2$.

Definition A.7. Let D be a densely defined and self-adjoint operator and $p \geq 1$. We define $\mathcal{B}_1(D,p)$ to be the completion of $\mathcal{B}_2(D,p)^2$ with respect to the family of norms $\{\mathcal{P}_n \, : \, n = 1, 2, \ldots\}$.

Definition A.8. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ relative to $(\mathcal{N}, \tau)$ is said to be finitely summable if there exists $s > 0$ such that for all $a \in \mathcal{A}$, $a(1 + D^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N}, \tau)$. In such a case we let

$$p = \inf\{s > 0 \, : \, \forall a \in \mathcal{A}, \, \tau(|a|(1 + D^2)^{-s/2}) < \infty\}$$

and call p the spectral dimension of $(\mathcal{A}, \mathcal{H}, D)$.

Note that $|a|(1 + D^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N}, \tau)$ by the polar decomposition $a = v|a|$, which does not require $|a|$ to be in $\mathcal{A}$. For the definition of spectral dimension to have meaning, we require that $\tau(a(1 + D^2)^{-s/2}) \geq 0$ for $a \geq 0$, a fact that follows from [12, Theorem 3]. For a semifinite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to be finitely summable with spectral dimension p, it is a necessary condition that $\mathcal{A} \subset \mathcal{B}_1(D,p)$ [22, Proposition 2.17].

Definition A.9. Given a densely-defined self-adjoint operator D, set $\mathcal{H}_\infty = \bigcap_{k \geq 0} \mathrm{Dom}(D^k)$. For an operator $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$, we define

$$\delta(T) = [|D|, T], \qquad L(T) = (1 + D^2)^{-1/2}[D^2, T], \qquad R(T) = [D^2, T](1 + D^2)^{-1/2}.$$

One has that (cf. [28, 25])

$$\bigcap_{n \geq 0} \mathrm{Dom}(L^n) = \bigcap_{n \geq 0} \mathrm{Dom}(R^n) = \bigcap_{k,l \geq 0} \mathrm{Dom}(L^k \circ R^l) = \bigcap_{n \geq 0} \mathrm{Dom}(\delta^n).$$

We see that to define $\delta^k(T)$, we require that $T : \mathcal{H}_k \to \mathcal{H}_k$ for $\mathcal{H}_k = \bigcap_{l=0}^k \mathrm{Dom}(D^l)$.

Definition A.10. Let D be a densely defined self-adjoint operator affiliated to $\mathcal{N}$ and $p \geq 1$. Then define for $k = 0, 1, \ldots$

$$\mathcal{B}_1^k(D,p) = \left\{ T \in \mathcal{N} \, \middle| \, T : \mathcal{H}_l \to \mathcal{H}_l \text{ and } \delta^l(T) \in \mathcal{B}_1(D,p) \, \forall l = 0, \ldots, k \right\},$$

$$\mathcal{B}_2^k(D,p) = \left\{ T \in \mathcal{N} \, \middle| \, T : \mathcal{H}_l \to \mathcal{H}_l \text{ and } \delta^l(T) \in \mathcal{B}_2(D,p) \, \forall l = 0, \ldots, k \right\}$$

as well as

$$\mathcal{B}_1^\infty(D,p) = \bigcap_{k=0}^\infty \mathcal{B}_1^k(D,p), \qquad \mathcal{B}_2^\infty(D,p) = \bigcap_{k=0}^\infty \mathcal{B}_2^k(D,p).$$

For any k (including ∞), we equip $\mathcal{B}_1^k(D,p)$ with the topology induced by the seminorms

$$\mathcal{P}_{n,l}(T) = \sum_{j=0}^l \mathcal{P}_n(\delta^j(T))$$

for $T \in \mathcal{N}$, $l = 0, \ldots, k$ and $n \in \mathbb{N}$.

If we are interested in index theory in the non-compact setting, we need to control the integrability of both functions and their derivatives. The noncommutative analogue of this regularity turns out to be a finitely summable spectral triple but with additional smoothness properties.

Definition A.11. Let $(\mathcal{A}, \mathcal{H}, D)$ be a semifinite spectral triple relative to $(\mathcal{N}, \tau)$. We say that $(\mathcal{A}, \mathcal{H}, D)$ is QC^k-summable if it is finitely summable with spectral dimension p and

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \mathcal{B}_1^k(D,p).$$

We say that $(\mathcal{A}, \mathcal{H}, D)$ is smoothly summable if it is QC^k-summable for all $k \in \mathbb{N}$, that is

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \mathcal{B}_1^\infty(D,p).$$

For a smoothly summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we can introduce the δ-φ topology on $\mathcal{A}$ by the seminorms

$$(24) \qquad \mathcal{A} \ni a \mapsto \mathcal{P}_{n,k}(a) + \mathcal{P}_{n,k}([D, a])$$

for $n, k \in \mathbb{N}$. The completion of $\mathcal{A}$ in the δ-φ topology is Fréchet and closed under the holomorphic functional calculus [22, Proposition 2.20]. We finish this section with a sufficient and checkable condition of finite summability of spectral triples.

Proposition A.12 ([22], Proposition 2.16)**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a semifinite spectral triple. If $\mathcal{A} \subset \mathcal{B}_1^\infty(D, p)$ for some $p \geq 1$, then $(\mathcal{A}, \mathcal{H}, D)$ is finitely summable with spectral dimension given by the infimum of such p's. More generally, if $\mathcal{A} \subset \mathcal{B}_2(D, p)\mathcal{B}_2^{\lfloor p \rfloor + 1}(D, p)$ for $p \geq 1$, then $(\mathcal{A}, \mathcal{H}, D)$ is finitely summable with spectral dimension given by the infimum of such p's.*

Lemma 5.6 offers a slight variation of this result in order to get sharp results on localisation. Lemma 5.6 has essentially the same conclusion, though different statement and proof, as [21, Proposition 6.6].

A.3. **The local index formula.** We now briefly recall the semifinite local index formula from [22], which is an extension of previous formulas, [28, 76, 25, 26], to non-unital and non-local semifinite (complex) spectral triples. We note that the local index formula requires a smoothly summable semifinite spectral triple of finite spectral dimension. This may seem restrictive, but turns out to be satisfied in our examples.

To define the resolvent cocycle, we first establish the notation $R_s(\lambda) = (\lambda - (1 + s^2 + D^2))^{-1}$.

Definition A.13 ([22, 25, 26])**.** Let $(\mathcal{A}, \mathcal{H}, D)$ be a smoothly summable complex semifinite spectral triple relative to $(\mathcal{N}, \tau)$ with spectral dimension p. For $a \in (0, 1/2)$, let ℓ be the vertical line $\ell = \{a + iv : v \in \mathbb{R}\}$. We define the resolvent cocycle $(\phi_m^r)_{m=0}^M$ for $\Re(r) > (1 - m)/2$ as

$$\phi_m^r(a_0, \ldots, a_m) = \frac{\eta_m}{2\pi i} \int_0^\infty s^m \tau\left(\gamma \int_\ell \lambda^{-p/2-r} a_0 R_s(\lambda)[D, a_1]R_s(\lambda) \cdots [D, a_m]R_s(\lambda) \, \mathrm{d}\lambda\right) \mathrm{d}s,$$

where

$$\eta_m = \left(-\sqrt{2i}\right)^\bullet 2^{m+1} \frac{\Gamma(m/2 + 1)}{\Gamma(m + 1)}$$

with $\bullet = 0, 1$ depending on whether the spectral triple is even or odd.

The integral over ℓ is well-defined by [22, Lemma 3.3]. The index formula is a pairing of a cocycle with an algebraic chain. If $e \in \mathcal{A}^\sim$ is a projection, we define $\mathrm{Ch}^0(e) = e$ and for $k \geq 1$,

$$\mathrm{Ch}^{2k}(e) = (-1)^k \frac{(2k)!}{k!} (e - 1/2) \otimes e \otimes \cdots \otimes e \in (\mathcal{A}^\sim)^{\otimes(2k+1)}.$$

If $u \in \mathcal{A}^\sim$ is a unitary, then we define for $k \geq 0$

$$\mathrm{Ch}^{2k+1}(u) = (-1)^k k! \, u^* \otimes u \otimes \cdots \otimes u^* \otimes u \in (\mathcal{A}^\sim)^{\otimes(2k+2)}.$$

We split up the theorem into odd and even cases.

Theorem A.14 ([28, 76, 22])**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be an odd smoothly summable complex semifinite spectral triple relative to $(\mathcal{N}, \tau)$ and with spectral dimension p. Let $N = \lfloor \frac{p}{2} \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the floor function, and let u be a unitary in the unitisation of $\mathcal{A}$. The semifinite index pairing can be computed with the resolvent cocycle*

$$\langle [u], [(\mathcal{A}, \mathcal{H}, D)] \rangle = \frac{-1}{\sqrt{2\pi i}} \operatorname*{res}_{r=(1-p)/2} \sum_{m=1, \text{odd}}^{2N-1} \phi_m^r(\mathrm{Ch}^m(u))$$

and the function $r \mapsto \sum_{m=1, \text{odd}}^{2N-1} \phi_m^r(\mathrm{Ch}^m(u))$ analytically continues to a deleted neighbourhood of $r = (1 - p)/2$.

Theorem A.15 ([28, 76, 22]). *Let $(\mathcal{A}, \mathcal{H}, D)$ be an even smoothly summable complex semifinite spectral triple relative to $(\mathcal{N}, \tau)$ and with spectral dimension p. Let $N = \lfloor \frac{p+1}{2} \rfloor$ and $e \in \mathcal{A}^{\sim}$ be a self-adjoint projection. The semifinite index pairing can be computed by the resolvent cocycle*

$$\langle [e] - [1_e], [(\mathcal{A}, \mathcal{H}, D)] \rangle = \operatorname*{res}_{r=(1-p)/2} \sum_{m=0,\text{even}}^{2N} \phi_m^r(\mathrm{Ch}^m(e) - \mathrm{Ch}^m(1_e))$$

and the function $r \mapsto \sum_{m=0,\text{even}}^{2N} \phi_m(\mathrm{Ch}^m(e))$ analytically continues to a deleted neighbourhood of $r = (1-p)/2$.

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