THE EXTENSION CLASS AND KMS STATES FOR CUNTZ–PIMSNER ALGEBRAS OF SOME BI-HILBERTIAN BIMODULES

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Abstract. For bi-Hilbertian A-bimodules, in the sense of Kajiwara–Pinzari–Watatani, we construct a Kasparov module representing the extension class defining the Cuntz–Pimsner algebra. The construction utilises a singular expectation which is defined using the C^*-module version of the Jones index for bi-Hilbertian bimodules. The Jones index data also determines a novel quasi-free dynamics and KMS states on these Cuntz–Pimsner algebras.

1. Introduction

The Cuntz–Pimsner algebras introduced in [35] have attracted enormous attention over the last fifteen years (see, for example, [1, 6, 7, 8, 9, 13, 18, 19, 22, 24, 26, 28, 32, 37]). They are at once quite tractable and very general, including models for crossed products and Cuntz–Krieger algebras [35], graph C^*-algebras [12], topological-graph C^*-algebras [21], Exel crossed products [4], C^*-algebras of self-similar actions [33] and many others.

Particularly important in the theory of Cuntz–Pimsner algebras is the natural Toeplitz extension $0 \to \mathrm{End}^0_A(F_E) \to \mathcal{T}_E \to \mathcal{O}_E \to 0$ of a Cuntz–Pimsner algebra by the compact endomorphisms of the associated Fock module. For example, Pimsner uses this extension in [35] to calculate the K-theory of a Cuntz–Pimsner algebra using that $\mathrm{End}^0_A(F_E)$ is Morita equivalent to A and $\mathcal{T}_E$ is KK-equivalent to A. It follows that the class of this extension is important in K-theory calculations, and a concrete Kasparov module representing this class could be useful, for example, in exhibiting Poincaré duality for appropriate classes of Cuntz–Pimsner algebras.

When E is an imprimitivity bimodule, this is relatively straightforward (see Section 3.1) because the Fock representation of $\mathcal{T}_E$ is the compression of a natural representation of $\mathcal{O}_E$ on a 2-sided Fock module. But for the general situation, there is no such 2-sided module. Pimsner sidesteps this issue in [35] by replacing the coefficient algebra A with the direct limit A_∞ of the algebras of compact endomorphisms on tensor powers of E, and E with the direct limit E_∞ of the modules of compact endomorphisms from $E^{\otimes n}$ to $E^{\otimes n+1}$. This is an excellent tool for computing the K-theory of $\mathcal{O}_E$: the module E_∞ (rather than E itself) induces the Pimsner–Voiculescu sequence in K-theory, and the Cuntz–Pimsner algebra of E_∞ is isomorphic to that of E. But at the level of KK-theory, replacing E with E_∞ changes things dramatically. The Toeplitz extension associated to E_∞ corresponds to an element of $KK^1(\mathcal{O}_E, A_\infty)$, rather than of $KK^1(\mathcal{O}_E, A)$, and the two are quite different: for example, if E is the 2-dimensional Hilbert space, then $A = \mathbb{C}$, whereas $A_\infty = M_{2^\infty}(\mathbb{C})$, the type-$2^\infty$ UHF algebra.

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In this paper we consider the situation where E is a finitely generated bi-Hilbertian bimodule, in the sense of Kajiwara–Pinzari–Watatani [18], over a unital C^*-algebra. Our main result is a construction of a Kasparov-module representative of the class in $KK^1(\mathcal{O}_E, A)$ corresponding to the extension of $\mathcal{O}_E$ by $\mathrm{End}^0_A(F_E)$. We assume our modules E are both full and injective. This situation is quite common, and we present a range of examples; but much that we do could be extended to more general finite-index bi-Hilbertian bimodules, [18].

After introducing some basic structural features of the modules we consider in Section 2, we give a range of examples. We then examine the important special case of self-Morita equivalence bimodules (SMEBs), which include crossed products by $\mathbb{Z}$. This case was first calculated by Pimsner [35] in order to show that A and $\mathcal{T}_E$ are KK-equivalent. We present the details here for completeness.

For SMEBs we can produce an unbounded representative of the extension

$$0 \to \mathrm{End}^0_A(F_E) \to \mathcal{T}_E \to \mathcal{O}_E \to 0$$

defining $\mathcal{O}_E$. Here E is our correspondence, F_E the (positive) Fock space, and $\mathcal{T}_E$, $\mathcal{O}_E$ are the Toeplitz–Pimsner and Cuntz–Pimsner algebras, respectively.

Having an unbounded representative can simplify the task of computing Kasparov products. Since products with the class of this extension define boundary maps in K-theory and K-homology exact sequences, this representative is a useful aid to computing K-theory via the Pimsner–Voiculescu exact sequence. An application of this technique to the quantum Hall effect appears in [3].

For the general case of (finitely generated) bi-Hilbertian bimodules, we do not obtain an unbounded representative, but the construction of the right A-module underlying the Kasparov module is novel. Using the bimodule structure, we construct a one-parameter family $\Phi_s : \mathcal{T}_E \to A$, $\Re(s) > 1$, of positive A-bilinear maps. Provided the residue at $s = 1$ exists, we obtain an expectation $\Phi_\infty = \mathrm{res}_{s=1} \Phi_s : \mathcal{T}_E \to A$, which vanishes on the covariance ideal, and so descends to $\mathcal{O}_E$. We use Φ_∞ to construct an A-valued inner product on $\mathcal{O}_E$, and thereby obtain the underlying C^*-module in our $\mathcal{O}_E$–A-Kasparov module representing the extension class. We provide a criterion for establishing the existence of the desired residue in Proposition 3.5. We show that this criterion is readily checkable in some key examples; in particular, we show in Example 3.8 that the residue exists when E is the bimodule associated to a finite primitive directed graph.

The bimodule structure and Jones–Watatani index are essential ingredients in the construction of Φ_∞. The (right) Jones–Watatani index also provides a natural and interesting one-parameter family of quasi-free automorphisms of $\mathcal{O}_E$, and we show that there is a natural family of KMS states on $\mathcal{O}_E$ parameterised by the states on A which are invariant for the dynamics encoded by E. This construction combines ideas from [25] and [5].

There are also corresponding dynamics arising from the left Jones–Watatani index, and the product of the left and right indices. The corresponding collections of KMS states would also be interesting, but we do not address them here. The key point is that many important Cuntz–Pimsner algebras arise from bi-Hilbertian modules, and this extra structure gives rise to new tools that are worthy of study.

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2. A class of bimodules

Throughout this paper, A will denote a separable, unital, nuclear C^*-algebra. Given a right Hilbert A-module E (written E_A when we want to remember the coefficient algebra), we denote the C^*-algebra of adjointable operators by $\mathrm{End}_A(E)$, the compact endomorphisms by $\mathrm{End}^0_A(E)$ and the finite-rank endomorphisms by $\mathrm{End}^{00}_A(E)$. The finite-rank endomorphisms are generated by rank one operators $\Theta_{e,f}$ with e, $f \in E$.

Definition 2.1. Let A be a unital C^*-algebra. Following [18], a bi-Hilbertian A-bimodule is a full right C^*-A-module with inner product $(\cdot \mid \cdot)_A$ which is also a full left Hilbert A-module with inner product ${}_A(\cdot \mid \cdot)$ such that the left action of A is adjointable with respect to $(\cdot \mid \cdot)_A$ and the right action of A is adjointable with respect to ${}_A(\cdot \mid \cdot)$.

If E is a bi-Hilbertian A-bimodule, then there are two Banach-space norms on E, arising from the two inner-products. The following straightforward lemma shows that these norms are automatically equivalent.

Lemma 2.2. *Let E be a bi-Hilbertian A-bimodule. Then there are constants $c, C \in \mathbb{R}$ such that $\|(e \mid e)_A\| \leq c\|{}_A(e \mid e)\|$ and $\|{}_A(e \mid e)\| \leq C\|(e \mid e)_A\|$ for all $e \in E$.*

Proof. By symmetry it suffices to find c. Suppose that no such c exists. Then there is a sequence $e_n \in E$ such that $\|(e_n \mid e_n)_A\| > n\|{}_A(e_n \mid e_n)\|$. By normalising, we may assume that each $\|(e_n \mid e_n)_A\| = 1$, and hence each $\|{}_A(e_n \mid e_n)\| < \frac{1}{n}$. So $e_n \to 0$ in E, and then continuity of $(\cdot \mid \cdot)_A$ forces $\|(0 \mid 0)_A\| = 1$, contradicting the inner-product axioms. □

Throughout the paper, if we say that A is a finitely generated projective bi-Hilbertian A-bimodule, we mean that it is finitely generated and projective both as a left and as a right A-module.

The next lemma characterises when a right A-module has a left inner product for a second algebra. It provides a noncommutative analogue of 'the trace over the fibres' for endomorphisms of vector bundles.

For us, a *frame* for a right-Hilbert module E_A is a sequence $(e_i)_{i\in\mathbb{N}}$ of elements such that the series $\sum \Theta_{e_i,e_i}$ converges strictly to Id_E; note that this would be called a countable right basis in the terminology of [18], or a standard normalised tight frame in the terminology of [14]. As discussed in [18, Section 1], every countably generated Hilbert module E over a σ-unital C^*-algebra A admits a frame (in our sense), and it admits a finite frame if and only if $\mathrm{End}^0_A(E) = \mathrm{End}_A(E)$. As discussed in the remark following [18, Proposition 1.2], if (e_i) is a frame for E, then the net $\sum_{i\in F} \Theta_{e_i,e_i}(f)$ indexed by finite subsets F of $\{e_i\}$ converges to f for all $f \in E$.

A less general version of the following basic lemma appears in [29, Lemma 3.23].

Lemma 2.3 (cf. [18, Lemma 2.6])**.** *Let E_A be a countably generated right-Hilbert A-module, and let $B \subset \mathrm{End}_A(E)$ be a C^*-subalgebra.*

(1) *Suppose that ${}_B(\cdot \mid \cdot)$ is a left B-valued inner product for which the right action of A is adjointable. Then there is a B-bilinear faithful positive map $\Phi : \mathrm{End}^{00}_A(E) \to B$ such that $\Phi(\Theta_{e,f}) := {}_B(e \mid f)$ for all $e, f \in E$. For any frame (e_i) for E, we have*

$$\Phi(T) = \textstyle\sum_i {}_B(Te_i \mid e_i) \quad \text{for all } T \in \mathrm{End}^{00}_A(E).$$

(2) *Suppose that* $\Phi : \mathrm{End}_A^0(E) \to B$ *is a* B*-bilinear faithful positive map. Then* ${}_B(e \mid f) := \Phi(\Theta_{e,f})$ *defines a left* B*-valued inner product on* E *for which the right action of* A *is adjointable.*

Proof. (1) Choose a frame (e_i) for E. Given a rank-one operator $\Theta_{e,f}$, using the frame property at the last equality, we calculate:

$$\begin{aligned}\sum_i {}_B(\Theta_{e,f}e_i \mid e_i) &= \sum_i {}_B(e(f \mid e_i)_A \mid e_i) = \sum_i {}_B(e \mid e_i(e_i \mid f)_A)\\ &= \sum_i {}_B(e \mid \Theta_{e_i,e_i}f) = {}_B(e \mid f).\end{aligned}$$

It follows that there is a well-defined linear map $\Phi : \mathrm{End}_A^{00}(E) \to B$ satisfying $\Phi(\Theta_{e,f}) = {}_B(e \mid f)$ as claimed. The remaining properties of Φ follow from straightforward calculations. For example,

$$\Phi(b_1\Theta_{e,f}b_2) = \Phi(\Theta_{b_1e,b_2^*f}) = {}_B(b_1e \mid b_2^*f) = b_1{}_B(e \mid f)b_2 = b_1\Phi(\Theta_{e,f})b_2,$$

so Φ is B-bilinear. Positivity and faithfulness follow from the corresponding properties of the inner product.

(2) Given $\Phi : \mathrm{End}_A^0(E) \to B$, we define

$${}_B(e \mid f) := \Phi(\Theta_{e,f}).$$

Since $(e, f) \mapsto \Theta_{e,f}$ is a left $\mathrm{End}_A^0(E)$-valued inner-product on E, and since Φ is faithful and B-linear, it is routine to check that ${}_B(\cdot \mid \cdot)$ is positive definite. Since Φ is positive, it is $*$-preserving, and so ${}_B(e \mid f) = {}_B(f \mid e)^*$. Write φ for the homomorphism $B \to \mathrm{End}_A(E)$ that implements the left action. Then B-linearity of Φ gives

$$b{}_B(e \mid f) = b\Phi(\Theta_{e,f}) = \Phi(\varphi(b)\Theta_{e,f}) = \Phi(\Theta_{b\cdot e,f}) = {}_B(b \cdot e \mid f).$$

So Φ is a left B-valued inner product. For adjointability of the right A-action, observe that

$${}_B(e \cdot a \mid f) = \Phi(\Theta_{e\cdot a,f}) = \Phi(\Theta_{e,f\cdot a^*}) = {}_B(e \mid f \cdot a^*). \qquad \square$$

Remark 2.4. Unlike the Hilbert space case, the preceding result does not give any automatic cyclicity properties for the map Φ (which we might otherwise be tempted to regard as an operator-valued trace): for $e, f \in E$ and $U \in \mathrm{End}_A(E)$ unitary, we have

$$\Phi(\Theta_{e,f}U) = {}_B(e \mid U^*f) \quad \text{and} \quad \Phi(U\Theta_{e,f}) = {}_B(Ue \mid f).$$

The adjoint U^* in the first expression is the adjoint with respect to the inner-product $(\cdot \mid \cdot)_A$, which is the inverse of U. However, it is not clear that U^{-1} is an adjoint for U with respect to ${}_B(\cdot \mid \cdot)$, even assuming that U is adjointable for ${}_B(\cdot \mid \cdot)$.

Remark 2.5. Consider the (very) special case where A is commutative, E is a symmetric A-bimodule in the sense that $a \cdot e = e \cdot a$ for all $e \in E$, and ${}_A(e \mid f) = (f \mid e)_A$. Then the operator-valued weight associated to ${}_A(\cdot \mid \cdot)$ is a trace: given $\Theta_{e,f}$ and $\Theta_{g,h}$,

$$\Phi(\Theta_{e,f}\Theta_{g,h}) = \Phi(\Theta_{e(f|g)_A,h}) = {}_A(e(f \mid g)_A \mid h)$$

and

$$\Phi(\Theta_{g,h}\Theta_{e,f}) = {}_A(g(h \mid e)_A \mid f).$$

The following computation shows that these are equal.

$$\begin{aligned}{}_A(e(f \mid g)_A \mid h) = (h \mid e(f \mid g)_A)_A = (h \mid e)_A(f \mid g)_A = (f(e \mid h)_A \mid g)_A \\ = {}_A(g \mid f(e \mid h)_A) = {}_A(g(h \mid e)_A \mid f).\end{aligned}$$

Thus for vector bundles we recover the trace over the fibres of endomorphisms.

Remark 2.6. If $T \in \mathrm{End}^0_A(E)$ commutes with all $b \in B$ then $\Phi(T) \in \mathcal{Z}(B)$, because

$$b\Phi(T) = \Phi(bT) = \Phi(Tb) = \Phi(T)b.$$

2.1. **Examples.** We devote the remainder of this section to showing that many familiar and important classes of correspondences give rise to bi-Hilbertian bimodules of the type we consider.

2.1.1. *Self-Morita equivalence bimodules (SMEBs).* The following examples all share the important property that both the left and right endomorphism algebras are isomorphic to the coefficient algebra (or its opposite). This will turn out to be an important hypothesis, and also covers many important examples.

Definition 2.7. Let A be a C^*-algebra. A self-Morita equivalence bimodule (SMEB) over A is a bi-Hilbertian A-bimodule E whose inner products are both full and satisfy the imprimitivity condition

$${}_A(e \mid f)g = e(f \mid g)_A, \qquad \text{for all } e,\, f,\, g \in E.$$

Recall from [36, Proposition 3.8] that if E_A is a self-Morita equivalence A-bimodule, then $A \cong \mathrm{End}^0_A(E)$.

Example 2.8 (Crossed products by $\mathbb{Z}$). Suppose that A is unital and nuclear, and let $\alpha : A \to A$ be an automorphism. Then the C^*-correspondence ${}_\alpha A_A$ with the usual right module structure, left action of A determined by α and left inner product ${}_A(a \mid b) = \alpha^{-1}(ab^*)$ is a SMEB. The imprimitivity condition follows from the calculation

$$a \cdot (b \mid c)_A = ab^*c = \alpha(\alpha^{-1}(ab^*))c = {}_A(a \mid b) \cdot c.$$

Example 2.9 (Line bundles). Suppose that A is unital and commutative, so that $A \cong C(X)$ for some second-countable compact Hausdorff space X. Given a complex line bundle $L \to X$, we obtain a SMEB over A by setting $E = \Gamma(L)$, the continuous sections of L. The left and right actions are by pointwise multiplication, and any Hermitian form $\langle \cdot, \cdot \rangle$ on L determines inner products by ${}_A(e \mid f)(x) := \langle e(x), f(x) \rangle =: (f \mid e)_A$.

The next result shows that for SMEBs, the map Φ of Lemma 2.3 is an expectation.

Lemma 2.10. *Suppose that E is a SMEB over a unital C^*-algebra A. The map $\Phi : \mathrm{End}^0_A(E) \to A$ of Lemma 2.3(1) satisfies $\Phi(\mathrm{Id}_E) = 1_A$.*

Proof. Choose a frame (e_i) for E. Since E is a SMEB, [36, Proposition 3.8] says that the map $\Theta_{x,y} \mapsto {}_A(x \mid y)$ determines an isomorphism $\psi : \mathrm{End}^0_A(E) \to A$. In particular, ψ is unital, and so

$$1_A = \psi(\mathrm{Id}_E) = \psi\big(\sum_i \Theta_{e_i,e_i}\big) = \sum_i {}_A(e_i \mid e_i) = \Phi(\mathrm{Id}_E). \qquad \square$$

Conversely, Corollary 4.14 of [18] shows that a bi-Hilbertian bimodule E satisfies $\Phi(\mathrm{Id}_E) = 1_A$ if and only E can be given a left inner product which makes E into a SMEB.

2.1.2. *Crossed products by injective endomorphisms.* Let A be a unital C^*-algebra and suppose that $\alpha : A \to A$ is an injective unital $*$-endomorphism. Assume there exists a faithful conditional expectation $\Phi : A \to \alpha(A)$. Then $L := \alpha^{-1} \circ \Phi$ is a *transfer operator* [11, Definition 2.1]; that is, $L : A \to A$ is a positive linear map satisfying

$$L(\alpha(a)b) = aL(b)$$

for all $a, b \in A$.

There is a bi-Hilbertian A-bimodule associated to the triple (A, α, L) as follows: A is a pre-Hilbert A-bimodule with

$$a \cdot e \cdot b := ae\alpha(b)$$

and

$$(e|f)_A := L(e^* f)$$

for $a, b, e, f \in A$. Denote by E the completion of A for the norm $\|e\|^2 = \|(e|e)_A\|$. By faithfulness of Φ, there is a left inner-product

$${}_A(e|f) = ef^*$$

which is left A-linear and for which the right action of A on E is adjointable. The associated Cuntz–Pimsner algebra satisfies

$$\mathcal{O}_E = A \rtimes_{\alpha,L} \mathbb{N}$$

where $A \rtimes_{\alpha,L} \mathbb{N}$ is as defined by Exel [11].

2.1.3. *Vector bundles.* If $E \to X$ is a complex vector bundle over a compact Hausdorff space, then the $C(X)$-module $\Gamma(E)$ of all continuous sections under pointwise multiplication is finitely generated and projective for any nondegenerate $C(X)$-valued inner products (left and right). If we alter the left action by composing with an automorphism, we also need to alter the left inner product as in Example 2.8. If E is rank one then we are back in the SMEB case of Example 2.9.

2.1.4. *Topological graphs.* A topological graph is a quadruple $G = (G^0, G^1, r, s)$ where G^0, G^1 are locally compact Hausdorff spaces, $r : G^1 \to G^0$ is a continuous map and $s : G^1 \to G^0$ is a local homeomorphism. For simplicity, we will assume that r and s are surjective. Given a topological graph G, Katsura [21] associates a right Hilbert module as follows. Let $A = C_0(G^0)$. Then, similarly to Section 2.1.5, $C_c(G_1)$ is a right pre-Hilbert A-module with left and right actions

$$(a \cdot e \cdot b)(g) := a(r(g))\, e(g)\, b(s(g)), \quad e \in C_c(G^1), \quad a, b \in A, \quad g \in G^1$$

and inner product

$$(e|f)_A(v) = \sum_{s(g)=v} \overline{e(g)} f(g), \qquad e, f \in C_c(G^1), \quad v \in G^0.$$

(Since s is a local homeomorphism, $\{g \in vG^1 : e(g) \neq 0\}$ is finite for $e \in C_c(G^1)$, so this formula for the inner-product makes sense.) We write E for the completion of $C_c(G^1)$ in the norm determined by the inner-product, and E is a right Hilbert A-module.

To impose a left Hilbert module structure on E, we restrict attention to topological graphs where r is also a local homeomorphism, and define

$${}_A(e \mid f)(v) = \sum_{r(g)=v} e(g)\overline{f(g)}, \qquad e, f \in C_c(G^1), \quad v \in G^0.$$

For the remainder of this section, suppose that G^0 and G^1 are compact. The following lemma and its proof are due to Mitch Hawkins, [16].

Lemma 2.11. *Suppose that r, $s : G^1 \to G^0$ are local homeomorphisms. For each $n \in \mathbb{N}$, the sets $\{v \in G^0 : |G^1 v| = n\}$ and $\{w \in G^0 : |vG^1| = n\}$ are compact open.*

Proof. We show that $\{v \in G^0 : |G^1 v| = n\}$ is compact open; symmetry does the rest. It suffices to show that $\{v \in G^0 : |G^1 v| \geq n\}$ is both closed and open.

First suppose that v satisfies $|G^1 v| \geq n$. Fix distinct $e_1, \ldots, e_n \in G^1 v$. Since G^1 is Hausdorff, we can pick disjoint open neighbourhoods U_i of e_i. Since s is a local homeomorphism, we can shrink the U_i so that $s(U_i) = s(U_j)$ for all $i, j \leq n$ and so that $s|_{U_i}$ is a homeomorphism for each i. Since s is a local homeomorphism, it is an open map, and so $V = s(U_1)$ is an open neighbourhood of v. For each $v' \in V$ each $U_i v'$ is a singleton, and the U_i are mutually disjoint, so $|G^1 v'| \geq n$. Hence $V \subseteq \{v \in G^0 : |G^1 v| \geq n\}$, and we deduce that the latter is open.

We now show that it is also closed. Suppose that v_m is a sequence in G^0 converging to v, and suppose that each $|G^1 v_m| \geq n$. For each m, choose distinct elements $e_{m,1}, \ldots, e_{m,n}$ of $G^1 v_m$. Since G^1 is compact, by passing to a subsequence we may assume that each sequence $e_{m,i}$ converges to some $e_i \in G^1$. By continuity of s, we have $s(e_i) = v$ for each i, so it suffices to show that $i \neq j$ implies $e_i \neq e_j$. For this, fix a neighbourhood U of e_i on which s is a homeomorphism. Since $e_{m,i} \to e_i$, the $e_{m,i}$ eventually belong to U. Since each $s(e_{m,j}) = v_j = s(e_{m,i})$ and each $e_{m,j} \neq e_{m,i}$, we see that $e_{m,j} \notin U_i$ for large m. Since $e_{m,j} \to e_j$, we deduce that $e_j \notin U$, and in particular $e_j \neq e_i$. □

Corollary 2.12. *For $m, n \in \mathbb{N}$, let*

$$G^1_{m,n} := \{e \in G^1 : |r(e)G^1| = m \text{ and } |G^1 s(e)| = n\}.$$

Then the $G^1_{m,n}$ are compact open sets, as are $s(G^1_{m,n})$ and $r(G^1_{m,n})$.

Proof. We have $G^1_{m,n} = s^{-1}(\{v : |G^1 v| = n\}) \cap r^{-1}(\{w : |wG^1| = m\})$. Lemma 2.11 and continuity of s and r imply that $G^1_{m,n}$ is clopen; since G^1 is compact, each $G^1_{m,n}$ then also compact. Since r, s are local homeomorphisms, they are open maps, so $r(G^1_{m,n})$ and $s(G^1_{m,n})$ are open. They are compact as they are continuous images of the compact set $G^1_{m,n}$. □

Since r, s are local homeomorphisms, each edge e has a neighbourhood U_e on which both s and r are homeomorphisms. By the preceding corollary, we may assume that each $U_e \subseteq G^1_{|r(e)G^1|,|G^1 s(e)|}$. The U_e cover G^1, so by compactness, there is a finite open cover $\mathcal{U}$ such that each $U \in \mathcal{U}$ is contained in some $G^1_{m(U),n(U)}$. Choose a partition of unity on G^1 subordinate to $\mathcal{U}$; say $\{f_U : U \in \mathcal{U}\}$. So $0 \leq f_U \leq 1$ and $f_U \in C_0(U)$ for each $U \in \mathcal{U}$, and $\sum_{U \in \mathcal{U}} f_U(e) = 1$ for all $e \in G^1$.

Lemma 2.13. *For each $U \in \mathcal{U}$, define $h_U \in C(G^1)$ by $h_U(e) := \sqrt{f_U(e)}$. The collection $\{h_U : U \in \mathcal{U}\}$ is a frame for both the left and the right inner-product on $C(G^1)$. We have $\Phi(\mathrm{Id}_E)(v) = |vG^1|$ for all $v \in G^0$.*

Proof. The situation is completely symmetrical in r and s, so we just have to show that the f_U form a frame for the right inner-product. For this, we fix $g \in C(G^1)$ and $e \in G^1$ and calculate

$$\sum_{U \in \mathcal{U}} (\theta_{h_U, h_U} g)(e) = \sum_U h_U(e)(h_U \mid g)_{C(G^0)}(s(e)) = \sum_U \sum_{s(e')=s(e)} \sqrt{f_U(e)}\overline{\sqrt{f_U(e')}} g(e') \tag{2.1}$$

Since s restricts to a homeomorphism on each $U \in \mathcal{U}$, we can only have $f_U(e)$ and $f_U(e')$ simultaneously nonzero in the sum on the right-hand side of (2.1) if $e = e'$. Since f_U is real-valued, we have $\overline{\sqrt{f_U(e)}} = \sqrt{f_U(e)}$, and so

$$\sum_{U\in\mathcal{U}}(\theta_{h_U,h_U}g)(e) = \sum_{U\in\mathcal{U}} h_U(e)^2 g(e) = \Big(\sum_U f_U(e)\Big)g(e) = g(e).$$

This proves that the h_U constitute a frame. For the final assertion, we calculate

$$\begin{aligned}\Phi(\mathrm{Id}_E)(v) &= \sum_U {}_{C(G^0)}(h_U \mid h_U)(v) = \sum_U \sum_{r(e)=v} h_U(e)\overline{h_U(e)}\\ &= \sum_{r(e)=v}\sum_U f_U(e) = \sum_{r(e)=v} 1 = |vG^1|.\end{aligned}$$ □

2.1.5. *Cuntz–Krieger algebras.* As a specific case of the example above, suppose that $G = (G^0, G^1, r, s)$ is a finite directed graph where G^0 and G^1 both have the discrete topology. We suppose for simplicity that G has no sources and no sinks. If B is the edge-adjacency matrix of G, then the Cuntz–Pimsner algebra $\mathcal{O}_E$ of the right Hilbert A-module E_A is the Cuntz–Krieger algebra O_B [35, Example 2, page 193]. If we think of the left Hilbert A-module ${}_AE$ as a right Hilbert A^{op} module $E^{\mathrm{op}}_{A^{\mathrm{op}}}$ with $e^{\mathrm{op}}\cdot a^{\mathrm{op}} = (a\cdot e)^{\mathrm{op}}$ and $(e^{\mathrm{op}} \mid f^{\mathrm{op}})_{A^{\mathrm{op}}} = \big({}_A(f \mid e)\big)^{\mathrm{op}}$, then the Cuntz–Pimsner algebra $\mathcal{O}_{E^{\mathrm{op}}}$ is the Cuntz–Krieger algebra O_{B^t} given by the transpose of the matrix B, which is given by the graph G^{op} defined by reversing the edges of G.

2.1.6. *Twisted topological graphs.* The following construction is due to Li [27]. Suppose that $G = (G^0, G^1, r, s)$ is a topological graph. Let $N = \{N_\alpha : \alpha \in \Lambda\}$ be an open cover of G^1. Given $\alpha_1, \dots, \alpha_n \in \Lambda$, write $N_{\alpha_1\dots\alpha_n} = \bigcap_{i=1}^n N_{\alpha_i}$. A collection of functions

$$S = \{s_{\alpha\beta} \in C(\overline{N_{\alpha\beta}}, \mathbb{T}) : \alpha, \beta \in \Lambda\}$$

is called a 1-*cocycle relative to* N if $s_{\alpha\beta}s_{\beta\gamma} = s_{\alpha\gamma}$ on $\overline{N_{\alpha\beta\gamma}}$.

Let

$$C_c(G, N, S) := \Big\{x \in \prod_{\alpha\in\Lambda} C(\overline{N_\alpha}) : x_\alpha = s_{\alpha\beta}x_\beta \text{ on } \overline{N_{\alpha\beta}} \text{ and } \overline{x_\alpha}x_\alpha \in C_c(E^1)\Big\}$$

For $x \in C_c(G, N, S)$ and $g \in G^1$, we write $x(g)$ for the tuple $\big(x(g)_\alpha\big)_{\alpha\in\Lambda}$, with the convention that $x(g)_\alpha = 0$ when $g \notin N_\alpha$. Choose for each $g \in E^1$ an element $\alpha(g)$ such that $g \in N_{\alpha(g)}$; since the $s_{\alpha\beta}$ are circle valued, for $x, y \in C_c(G, N, S)$, the map $g \mapsto \overline{x(g)_{\alpha(g)}}y(g)_{\alpha(g)}$ does not depend on our choice of the assignment $g \mapsto \alpha(g)$. Now $C_c(G, N, S)$ becomes a pre-right-Hilbert $C_0(G^0)$-module under the operations

$$\begin{aligned}(x\cdot a)(g)_\alpha &= x(g)_\alpha a(s(g)),\\ (x \mid y)_A(v) &= \sum_{s(g)=v} \overline{x(g)}_{\alpha(g)}y(g)_{\alpha(g)}, \text{ and}\\ (a\cdot x)(g)_\alpha &= a(r(g))x(g)_\alpha\end{aligned}$$

for $x, y \in C_c(G, N, S)$, $a \in A$, $\alpha \in \Lambda$ and $v \in G^0$. Theorem 3.3 of [27] ensures that the completion $E(G, N, S)$ of $C_c(G, N, S)$ is a right-Hilbert A-bimodule.

If $r : G^1 \to G^0$ is a local homeomorphism, then there is a left inner-product on $E(G, N, S)$ satisfying

$$ {}_A(x \mid y)(v) = \sum_{r(g)=v} x(g)_{\alpha(g)}\overline{y(g)}_{\alpha(g)}, $$

which again does not depend on our choice of assignment $g \mapsto \alpha(g)$. The right action is adjointable for this left inner-product, and $E(G, N, S)$ is then a bi-Hilbertian A-bimodule.

3. A Kasparov module representing the extension class

We now show how to represent the Kasparov class arising from the defining extension of a Cuntz–Pimsner algebra of a bimodule. The easy case turns out to be the SMEB case, which we treat first, since in this case we can also obtain more information in the form of an unbounded representative of the Kasparov module.

The SMEB case does not immediately show how to proceed in the general case: the dilation of the representation-mod-compacts of $\mathcal{O}_E$ on the Fock module to an actual representation of $\mathcal{O}_E$ is easily achieved in the SMEB case by using a two-sided Fock module.

Utilising the extra information coming from the bi-Hilbertian bimodule structure allows us to handle the general case, by constructing an A module with a representation of $\mathcal{O}_E$.

3.1. **The SMEB case.** The following theorem summarises the situation when $\Phi(\mathrm{Id}_E) = 1_A$. The bounded Kasparov module representing the extension in this case is implicit in Pimsner [35], and numerous similar constructions of Kasparov modules associated to circle actions have appeared in [2, 5, 30, 34] amongst others. Similar results for the unbounded Kasparov module were obtained in [15]. The Fock module associated to C^*-bimodules E over a C^*-algebra A is defined as

$$ F_E := \bigoplus_{n\geq 0} E^{\otimes n} $$

with $E^{\otimes 0} := A$, where the internal product $E^{\otimes n}$ is taken regarding E as a right A-module with a left action of A. We let [ext] denote the class of the extension

$$ 0 \to \mathrm{End}^0_A(F_E) \to T_E \to \mathcal{O}_E \to 0 $$

in $KK^1(\mathcal{O}_E, \mathrm{End}^0_A(E))$, and $[F_E] \in KK(\mathrm{End}^0_A(F_E), A)$ the class of the Morita equivalence.

Theorem 3.1. *Let E be a SMEB over A. For $\mathbb{Z} \ni n < 0$, define $E^{\otimes n} := \overline{E}^{\otimes|n|}$. Let $F_{E,\mathbb{Z}}$ denote the Hilbert-bimodule direct sum*

$$ F_{E,\mathbb{Z}} := \bigoplus_{n\in\mathbb{Z}} E^{\otimes n}, $$

and define an operator N on the algebraic direct sum $\bigcup_{n=1}^{\infty}\bigoplus_{m=-n}^{n} E^{\otimes m} \subseteq F_{E,\mathbb{Z}}$ by $N\xi := n\xi$ for $\xi \in E^{\otimes n}$. There is a homomorphism $\rho : \mathcal{O}_E \to \mathrm{End}_A(F_{E,\mathbb{Z}})$ such that $\rho(s_e)\xi = e \otimes \xi$ for all $e \in E$ and $\xi \in \bigcup E^{\otimes n}$. The triple

$$ (\mathcal{O}_E, (F_{E,\mathbb{Z}})_A, N) $$

is an unbounded Kasparov $\mathcal{O}_E$–A module that represents the class [ext] $\otimes_{\mathrm{End}^0_A(F_E)}$ $[F_E] \in KK^1(\mathcal{O}_E, A)$.

Proof. If E is a SMEB, then the conjugate module $\overline{E}$ is also a SMEB, and we have $E\otimes_A\overline{E}\cong \mathrm{End}^0_A(E)\cong A$, and similarly $\overline{E}\otimes_A E\cong A$. This shows that the coefficient algebra A is the fixed point algebra for the gauge action, and that the spectral subspaces for the gauge action are full. Then by [5, Proposition 2.9], $(\mathcal{O}_E,(F_{E,\mathbb{Z}})_A,N)$ is an unbounded Kasparov module.

The corresponding bounded Kasparov module is determined by the non-negative spectral projection of N, denoted P_+, [20, Section 7]. Since $P_+F_{E,\mathbb{Z}}$ is canonically isomorphic to F_E and compression by P_+ implements a positive splitting for the quotient map $q:\mathcal{T}_E\to\mathcal{O}_E$, we deduce that $(\mathcal{O}_E,F_{E,\mathbb{Z}},2P_+-1)$ represents [ext], and hence $(\mathcal{O}_E,F_{E,\mathbb{Z}},N)$ does too. □

3.2. **An operator-valued weight.** Our next goal is to construct a Kasparov module representing the extension class in the case when E is not a SMEB. To do so, we seek to dilate the Fock representation of $\mathcal{T}_E$ to a representation of $\mathcal{O}_E$, but we cannot do this using the module $F_{E,\mathbb{Z}}$ above when E is not a SMEB; the 2-sided direct sum does not carry a representation of $\mathcal{O}_E$ by translation operators. In [35], Pimsner circumvents this problem by enlarging E to a module E_∞ over the core $\mathcal{O}_E^\gamma$, and enlarging the Fock module accordingly. This has the disadvantage, however, that the resulting exact sequence

$$0\to \mathrm{End}^0_{\mathcal{O}_E^\gamma}(F_{E_\infty})\to\mathcal{T}_{E_\infty}\to\mathcal{O}_{E_\infty}\cong\mathcal{O}_E\to 0$$

is very different from the sequence $0\to\mathrm{End}^0_A(F_E)\to\mathcal{T}_E\to\mathcal{O}_E\to 0$ in which we were originally interested. For example, if $A=\mathbb{C}$ and $E=\mathbb{C}^2$, then $\mathrm{End}^0_A(F_E)\cong\mathbb{K}$, whereas $\mathrm{End}_{\mathcal{O}_E^\gamma}(F_{E^\infty})$ is Morita equivalent to the UHF algebra $M_{2^\infty}(\mathbb{C})$.

In this subsection, we show how to dilate the Fock representation without changing coefficients when E is a finitely generated bi-Hilbertian bimodule satisfying an analytic hypothesis, and present some examples of this situation. This will require some set-up building on the tools developed in Section 2. We construct the desired Kasparov module in subsection 3.3.

Fix a bi-Hilbertian A-bimodule E, and let $\{e_i\}$ be a frame for the right module E_A. Given a multi-index $\rho=(\rho_1,\dots,\rho_k)$ we write $e_\rho=e_{\rho_1}\otimes\cdots\otimes e_{\rho_k}$ for the corresponding element of the natural frame of $E^{\otimes k}$. We define

$$e^{\beta_k}=\sum_{|\rho|=k}{}_A(e_\rho\mid e_\rho)=\Phi_k(\mathrm{Id}_{E^{\otimes k}}),\tag{3.1}$$

and just write e^β for e^{β_1}. Provided that E_A is full and finitely generated, e^{β_k} is a positive, invertible and central element of A, [18], so that $\beta_k\in A$ is the logarithm of $\Phi_k(\mathrm{Id}_{E^{\otimes k}})$. Since E will always be clear from context, this justifies our notation

$$\beta:=\log(\Phi(\mathrm{Id}_E)),\qquad \beta_k:=\log(\Phi(\mathrm{Id}_{E^{\otimes k}})).\tag{3.2}$$

We write $\rho=\underline{\rho}\overline{\rho}$ for the decomposition of a multi-index ρ into its initial and final segments, whose lengths will be clear from context. From the formula

$${}_A(e_\rho\mid e_\rho)={}_A(e_{\underline{\rho}}\,{}_A(e_{\overline{\rho}}\mid e_{\overline{\rho}})\mid e_{\underline{\rho}})$$

we see that for $0\le n\le k$

$$e^{\beta_k}=\sum_{|\underline{\rho}|=k-n}{}_A(e_{\underline{\rho}}e^{\beta_n}\mid e_{\underline{\rho}})\le\|e^{\beta_n}\|\,e^{\beta_{k-n}}.$$

Lemma 3.2. *Let E be a finitely generated bi-Hilbertian A-bimodule and for $k \geq 0$, define $\Phi_k : \mathrm{End}_A(E^{\otimes k}) \to A$ by*

$$\Phi_k(T) := \sum_{|\rho|=k} {}_A(Te_\rho \mid e_\rho), \tag{3.3}$$

where ${}_A(\cdot \mid \cdot)$ is the left A-valued inner product on F_E. For $n \leq k$ and $\xi, \eta \in E^{\otimes n}$ we have

$$\Phi_k(\Theta_{\xi,\eta} \otimes \mathrm{Id}_{k-n}) = {}_A(\xi \mid \eta e^{\beta(k-|\eta|)}).$$

Proof. We calculate, using centrality of e^{β_k} in A at the fifth step,

$$\begin{aligned}
\Phi_k(\Theta_{\xi,\eta} \otimes \mathrm{Id}_{k-n}) &= \sum_{|\rho|=k} {}_A(\Theta_{\xi,\eta}e_\rho \mid e_\rho) = \sum_{|\rho|=k} {}_A(\xi \cdot (\eta \mid e_{\underline{\rho}})_A\, e_{\overline{\rho}} \mid e_\rho) \\
&= \sum_{|\rho|=k} {}_A(\xi \cdot (\eta \mid e_{\underline{\rho}})_A\, {}_A(e_{\overline{\rho}} \mid e_{\overline{\rho}}) \mid e_{\underline{\rho}}) = \sum_{|\underline{\rho}|=|\eta|} {}_A(\xi \cdot (\eta \mid e_{\underline{\rho}})_A\, e^{\beta(k-|\eta|)} \mid e_{\underline{\rho}}) \\
&= \sum_{|\underline{\rho}|=|\eta|} {}_A(\xi \cdot e^{\beta(k-|\eta|)}(\eta \mid e_{\underline{\rho}})_A \mid e_{\underline{\rho}}) = \sum_{|\underline{\rho}|=|\eta|} {}_A(\xi \cdot e^{\beta(k-|\eta|)} \mid e_{\underline{\rho}} \cdot (e_{\underline{\rho}} \mid \eta)_A) \\
&= {}_A(\xi \cdot e^{\beta(k-|\eta|)} \mid \eta) = {}_A(\xi \mid \eta \cdot e^{\beta(k-|\eta|)}).
\end{aligned}$$

□

Lemma 3.3. *Let E be a finitely generated bi-Hilbertian A-bimodule, and for each $k \geq 0$, let $\Phi_k : \mathrm{End}^0(E^{\otimes k}) \to A$ be the positive map of Lemma 2.4(1). For $0 \leq T \in \mathcal{T}_E$, and for $\Re(s) > 1$, the series*

$$\sum_{k=0}^{\infty} \Phi_k(T)e^{-\beta_k}(1+k^2)^{-s/2} \tag{3.4}$$

converges to an element $\Phi^s_\infty(T)$ of A which is positive for s real.

Proof. By definition, we have $\Phi_k(\mathrm{Id}_{E^{\otimes k}})e^{-\beta_k} = 1_A$. Thus for $\Re(s) > 1$, the series

$$\sum_{k=0}^{\infty} \Phi_k(\mathrm{Id}_{E^{\otimes k}})e^{-\beta_k}(1+k^2)^{-s/2} = \Big(\sum_{k=0}^{\infty}(1+k^2)^{-s/2}\Big)1_A$$

converges in norm. The function $s \mapsto \sum \Phi_k(\mathrm{Id}_{E^{\otimes k}})e^{-\beta k}(1+k^2)^{-s/2}$ has well-defined residue 1_A at $s = 1$. Since $\mathcal{T}_E$ can be regarded as a subalgebra of $\mathrm{End}_A(F_E)$, the formula (3.3) makes sense for $T \in \mathcal{T}_E$. Indeed, if $P_k : F_E \to E^{\otimes k}$ is the projection, then $P_k \,\mathrm{End}_A(F_E)P_k \cong \mathrm{End}_A(E^{\otimes k})$, and then $\Phi_k(T) = \Phi_k(P_kTP_k)$ for all $T \in \mathcal{T}_E$.

For $0 \leq T \in \mathcal{T}_E$, the inequality $T \leq \|T\| \,\mathrm{Id}_{\mathcal{T}_E}$ shows that

$$\Phi_k(T)e^{-\beta_k} \leq \|T\|\, 1_A. \tag{3.5}$$

So for $s \in \mathbb{C}$ with $\Re(s) > 1$, the series (3.4) converges in norm. □

We now construct an A-valued map on $\mathcal{T}_E$ by taking the residue at $s = 1$ of the map Φ^s_∞ of Lemma 3.3, and then show that this residue functional factors through $\mathcal{O}_E$.

Recall that, given sequences (x_n), (y_n) of real numbers, we write $x_n \in O(y_n)$ if there is a constant C such that $x_n \leq Cy_n$ for large n.

Lemma 3.4. *Let E be a finitely generated bi-Hilbertian A-bimodule, take $\eta \in E^{\otimes n}$ and suppose that the sequence $\left(e^{-\beta_k}\eta e^{\beta(k-|\eta|)}\right)_k$ converges; write $\tilde{\eta}$ for its limit. Suppose that there exists $\delta > 0$ such that*

$$\|e^{-\beta_k}\eta e^{\beta(k-|\eta|)} - \tilde{\eta}\| \in O(k^{-\delta}).$$

For $\xi \in F_E$ the function $s \mapsto \Phi_\infty^s(T_\xi T_\eta^*)$ has a well-defined residue $\Phi_\infty(T_\xi T_\eta^*)$ at $s = 1$, and we have $\Phi_\infty(T_\xi T_\eta^*) = {}_A(\xi \mid \tilde{\eta})$, where the inner-product is taken in F_E.

Proof. For $k > |\eta|$, and for a multi-index $\rho = \underline{\rho}\overline{\rho}$ of length k with $|\underline{\rho}| = |\eta|$, we have $T_\xi T_\eta^* e_\rho = \Theta_{\xi,\eta}(e_{\underline{\rho}}) \otimes e_{\overline{\rho}}$. So for $k > |\eta|$,

$$\begin{aligned}
\Phi_k(T_\xi T_\eta^*) &= \delta_{|\xi|,|\eta|} \sum_{|\rho|=k} {}_A(\Theta_{\xi,\eta}(e_{\underline{\rho}}) \otimes e_{\overline{\rho}} \mid e_\rho) \\
&= \delta_{|\xi|,|\eta|} \sum_{|\rho|=k} {}_A(\Theta_{\xi,\eta}(e_{\underline{\rho}}) \cdot {}_A(e_{\overline{\rho}} \mid e_{\overline{\rho}}) \mid e_{\underline{\rho}}) \\
&= \delta_{|\xi|,|\eta|} \sum_{\underline{\rho}} {}_A(\Theta_{\xi,\eta}(e_{\underline{\rho}}) \cdot e^{-\beta_{k-|\eta|}} \mid e_{\underline{\rho}}). \qquad (3.6)
\end{aligned}$$

Since $e^{-\beta_{k-|\eta|}}$ is central and self-adjoint, we have

$$\Theta_{\xi,\eta}(e_{\underline{\rho}}) \cdot e^{-\beta_{k-|\eta|}} = \xi \cdot (\eta \mid e_{\underline{\rho}})_A e^{-\beta_{k-|\eta|}} = \xi \cdot e^{-\beta_{k-|\eta|}}(\eta \mid e_{\underline{\rho}})_A = \xi \cdot (\eta \cdot e^{-\beta_{k-|\eta|}} \mid e_{\underline{\rho}})_A.$$

So (3.6) gives

$$\Phi_k(T_\xi T_\eta^*) = \delta_{|\xi|,|\eta|}\Phi_{|\eta|}(\Theta_{\xi,\eta\cdot e^{\beta(k-|\eta|)}}) = \delta_{|\xi|,|\eta|A}(\xi \mid \eta \cdot e^{\beta(k-|\eta|)}).$$

Since the summands of F_E are, by definition, mutually orthogonal in its inner product, we deduce that for any ξ, η and k, we have

$$\begin{aligned}
\Phi_k(T_\xi T_\eta^*)e^{-\beta_k} &= \chi_{\{1,\dots,k\}}(|\eta|)_A(\xi \mid \eta \cdot e^{\beta(k-|\eta|)})e^{-\beta_k} \\
&= \chi_{\{1,\dots,k\}}(|\eta|)_A(\xi \mid \tilde{\eta}) + {}_A(\xi \mid e^{-\beta_k} \cdot \eta \cdot e^{\beta(k-|\eta|)} - \tilde{\eta}),
\end{aligned}$$

and so $\|\Phi_k(T_\xi T_\eta^*)e^{-\beta k} - {}_A(\xi \mid \tilde{\eta})\| \in O(k^{-\delta})$. In particular, $\operatorname{res}_{s=1}\Phi_\infty^s(T_\xi T_\eta^*)$ exists and is equal to ${}_A(\xi \mid \tilde{\eta})$ as claimed. □

Proposition 3.5. *Let E be a finitely generated bi-Hilbertian A-bimodule. Suppose that for every $\eta \in F_E$ the limit $\tilde{\eta} := \lim_{k\to\infty} e^{-\beta_k}\eta e^{\beta(k-|\eta|)}$ exists and that for each η there is a δ such that*

$$\|e^{-\beta_k}\eta e^{\beta(k-|\eta|)} - \tilde{\eta}\| \in O(k^{-\delta}).$$

Then there is a conditional expectation $\Phi_\infty : \mathcal{T}_E \to A$ such that

$$\Phi_\infty(T_\xi T_\eta^*) = \operatorname{res}_{s=1}\Phi_\infty^s(T_\xi T_\eta^*),$$

and this Φ_∞ descends to a well-defined functional $\Phi_\infty : \mathcal{O}_E \to A$.

Proof. For $T \in \operatorname{End}_A^0(E^{\otimes k})$ self-adjoint, we have

$$\Phi_k(T)e^{-\beta_k} = \Phi_k(P_kTP_k)e^{-\beta_k} \le \Phi_k(\|T\|P_k)e^{-\beta_k} = \|T\|.$$

So for a self-adjoint finite sum $\sum_i T_{\xi_i}T_{\eta_i}^*$, $\|\sum_i \Phi_\infty^s(T_{\xi_i}T_{\eta_i}^*)\| \le \|\sum_i T_{\xi_i}T_{\eta_i}^*\|\sum_{k=0}^\infty (1+k^2)^{-s/2}$. Since every T can be expressed as a sum of two self-adjoints, we deduce that Φ_∞ is bounded on $\operatorname{span}\{T_\xi T_\eta^* : \xi, \eta \in F_E\}$. It follows that Φ_∞ extends by linearity to a bounded linear map on $\operatorname{span}\{T_\xi T_\eta^* : \xi, \eta \in F_E\}$, and so extends to $\mathcal{T}_E$.

It is routine to check from the defining formula that Φ_∞ is positive, idempotent and A-linear.

For the last assertion, we compute:

$$\begin{aligned}\Phi_\infty(a - \sum_j \Theta_{e_j,e_j} a) &= \operatorname{res}_{s=1} \sum_{k=0}^{\infty} (a - \sum_j {}_A(e_j \mid e_j \cdot e^{\beta_{(k-1)}}) e^{-\beta_k} a)(1+k^2)^{-s/2} \\ &= \operatorname{res}_{s=1} \sum_{k=0}^{\infty} (a - e^{\beta_k} e^{-\beta_k} \cdot a)(1+k^2)^{-s/2} \\ &= \operatorname{res}_{s=1} \sum_{k=0}^{\infty} (a-a)(1+k^2)^{-s/2} = 0.\end{aligned}$$

Hence Φ_∞ vanishes on the covariance ideal, and so descends to the quotient $\mathcal{O}_E$. □

Example 3.6 (Cuntz algebras). Fix $N \geq 1$. Let E be the Hilbert space $\mathbb{C}^N = \operatorname{span}\{e_i : 1 \leq i \leq N\}$, which is a bi-Hilbertian $\mathbb{C}$-bimodule in the obvious way. Then $\mathcal{O}_E \cong \mathcal{O}_N$. We have $e^{\beta_k} = N^k$ for $k \geq 1$. If $\eta \in E^{\otimes n}$ and $k \geq n$ then

$$e^{-\beta_k} \cdot \eta \cdot e^{\beta_{k-n}} = N^{-k} \eta N^{k-n} = N^{-n} \eta$$

and so the hypotheses of Proposition 3.5 are satisfied and Φ_∞ exists. In fact, we have

$$\Phi_\infty(S_\eta S_\zeta^*) = \frac{1}{N^{|\eta|}} \delta_{\eta,\zeta},$$

and Φ_∞ is the usual KMS state for the gauge action on $\mathcal{O}_N$.

For the next example, we need to recall a bit of Perron–Frobenius theory and state an elementary lemma. If A is a primitive nonnegative matrix, then the Perron–Frobenius theorem (see, for example [31, Chapter 8]) says that A has a unique eigenvector x with nonnegative entries and unit 2-norm. The entries of x are in fact all strictly positive, and $Ax = r_\sigma(A)x$ where $r_\sigma(A)$ is the spectral radius of A. (We avoid the usual notation, $\rho(A)$, for the spectral radius because the symbol ρ is used extensively as a multi-index elsewhere in the paper.) The sequence $r_\sigma(A)^{-k}A^k$ converges in norm to the rank-one projection P onto $\mathbb{C}x$, which commutes with A. The following elementary lemma describes the rate of convergence of this sequence.

Lemma 3.7. *Let A be a primitive nonnegative matrix, x its 2-norm-unimodular Perron–Frobenius eigenvector, and P the projection onto $\mathbb{C}x$. Then there exist $C > 0$ and $\alpha < 1$ such that $\|r_\sigma(A)^{-k}A^k - P\| \leq C\alpha^k$ for all k.*

Proof. Since P commutes with A, we have $A^k = PA^kP + (1-P)A^k(1-P) = (PAP)^k + ((1-P)A(1-P))^k$ for all k. Since $PA^kP = r_\sigma(A)^k P$ for all k, we then have $r_\sigma(A)^{-k}A^k - P = r_\sigma(A)^{-k}(1-P)A^k(1-P)$. So $\|r_\sigma(A)^{-k}A^k - P\| = r_\sigma(A)^{-k}\|(1-P)A^k(1-P)\|$ for all k. Let $\lambda := r_\sigma((1-P)A(1-P))$. Then λ is an eigenvalue of A and hence the Perron–Frobenius theorem gives $|\lambda| < r_\sigma(A)$. The spectral-radius formula then gives $\|(1-P)A^k(1-P)\|^{1/k} \to |\lambda| < r_\sigma(A)$, and so there exists l such that $\|(1-P)A^l(1-P)\| < r_\sigma(A)^l$. So $\alpha := \|(1-P)A^l(1-P)\|^{1/l} r_\sigma(A)^{-l} < 1$. For every k, we have $r_\sigma(A)^{-kl}\|(1-P)A^{kl}(1-P)\| \leq$

$r_\sigma(A)^{-kl}\|(1-P)A^l(1-P)\|^k < \alpha^{kl}$. Now for any $p < l$, we have

$$\begin{aligned} r_\sigma(A)^{-kl-p}\|(1-P)A^{kl+p}(1-P)\| &\le r_\sigma(A)^{-p}\|(1-P)A^p(1-P)\|\alpha^{kl} \\ &= r_\sigma(A)^{-p}\alpha^{-p}\|(1-P)A^p(1-P)\|\alpha^{kl+p}, \end{aligned}$$

and so any $C \ge \max_{p<l} r_\sigma(A)^{-p}\alpha^{-p}\|(1-P)A^p(1-P)\|$ does the job. □

Example 3.8 (C^*-algebras of primitive graphs). Fix a finite primitive directed graph G and let $E(G)$ be the associated $\mathbb{C}^{G^0}$-module. Write $A = A_G \in M_{G^0}(\mathbb{Z})$ for the vertex adjacency matrix of G. For $k \ge 1$ we have

$$e^{\beta_k} = \sum_{v\in G^0} |vG^k|\delta_v = \sum_{v,w\in G^0} A^k(v,w)\delta_v$$

Fix $\lambda \in G^n$. We write $\delta_\lambda = \delta_{\lambda_1} \otimes \cdots \otimes \delta_{\lambda_n} \in E(G)^{\otimes n}$. For $k > n$ we have

$$e^{-\beta_k}\cdot\delta_\lambda\cdot e^{\beta_{k-n}} = \frac{|s(\lambda)G^{k-n}|}{|r(\lambda)G^k|}\delta_\lambda = \frac{\|(A^t)^{k-n}\delta_{s(\lambda)}\|_1}{\|(A^t)^k\delta_{r(\lambda)}\|_1}\delta_\lambda$$

We show first that $\lim_{k\to\infty} e^{-\beta_k}\cdot\delta_\lambda\cdot e^{\beta_{k-n}}$ exists. Since G is a finite primitive directed graph, we can apply Perron–Frobenius theory to the transpose A^t of its vertex-adjacency matrix. Let $x \in \mathbb{C}^{G^0}$ be the 2-norm-unimodular Perron–Frobenius eigenvector for A^t; so $A^t x = r_\sigma(A^t)x$, and $\|x\|_2 = 1$. Let P be the projection onto $\mathbb{C}x$. By Lemma 3.7, there exist $\alpha < 1$ and $C > 0$ such that $\|r_\sigma(A^t)^{-k}(A^t)^k - P\| \le C\alpha^k$ for all k. Since x has real entries, we have

$$\lim_{k\to\infty} r_\sigma(A^t)^{-k}(A^t)^k\delta_v = P\delta_v = x\langle\delta_v, x\rangle = x_v x$$

for each $v \in E^0$. Hence

$$\begin{aligned} \lim_{k\to\infty} e^{-\beta_k}\cdot\delta_\lambda\cdot e^{-\beta_{k-n}} &= \lim_{k\to\infty}\frac{\sum_{v\in E^0} A^{k-n}(s(\lambda),v)}{\sum_{w\in E^0} A^k(r(\lambda),w)}\delta_\lambda \\ &= \lim_{k\to\infty}\frac{r_\sigma(A^t)^{k-n}}{r_\sigma(A^t)^k}\left(\frac{\|\lim_{k\to\infty}\frac{1}{r_\sigma(A^t)^{k-n}}(A^t)^{k-n}\delta_{s(\lambda)}\|_1}{\|\lim_{k\to\infty}\frac{1}{r_\sigma(A^t)^k}(A^t)^k\delta_{r(\lambda)}\|_1}\right)\delta_\lambda \\ &= \frac{1}{r_\sigma(A^t)^n}\frac{x_{s(\lambda)}}{x_{r(\lambda)}}\delta_\lambda. \end{aligned} \tag{3.7}$$

To calculate the rate of convergence, we use Equation (3.7) to write

$$\left|r_\sigma(A^t)^n\Big(e^{-\beta_k}\cdot\delta_\lambda\cdot e^{-\beta_{k-n}} - \frac{1}{r_\sigma(A^t)^n}\frac{x_{s(\lambda)}}{x_{r(\lambda)}}\delta_\lambda\Big)\right| = \left|\frac{\|r_\sigma(A^t)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1}{\|r_\sigma(A^t)^{-k}(A^t)^k\delta_{r(\lambda)}\|_1} - \frac{\|x_{s(\lambda)}x\|_1}{\|x_{r(\lambda)}x\|_1}\right|.$$

We have $\|r_\sigma(A^t)^{-k}(A^t)^k\delta_{r(\lambda)} - x_{r(\lambda)}x\|_1 \le C\alpha^k\|x\|_1$ for all k. For $k \in \mathbb{N}$ we have

$$(1-C\alpha^k)\|x_{r(\lambda)}x\|_1 \le \|r_\sigma(A^t)^{-k}(A^t)^k\delta_{r(\lambda)}\|_1 \le (1+C\alpha^k)\|x_{r(\lambda)}x\|_1,$$

and hence

$$\begin{aligned} (1+C\alpha^k)^{-1}\frac{\|r_\sigma(A)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1}{\|x^G_{r(\lambda)}x^G\|_1} &\le \frac{\|r_\sigma(A)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1}{\|r_\sigma(A)^{-k}(A^t)^k\delta_{r(\lambda)}\|_1} \\ &\le (1-C\alpha^k)^{-1}\frac{\|r_\sigma(A)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1}{\|x^G_{r(\lambda)}x^G\|_1}. \end{aligned}$$

Consequently

$$\begin{aligned}\left|\frac{\|r_\sigma(A^t)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1}{\|r_\sigma(A^t)^{-k}(A^t)^k\delta_{r(\lambda)}\|_1}-\frac{\|x_{s(\lambda)}x\|_1}{\|x_{r(\lambda)}x\|_1}\right| \\ \leq \max\left\{\left|\frac{(1+C\alpha^k)^{-1}\|r_\sigma(A^t)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1-\|x_{s(\lambda)}x\|_1}{\|x_{r(\lambda)}x\|_1}\right|,\right. \\ \left.\left|\frac{(1-C\alpha^k)^{-1}\|r_\sigma(A^t)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1-\|x_{s(\lambda)}x\|_1}{\|x_{r(\lambda)}x\|_1}\right|\right\}.\end{aligned}$$

Using the identity $(1+C\alpha^k)^{-1}=1-C\alpha^k(1+C\alpha^k)^{-1}$, we see that

$$\begin{aligned}\left|(1+C\alpha^k)^{-1}\|r_\sigma(A^t)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1-\|x_{s(\lambda)}x\|_1\right| \\ \leq \left|\|r_\sigma(A^t)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1-\|x_{s(\lambda)}x\|_1\right| \\ +\left|C\alpha^k(1+C\alpha^k)^{-1}\|r_\sigma(A^t)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1\right|.\end{aligned}$$

The first term is in $O(\alpha^k)$ by the reverse triangle inequality and Lemma 3.7. The second term is in $O(\alpha^k)$ because the sequences $(1+C\alpha^{-k})^{-1}$ and $\|r_\sigma(A^t)^{n-k}(A^t)^{k-n}\delta_{s(\lambda)}\|_1$ are convergent, and hence bounded. Similar estimates show that $\left|(1-C\alpha^k)^{-1}\|r_\sigma(A^t)^{k-n}(A^t)^{n-k}\delta_{s(\lambda)}\|_1-\|x_{s(\lambda)}x\|_1\right|$ is in $O(\alpha^k)$. Hence

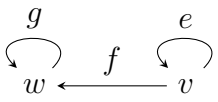
$$\left\|e^{-\beta_k}\cdot\delta_\lambda\cdot e^{\beta_{k-n}}-\frac{1}{r_\sigma(A^t)^n}\frac{x_{s(\lambda)}}{x_{r(\lambda)}}\delta_\lambda\right\|\in O(\alpha^k).$$

Since the δ_λ span $E^{\otimes n}$, it follows that $\|e^{-\beta_k}\cdot\eta\cdot e^{\beta_{k-n}}-\tilde{\eta}\|\in O(\alpha^k)$ for each $\eta\in E^{\otimes n}$. Since every $\delta>1$ satisfies $k^{-\delta}>\alpha^k$ for large k, it follows that the module E satisfies the hypotheses of Proposition 3.5.

Remark 3.9. Since the Cuntz–Krieger algebra of a $\{0,1\}$-matrix A is isomorphic to the graph C^*-algebra of the graph with adjacency matrix A [23, Proposition 4.1], the preceding example shows that Proposition 3.5 covers the situation of Cuntz-Krieger algebras associated to primitive matrices.

The following example is the graph C^*-algebraic realisation of the C^*-algebra of $SU_q(2)$ [17]. Since the graph in question is not primitive, the analysis of the preceding example does not apply, but we can check the hypotheses of Proposition 3.5 by hand.

Example 3.10. Consider the following graph G.



The module E is a copy of $\mathbb{C}^3$ which we write as $\operatorname{span}\{\delta_e,\delta_f,\delta_g\}$. The left action of the projection p_v is by 1 on δ_e and zero elsewhere, and $p_w=1-p_v$. The right action has p_v acting by 1 on both δ_e and δ_f and by zero on δ_g. Schematically,

$$E=\begin{pmatrix}\delta_e\\ \delta_f\\ \delta_g\end{pmatrix},\quad L(p_v)=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad R(p_v)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}.$$

Hence $e^\beta = p_v + 2p_w$. We have

$$E^{\otimes n} = \begin{pmatrix} \delta_{e^{\otimes n}} \\ \delta_{f\otimes e^{\otimes n-1}} \\ \delta_{g\otimes f\otimes e^{\otimes n-2}} \\ \vdots \\ \delta_{g^{\otimes n-1}\otimes f} \\ \delta_{g^{\otimes n}} \end{pmatrix}$$

and the left action of p_v is nonzero only on $\delta_{e^{\otimes n}}$, while the right action of p_w is nonzero only on $g^{\otimes n}$. Hence $e^{\beta_n} = p_v + (n+1)p_w$. So for a path $\lambda \in G^n$

$$e^{-\beta_k}\delta_\lambda e^{\beta(k-n)} = \begin{cases} \delta_\lambda & \lambda = e^n \\ \frac{k+1-n}{k+1}\delta_\lambda & \lambda = g^n \\ \frac{1}{k+1}\delta_\lambda & \text{otherwise.} \end{cases}$$

Hence the hypotheses of Proposition 3.5 are satisfied and Φ_∞ is well-defined for the algebra of this graph.

Remark 3.11. The boundedness of the sequence $\Phi_k(T)e^{-\beta_k}$, for $T \in T_E$, suggests that we could extend the definition of Φ_∞ using Dixmier trace methods. The difficulty is with the meaning of ω-limits in the C^*-algebra A. Victor Gayral has pointed out to us that in any representation $\pi : A \to \mathcal{B}(\mathcal{H})$, for any vectors ξ, $\eta \in \mathcal{H}$, and for a suitable generalised limit $\omega \in (L^\infty([0,\infty)))^*$, the functional

$$T \mapsto \omega\text{-}\lim_N \left(\sum_{k=0}^N \langle \xi, \Phi_k(T)e^{-\beta_k}\eta\rangle (1+k^2)^{-1/2} \right)$$

is well-defined and so we can make sense of 'weak ω-limits'. Unfortunately, however, the resulting limits in general lie in A'' rather than A, so they are not well suited to our purposes.

In the special case where $\beta f = f\beta$ for all $f \in E$, the limits above always exist.

Lemma 3.12. *If $\beta f = f\beta$ for all $f \in E$, then*

$$e^{\beta_n} = e^{\beta n}$$

where $\beta = \beta_1$. Consequently $e^{-\beta_k}\eta e^{\beta_{k-|\eta|}} = e^{-\beta|\eta|}\eta$ for all $\eta \in F_E$ and all k.

Proof. This is just from the definition: for each multi-index ρ of length n, we write $\rho = (\underline{\rho}, \rho_n)$ where $\underline{\rho}$ has length $n-1$, and calculate:

$$\begin{aligned} e^{\beta_n} = \sum_{|\rho|=n} {}_A(e_\rho \mid e_\rho) &= \sum_{|\rho|=n} {}_A(e_{\underline{\rho}}A(e_{\rho_n} \mid e_{\rho_n}) \mid e_{\underline{\rho}}) = \sum_{|\rho|=n} {}_A(e_{\underline{\rho}}e^\beta \mid e_{\underline{\rho}}) \\ &= e^\beta \sum_{|\rho|=n} {}_A(e_{\underline{\rho}} \mid e_{\underline{\rho}}) = e^\beta e^{\beta_{n-1}}. \end{aligned}$$

□

3.3. **The Kasparov module representing the extension.** For this section we assume that the bimodule E satisfies the hypotheses of Proposition 3.5, so that the expectation $\Phi_\infty : \mathcal{O}_E \to A$ is defined.

The functional Φ_∞ is, by construction, gauge invariant in the sense that if $\xi \in E^{\otimes k}$ and $\eta \in E^{\otimes n}$ with $k \neq n$ then $\Phi_\infty(T_\xi T_\eta^*) = 0$. We define an A-valued inner product on $\mathcal{O}_E$ by

$$(S_1 \mid S_2)_A := \Phi_\infty(S_1^* S_2), \qquad S_1,\, S_2 \in \mathcal{O}_E. \tag{3.8}$$

We observe that $\mathcal{N} = \{T \in \mathcal{T}_E : \Phi_\infty(T^*T) = 0\}$ is an A-bimodule: it carries a right action because Φ_∞ is A-bilinear and it carries a left action because $\Phi_\infty(T^*a^*aT) \leq \|a\|^2\Phi_\infty(T^*T)$. Similarly $\mathcal{N}$ is a left $\mathcal{T}_E$ module, and as Φ_∞ vanishes on $\mathrm{End}^0_A(F_E)$ we have $k \cdot \mathcal{T}_E \subset \mathcal{N}$ for all $k \in \mathrm{End}^0_A(F_E)$. These observations justify the following definition.

Definition 3.13. We let $(\mathcal{O}_E)^\Phi_A$ denote the right C^*-A-module obtained as the completion of $\mathcal{O}_E/\mathcal{N}$ in the norm $\|S + \mathcal{N}\| := \|\Phi_\infty(S^*S)\|_A$. We denote the class of $S_\mu S_\nu^*$ in $(\mathcal{O}_E)^\Phi_A$ by $W_{\mu,\nu}$, and as a notational shortcut, we write W_μ for the class of S_μ instead of $W_{\mu,1}$.

Our notational ambiguity will not cause problems: we have written Φ_∞ for the functional $\mathcal{T}_E \to A$ obtained from Proposition 3.5, and also for the functional on $\mathcal{O}_E$ to which this functional descends. So we can form a Hilbert bimodule either by using the former Φ_∞ to define an A-valued sesquilinear form on $\mathcal{T}_E$, or by using the latter Φ_∞ to define one on $\mathcal{O}_E$. But since Φ_∞ vanishes on the covariance ideal, these two modules coincide, and the canonical representation of $\mathcal{T}_E$ on the former induced by multiplication actually descends to the corresponding representation of $\mathcal{O}_E$ on the latter.

In particular, the module $(\mathcal{O}_E)^\Phi_A$ carries a representation of $\mathcal{O}_E$, defined by left multiplication, and so, for instance, $S_\mu S_\nu^* \cdot W_{\sigma,\rho} = W_{\mu\cdot(e_\nu \mid e_{\underline{\sigma}})_A \cdot \overline{\sigma},\rho}$ when $|\nu| \leq |\sigma|$.

Using the module $(\mathcal{O}_E)^\Phi_A$, we can now produce a Kasparov module representing the extension class $[\mathrm{ext}] \otimes_{\mathrm{End}^0(F_E)} [F_E]$.

Theorem 3.14. *If E_A is a finitely generated projective A-bimodule satisfying the hypotheses of Proposition 3.5, and $e_1, \ldots, e_N$ is a frame for E_A, then the series*

$$\sum_{k\geq 0} \sum_{|\rho|=k} \Theta_{W_{e_\rho}, W_{e_\rho}}$$

converges strictly to a projection $P \in \mathrm{End}_A((\mathcal{O}_E)^\Phi_A)$. The map $\xi \mapsto W_\xi$ is an isometric isomorphism of F_E onto $P(\mathcal{O}_E)^\Phi_A$. The left action of $\mathcal{O}_E$ on $(\mathcal{O}_E)^\Phi_A$ has compact commutators with P. The triple

$$(\mathcal{O}_E, (\mathcal{O}_E)^\Phi_A, 2P - 1)$$

is a Kasparov module which represents the class $[\mathrm{ext}] \otimes_{\mathrm{End}^0(F_E)} [F_E]$.

Proof. The map $\iota : (F_E)_A \to (\mathcal{O}_E)^\Phi_A$ given by $\xi \mapsto \iota(\xi) := W_\xi$ is isometric. This follows from the computation

$$(W_\xi | W_\xi)_A = \Phi_\infty(W_\xi^* W_\xi) = \Phi_\infty((\xi|\xi)_A) = (\xi|\xi)_A,$$

and polar decomposition. We now define projections P_k for $k \geq 0$ by

$$P_k := \sum_{|\rho|=k} \Theta_{W_{e_\rho}, W_{e_\rho}}.$$

The P_k are adjointable because they are finite sums of rank one operators for which $\Theta^*_{\xi,\eta} = \Theta_{\eta,\xi}$, and this formula then shows that the P_k are self-adjoint. The projection property

follows from the computation

$$\begin{aligned}P_kP_\ell &= \Big(\sum_{|\rho|=k}\Theta_{W_{e_\rho},W_{e_\rho}}\Big)\Big(\sum_{|\sigma|=\ell}\Theta_{W_{e_\sigma},W_{e_\sigma}}\Big) = \sum_{|\rho|=k,\,|\sigma|=\ell}\Theta_{W_{e_\rho}(W_{e_\rho}|W_{e_\sigma}),W_{e_\sigma}}\\ &= \begin{cases} 0 & k\neq\ell\\ \sum_{|\rho|=k}\Theta_{W_{e_\rho},W_{e_\rho}} & k=\ell\end{cases}\\ &= \delta_{k,\ell}P_k\end{aligned}$$

which also shows that the various P_k are mutually orthogonal. From these computations, it is immediate that $P := \sum_{k\geq 0}P_k$ is a projection. Next observe that the image of P is contained in $\iota(F_E)\subset(\mathcal{O}_E)_A^\Phi$, since

$$\begin{aligned}PW_{\mu,\nu} &= \sum_{0\leq|\rho|\leq|\mu|} W_{e_\rho}\Phi_\infty(S_{e_\rho}^*S_\mu S_\nu^*) = \sum_{0\leq|\rho|\leq|\mu|} W_{e_\rho}\Phi_\infty\big((e_\rho\mid\underline{\mu})_A S_{\overline{\mu}}S_\nu^*\big)\\ &= \lim_{k\to\infty}\sum_{0\leq|\rho|\leq|\mu|} W_{e_\rho}(e_\rho\mid\underline{\mu})_{AA}(\overline{\mu}\mid\nu e^{\beta_{(k-|\nu|)}})e^{-\beta_k}\\ &= \lim_{k\to\infty}\sum_{0\leq|\rho|\leq|\mu|} W_{e_\rho(e_\rho|\underline{\mu})_A\,A}(\overline{\mu}\mid\nu e^{\beta_{(k-|\nu|)}})e^{-\beta_k}.\end{aligned}$$

If $|\overline{\mu}|\neq|\nu|$, then ${}_A(\overline{\mu}\mid\nu e^{\beta_{(k-|\nu|)}}) = \Phi_\infty(S_{\overline{\mu}}S^*_{\nu e^{\beta_{k-|\nu|}}}) = 0$, and so we see that $PW_{\mu,\nu}=0$ if $|\mu|<|\nu|$, and

$$PW_{\mu,\nu} = W_{\underline{\mu}}\Phi_\infty(S_{\overline{\mu}}S_\nu^*)\quad\text{ if } |\mu|\geq|\nu| \text{ and } \mu=\underline{\mu}\otimes\overline{\mu} \text{ with } |\overline{\mu}|=|\nu|.$$

Thus $PW_{\mu,\nu}\in\iota(F_E)\cdot A\subset\iota(F_E)$. Since the image of P can easily be seen to contain W_{e_ρ} for all multi-indices ρ, it follows that the image of P equals F_E. Thus we also learn that F_E is a complemented sub-module of $(\mathcal{O}_E)_A^\Phi$ and that the isometric inclusion $\iota:F_E\hookrightarrow(\mathcal{O}_E)_A^\Phi$ is also adjointable. It is straightforward to check that the map ι also intertwines the actions of $\mathcal{T}_E$ on these copies of F_E. Thus the compression of the action of $\mathcal{O}_E$ on $(\mathcal{O}_E)_A^\Phi$ to the subspace $P(\mathcal{O}_E)_A^\Phi$ gives a positive splitting of the quotient map $\mathcal{T}_E\xrightarrow{q}\mathcal{O}_E$.

To see that $(\mathcal{O}_E,(\mathcal{O}_E)_A^\Phi,2P-1)$ is a Kasparov module, we must verify the compactness of the commutators $[P,S_\mu]$. Fix elementary tensors ρ, σ, $\mu\in F_E$, and observe that

$$PS_\mu W_{\rho,\sigma}=0\quad\text{if } |\mu|+|\rho|-|\sigma|<0,\qquad\text{and}\qquad S_\mu PW_{\rho,\sigma}=0\quad\text{if } |\rho|-|\sigma|<0.$$

In the following, if $|\mu|+|\rho|-|\sigma|\geq 0$ then we split $\mu=\underline{\mu}\otimes\overline{\mu}$ so that $|\overline{\mu}|+|\rho|-|\sigma|=0$. Then to complete the proof, first observe the easy relation (proved above)

$$PW_\mu\cdot c = W_\mu\cdot\Phi_\infty(c),\quad c\in\operatorname{span}\{S_\alpha S_\beta^*:|\alpha|=|\beta|\}.$$

Then

$$\begin{aligned}&PS_\mu W_{\rho,\sigma}-S_\mu PW_{\rho,\sigma}\\ &= \begin{cases} PW_{\mu\rho,\sigma}-W_{\mu\underline{\rho}}\Phi_\infty(S_{\overline{\rho}}S_\sigma^*) & |\rho|-|\sigma|\geq 0\\ PW_{\mu\rho,\sigma} & |\rho|-|\sigma|<0\end{cases} = \begin{cases} 0 & |\rho|-|\sigma|\geq 0\\ W_{\mu\rho,\sigma} & -|\mu|\leq|\rho|-|\sigma|<0\\ 0 & |\rho|-|\sigma|<-|\mu|\end{cases}\\ &= \sum_{j=-|\mu|}^{-1}P_jS_\mu W_{\rho,\sigma}\end{aligned}$$

which is explicitly the action of a finite sum of finite rank operators, and so certainly compact.

Since $(2P-1)^2 = 1$, $(2P-1)^* = (2P-1)$ and $[2P-1, S_\mu S_\nu^*]$ is compact for all vectors $\mu, \nu \in F_E$, we have completed the proof that we obtain an odd Kasparov module. So it suffices to show that the Busby invariant agrees with that of the class $[\mathrm{ext}] \otimes_{\mathrm{End}^0_A(F_E)} [F_E]$. We have seen that the representation of $\mathcal{T}_E$ on $P(\mathcal{O}_E)^\Phi_A$ is isomorphic to the Fock representation, so we just need to show that the representation $\pi : \mathcal{O}_E \to \mathrm{End}_A((\mathcal{O}_E)^\Phi_A)$ induced by multiplication is faithful. Since Φ_∞ is the identity map on A, the image of A in the module $(\mathcal{O}_E)^\Phi_A$ is a copy of the standard module ${}_AA_A$, and so π is faithful on A. As discussed above, the gauge action on $\mathcal{O}_E$ determines a unitary action of $\mathbb{T}$ on $(\mathcal{O}_E)^\Phi_A$, and this unitary action induces an action β of $\mathbb{T}$ on $\mathrm{End}_A((\mathcal{O}_E)^\Phi_A)$. It is routine to check that $\beta_z \circ \pi = \pi \circ \gamma_z$ for all z. So the gauge-invariant uniqueness theorem [21, Theorem 4.5] shows that π is injective. □

Corollary 3.15. *Let E be a finitely generated bi-Hilbertian A-bimodule. Then the boundary maps in the K-theory and K-homology exact sequences for Cuntz–Pimsner algebras, labelled (1) and (2) in* [35, Theorem 4.1]*, are given by the Kasparov product with $(\mathcal{O}_E, (\mathcal{O}_E)^\Phi_A, 2P-1)$.*

Remark 3.16. If E is a SMEB then the class $(\mathcal{O}_E, (\mathcal{O}_E)^\Phi_A, 2P-1)$ is just the class of the Fock module presented earlier.

4. KMS functionals for bimodule dynamics

Given a bi-Hilbertian bimodule E over a unital algebra A, we have seen that the (right) Jones–Watatani index $\Phi(\mathrm{Id}_E) = e^\beta \in \mathcal{Z}(A)$ carries useful structural information about the associated Cuntz-Pimsner algebra. The Jones–Watatani index can be defined for a much wider class of bimodules than those considered here, and we refer to [18] for further examples, the general framework, and relations to conjugation theory.

Our aim in this final section is to show that the Jones–Watatani index of the bimodule E also determines a natural one-parameter group of automorphisms of $\mathcal{O}_E$ that often admits a natural KMS state. The dynamics and most of the ingredients of the KMS states we construct arise from the bimodule alone, but we require one additional ingredient: a state on A which is invariant for E in an appropriate sense.

Definition 4.1. Let E be a bi-Hilbertian bimodule over a unital C^*-algebra A. A state $\phi : A \to \mathbb{C}$ is E-invariant if for all e_1, $e_2 \in E$ we have $\phi((e_1|e_2)_A) = \phi({}_A(e_2|e_1))$.

Lemma 4.2. *If a state $\phi : A \to \mathbb{C}$ is E-invariant, then it is a trace.*

Proof. For $e, f \in E$ and $a \in A$, we have

$$\phi((e \mid f)_A a) = \phi((e \mid fa)_A) = \phi({}_A(fa \mid e)) = \phi({}_A(f \mid ea^*)) = \phi((ea^* \mid f)_A) = \phi(a(e \mid f)_A).$$

So in particular ϕ is tracial on the range of $(\cdot \mid \cdot)_A$. Since the right inner-product is full, this completes the proof. □

Before proceeding, we present some examples that demonstrate that the existence of an invariant trace is not a prohibitively restrictive hypothesis.

Example 4.3 (Crossed products by $\mathbb{Z}$). Let E be the module A with left action given by an automorphism α, as in Example 2.8. Then the definition of an E-invariant state $\phi : A \to \mathbb{C}$ immediately says that ϕ is α-invariant. When $A = C(X)$ is abelian, this is of course just an α-invariant measure.

Example 4.4 (Directed graphs). Let $G = (G^0, G^1, r, s)$ be a finite directed graph with no sinks or sources. Let $A = \mathbb{C}^{|G^0|}$ and let $E = \overline{C_c(G^1)}$ be the bi-Hilbertian A-bimodule from Example 2.1.5. Define $A \to \mathbb{C}$ by

$$\varphi(f) = \sum_{v \in G^0} f(v)$$

where $f \in C(G^0)$. If $\xi, \eta \in C_c(G^1)$ we have

$$\varphi((\xi|\eta)_A) = \sum_{v \in G^0} (\xi|\eta)_A(v) = \sum_{v \in G^0} \sum_{s(e)=v} \overline{\xi(e)}\eta(e) = \sum_{v \in G^0} \sum_{r(e)=v} \eta(e)\overline{\xi(e)} = \varphi({}_A(\eta|\xi))$$

so φ is E-invariant.

Example 4.5 (Topological graphs). Let $G = (G^0, G^1, r, s)$ be a topological graph with $r : G^1 \to G^0$ a local homeomorphism. Let E be the bi-Hilbertian bimodule over $A = C_0(G^0)$ from Example 2.1.4. Suppose that μ is a probability measure on G^0 satisfying

$$\int_{r(\operatorname{supp}\xi)} \xi d\mu = \int_{s(\operatorname{supp}\xi)} \xi d\mu$$

whenever $\xi \in C_c(G^1)$ with r and s bijective on $\operatorname{supp}\xi$. Given $f \in A$ define

$$\varphi(f) = \int f d\mu.$$

Then φ is E-invariant.

It is fairly clear that the preceding example can be further generalised to the twisted topological graph algebras of Li [27] (see Example 2.1.6).

We now show how an E-invariant trace can be used to construct a KMS state for a dynamics on $\mathcal{O}_E$ determined by the Jones–Watatani index of the module.

Lemma 4.6. *Let E be a finitely generated bi-Hilbertian A-bimodule, and let (T, π) denote the universal generating Toeplitz representation of E in $\mathcal{T}_E$. There is a dynamics $\gamma : \mathbb{R} \to \operatorname{Aut}(\mathcal{T}_E)$ such that*

$$\gamma_t(T_e) := \pi(e^{i\beta t})T_e, \quad e \in E, \quad \text{and} \quad \gamma_t(\pi(a)) = \pi(a), \quad a \in A.$$

Moreover, this dynamics descends to a dynamics, also denoted γ, on $\mathcal{O}_E$.

Proof. Fix $t \in \mathbb{R}$ and define $R : E \to \mathcal{T}_E$ by $R_e := e^{i\beta t}T_e$. Then R is a linear map, and since $e^{i\beta t}$ is central in A, we have

$$\pi(a)R_e = \pi(ae^{i\beta t})T_e = \pi(e^{i\beta t}a)T_e = R_{a\cdot e}.$$

We have $R_e\pi(a) = R_{e\cdot a}$ by associativity of multiplication. For $e, f \in E$, we have $R_e^*R_f = T_e^*\pi(e^{-i\beta t}e^{i\beta t})T_f$. Since e^β is invertible and positive, $e^{i\beta t}$ is unitary, with adjoint $e^{-i\beta t}$, and so $R_e^*R_f = T_e^*T_f = \pi((e \mid f)_A)$. So (R, π) is a Toeplitz representation.

Now the universal property of $\mathcal{T}_E$ shows that there is a homomorphism $\gamma_t : \mathcal{T}_E \to \mathcal{T}_E$ satisfying the desired formulae. Clearly $\gamma_s \circ \gamma_t = \gamma_{s+t}$ and $\gamma_e = \mathrm{Id}_{\mathcal{T}_E}$ on generators, and it follows that $t \mapsto \gamma_t$ is a homomorphism of $\mathbb{R}$ into $\operatorname{Aut}(\mathcal{T}_E)$. A routine $\varepsilon/3$-argument shows that this homomorphism is strongly continuous, completing the proof.

To see that γ descends to a dynamics on $\mathcal{O}_E$, observe that with (R, π) as above, for $e, f \in E$, we have

$$R^{(1)}(\theta_{e,f}) = R_eR_f^* = \pi(e^{i\beta t})T_eT_f^*\pi(e^{-i\beta t}).$$

So for $a \in A$, writing $\phi(a) \in \mathrm{End}^0_A(E)$ for the compact operator given by $\phi(a)e = a \cdot e$ for $e \in E$, we have $R^{(1)}(\phi(a)) = \pi(e^{i\beta t})T^{(1)}(\phi(a))\pi(e^{-i\beta t})$. Since $e^{i\beta t}$ is a central unitary in A, we have $\pi(a) = \pi(e^{i\beta t}ae^{-i\beta t})$ for all $a \in A$, and hence

$$\begin{aligned}\gamma_t(T^{(1)}(\phi(a)) - \pi(a)) &= R^{(1)}(\phi(a)) - \pi(a)\\ &= \pi(e^{i\beta t})T^{(1)}(\phi(a))\pi(e^{-i\beta t}) - \pi(e^{i\beta t}ae^{-i\beta t})\\ &= \pi(e^{i\beta t})\big(T^{(1)}(\phi(a)) - \pi(a)\big)\pi(e^{-i\beta t}).\end{aligned}$$

So each γ_t preserves the covariance ideal and therefore descends to $\mathcal{O}_E$ as claimed. □

Note that, in general, $e^{i\beta t}f \neq fe^{i\beta t}$ for $f \in E$. So we typically have

$$\gamma_t(S_{e_\rho}S^*_{e_\gamma}) \neq e^{i\beta|\rho|t}S_{e_\rho}S^*_{e_\gamma}e^{-i\beta|\gamma|t}.$$

The dynamics on $\mathcal{T}_E$ described in Lemma 4.6 is implemented by the second quantisation of the one parameter unitary group $t \mapsto U_t = e^{i\beta t}$, [25]. The second quantisation is given by $\Gamma(U_t) = \mathrm{Id}_A \oplus U_t \oplus (U_t \otimes U_t) \oplus \cdots$. We let

$$\mathcal{D} = \oplus_{n\in\mathbb{N}}\big(\beta \otimes \mathrm{Id}_{E^{\otimes n-1}} + \mathrm{Id}_E \otimes \beta \otimes \mathrm{Id}_{E^{\otimes n-2}} + \cdots + \mathrm{Id}_{E^{\otimes n-1}} \otimes \beta\big)$$

be the (self-adjoint, regular) generator of the unitary group $\Gamma(U_t)$. Combining ideas from [5, 25] we can construct a KMS state for γ. Recall from [25, Theorem 1.1] that if ϕ is a trace on A, and M is a right-Hilbert A-module, then there is a norm lower semicontinuous semifinite trace Tr_ϕ on $\mathrm{End}^0(M)$ such that

$$\mathrm{Tr}_\phi(\Theta_{\xi,\eta}) = \phi((\eta \mid \xi)_A) \quad \text{for all } \xi, \eta \in M. \tag{4.1}$$

Note that if M is finitely generated, then Tr_ϕ is finite.

Proposition 4.7. *Let E be a finitely generated bi-Hilbertian A-bimodule, ϕ an E-invariant trace on A, and $\beta \in \mathcal{Z}(A)$ as defined in Equation (3.2). Let N denote the number operator on the Fock space. Then there is a state $\phi_{\mathcal{D}}$ on $\mathcal{T}_E$ such that*

$$\phi_{\mathcal{D}}(T_\xi T^*_\eta) = \mathrm{res}_{s=1}\,\mathrm{Tr}_\phi(e^{-\mathcal{D}}T_\xi T^*_\eta(1+N^2)^{-s/2}) \quad \text{for all } \xi, \eta \in \textstyle\bigcup_n E^{\otimes n}.$$

This $\phi_{\mathcal{D}}$ vanishes on $\mathrm{End}^{00}(F_E)$, and descends to a linear functional, still denoted $\phi_{\mathcal{D}}$ on $\mathcal{O}_E$. Moreover, $\phi_{\mathcal{D}}$ is a KMS$_1$-state of $\mathcal{O}_E$ for γ.

Proof. Let Tr_ϕ be the functional obtained from (4.1) with $M = F_E$, and for each k, let $\mathrm{Tr}_{\phi,k}$ be the functional obtained from (4.1) with $M = E^{\otimes k}$. If $\eta \in E^{\otimes k}$ and $\zeta \in E^{\otimes l}$ with $k \neq l$, then $(\eta \mid \xi)_A = 0$ in F_E, and so (4.1) gives $\mathrm{Tr}_\phi(\Theta_{\xi,\eta}) = 0$; and if $\xi, \eta \in E^{\otimes k}$ then (4.1) gives $\mathrm{Tr}_\phi(\Theta_{\xi,\eta}) = \mathrm{Tr}_{\phi,k}(\Theta_{\xi,\eta})$.

For each n, let $P_n \in \mathrm{End}_A(F_E)$ be the projection onto $E^{\otimes n}$. For $\xi, \eta \in \bigcup_k E^{\otimes k}$, we have

$$e^{-\mathcal{D}}T_\xi T^*_\eta(1+N^2)^{-s/2} = \sum_{n,m=0}^{\infty} P_n e^{-\mathcal{D}}T_\xi T^*_\eta(1+m^2)^{-s/2}P_m,$$

and so

$$\mathrm{Tr}_\phi(e^{-\mathcal{D}}T_\xi T^*_\eta(1+N^2)^{-s/2}) = \sum_{n=0}^{\infty}\mathrm{Tr}_{\phi,n}(P_n e^{-\mathcal{D}}T_\xi T^*_\eta(1+n^2)^{-s/2}P_n).$$

Since $\xi, \eta \in E^{\otimes k}$,

$$\mathrm{Tr}_{\phi,n}(P_n e^{-\mathcal{D}}T_\xi T^*_\eta P_n) = \begin{cases} 0 & n < k \\ \mathrm{Tr}_{\phi,n}(e^{-\mathcal{D}}\Theta_{\xi,\eta} \otimes \mathrm{Id}_{E^{\otimes n-k}}) & n \geq k \end{cases}$$

Fix $n \geq k$, let $\{e_\rho\}$ be a frame for $E^{\otimes n-k}$, and compute:

$$\begin{aligned}
\operatorname{Tr}_{\phi,n}(e^{-\mathcal{D}}\Theta_{\xi,\eta}\otimes \mathrm{Id}_{E^{\otimes n-k}}) &= \sum_{|\rho|=n-k} \operatorname{Tr}_{\phi,n}(e^{-\mathcal{D}}\Theta_{\xi\otimes e_\rho,\eta\otimes e_\rho}) \\
&= \sum_{|\rho|=n-k} \phi\big((\eta\otimes e_\rho \mid e^{-\mathcal{D}}\xi\otimes e_\rho)_A\big) \\
&= \sum_{|\rho|=n-k} \phi\big((e_\rho \mid (\eta \mid e^{-\mathcal{D}}\xi)_A e^{-\mathcal{D}}e_\rho)_A\big) \\
&= \sum_{|\rho|=n-k} \phi\big((\otimes_{j=1}^{n-k} e^{-\beta/2}e_{\rho_j} \mid \big(\eta \mid e^{-\mathcal{D}}\xi\big)_A \cdot \big(\otimes_{j=1}^{n-k} e^{-\beta/2}e_{\rho_j}\big))_A\big) \\
&= \sum_{|\rho|=n-k} \phi\big({}_A\big((\eta \mid e^{-\mathcal{D}}\xi)_A \cdot \big(\otimes_{j=1}^{n-k} e^{-\beta/2}e_{\rho_j}\big) \mid \otimes_{j=1}^{n-k} e^{-\beta/2}e_{\rho_j}\big)\big) \\
&= \sum_{|\rho|=n-k} \phi\big((\eta \mid e^{-\mathcal{D}}\xi)_A\, {}_A(\otimes_{j=1}^{n-k} e^{-\beta/2}e_{\rho_j} \mid \otimes_{j=1}^{n-k} e^{-\beta/2}e_{\rho_j})\big).
\end{aligned}$$

We have $\sum {}_A(\otimes_{j=1}^{n-k} e^{-\beta/2}e_{\rho_j} \mid \otimes_{j=1}^{n-k} e^{-\beta/2}e_{\rho_j}) = 1_A$ by the calculations of Lemma 3.2. Hence

$$\operatorname{Tr}_{\phi,n}(e^{-\mathcal{D}}\Theta_{\xi,\eta}\otimes \mathrm{Id}_{E^{\otimes n-k}}) = \phi((\eta|e^{-\mathcal{D}}\xi)_A).$$

Hence

$$\operatorname{Tr}_\phi(e^{-\mathcal{D}}T_\xi T_\eta^*(1+N^2)^{-s/2}) = \sum_{n=k}^{\infty} \phi((\eta \mid e^{-\mathcal{D}}\xi)_A)(1+n^2)^{-s/2},$$

and we see that $\phi_{\mathcal{D}}(T_\xi T_\eta^*) := \operatorname{res}_{s=1}\operatorname{Tr}_\phi(e^{-\mathcal{D}}T_\xi T_\eta^*(1+N^2)^{-s/2})$ is well-defined, and

$$\phi_{\mathcal{D}}(T_\xi T_\eta^*) = \phi((\eta \mid e^{-\mathcal{D}}\xi)_A). \tag{4.2}$$

Fix $a \in A$. By the calculation of Lemma 3.2, and centrality of β,

$$\begin{aligned}
\phi_{\mathcal{D}}(a) &= \phi_{\mathcal{D}}(a1_{\mathcal{T}_E}) \\
&= \operatorname{res}_{s=1}\sum_n \operatorname{Tr}_{\phi,n}(e^{-\mathcal{D}}a\mathrm{Id}_{E^{\otimes n}})(1+n^2)^{-s/2} \\
&= \operatorname{res}_{s=1}\sum_n \phi\Big(\sum_{|\rho|=n}(e_\rho|e^{-\mathcal{D}}ae_\rho)_A\Big)(1+n^2)^{-s/2} \\
&= \operatorname{res}_{s=1}\sum_n \phi\Big(\sum_{|\rho|=n} {}_A\Big(a\cdot\big(\otimes_{j=1}^n e^{-\beta/2}e_{\rho_j}\big) \;\Big|\; \otimes_{j=1}^n e^{-\beta/2}e_{\rho_j}\Big)\Big)(1+n^2)^{-s/2} \\
&= \operatorname{res}_{s=1}\sum_n \phi(a1_A)(1+n^2)^{-s/2} = \phi(a).
\end{aligned} \tag{4.3}$$

To check that $\phi_{\mathcal{D}}$ extends to a norm-decreasing linear map on $\mathcal{T}_E$, apply (4.3) to $a = 1_A \in A$ to see that $\phi_{\mathcal{D}}(1_{\mathcal{T}_E}) = \phi(1_A) = 1$. Equation (4.2) shows that the formula $\sum_i T_{\xi_i}T_{\eta_i}^* \mapsto \sum_i \phi_{\mathcal{D}}(T_{\xi_i}T_{\eta_i}^*)$ carries positive elements of $\operatorname{span}\{T_\xi T_\eta^* : \xi,\eta \in \bigcup_k E^{\otimes k}\}$ to $[0,\infty)$. Hence, for $T = \sum_i T_{\xi_i}T_{\eta_i}^*$ self-adjoint, $|\phi_{\mathcal{D}}(T)| \leq \|T\|\phi_{\mathcal{D}}(1_{\mathcal{T}_E}) = \|T\|$. So the formula $\phi_{\mathcal{D}}(\sum_i T_{\xi_i}T_{\eta_i}^*)$ is well-defined and bounded, so extends to a bounded linear functional on $\mathcal{T}_E$ satisfying $\phi_{\mathcal{D}}(1) = 1$; that is, a state.

A calculation like (4.2) shows that $\phi_{\mathcal{D}}\big(a\sum_{j=1}^{N}\Theta_{e_j,e_j}\big)=\phi(a)$ as well. So $\phi_{\mathcal{D}}$ vanishes on the covariance ideal, and hence descends to a state on $\mathcal{O}_E$.

We check the KMS condition. For $S_1,\ S_2\in\overline{\text{span}}\{S_\xi S_\eta^*:\xi,\eta\in\bigcup_k E^{\otimes k}\}$, we have

$$\mathrm{Tr}_\phi(e^{-\mathcal{D}}S_1S_2(1+N^2)^{-s/2})=\mathrm{Tr}_\phi(e^{-\mathcal{D}}(e^{\mathcal{D}}S_2e^{-\mathcal{D}})S_1(1+N^2)^{-s/2}+e^{-\mathcal{D}}S_1[S_2,(1+N^2)^{-s/2}]).$$

Now we show that the commutator $[S_2,(1+N^2)^{-s/2}]$ is 'trace-class'. For $0<s<2$ we employ the integral formula for fractional powers to find

$$[S_2,(1+N^2)^{-s/2}]=\frac{\sin(s\pi/2)}{\pi}\int_0^\infty\lambda^{-s/2}(1+\lambda+N^2)^{-1}[N^2,S_2](1+\lambda+N^2)^{-1}d\lambda.$$

We claim that the integral on the right converges in norm for all $2>s>0$, proving finiteness of the sum defining $\mathrm{Tr}_\phi(e^{-\mathcal{D}}S_1[S_2,(1+N^2)^{-s/2}])$ at $s=1$. To see this, observe that gauge invariance says that we need only consider the case when S_1S_2 is homogenous of degree 0 for the gauge action. If S_2 is of degree zero, then there is nothing to prove. For S_2 homogenous of degree m we have

$$\begin{aligned}\mathrm{Tr}_\phi(e^{-\mathcal{D}}S_1[S_2,(1+N^2)^{-s/2}])&=\sum_{n=0}^\infty\mathrm{Tr}_{\phi,n}(P_ne^{-\mathcal{D}}S_1[S_2,(1+N^2)^{-s/2}]P_n)\\&=\sum_{n=0}^\infty\mathrm{Tr}_{\phi,n}\Big(P_ne^{-\mathcal{D}}S_1\frac{\sin(s\pi/2)}{\pi}\\&\qquad\times\int_0^\infty\lambda^{-s/2}(1+\lambda+N^2)^{-1}(N[N,S_2]+[N,S_2]N)(1+\lambda+N^2)^{-1}P_n\,d\lambda\Big)\\&=m\sum_{n=0}^\infty\mathrm{Tr}_\phi\left(P_ne^{-\mathcal{D}}S_1S_2P_n\right)\frac{\sin(s\pi/2)}{\pi}f_n(s)\end{aligned}$$

where

$$f_n(s)=\int_0^\infty\lambda^{-s/2}\Big((1+\lambda+(n+m)^2)^{-1}(n+m)(1+\lambda+n^2)^{-1}+(1+\lambda+(n+m)^2)^{-1}n(1+\lambda+n^2)^{-1}\Big)\,d\lambda.$$

Now observe that $\left|\mathrm{Tr}_\phi\left(P_ne^{-\mathcal{D}}S_1S_2P_n\right)\right|<\|S_1S_2\|$ for all n, and use the elementary estimates

$$(1+\lambda+n^2)^{-1}n\le\frac{1}{\sqrt{2}}(1+\lambda)^{-1/2},\quad(1+\lambda+n^2)^{-1}\le(1/2+\lambda)^{-\epsilon}(1/2+n^2)^{-(1-\epsilon)}\text{ for }1>\epsilon>0$$

to see that

$$f_n(s)\le\frac{1}{\sqrt{2}}\Big((1/2+n^2)^{-(1-\epsilon)}+(1/2+(n+m)^2)^{-(1-\epsilon)}\Big)\int_0^\infty\lambda^{-s/2}(1+\lambda)^{-1/2}(1/2+\lambda)^{-\epsilon}\,d\lambda.$$

These estimates, along with a suitable choice of ϵ, show that $\mathrm{Tr}_\phi(e^{-\mathcal{D}}S_1[S_2,(1+N^2)^{-s/2}])$ is finite for all $2>s>0$, and more generally for $2>\Re(s)>0$.

Taking the derivative with respect to s of the function $s\mapsto\mathrm{Tr}_\phi(e^{-\mathcal{D}}S_1[S_2,(1+N^2)^{-s/2}])$ and repeating the estimate (now with an extra $\log(\lambda)$ in the integral defining $f_n(s)$) shows that the derivative is also finite at $s=1$, proving holomorphicity. Hence the residue at $s=1$ of $s\mapsto\mathrm{Tr}_\phi(e^{-\mathcal{D}}S_1[S_2,(1+N^2)^{-s/2}])$ vanishes and $\phi_{\mathcal{D}}$ is KMS. □

Remark 4.8. It is tempting to consider a dynamics on $\mathcal{O}_E$ coming from the unitary group $W_t = \oplus_{k\geq 0} e^{i\beta_k t}$, where $\beta_k = \log(\Phi_k(\mathrm{Id}_{E^{\otimes k}}))$. There is a dynamics σ on $\mathrm{End}_A(F_E)$ given by $\sigma_t(T) = W_t T W_t^*$, and it is natural to ask whether σ restricts to a dynamics on $\mathcal{T}_E$.

Observe, however, that since $e^{\beta_0} = 1_A$ and $e^{\beta_1} = e^{\beta}$, the dynamics σ agrees on the generators of $\mathcal{T}_E$ with the dynamics defined in Lemma 4.6. So if σ does indeed extend to $\mathcal{T}_E$, then it agrees with γ, and the analysis above applies. That is, there is nothing new to be gained by considering the dynamics σ, at least for the algebra $\mathcal{T}_E$.

Proposition 4.7 combined with the results of [5] yields the following.

Corollary 4.9. *Let E be a bi-Hilbertian module over A and $\phi : A \to \mathbb{C}$ an E-invariant state. Let $\mathcal{H} = L^2(\mathcal{O}_E, \phi_{\mathcal{D}})$ be the GNS space of $\phi_{\mathcal{D}} : \mathcal{O}_E \to \mathbb{C}$, and $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ the weak closure of the algebra generated by $\mathcal{O}_E$ and the spectral projections P_k for the unitary extension of the dynamics γ to $\mathcal{H}$. Then $(\mathcal{O}_E, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi_{\mathcal{D}})$ is a modular spectral triple.*

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