

THE K -THEORETIC BULK-EDGE CORRESPONDENCE FOR TOPOLOGICAL INSULATORS

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ABSTRACT. We study the application of Kasparov theory to topological insulator systems and the bulk-edge correspondence. We consider observable algebras as modelled by crossed products, where bulk and edge systems may be linked by a short exact sequence. We construct unbounded Kasparov modules encoding the dynamics of the crossed product. We then link bulk and edge Kasparov modules using the Kasparov product. Because of the anti-linear symmetries that occur in topological insulator models, real C^* -algebras and KKO -theory must be used.

Keywords: Topological insulators, KK -theory, bulk-edge correspondence

Subject classification: Primary 81R60, Secondary 19K35

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1. INTRODUCTION

The bulk-edge correspondence is fundamental for topological insulators. Indeed, insulators in non-trivial topological phases are characterised by the existence of edge states

Date: May 14, 2023.

at the Fermi energy which are robust against perturbations coming from disorder. This is a topological bulk-boundary effect, as explained in all details for complex topological insulators in [52] (see Corollary 6.5.5). The theory of [52] is based on the noncommutative topology (K -theory, cyclic cohomology and index theory) of complex C^* -algebras, an approach to solid state systems proposed and developed by Bellissard [5, 6].

Complex topological insulators are topological insulators which are not invariant under a symmetry implemented by anti-linear operators, such as time reversal symmetry. The class of insulators with anti-linear symmetries require noncommutative topology for real algebras and here the bulk-edge correspondence is still in development.

The bulk-edge correspondence in its noncommutative formulation has two sides. One side is the K -theoretic correspondence, where the boundary map of K -theory yields an equality between K -group elements, one of the K -group of the observable (bulk) algebra, and the other of the K -group of the edge algebra. The other side is “dual to K -theory”, namely a correspondence between elements of a theory which provides us with functionals on K -theory. The pairing of K -group elements with the dual theory can be used to obtain topological numbers from the K -theory elements and, possibly, with a physical explanation of these numbers as topologically quantised entities (non-dissipative transport coefficients).

In the complex case this has been achieved using cyclic cohomology [32, 34, 52]. The result of applying a functional coming from cyclic cohomology (one speaks of a pairing between cyclic cocycles and K -group elements) is a complex number and since the functional is additive the result is necessarily 0 on elements of finite order. The most exciting topological invariants, however, lead to K -group elements which have order 2, the Kane–Mele invariant being such an example. This is why we look into another theory dual to K -group theory, namely K -homology (its proper dual in the algebraic sense). It leads to functionals which applied to K -group elements are Clifford index valued and do not vanish on finite groups. *Our specific aim in this paper is to describe a computable version of the bulk-edge correspondence for K -homology.*

We actually work in the more general setting of Kasparov’s KK -theory of which K -theory and K -homology are special cases. In doing so we generalise the approach using Kasparov’s theory for the quantum Hall effect [8] to the case of all topological insulators.

Whereas the specific details of the insulator feed into the construction of K -group elements, the K -homology class appears to be stable for systems of the same dimension. We call it the fundamental K -cycle as it is constructed similarly to Kasparov’s fundamental class for oriented manifolds [28, 41]. It may well be a key feature of these kinds of condensed matter models that the physics (alternatively the geometry and topology) is governed by classes of this type, which is effectively the fundamental class of the momentum space.

An important aspect of the K -cycle is that it involves the Dirac operator in momentum space $\sum_{j=1}^d X_j \gamma^j$ with X_j the components of the position operator in d dimensions, and γ^j the gamma matrices acting in a physical representation of the relevant algebra on $\ell^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ or $\ell^2(\mathbb{Z}^{d-1}, \mathbb{C}^\nu)$ (we work in the tight binding approximation). This operator, or rather its phase, has played a fundamental role ever since it was employed in the integer quantum Hall effect (the phase induces a singular gauge transformation sending one charge to infinity, a process which is at the heart of Laughlin’s Gedankenexperiment). The associated index formula has recently been generalised to all topological insulators [21].

Computing boundary maps in K -homology is notoriously difficult. It comes down to realising K -homology groups of A as Kasparov $KKO_i(A, \mathbb{R})$ -groups [27] and determining

the Kasparov product with the so-called extension class. Until recently this seemed impossible in general. But recent progress on the formulation of KK -theory via unbounded Kasparov cycles [4, 10, 16, 18, 24, 38, 45, 47] make it feasible in our case. We determine an unbounded Kasparov cycle for the class of the Toeplitz extension of the observable algebra of the insulator. The task is then to compute the Kasparov product of the KK -class of this extension, denoted $[\text{ext}]$, with the class of the fundamental K -cycle of the edge algebra, denoted λ_e . One of the important results of this paper is that the product is, up to sign, the KK -class of the bulk fundamental K -cycle λ_b ,

$$[\text{ext}] \hat{\otimes}_{A_e} [\lambda_e] = (-1)^{d-1} [\lambda_b],$$

where $[\lambda_\bullet]$ denotes the class of the Kasparov module in KK -theory. This is the dual side to the bulk-boundary correspondence, showing how to relate the fundamental K -cycles of the bulk and edge theories. We also emphasise that by working in the unbounded setting, our computations are explicit and have direct link to the underlying physics and geometry of the system.

We can relate our results about fundamental K -cycles to topological phases by identifying the (bulk) invariants of topological phases as a pairing/product of the K -cycle λ_b with a K -theory class $[x_b]$. We do not prescribe what the K -group element $[x_b]$ of the bulk has to be. It can be the homotopy class of a symmetry compatible Hamiltonian (translated from van Daele K -theory to KK -theory) or the class of the symmetry of the insulator constructed in [9]. Leaving this flexible gives us the possibility to consider insulators systems of quite general symmetry type without affecting the central correspondence. It also allows our approach to be adapted to different experimental arrangements, an important feature given the difficulties of measuring \mathbb{Z}_2 -labelled phases experimentally.

The K -theory side of the bulk-edge correspondence can be obtained by realising the K -groups of the bulk algebra A as Kasparov $KKO_j(\mathbb{R}, A)$ -groups so that the boundary map applied to a KK -class $[x]$ of A is given by the Kasparov product $[x_b] \hat{\otimes}_A [\text{ext}]$. The bulk-boundary correspondence between topological quantised numbers is then a direct consequence of the associativity of the Kasparov product

$$([x_b] \hat{\otimes}_A [\text{ext}]) \hat{\otimes}_{A_e} [\lambda_e] = [x_b] \hat{\otimes}_A ([\text{ext}] \hat{\otimes}_{A_e} [\lambda_e]).$$

The equation says that the two topological quantised entities, that for the bulk $[x_b] \hat{\otimes}_A [\lambda_b]$, and that for the edge $[x_e] \hat{\otimes}_A [\lambda_e]$, where $[x_e] = [x_b] \hat{\otimes}_A [\text{ext}]$ corresponds to the image of the boundary map on $[x_b]$, are equal. Having said that, the result of the pairing lies in $KKO_{i+j+1}(\mathbb{R}, \mathbb{R})$ which is a group generated by one element and not a number, and so still needs interpretation. While this number can sometimes be interpreted as an index (modulo 2 valued, in certain degrees) we lack a better understanding, which had been possible in the complex case due to the use of derivations which make up the cyclic cohomology classes.

Apart from [8], our work relies most substantially on [9] which used real Kasparov theory to derive the groups that appear in the ‘periodic table of topological insulators and superconductors’ as outlined by Kitaev [35]. Our work also builds on and complements that in [17] for the commutative case and [21, 29, 37, 58] for the noncommutative approach. We consider systems with weak disorder (that is, disordered Hamiltonians retaining a spectral gap). The substantial problem of strong disorder and localisation will not be treated here.

1.1. Relation to other work. There have been several mathematical papers detailing aspects of the bulk-edge correspondence for topological insulators with anti-linear symmetries. Graf and Porta prove a bulk-edge correspondence for two-dimensional Hamiltonians with odd time-reversal symmetry using Bloch bundles and without reference to K -theory [20]. Similar results are obtained by Avilla, Schulz-Baldes and Villegas-Blas using spin Chern numbers and an argument involving transfer matrices [3, 56]. Spin Chern numbers are related to the noncommutative approach to the quantum Hall effect and, as such, allow samples with disorder to be considered.

An alternative approach is taken by Loring, who derives the invariants of topological phases by considering almost commuting Hermitian matrices and their Clifford pseudospectrum [42]. Such a viewpoint gives expressions for the invariants of interest that are amenable to numerical simulation. Loring also relates indices associated to d and $d + 1$ dimensional systems, a bulk-edge correspondence [42, Section 7]. What is less clear is the link between Loring’s results and earlier results on the bulk-edge correspondence for the quantum Hall effect, particularly [32, 34].

Papers by Mathai and Thiang establish a K -theoretic bulk-edge correspondence for time-reversal symmetric systems [43, 44], and more recently with Hannabuss [22] consider the general case of topological insulators. Mathai and Thiang show that the invariants of interest in K -theory pass from bulk to edge under the boundary map of the real or complex Pimsner–Voiculescu sequence. Mathai and Thiang also show that, under T -duality, the boundary map in K -theory of tori can be expressed as the conceptually simpler restriction map. Similar results also appear in the work of Li, Kaufmann and Wehefritz-Kaufmann, who consider time-reversal symmetric systems and their relation to the KO , KR and KQ groups of \mathbb{T}^d [40]. A bulk-edge correspondence then links the topological K -groups associated to the bulk and boundary using the Baum–Connes isomorphism and Poincaré duality.

Recent work by Kubota establishes a bulk-edge correspondence in K -theory for topological phases of quite general type [37]. Kubota follows the general framework of [17, 58] and considers twisted equivariant K -groups of uniform Roe algebras, which can be computed using the coarse Mayer–Vietoris exact sequence and coarse Baum–Connes map. Classes in such groups are associated to gapped Hamiltonians compatible with a twisted symmetry group. An edge invariant is also defined and is shown to be isomorphic to the bulk class under the boundary map of the coarse Mayer–Vietoris exact sequence in K -theory [36, 37]. The use of Roe algebras and coarse geometry means that there is the potential for systems with impurities and uneven edges to be considered.

1.2. Outline of the paper. We begin with a short review of unbounded Kasparov theory in Section 2. We particularly focus on real Kasparov theory as it is relatively understudied in the literature.

The central mathematical content of this paper is in Section 3, where we construct fundamental K -cycles $\lambda^{(d)}$ for (possibly twisted) \mathbb{Z}^d -actions of unital C^* -algebras. We can then link actions of different order by the Pimsner–Voiculescu short exact sequence [50], which we represent with an unbounded Kasparov module. Finally we use the unbounded product to show that the fundamental K -cycle $\lambda^{(d)}$ can be factorised into the product of the extension Kasparov module and $\lambda^{(d-1)}$ (up to a basis ordering of Clifford algebra elements, which may introduce a minus sign). These results are then applied to C^* -algebraic models of disordered or aperiodic media in Section 4, where the (bulk) algebra of interest is the twisted crossed product $C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d$.

In Section 5 we relate our result to topological phases by pairing the K -cycle $\lambda^{(d)}$ with a K -theory class $[x_b]$ related to gapped Hamiltonians with time reversal and/or particle-hole and/or chiral symmetry. We also include a detailed discussion on the computation of the bulk and edge pairings, using both Clifford modules and the Atiyah–Bott–Shapiro construction as well as semifinite spectral triples and the semifinite local index formula. Limitations and open questions are also considered.

Acknowledgements. All authors thank the Hausdorff Research Institute for Mathematics for support. CB and AR thank Alan Carey, Magnus Goffeng and Guo Chuan Thiang for useful discussions. CB and AR acknowledge the support of the Australian Research Council.

2. PRELIMINARIES ON REAL KASPAROV THEORY

For the convenience of the reader and to establish notation we briefly summarise the results in real Kasparov theory of use to us for the bulk-edge correspondence. The reader may consult [27, 55] or [9, Appendix A] for more information.

2.1. KKO -groups. The use of Clifford algebras to define higher KK -groups makes it important to work with \mathbb{Z}_2 -graded C^* -algebras and \mathbb{Z}_2 -graded tensor products, $\hat{\otimes}$. A basic reference is [7, §14]. Given a real C^* -module E_B (written E when B is clear) over a \mathbb{Z}_2 -graded C^* -algebra B , we denote by $\text{End}_B(E)$ the algebra of endomorphisms of E which are adjointable with respect to the B -valued inner-product $(\cdot | \cdot)_B$ on E_B . The algebra of compact endomorphisms $\text{End}_B^0(E)$ is generated by the operators Θ_{e_1, e_2} for $e_1, e_2 \in E$ such that for $e_3 \in E$

$$\Theta_{e_1, e_2}(e_3) = e_1 \cdot (e_2 | e_3)_B$$

with $e \cdot b$ the (graded) right-action of B on E .

Definition 2.1. Let A and B be \mathbb{Z}_2 -graded C^* -algebras. A real unbounded Kasparov module $(\mathcal{A}, {}_\pi E_B, D)$ is a \mathbb{Z}_2 -graded real C^* -module E_B , a dense $*$ -subalgebra $\mathcal{A} \subset A$ with graded real homomorphism $\pi : \mathcal{A} \rightarrow \text{End}_B(E)$, and an unbounded regular odd operator D such that for all $a \in \mathcal{A}$

$$[D, \pi(a)]_\pm \in \text{End}_B(E), \quad \pi(a)(1 + D^2)^{-1/2} \in \text{End}_B^0(E).$$

Where unambiguous, we will omit the representation π and write unbounded Kasparov modules as (\mathcal{A}, E_B, D) . The results of Baaj and Julg [4] continue to hold for real Kasparov modules, so given an unbounded module (\mathcal{A}, E_B, D) we apply the bounded transformation $F_D = D(1 + D^2)^{-1/2}$ to obtain the real (bounded) Kasparov module (A, E_B, F_D) , where A is the C^* -closure of the dense subalgebra \mathcal{A} .

One can define notions of unitary equivalence, homotopy and degenerate Kasparov modules in the real setting (see [27, §4]). Hence we can define the group $KKO(A, B)$ as the equivalence classes of real (bounded) Kasparov modules modulo the equivalence relation generated by these relations.

Clifford algebras are used to define higher KKO -groups and encode periodicity. In the real setting, we define $C\ell_{p,q}$ to be the real span of the mutually anti-commuting generators $\gamma^1, \dots, \gamma^p$ and ρ^1, \dots, ρ^q such that

$$(\gamma^i)^2 = 1, \quad (\gamma^i)^* = \gamma^i, \quad (\rho^i)^2 = -1, \quad (\rho^i)^* = -\rho^i.$$

We now recall the relation between real KK -groups and real K -theory.

Proposition 2.2 ([27], §6, Theorem 3). *For trivially graded, σ -unital real algebras A , there is an isomorphism $KKO(C\ell_{n,0}, A) \cong KO_n(A)$.*

Each short exact sequence of real C^* -algebras $0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$ with ideal B and quotient algebra A gives rise to an element of the extension group $\text{Ext}_{\mathbb{R}}(A, B)$ and this group is related to the real Kasparov KK -groups.

Proposition 2.3 ([27], §7). *If A and B are separable real C^* -algebras, then*

$$\text{Ext}_{\mathbb{R}}^{-1}(A, B) \cong KKO(A \hat{\otimes} C\ell_{0,1}, B) \cong KKO(A, B \hat{\otimes} C\ell_{1,0}).$$

2.2. The product. The generality of the constructions and proofs in [27] mean that all the central results in complex KK -theory carry over into the real (and Real) setting. In particular, the intersection product

$$KKO(A, B) \times KKO(B, C) \rightarrow KKO(A, C)$$

is still a well-defined map and other important properties such as stability

$$KKO(A \hat{\otimes} \mathcal{K}(\mathcal{H}), B) \cong KKO(A, B)$$

continue to hold, where $\mathcal{K}(\mathcal{H})$ is the algebra of real compact operators on a separable real Hilbert space.

Let $(\mathcal{A}, E_B^1, D_1)$ and $(\mathcal{B}, E_C^2, D_2)$ be unbounded real A - B and B - C Kasparov modules. We would like to take the product at the unbounded level. Naively one would like to use the formula

$$(\mathcal{A}, (E^1 \hat{\otimes}_B E^2)_C, D_1 \hat{\otimes} 1 + 1 \hat{\otimes} D_2),$$

where $(E^1 \hat{\otimes}_B E^2)_C$ is the B -balanced \mathbb{Z}_2 -graded tensor product. This does not make sense as D_2 is not \mathcal{B} -linear (nor B -linear) and so $1 \hat{\otimes} D_2$ does not descend to the balanced tensor product. Instead one needs to choose a connection on E_B^1 to correct the naive formula for $1 \hat{\otimes} D_2$. First define the ‘ D_2 one forms’ by

$$\Omega_{D_2}^1(\mathcal{B}) = \left\{ \sum_j b_j [D_2, c_j] \in \text{End}_C(E^2) : b_j, c_j \in \mathcal{B} \right\}.$$

Then a D_2 connection on E^1 is a choice of dense \mathcal{B} submodule $\mathcal{E}^1 \subset E^1$ and a linear map

$$\nabla : \mathcal{E}^1 \rightarrow \mathcal{E}^1 \hat{\otimes}_B \Omega_{D_2}^1(\mathcal{B}) \quad \text{such that} \quad \nabla(eb) = \nabla(e)b + e \hat{\otimes} [D_2, b].$$

Setting $m : \Omega_{D_2}^1(\mathcal{B}) \hat{\otimes} E^2 \rightarrow E^2$ to be $m(a[D_2, b] \hat{\otimes} f) = a[D_2, b]f$, we then define

$$(1 \hat{\otimes}_{\nabla} D_2)(e \hat{\otimes} f) = (-1)^{|e|} e \hat{\otimes} D_2 f + (1 \hat{\otimes} m)(\nabla(e) \hat{\otimes} f),$$

with $|e|$ the degree of e in the \mathbb{Z}_2 -graded module E_B^1 . A short but illuminating calculation shows that $1 \hat{\otimes}_{\nabla} D_2$ is well-defined on the balanced tensor product, and it is reasonable to hope that the formula

$$(1) \quad D := D_1 \hat{\otimes} 1 + 1 \hat{\otimes}_{\nabla} D_2$$

would define an operator on $E^1 \hat{\otimes}_B E^2$ such that $(\mathcal{A}, (E^1 \hat{\otimes}_B E^2)_C, D)$ represents the product class.

In fact this is true in very many cases as proved in [24, 41, 45, 47]. These papers provide very general settings where the formula (1) can be guaranteed to produce an unbounded Kasparov module representing the product. These proofs, all in the complex case, proceed by showing that the conditions of Kucerovsky’s theorem are satisfied. We will not develop such a general framework, but concretely construct potential representatives according to the recipe in Equation (1), and check Kucerovsky’s conditions directly. Importantly, Kucerovsky’s theorem is valid in the real case.

To state Kucerovsky’s conditions, we start by defining a creation operator. Given $e_1 \in E_B^1$ and a $*$ -homomorphism $\psi : B \rightarrow \text{End}_C(E^2)$, we let $T_{e_1} \in \text{Hom}_C(E^2, E^1 \otimes_B E^2)$

be given by $T_{e_1}e_2 = e_1 \hat{\otimes} e_2$. One can check that T_{e_1} is adjointable with $T_{e_1}^*(f_1 \hat{\otimes} e_2) = \psi((e_1|f_1)_B)e_2$.

Theorem 2.4 (Kucerovsky's criteria [38], Theorem 13). *Let $(\mathcal{A}, \pi_1 E_B^1, D_1)$ and $(\mathcal{B}, \pi_2 E_C^2, D_2)$ be unbounded Kasparov modules. Write $E := E^1 \hat{\otimes}_B E^2$. Suppose that $(\mathcal{A}, \pi_1 E_C, D)$ is an unbounded Kasparov module such that*

Connection condition: *For all e_1 in a dense subspace of $\pi_1(A)E^1$, the commutators*

$$\left[\begin{pmatrix} D & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_{e_1} \\ T_{e_1}^* & 0 \end{pmatrix} \right]$$

are bounded on $\text{Dom}(D \oplus D_2) \subset E \oplus E^2$;

Domain condition: $\text{Dom}(D) \subset \text{Dom}(D_1 \hat{\otimes} 1)$;

Positivity condition: *For all $e \in \text{Dom}(D)$,*

$$((D_1 \hat{\otimes} 1)e|De) + (De|(D_1 \hat{\otimes} 1)e) \geq K(e|e)$$

for some $K \in \mathbb{R}$.

Then the class of $(\mathcal{A}, \pi_1 E_C, D)$ in $KK(A, C)$ represents the Kasparov product.

In fact, if ∇ satisfies an extra Hermiticity condition (which can always be achieved), an operator of the form $D = D_1 \hat{\otimes} 1 + 1 \hat{\otimes} \nabla D_2$ always satisfies the domain condition and connection condition. Under mild hypotheses, the operator $1 \hat{\otimes} \nabla D_2$ will be self-adjoint [47, Theorem 3.17] and the sum will have locally compact resolvent (that is, $\pi_1(a)(1 + D^2)^{-1/2}$ compact for $a \in \mathcal{A}$) [24, Theorem 6.7]. The self-adjointness of the sum and the boundedness of commutators $[D, \pi_1(a)]$ needs to be checked directly, as does the positivity condition. In our examples all these extra conditions are satisfied and so the task of checking that we have a spectral triple representing the product is relatively straightforward.

2.3. Semifinite theory. An unbounded A - \mathbb{C} or A - \mathbb{R} Kasparov module is precisely a complex or real spectral triple as defined by Connes. Complex spectral triples satisfying additional regularity properties have the advantage that the local index formula by Connes and Moscovici [14] gives computable expressions for the index pairing with K -theory, a special case of the Kasparov product

$$K_*(A) \times KK^*(A, \mathbb{C}) \rightarrow K_0(\mathbb{C}) \cong \mathbb{Z}.$$

We would like to find computable expressions for more general Kasparov products. To do this, we generalise the definition of spectral triple using more general semifinite von Neumann algebras in place of the bounded operators on Hilbert space.

Let τ be a fixed faithful, normal, semifinite trace on a von Neumann algebra \mathcal{N} . We let $\mathcal{K}_{\mathcal{N}}$ be the τ -compact operators in \mathcal{N} (that is, the norm closed ideal generated by the projections $P \in \mathcal{N}$ with $\tau(P) < \infty$).

Definition 2.5 ([12]). A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ relative to (\mathcal{N}, τ) is given by a \mathbb{Z}_2 -graded Hilbert space \mathcal{H} , a graded $*$ -algebra $\mathcal{A} \subset \mathcal{N}$ with (graded) representation on \mathcal{H} and a densely defined odd unbounded self-adjoint operator D affiliated to \mathcal{N} such that

- (1) $[D, a]_{\pm}$ is well-defined on $\text{Dom}(D)$ and extends to a bounded operator on \mathcal{H} for all $a \in \mathcal{A}$,
- (2) $a(1 + D^2)^{-1/2} \in \mathcal{K}_{\mathcal{N}}$ for all $a \in \mathcal{A}$.

A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is QC^∞ if for $\partial = [(1 + D^2)^{1/2}, \cdot]$ and $b \in \mathcal{A} \cup [D, \mathcal{A}]$, $\partial^j(b) \in \mathcal{N}$ for all $j \in \mathbb{N}$.

A unital semifinite spectral triple is p -summable if $(1 + D^2)^{s/2}$ is τ -trace-class for all $s > p$.

If we take $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and $\tau = \text{Tr}$, then we recover the usual definition of a spectral triple.

Theorem 2.6 ([25]). *Let $(\mathcal{A}, \mathcal{H}, D)$ be a complex semifinite spectral triple associated to (\mathcal{N}, τ) with \mathcal{A} separable and \mathcal{A} its C^* -completion. Then $(\mathcal{A}, \mathcal{H}, D)$ determines a class in $KK(A, C)$ with C a separable C^* -subalgebra of $\mathcal{K}_{\mathcal{N}}$.*

Semifinite spectral triples can also be paired with K -theory elements by the following composition

$$(2) \quad K_0(A) \times KK(A, C) \rightarrow K_0(C) \xrightarrow{\tau_*} \mathbb{R},$$

with the class in $KK(A, C)$ coming from Theorem 2.6. The range of the semifinite index pairing is a discrete subset of \mathbb{R} and can potentially detect finer invariants than the usual index pairing. Importantly, the local index formula can be generalised to QC^∞ and p -summable semifinite spectral triples for $p \geq 1$ [12, 13]. Hence the semifinite local index formula may be used to compute the map in Equation (2).

Suppose that (\mathcal{A}, E_B, D) is an unbounded Kasparov module and the algebra B possesses a faithful semifinite norm-lower semicontinuous trace. Then one can often construct a semifinite spectral triple (see for example [49]). The passage from Kasparov module to semifinite spectral triple is advantageous as the algebra B is usually more closely linked to the example or problem under consideration than the algebra C from Theorem 2.6. If a sufficiently regular (complex) semifinite spectral triple can be constructed, then we may use the semifinite local index formula to compute the map given in Equation (2) (with C replaced by B). Therefore semifinite spectral triples and index theory can be employed in order to compute pairings of complex K -theory classes with unbounded Kasparov modules.

Remark 2.7 (Real semifinite spectral triples and the local index formula). While the definition of a semifinite spectral triple and semifinite index pairing can be extended to real algebras, the semifinite local index formula can not be used to detect torsion invariants as it involves a mapping to cyclic cohomology. However, one may naturally ask whether the semifinite local index formula can be used to detect integer invariants in the real setting (e.g. arising from $K_0(\mathbb{R})$ or $K_4(\mathbb{R})$). We are of the opinion that a local formula can be employed to access real integer invariants as in [12, 13], though the details of the proof given for the complex case need to be checked for the real case.

3. KASPAROV MODULES AND BOUNDARY MAPS OF TWISTED \mathbb{Z}^d -ACTIONS

Our motivation is to study topological invariants of disordered or aperiodic topological states in the tight-binding approximation. Suppose A is an observable algebra acting on $\ell^2(\mathbb{Z}^d)$ and is such that $A \cong C \rtimes_{\beta} \mathbb{Z}$ with C acting on $\ell^2(\mathbb{Z}^{d-1})$. We consider the Toeplitz extension [50, 55]

$$0 \rightarrow C \otimes \mathcal{K}(\ell^2(\mathbb{N})) \rightarrow \mathcal{T}(\beta) \rightarrow A \rightarrow 0.$$

As in [32, 52] and Section 4.1, the algebra $\mathcal{T}(\beta)$ acts on $\ell^2(\mathbb{Z}^{d-1} \times \mathbb{N})$ and can be interpreted as the algebra of observables on the space with boundary. The ideal $C \otimes \mathcal{K}$ is thought of as the observables concentrated at the boundary $\ell^2(\mathbb{Z}^{d-1} \times \{0\})$ with compact decay into the interior of the sample. The quotient A acts on a space without boundary (approximating

the interior of a sample) and is often called the bulk algebra. It follows from Kasparov's work that any short exact sequence of (trivially graded) C^* -algebras gives rise to a long exact sequence in K -homology whose boundary map is given by the internal Kasparov product with the extension class defined by the Toeplitz extension. Hence topological properties of the algebra without boundary A can be related to the edge algebra C by the boundary map of K -theory or K -homology (or KK -theory more generally).

Boundary maps in KK -theory are in general very hard to compute. In our case we use the fact that for disordered media our bulk and edge algebras arise as (twisted) crossed products. Therefore we consider $A \cong B \rtimes_{\alpha, \theta} \mathbb{Z}^d$ and $C \cong B \rtimes_{\alpha_{\parallel}, \theta} \mathbb{Z}^{d-1}$ for B a separable and unital (real or complex) C^* -algebra. The dynamics of the crossed product allows us to build explicit unbounded Kasparov modules for the A and C , which are then shown to be directly related under the boundary map in KK -theory. Our formula is explicit and avoids homotopy arguments.

In Section 4 we will specialise to the physically interesting case of $B = C(\Omega)$, where $C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d$ is the (bulk) observable algebra of a disordered or aperiodic solid state system [5, 52].

3.1. Fundamental K -cycles for \mathbb{Z}^d -actions. We construct an unbounded Kasparov module encoding a (twisted) \mathbb{Z}^d -action on a real or complex C^* -algebra B . We remark that similar constructions appear in [23, 49, 51] in the complex setting.

Let B be a separable and unital C^* -algebra with action α of \mathbb{Z}^d and twisting cocycle $\theta : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathcal{U}(B)$, with $\mathcal{U}(B)$ the unitaries of B . The twisted crossed product $A := B \rtimes_{\alpha, \theta} \mathbb{Z}^d$ is the universal C^* -completion of the algebraic crossed product $\mathcal{A} := B_{\alpha, \theta} \mathbb{Z}^d$ given by finite sums $\sum_{n \in \mathbb{Z}^d} S^n b_n$ where $b_n \in B$, $n \in \mathbb{Z}^d$ is a multi-index and $S^n = S_1^{n_1} \cdots S_d^{n_d}$ is a product of powers of d abstract unitary elements S_i subject to the multiplication extending that of B by

$$S_i b = \alpha_i(b) S_i, \quad S_i S_j = \theta_{ij} S_j S_i, \quad S_i^* = S_i^{-1}.$$

The map α_i is the automorphism corresponding to the action of $e_i \in \mathbb{Z}^d$ for e_i the standard generators of \mathbb{Z}^d . The elements θ_{ij} belong to B and can be obtained from the cocycle θ .

As an example, let us consider the case $d = 1$ in which $\theta = 1$. Given a left action ρ of B on a module M one obtains a left action π of $B_\alpha \mathbb{Z}$ on the module given by the algebraic tensor product $\ell^2(\mathbb{Z}) \otimes M$. The action π is given on elementary tensors by

$$(3) \quad \pi(b)(\delta_j \otimes \xi) = \delta_j \otimes \rho(\alpha^{-j}(b))\xi, \quad \pi(S)(\delta_j \otimes \xi) = \delta_{j+1} \otimes \xi$$

with $\{\delta_j\}_{j \in \mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$ and $\xi \in M$. We will be interested in the case where M is a C^* -module so that we can expect the action to extend to an action of the C^* -crossed product $B \rtimes_\alpha \mathbb{Z}$ on the completion of the algebraic tensor product $\ell^2(\mathbb{Z}) \otimes M$ which we denote by $\ell^2(\mathbb{Z}, M)$.

The above procedure iterated for higher-order crossed products yields an action of $A = B \rtimes_{\alpha, \theta} \mathbb{Z}^d$ on $\ell^2(\mathbb{Z}^d, M)$ if A can be rewritten as a d -fold iterated crossed product with \mathbb{Z} . This is always the case if we allow B to be replaced by its stabilisation [48], but the left-action may also be modified to yield directly a representation of $B \rtimes_{\alpha, \theta} \mathbb{Z}^d$ [48].

We consider the left regular representation of B on the C^* -module B_B and obtain a representation of the crossed product $B \rtimes_{\alpha, \theta} \mathbb{Z}^d$ on the C^* -module $\ell^2(\mathbb{Z}^d) \otimes B =: \ell^2(\mathbb{Z}^d, B)$. More precisely, the map $\iota : B_{\alpha, \theta} \mathbb{Z}^d \rightarrow \ell^2(\mathbb{Z}^d, B)$, $\iota(S^n b) = \delta_n \otimes b$ for $\{\delta_n\}_{n \in \mathbb{Z}^d}$ the canonical basis of $\ell^2(\mathbb{Z}^d)$ is an inclusion of the dense subalgebra \mathcal{A} into the C^* -module such that the representation of $\mathcal{A} = B_{\alpha, \theta} \mathbb{Z}$ on the image of ι is given by left multiplication:

$$S^n b_1 \cdot (\delta_m \otimes b_2) = \iota(S^n b_1 S^m b_2) = \delta_{n+m} \otimes \alpha^{-m-n}(\theta(m, n)) \alpha^{-m}(b_1) b_2$$

with multi-index notation $\alpha^m = \alpha_1^{m_1} \cdots \alpha_d^{m_d}$. The C^* -module is the completion of $\iota(\mathcal{A})$ with respect to the B -valued inner product

$$(\lambda_1 \otimes b_1 \mid \lambda_2 \otimes b_2)_B = \langle \lambda_1, \lambda_2 \rangle_{\ell^2(\mathbb{Z}^d)} b_1^* b_2$$

and norm $\|\xi\|^2 = \|(\xi \mid \xi)_B\|_B$. The C^* -module also carries a right-action of B by right-multiplication. Note that on $\iota(\mathcal{A})$ the inner product may be written as

$$(\iota(a_1) \mid \iota(a_2))_B = \Phi_0(a_1^* a_2), \quad \Phi_0(S^n b) = \begin{cases} b, & n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

As the conditional expectation $\Phi_0 : A \rightarrow B$ is positive, $\Phi_0(b^* a b) \leq \|a\|_A \Phi_0(b^* b)$ for any positive $a \in A$ and $b \in B$.

Proposition 3.1. *The left-action of \mathcal{A} on $\ell^2(\mathbb{Z}^d, B)$ extends to an adjointable representation of $A = B \rtimes_{\alpha, \theta} \mathbb{Z}^d$.*

Proof. We first check adjointability of the elements of \mathcal{A} acting on $\iota(\mathcal{A})$ by computing

$$\begin{aligned} (a \cdot \iota(a_1) \mid \iota(a_2))_B &= \Phi_0((a a_1)^* a_2) \\ &= \Phi_0(a_1 a^* a_2) = (\iota(a_1) \mid a^* \cdot \iota(a_2))_B \end{aligned}$$

Furthermore, the action of \mathcal{A} on $\iota(\mathcal{A})$ is uniformly bounded:

$$\begin{aligned} \|a\|_{\text{End}}^2 &= \sup_{\substack{a' \in \mathcal{A} \\ \|\iota(a')\|=1}} \|(a \cdot \iota(a') \mid a \cdot \iota(a'))_B\| \\ &\leq \sup_{\substack{a' \in \mathcal{A} \\ \|\iota(a')\|=1}} \|a^* a\|_A \|\Phi_0(a'^* a')\| = \|a^* a\|_A. \end{aligned}$$

Hence the action extends by continuity, first from \mathcal{A} on $\iota(\mathcal{A})$ to a bounded action of \mathcal{A} on $\ell^2(\mathbb{Z}^d, B)$ and then to a bounded action of the whole C^* -algebra A on $\ell^2(\mathbb{Z}^d, B)$. It then follows that all elements of A are adjointable. \square

Inside $\ell^2(\mathbb{Z}^d, B)$ we consider the elements $e_m = \delta_m \otimes 1_B$ for $m \in \mathbb{Z}^d$. We compute that for $b \in B$

$$\begin{aligned} \Theta_{e_l, e_m} e_n b &= \Theta_{e_l, e_m} (\delta_n \otimes b) = (\delta_l \otimes 1_B) \cdot (\delta_m \otimes 1_B \mid \delta_n \otimes b)_B \\ &= \delta_{m, n} \delta_l \otimes b = \delta_{m, n} e_l \cdot b. \end{aligned}$$

In particular we note that

$$\sum_{m \in \mathbb{Z}^d} \Theta_{e_m, e_m} = \text{Id}_{\ell^2(\mathbb{Z}^d, B)}$$

and so $\{e_m\}_{m \in \mathbb{Z}^d}$ is a frame for the module $\ell^2(\mathbb{Z}^d, B)$.

The last ingredient we need for a Kasparov module is a Dirac-like operator, which we construct using the position operators, $X_j : \text{Dom}(X_j) \rightarrow \ell^2(\mathbb{Z}^d, B)$ for $j \in \{1, \dots, d\}$ such that $X_j(\delta_m \otimes b) = m_j(\delta_m \otimes b)$ for any $m \in \mathbb{Z}^d$. We construct the Dirac-like operator by employing an irreducible Clifford representation. On the graded vector space $\bigwedge^* \mathbb{R}^d$ (we denote the grading by $\gamma_{\bigwedge^* \mathbb{R}^d}$) there is a representation of $C\ell_{d,0}$ and a representation of $C\ell_{0,d}$. The generators γ^j of $C\ell_{d,0}$ and the generators ρ^j of $C\ell_{0,d}$ act by

$$\gamma^j(w) = e_j \wedge w + \iota(e_j)w, \quad \rho^j(w) = e_j \wedge w - \iota(e_j)w,$$

for $\{e_j\}_{j=1}^d$ the standard basis of \mathbb{R}^d , $w \in \bigwedge^* \mathbb{R}^d$ and $\iota(v)w$ the contraction of w along v . These two actions graded-commute. On the tensor product space $\ell^2(\mathbb{Z}^d, B) \otimes \bigwedge^* \mathbb{R}^d$ we define

$$D := \sum_{j=1}^d X_j \otimes \gamma^j.$$

A simple check shows that D is odd, self-adjoint and regular on $\ell^2(\mathbb{Z}^d, B) \otimes \bigwedge^* \mathbb{R}^d$.

Proposition 3.2. *Consider a possibly twisted \mathbb{Z}^d -action α, θ on a separable and unital C^* -algebra B . Let A be the associated crossed product with dense subalgebra $\mathcal{A} = B_{\alpha, \theta} \mathbb{Z}^d$. The data*

$$\lambda^{(d)} = \left(\mathcal{A} \hat{\otimes} C\ell_{0,d}, \ell^2(\mathbb{Z}^d, B)_B \otimes \bigwedge^* \mathbb{R}^d, \sum_{j=1}^d X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^d} \right)$$

defines an unbounded $A \hat{\otimes} C\ell_{0,d}$ - B Kasparov module. The $C\ell_{0,d}$ -action is generated by the operators ρ^j . In the complex case we have \mathbb{C} in place of \mathbb{R} in the above formula.

Proof. The left-action of A is adjointable by Proposition 3.1. By construction D graded-commutes with the left Clifford representation. What remains to be checked is that $[D, a] = \sum_j [X_j, a] \otimes \gamma^j$ is adjointable for $a \in \mathcal{A}$ and $(1 + D^2)^{-1/2}$ is compact. It is directly verified that

$$[X_j, S^m b] = n_j S^m b$$

and so we see that $[X_j, a] \in \mathcal{A}$ for all $a \in \mathcal{A}$. In particular the commutator is adjointable. Furthermore, $D^2 = \sum_j X_j^2 \otimes 1_{\bigwedge^* \mathbb{R}^d}$ and hence $(1 + D^2)^{-1/2} = (1 + |X|^2)^{-1/2} \otimes 1_{\bigwedge^* \mathbb{R}^d}$, where $|X|^2 = \sum_j X_j^2$. Using the frame $\{e_m\}_{m \in \mathbb{Z}^d}$, we note that $(1 + |X|^2)e_m = (1 + |m|^2)e_m$ and so

$$(1 + D^2)^{-\frac{1}{2}} = \sum_{m \in \mathbb{Z}^d} (1 + |m|^2)^{-\frac{1}{2}} \Theta_{e_m, e_m} \otimes 1_{\bigwedge^* \mathbb{R}^d},$$

which is a norm-convergent sum of finite-rank operators and so is compact. \square

We call $\lambda^{(d)}$ the fundamental K -cycle of the \mathbb{Z}^d -action because of its similarity to Kasparov's fundamental class [28].

3.2. The extension K -cycle. Under mild assumptions on the twist θ (see [31]), we can unwind the crossed product $A = B \rtimes_{\alpha, \theta} \mathbb{Z}^d$ such that, for $\alpha = (\alpha^\parallel, \alpha_d)$ and α^\parallel the restricted action of \mathbb{Z}^{d-1} ,

$$(4) \quad A = (B \rtimes_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1}) \rtimes_{\alpha_d} \mathbb{Z} = C \rtimes_{\alpha_d} \mathbb{Z}$$

where $C = B \rtimes_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1}$. We link C and $C \rtimes_{\alpha_d} \mathbb{Z}$ by the Toeplitz extension, which we briefly recall.

Very similar to the construction of the crossed product $C \rtimes_{\alpha_d} \mathbb{Z}$, we can consider $C_{\alpha_d} \mathbb{N}$ the algebra given by finite sums $\sum_{k \in \mathbb{N}} \tilde{S}_d^k c_k + (\tilde{S}_d^*)^k c'_k$, where $c_k, c'_k \in C$ and \tilde{S}_d is the operator such that

$$\tilde{S}_d b = \alpha_d(b) \tilde{S}_d, \quad \tilde{S}_d^* b = \alpha_d^{-1}(b) \tilde{S}_d, \quad \tilde{S}_d^* \tilde{S}_d = 1, \quad \tilde{S}_d \tilde{S}_d^* = 1 - p$$

with $p = p^* = p^2$ a projection. Thus \tilde{S}_d is no longer unitary but an isometry. There is a unique $*$ -algebra morphism $q : C_{\alpha_d} \mathbb{N} \rightarrow C_{\alpha_d} \mathbb{Z}$ determined by $q(\tilde{S}_d) = S_d$ and is the identity on C . Its kernel is the ideal generated by p which can easily be seen to be isomorphic to $F \otimes C$ where F is the algebra of the finite rank operators. The exact sequence

$$0 \rightarrow F \otimes C \rightarrow C_{\alpha_d} \mathbb{N} \xrightarrow{q} C_{\alpha_d} \mathbb{Z} \rightarrow 0$$

is the algebraic version of the Toeplitz extension, the C^* -version is obtained by taking the universal C^* -closures. The C^* -closure of $C_{\alpha_d}\mathbb{N}$, denoted by $\mathcal{T}(\alpha_d)$, is the Toeplitz algebra of the \mathbb{Z} -action α_d and the closure of $F \otimes C$ is $\mathcal{K} \otimes C$, with \mathcal{K} the algebra of complex operators on a separable (real or complex) Hilbert space. The short exact sequence

$$(5) \quad 0 \rightarrow \mathcal{K} \otimes C \rightarrow \mathcal{T}(\alpha_d) \rightarrow C \rtimes_{\alpha_d} \mathbb{Z} \rightarrow 0$$

gives rise to a class $[\text{ext}]$ in the extension group $\text{Ext}^{-1}(A, C)$, which by Proposition 2.3 is the same as the group $KKO^1(A, C)$ (or in the complex case $KK^1(A, C)$).

The extension class $[\text{ext}]$ serves to compute boundary maps in K -theory and K -homology, namely by taking Kasparov products with it. In order to make these maps computable, we construct an unbounded representative of $[\text{ext}]$.

Proposition 3.3. *Let C be a separable and unital C^* -algebra and $A = C \rtimes_{\alpha_d} \mathbb{Z}$. The extension class of the Toeplitz extension of Equation (5) is represented by the fundamental K -cycle of the \mathbb{Z} -action,*

$$(6) \quad \left(C_{\alpha_d} \mathbb{Z} \hat{\otimes} C \ell_{0,1}, \ell^2(\mathbb{Z}, C)_C \otimes \bigwedge^* \mathbb{R}, D = X_1 \otimes \gamma^1, \gamma_{\wedge^* \mathbb{R}} \right).$$

There is an analogous result for complex algebras.

It serves the clarity of arguments further down to denote $D = N \otimes \gamma_{\text{ext}}$ for the extension module.

Proof. The Kasparov module from Equation (6) defines the same KK -class as the bounded and ungraded cycle

$$(C \rtimes_{\alpha_d} \mathbb{Z}, \ell^2(\mathbb{Z}, C)_C, 2P - 1)$$

where $P = \chi_{[0,\infty)}(N)$. From [54, Proposition 3.14] we know that this class is the extension class of the short exact sequence

$$0 \rightarrow \text{End}_C^0[P(\ell^2(\mathbb{Z}, C))] \rightarrow C^*(PC_{\alpha_d} \mathbb{Z} P, \text{End}_C^0[P(\ell^2(\mathbb{Z}, C))]) \rightarrow Q \rightarrow 0,$$

where $C^*(PC_{\alpha_d} \mathbb{Z} P, \text{End}_C^0[P(\ell^2(\mathbb{Z}, C))])$ is a closed subalgebra of the adjointable operators, $\text{End}_C[P(\ell^2(\mathbb{Z}, C))]$, and Q the quotient. Now P is the projection onto $\ell^2(\mathbb{N}, C)$ and hence $PC_{\alpha_d} \mathbb{Z} P = C_{\alpha_d} \mathbb{N}$. Moreover, using [7, §13]

$$\text{End}_C^0[P(\ell^2(\mathbb{Z}, C))] \cong \mathcal{K}(\ell^2(\mathbb{N})) \otimes C.$$

Hence $C^*(PC_{\alpha_d} \mathbb{Z} P, \text{End}_C^0[P(\ell^2(\mathbb{Z}, C))])$ is the completion of $C_{\alpha_d} \mathbb{N}$ and Q is the completion of $C_{\alpha_d} \mathbb{Z}$ as required. \square

3.3. The product with the extension K -cycle. Suppose $d \geq 2$, then we can construct the fundamental K -cycles $\lambda^{(d-1)}$ and $\lambda^{(d)}$ representing classes in $KKO^{d-1}(B \rtimes_{\alpha_{\parallel, \theta}} \mathbb{Z}^{d-1}, B)$ and $KKO^d(B \rtimes_{\alpha_{\parallel, \theta}} \mathbb{Z}^d, B)$ respectively. The extension class representing Equation (5) gives a well-defined map

$$KKO^1(B \rtimes_{\alpha_{\parallel, \theta}} \mathbb{Z}^d, C) \times KKO^{d-1}(C, B) \rightarrow KKO^d(B \rtimes_{\alpha_{\parallel, \theta}} \mathbb{Z}^d, B)$$

(or complex). Our central result is that this Kasparov product of the extension K -cycle with the edge K -cycle $\lambda^{(d-1)}$ gives, up to a possible sign, $\lambda^{(d)}$, the fundamental K -cycle of our original crossed product $A = B \rtimes_{\alpha_{\parallel, \theta}} \mathbb{Z}^d$.

Theorem 3.4. *Let B be a separable and unital real or complex C^* -algebra with fundamental K -cycles $\lambda^{(d)}$ and $\lambda^{(d-1)}$ for (possibly twisted) \mathbb{Z}^d and \mathbb{Z}^{d-1} -actions. Then the unbounded Kasparov product of the extension Kasparov module from Proposition 3.3 with*

$\lambda^{(d-1)}$ gives, up to a cyclic permutation of the Clifford generators, the fundamental K -cycle $\lambda^{(d)}$. On the level of KK -classes this means

$$[\text{ext}] \hat{\otimes}_C [\lambda^{(d-1)}] = (-1)^{d-1} [\lambda^{(d)}],$$

where $-[x]$ denotes the inverse of the KK -class.

Proof. We will focus on the real setting though note that the case of complex algebras and modules follows the same argument. We denote by $C = B \rtimes_{\alpha \parallel, \theta} \mathbb{Z}^{d-1}$ and $A = B \rtimes_{\alpha, \theta} \mathbb{Z}^d \cong C \rtimes_{\alpha_d} \mathbb{Z}$ with dense subalgebras $\mathcal{C} = B_{\alpha \parallel, \theta} \mathbb{Z}^{d-1}$ and $\mathcal{A} = B_{\alpha, \theta} \mathbb{Z}^d$. We are taking the product of an $A \hat{\otimes} C \ell_{0,1}$ - C Kasparov module with a $C \hat{\otimes} C \ell_{0,d-1}$ - B Kasparov module. Our first step is an external product of the $A \hat{\otimes} C \ell_{0,1}$ - C Kasparov module with the identity class in $KKO(C \ell_{0,d-1}, C \ell_{0,d-1})$. This class can be represented by the Kasparov module

$$\left(C \ell_{0,d-1}, (C \ell_{0,d-1})_{C \ell_{0,d-1}}, 0, \gamma_{C \ell_{0,d-1}} \right)$$

with right and left actions given by right and left Clifford multiplication. The external product results in the $A \hat{\otimes} C \ell_{0,d} \hat{\otimes} C \ell_{0,d-1}$ Kasparov module represented by

$$\left(\mathcal{A} \hat{\otimes} C \ell_{0,d}, \left(\ell^2(\mathbb{Z}, C) \otimes \bigwedge^* \mathbb{R} \hat{\otimes} C \ell_{0,d-1} \right)_{C \hat{\otimes} C \ell_{0,d-1}}, N \otimes \gamma_{\text{ext}} \hat{\otimes} 1, \gamma_{\bigwedge^* \mathbb{R}} \hat{\otimes} \gamma_{C \ell_{0,d-1}} \right).$$

We now take the internal product of this module with the K -cycle for twisted \mathbb{Z}^{d-1} -actions, $[\lambda^{(d-1)}] \in KKO(C \hat{\otimes} C \ell_{0,d-1}, B)$ and represented by the unbounded Kasparov module

$$\left(\mathcal{C} \hat{\otimes} C \ell_{0,d-1}, \ell^2(\mathbb{Z}^{d-1}, B) \otimes \bigwedge^* \mathbb{R}^{d-1}, \sum_{j=1}^{d-1} X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^{d-1}} \right).$$

We start with the balanced tensor product of C^* -modules, where

$$\begin{aligned} & \left(\ell^2(\mathbb{Z}, C) \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R} \hat{\otimes}_{\mathbb{R}} C \ell_{0,d-1} \right) \hat{\otimes}_{C \hat{\otimes} C \ell_{0,d-1}} \left(\ell^2(\mathbb{Z}^{d-1}, B) \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R}^{d-1} \right) \\ & \cong \left(\ell^2(\mathbb{Z}, C) \otimes_C \ell^2(\mathbb{Z}^{d-1}, B) \right) \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R} \hat{\otimes}_{\mathbb{R}} \left(C \ell_{0,d-1} \cdot \bigwedge^* \mathbb{R}^{d-1} \right) \\ & \cong \left(\ell^2(\mathbb{Z}, C) \otimes_C \ell^2(\mathbb{Z}^{d-1}, B) \right) \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R} \hat{\otimes}_{\mathbb{R}} \bigwedge^* \mathbb{R}^{d-1} \end{aligned}$$

as the action of $C \ell_{0,d-1}$ on $\bigwedge^* \mathbb{R}^{d-1}$ by left-multiplication is nondegenerate.

Next we define $1 \otimes_{\nabla} X_j$ on the dense submodule $\ell^2(\mathbb{Z}, \mathcal{C}) \otimes_C \ell^2(\mathbb{Z}^{d-1}, B)$ for all $j \in \{1, \dots, d-1\}$, where

$$(7) \quad (1 \otimes_{\nabla} X_j) (\delta_k \otimes c \otimes \delta_n \otimes b) = \delta_k \otimes c \otimes X_j \delta_n \otimes b + \delta_k \otimes 1 \otimes [X_j, c] \delta_n \otimes b$$

with $\{\delta_k\}_{k \in \mathbb{Z}}$ an orthonormal basis of $\ell^2(\mathbb{Z})$, $c \in \mathcal{C}$, $\{\delta_n\}_{n \in \mathbb{Z}^{d-1}}$ an orthonormal basis for $\ell^2(\mathbb{Z}^{d-1})$ and $b \in B$. The operator is well-defined as $[X_j, c] \in \mathcal{C}$ for $c \in \mathcal{C}$. Then

$$(8) \quad \left(\mathcal{A} \hat{\otimes} C \ell_{0,1} \hat{\otimes} C \ell_{0,d-1}, \left(\ell^2(\mathbb{Z}, C) \otimes_C \ell^2(\mathbb{Z}^{d-1}, B) \right) \otimes \bigwedge^* \mathbb{R} \hat{\otimes}_{\mathbb{R}} \bigwedge^* \mathbb{R}^{d-1}, \right. \\ \left. N \otimes 1 \otimes \gamma_{\text{ext}} \hat{\otimes} 1 + \sum_{j=1}^{d-1} (1 \otimes_{\nabla} X_j) \otimes 1 \hat{\otimes} \gamma^j \right)$$

is a candidate for the unbounded Kasparov module representing the product, where the Clifford actions take the form

$$\begin{aligned} \rho_{\text{ext}} \hat{\otimes} 1 (\omega_1 \hat{\otimes} \omega_2) &= (e_1 \wedge \omega_1 - \iota(e_1) \omega_1) \hat{\otimes} \omega_2 \\ 1 \hat{\otimes} \rho^j (\omega_1 \hat{\otimes} \omega_2) &= (-1)^{|\omega_1|} \omega_1 \hat{\otimes} (e_j \wedge \omega_2 - \iota(e_j) \omega_2), \end{aligned}$$

for $j \in \{1, \dots, d-1\}$ with $|\omega_1|$ is the degree of the form ω_1 . Similar equations hold for $\gamma_{\text{ext}} \hat{\otimes} 1$ and $1 \hat{\otimes} \gamma^j$. Arguments very similar to the proof of Proposition 3.2 show that Equation (8) is a real or complex Kasparov module depending on what setting we are in. We now briefly check Kucerovsky's criterion (Theorem 2.4). The connection criterion holds precisely because we have used a connection ∇ to construct $1 \otimes_{\nabla} X_j$. The domain condition is a simple check and the positivity condition is explicitly checkable as the operators of interest act as number operators. Therefore the unbounded Kasparov module of Equation (8) is an unbounded representative of the product $[\text{ext}] \hat{\otimes}_C [\lambda^{(d-1)}]$ at the level of KK -classes.

Our next task is to relate the Kasparov module of Equation (8) to $\lambda^{(d)}$. We first identify $\bigwedge^* \mathbb{R} \hat{\otimes}_{\mathbb{R}} \bigwedge^* \mathbb{R}^{d-1} \cong \bigwedge^* \mathbb{R}^d$ and use the graded isomorphism $C\ell_{p,q} \hat{\otimes} C\ell_{r,s} \cong C\ell_{p+r, q+s}$ from [27, §2.16] on the left and right Clifford generators by the mapping

$$\begin{aligned} \rho_{\text{ext}} \hat{\otimes} 1 &\mapsto \rho^1, & 1 \hat{\otimes} \rho^j &\mapsto \rho^{j+1}, \\ \gamma_{\text{ext}} \hat{\otimes} 1 &\mapsto \gamma^1, & 1 \hat{\otimes} \gamma^j &\mapsto \gamma^{j+1}. \end{aligned}$$

Therefore applying this isomorphism gives the unbounded Kasparov module

$$\left(\mathcal{A} \hat{\otimes} C\ell_{0,d}, (\ell^2(\mathbb{Z}, C) \otimes_C \ell^2(\mathbb{Z}^{d-1}, B)) \otimes \bigwedge^* \mathbb{R}^d, N \otimes 1 \otimes \gamma^1 + \sum_{j=1}^{d-1} (1 \otimes_{\nabla} X_j) \otimes \gamma^{j+1} \right)$$

with $C\ell_{0,d}$ -action generated by $\rho^j(\omega) = e_j \wedge \omega - \iota(e_j)\omega$ and $C\ell_{d,0}$ -action generated by $\gamma^j(\omega) = e_j \wedge \omega + \iota(e_j)\omega$ for $\omega \in \bigwedge^* \mathbb{R}^d$ and $\{e_j\}_{j=1}^d$ the standard basis of \mathbb{R}^d .

Next we define the map $\ell^2(\mathbb{Z}, C) \otimes_C \ell^2(\mathbb{Z}^{d-1}, B) \rightarrow \ell^2(\mathbb{Z}^d, B)$ on generators as

$$\delta_k \otimes c \otimes \delta_n \otimes b \mapsto \alpha_d^k(c) \cdot \delta_{(n,k)} \otimes b$$

for $k \in \mathbb{Z}$, $n \in \mathbb{Z}^{d-1}$ and α_d the automorphism on C such that $A = C \rtimes_{\alpha_d} \mathbb{Z}$. One easily checks that this map is compatible with the right-action of B and B -valued inner product and, thus, is a unitary map on C^* -modules. We check compatibility with the left action by $A \cong C^*(C, S_d)$, where for $c_1, c_2 \in C$,

$$\begin{aligned} c_1(\delta_k \otimes c_2 \otimes \delta_n \otimes b) &= \delta_k \otimes \alpha_d^{-k}(c_1)c_2 \otimes \delta_n \otimes b \\ &\mapsto \alpha_d^k(\alpha_d^{-k}(c_1)c_2) \cdot \delta_{(n,k)} \otimes b \\ &= c_1(\alpha_d^k(c_2) \cdot \delta_{(n,k)} \otimes b). \end{aligned}$$

Furthermore,

$$\begin{aligned} S_d(\delta_k \otimes c \otimes \delta_n \otimes b) &= \delta_{k+1} \otimes c \otimes \delta_n \otimes b \\ &\mapsto \alpha_d^{k+1}(c) \cdot \delta_{(n,k+1)} \otimes b \\ &= S_d \alpha_d^k(c) S_d^* \cdot \delta_{(n,k+1)} \otimes b \\ &= S_d(\alpha_d^k(c) \cdot \delta_{(n,k)} \otimes b) \end{aligned}$$

and so the generators of the left-action are compatible with the isomorphism. It is then straightforward to check that the compatibility of the left-action extends to the C^* -completion. Next we compute that

$$\begin{aligned} N(\delta_k \otimes c \otimes \delta_n \otimes b) &= k\delta_k \otimes c \otimes \delta_n \otimes b \\ &\mapsto k\alpha_d^k(c) \cdot \delta_{(n,k)} \otimes b = X_d(\alpha_d^k(c) \cdot \delta_{(n,k)} \otimes b) \end{aligned}$$

as $\alpha_d^k(c)$ commutes with X_d . Lastly we use Equation (7) to check that

$$\begin{aligned} (1 \otimes_{\nabla} X_j)(\delta_k \otimes c \otimes \delta_n \otimes b) &= \delta_k \otimes (c + [X_j, c]) \otimes \delta_n \otimes b \\ &\mapsto \alpha_d^k(c + [X_j, c]) \cdot \delta_{(n,k)} \otimes b \\ &= \alpha_d^k(X_j c) \cdot \delta_{(n,k)} \otimes b \\ &= X_j (\alpha_d^k(c) \cdot \delta_{(n,k)} \otimes b) \end{aligned}$$

as $[X_j, c] \in \mathcal{C}$ for $c \in \mathcal{C}$ and $\alpha_d^k(X_j) = X_j$ for $j \in \{1, \dots, d-1\}$. We note that the identification $1 \otimes_{\nabla} X_j \mapsto X_j$ requires us to use the smooth subalgebra $\mathcal{C} \subset C$ in order for $[X_j, c]$ to be a well-defined element of C . This is typical of the unbounded product and why we need to have the equality $\overline{\mathcal{C} \cdot \ell^2(\mathbb{Z}^{d-1}, B)} = \ell^2(\mathbb{Z}^{d-1}, B)$.

To summarise, the unbounded Kasparov module representing the product is unitarily equivalent to

$$(9) \quad \left(\mathcal{A} \hat{\otimes} C\ell_{0,d}, \ell^2(\mathbb{Z}^d, B)_B \otimes \bigwedge^* \mathbb{R}^d, X_d \otimes \gamma^1 + \sum_{j=1}^{d-1} X_j \otimes \gamma^{j+1}, \gamma_{\bigwedge^* \mathbb{R}^d} \right)$$

with left and right Clifford actions as previously. This is almost the same as the fundamental K -cycle $\lambda^{(d)}$, the only difference being the labelling of the Clifford basis. The map $\xi(\gamma^j) = \gamma^{\sigma(j)}$ and $\xi(\rho^j) = \rho^{\sigma(j)}$ for $\sigma(j) = (j-1) \bmod d$ is potentially an orientation reversing map on Clifford algebras. Taking the canonical orientation $\omega_{C\ell_{0,d}} = \rho^1 \cdots \rho^d$ on $C\ell_{0,d}$,

$$\xi(\omega_{C\ell_{0,d}}) = \rho^d \rho^1 \cdots \rho^{d-1} = (-1)^{d-1} \rho^1 \cdots \rho^d = (-1)^{d-1} \omega_{C\ell_{0,d}},$$

and similarly for the γ^j and $C\ell_{d,0}$. Using [27, §5, Theorem 3], such a map on Clifford algebras will send the KK -class of the Kasparov module of Equation (9) to its inverse if d is even or leaves the KK -class invariant if d is odd. Therefore at the level of KK -classes, $[\text{ext}] \hat{\otimes}_C [\lambda^{(d-1)}] = (-1)^{d-1} [\lambda^{(d)}]$ as required. \square

Remark 3.5. It would be preferable to have an explicit unitary equivalence implementing the coordinate permutation of Clifford indices as existing proofs demonstrating the signed equality arising from a permutation require (explicit) homotopies. The difficulty of defining reasonable equivalence relations on unbounded Kasparov modules stronger than ‘unitary equivalence modulo bounded perturbation’ means that such homotopies compromise the computability and any interpretation stronger than mere (signed) equality of KK -classes.

Corollary 3.6. *The pairing of a K -theory class $[z] \in KO_j(B \rtimes_{\alpha, \theta} \mathbb{Z}^d)$ (or complex) with $\lambda^{(d)}$ is, up to a sign, the same as the pairing of $\partial[z] \in KO_{j-1}(B \rtimes_{\alpha, \theta} \mathbb{Z}^{d-1})$ with $\lambda^{(d-1)}$.*

Proof. Using Theorem 3.4 and associativity of the Kasparov product,

$$\begin{aligned} [z] \hat{\otimes}_A [\lambda^{(d)}] &= (-1)^{d-1} [z] \hat{\otimes}_A ([\text{ext}] \hat{\otimes}_C [\lambda^{(d-1)}]) \\ &= (-1)^{d-1} ([z] \hat{\otimes}_A [\text{ext}]) \hat{\otimes}_C [\lambda^{(d-1)}] \\ &= (-1)^{d-1} \partial[z] \hat{\otimes}_C [\lambda^{(d-1)}] \end{aligned}$$

as the product with $[\text{ext}]$ implements the boundary map in K -theory. \square

Similarly if $d = 1$ and $A = B \rtimes_{\beta} \mathbb{Z}$, then $[\lambda^{(1)}] = [\text{ext}]$ and $[z] \hat{\otimes}_A [\lambda^{(1)}] = \partial[z]$ which is in $KO_{j-1}(B \otimes \mathcal{K})$, the K -theory of the ‘0-dimensional boundary’. We will return to such pairings in Section 5.2.

4. THE DISORDERED BULK-EDGE CORRESPONDENCE IN K -HOMOLOGY

We now apply our results about fundamental K -cycles of crossed products and the Pimsner–Voiculescu exact sequence to the case of C^* -algebras modelling disordered or aperiodic topological phases.

4.1. The algebras and the extension for topological insulators.

4.1.1. The bulk algebra. The bulk algebra is the observable algebra of the solid seen as infinite and without boundary. We denote it by A_b . It is also referred to as the noncommutative Brillouin zone and was developed in [5, 6]. We work in the tight binding approximation where the configuration space is \mathbb{Z}^d . In this case the bulk algebra is $A_b = C(\Omega, M_N) \rtimes_{\alpha, \theta} \mathbb{Z}^d$, the twisted crossed product algebra of matrix-valued continuous functions over a compact space Ω by the group \mathbb{Z}^d .

The space Ω may be thought of as a space of disorder configurations. It is equipped with a topological structure and a continuous action α of \mathbb{Z}^d by shift of the configuration. One usually assumes that Ω possesses a probability measure that is invariant and ergodic under the \mathbb{Z}^d -action to obtain expectation values. The space Ω could be taken to be a point, or contractible to a point. The latter point of view is taken in [52] but this excludes the interesting possibility of describing for instance quasicrystals. The algebra of $N \times N$ matrices, M_N , is used to incorporate internal degrees of freedom like spin. Whereas these internal components are important for the implementation of certain symmetries, for instance odd time reversal, they do not interfere with the main topological constructions which will follow and, in order not to overburden the notation, we will suppress them.

An external magnetic field can be incorporated by a two cocycle $\theta : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathcal{U}(C(\Omega))$ to twist the action, but note that the existence of anti-linear symmetries puts restrictions on the form of such a magnetic field. Indeed, for systems with symmetries which are implemented by anti-linear operators we are bound to look at a real subalgebra of the above algebra. In this case we consider therefore $C(\Omega)$ as the algebra of real valued continuous functions and exclude the possibility of complex-valued twisting cocycles. The presence of a boundary will impose further constraints on any possible cocycle.

Of importance for our application to physics is the family of representations $\{\pi_\omega\}_{\omega \in \Omega}$ which are induced by the evaluation representations $\text{ev}_\omega : C(\Omega) \rightarrow \mathbb{R}$ (or \mathbb{C}). These π_ω are thus representations of $C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d$ on $\ell^2(\mathbb{Z}^d)$. They look as follows on $C(\Omega)_{\alpha, \theta} \mathbb{Z}^d$, a dense subalgebra of the bulk algebra,

$$\pi_\omega(f)\delta_n = f(\alpha^n(\omega))\delta_n, \quad \pi_\omega(S_i)\delta_n = \theta(n, e_i)(\omega)\delta_{n+e_i}$$

with $f \in C(\Omega)$ and $\{S_i\}_{i=1}^d$ the unitary generators of the twisted \mathbb{Z}^d -action with $\{e_i\}_{i=1}^d$ the generators of \mathbb{Z}^d . A tight binding Hamiltonian is a self-adjoint element of the algebraic crossed product

$$h = \sum_{n \in \mathbb{Z}^d} S^n v_n.$$

Here the functions $v_n \in C(\Omega)$ are chosen such that the expression is self-adjoint and the sum has only finitely many non-zero terms. If we take internal degrees of freedom into account then the v_n are matrix valued and $h \in M_k(C(\Omega)_{\alpha, \theta} \mathbb{Z}^d)$. The representations π_ω give rise to a family of Hamiltonians $H_\omega = \pi_\omega(h)$ which are unitarily equivalent for points ω in the same orbit.

4.1.2. *The Toeplitz extension of the bulk algebra.* We require that the twisting cocycle θ can be arranged in such a way that $S_i S_d = S_d S_i$ for all $i \in \{1, \dots, d\}$, something that often reduces to a choice of magnetic gauge. This assumption allows us to decompose the twisted \mathbb{Z}^d -action, $\alpha = (\alpha^\parallel, \alpha_d)$ with α^\parallel a twisted \mathbb{Z}^{d-1} -action and α_d an untwisted \mathbb{Z} -action, i.e.,

$$C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d \cong (C(\Omega) \rtimes_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1}) \rtimes_{\alpha_d} \mathbb{Z}.$$

The bulk algebra can then be seen as a quotient algebra of the Toeplitz algebra $\mathcal{T}(\alpha_d)$, which is interpreted as the observables on a system with boundary. Indeed, the evaluation representation $\text{ev}_\omega : C(\Omega) \rightarrow \mathbb{R}$ (or \mathbb{C}) induces a representation of $(C(\Omega)_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1})_{\alpha_d} \mathbb{N}$ on $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z}^{d-1})$. The tensor product Hilbert space is dense in $\ell^2(\mathbb{Z}^{d-1} \times \mathbb{N})$ and $\mathbb{Z}^{d-1} \times \mathbb{N}$ is the configuration space for the system on a half-space. That is, the insulator seen as infinite with a boundary, the so-called edge, which is at $\mathbb{Z}^{d-1} \times \{0\}$. The representation of $(C(\Omega)_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1})_{\alpha_d} \mathbb{N}$ looks far away from the edge like that of $(C(\Omega)_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1})_{\alpha_d} \mathbb{Z} = C(\Omega)_{\alpha} \mathbb{Z}^d$, but at the edge the translation \tilde{S}_d is truncated. The algebra $(C(\Omega)_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1})_{\alpha_d} \mathbb{N}$, or rather its closure $\mathcal{T}(\alpha_d)$ can therefore be seen as the observable algebra of the insulator with boundary.

4.1.3. *The edge algebra.* The projection $p = 1 - \tilde{S}_d \tilde{S}_d^*$ generates an ideal in $\mathcal{T}(\alpha_d)$. In the evaluation representation this projection becomes the projection onto the subspace $\ell^2(\mathbb{Z}^{d-1} \times \{0\})$, and hence is naturally associated to the edge. The algebraic (non-closed) ideal is given by $F \otimes C(\Omega)_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1}$ with F the finite-rank operators perpendicular to the edge. Its C^* -closure $\mathcal{K} \otimes C(\Omega) \rtimes_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1}$ is represented as operators which are localised near the edge and so is called the edge algebra. We will use the notation $A_e = C(\Omega) \rtimes_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1}$ so that the edge algebra is the stabilisation of A_e .

To summarize, the extension which gives rise to the bulk boundary correspondence for topological insulators is

$$0 \rightarrow \mathcal{K} \otimes C(\Omega) \rtimes_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1} \rightarrow \mathcal{T}(\alpha_d) \rightarrow C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d \rightarrow 0.$$

4.2. **The fundamental K -cycles for the bulk and the edge.** We apply the construction of Proposition 3.2 to $A_b = C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d$ and to $A_e = C(\Omega) \rtimes_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1}$ with their dense subalgebras $\mathcal{A}_b = C(\Omega)_{\alpha, \theta} \mathbb{Z}^d$ and $\mathcal{A}_e = C(\Omega)_{\alpha^\parallel, \theta} \mathbb{Z}^{d-1}$. As a result we obtain the fundamental K -cycle for the bulk

$$(10) \quad \lambda_b = \left(\mathcal{A}_b \hat{\otimes} C\ell_{0,d}, \ell^2(\mathbb{Z}^d, C(\Omega))_{C(\Omega)} \otimes \bigwedge^* \mathbb{R}^d, D_b, \gamma_{\bigwedge^* \mathbb{R}^d} \right), \quad D_b = \sum_{j=1}^d X_j \otimes \gamma^j,$$

and the fundamental K -cycle of the edge

$$\lambda_e = \left(\mathcal{A}_e \hat{\otimes} C\ell_{0,d-1}, \ell^2(\mathbb{Z}^{d-1}, C(\Omega))_{C(\Omega)} \otimes \bigwedge^* \mathbb{R}^{d-1}, D_e, \gamma_{\bigwedge^* \mathbb{R}^{d-1}} \right), \quad D_e = \sum_{j=1}^{d-1} X_j \otimes \gamma^j.$$

Hence we obtain unbounded representatives of elements $[\lambda_b] \in KKO^d(A_b, C(\Omega))$ and $[\lambda_e] \in KKO^{d-1}(A_e, C(\Omega))$ (in the real case, otherwise it's KK).

We apply Proposition 3.3 to $A_b = A_e \rtimes_{\alpha_d} \mathbb{Z}$ to obtain the extension cycle

$$\text{ext}_{be} = \left(\mathcal{A}_b \hat{\otimes} C\ell_{0,1}, \ell^2(\mathbb{Z}, A_e)_{A_e} \otimes \bigwedge^* \mathbb{R}, D = X_1 \otimes \gamma^1, \gamma_{\bigwedge^* \mathbb{R}} \right).$$

Theorem 3.4 then yields the factorisation of the fundamental K -cycles for topological insulators,

$$[\text{ext}_{be}] \hat{\otimes} [\lambda_e] = (-1)^{d-1} [\lambda_b].$$

4.3. From Kasparov module to spectral triple. Recall the map $\text{ev}_\omega : C(\Omega) \rightarrow \mathbb{F}$ (for $\mathbb{F} = \mathbb{R}$ or \mathbb{C}), which gives a family of representations $\{\pi_\omega\}_{\omega \in \Omega}$ of the crossed product $C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d$ on $\ell^2(\mathbb{Z}^d)$ subject to the covariance condition, $S^n \pi_\omega(a) (S^*)^n = \pi_{\alpha^n(\omega)}(a)$. The representations π_ω were used in [21, 9] to define a real or complex spectral triple

$$(11) \quad \lambda_b(\omega) = \left(\mathcal{A} \hat{\otimes} C\ell_{0,d}, \pi_\omega \ell^2(\mathbb{Z}^d) \otimes \bigwedge^* \mathbb{R}^d, \sum_{j=1}^d X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^d} \right).$$

The spectral triple gives a K -homology class for the crossed-product algebra, which by the covariance relation is independent under a fixed α -orbit of Ω .

Let us link the bulk spectral triple of Equation (11) to the fundamental K -cycle of the bulk algebra from Equation (10). We take the trivially graded Kasparov module

$$\text{ev}_\omega = (C(\Omega), {}_{\text{ev}_\omega} \mathbb{F}, 0)$$

with ev_ω the evaluation representation of $C(\Omega)$. The Kasparov module ev_ω represents a class in $KKO(C(\Omega), \mathbb{R})$ (or $KK(C(\Omega), \mathbb{C})$) that can be paired with our fundamental K -cycle.

Proposition 4.1. *The spectral triple $\lambda_b(\omega)$ is unitarily equivalent to the internal product of the unbounded bulk Kasparov module λ_b with ev_ω .*

Proof. Because the evaluation Kasparov module is very simple, we can easily compute the internal product

$$\begin{aligned} & \left(\mathcal{A} \hat{\otimes} C\ell_{0,d}, \ell^2(\mathbb{Z}^d, C(\Omega)) \otimes \bigwedge^* \mathbb{R}^d, \sum_{j=1}^d X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^d} \right) \hat{\otimes}_{C(\Omega)} (C(\Omega), {}_{\text{ev}_\omega} \mathbb{F}, 0) \\ & \cong \left(\mathcal{A} \hat{\otimes} C\ell_{0,d}, \mathcal{H} \otimes \bigwedge^* \mathbb{R}^d, \sum_{j=1}^d X_j \otimes 1 \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^d} \right) \end{aligned}$$

with $\mathcal{H} = \ell^2(\mathbb{Z}^d, C(\Omega)) \otimes_{{}_{\text{ev}_\omega} \mathbb{F}}$. We identify $\ell^2(\mathbb{Z}^d, C(\Omega)) \otimes_{{}_{\text{ev}_\omega} \mathbb{F}}$ with $\ell^2(\mathbb{Z}^d)$ under which $X_j \otimes 1 \mapsto X_j$ and the left-action of \mathcal{A} takes the form

$$S^m g \cdot \delta_n = \theta(n, m) g(\alpha^n(\omega)) \delta_{n+m} = \pi_\omega(S^m g) \delta_n.$$

Hence we recover $\lambda_b(\omega)$. □

Similarly we obtain a spectral triple for the edge algebra

$$\lambda_e(\omega) = \left(\mathcal{A}_e \hat{\otimes} C\ell_{0,d-1}, \pi_\omega \ell^2(\mathbb{Z}^{d-1}) \otimes \bigwedge^* \mathbb{R}^{d-1}, \sum_{j=1}^{d-1} X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^{d-1}} \right)$$

with $[\lambda_e(\omega)] = [\lambda_e] \hat{\otimes}_{C(\Omega)} [\text{ev}_\omega]$. Therefore by Theorem 3.4 we obtain a factorisation of the bulk spectral triple

$$[\text{ext}_{be}] \hat{\otimes} [\lambda_e(\omega)] = (-1)^{d-1} [\lambda_b(\omega)].$$

5. K -THEORY, PAIRINGS AND COMPUTATIONAL CHALLENGES

In this section we outline how our rather general results about Kasparov modules and twisted \mathbb{Z}^d -actions can be related to topological phases and the bulk-edge correspondence.

As mentioned in the introduction, the measured quantities in insulator models involve the pairing of a K -theory class with a dual theory, be it cyclic cohomology, K -homology or the more general Kasparov product.

Symmetry type	K -theory group
even TRS	$KO_0(A)$
even TRS, even PHS	$KO_1(A)$
even PHS	$KO_2(A)$
even PHS, odd TRS	$KO_3(A)$
odd TRS	$KO_4(A)$
odd TRS, odd PHS	$KO_5(A)$
odd PHS	$KO_6(A)$
even TRS, odd PHS	$KO_7(A)$
N/A	$K_0(A)$
chiral	$K_1(A)$

TABLE 1. Symmetry types and the corresponding K -theory group of the ungraded algebra $A = C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d$ (real or complex). See Proposition 5.1.

Computing these pairings gives rise to analytic index formulas, which are in general a Clifford module valued index in the sense of Atiyah–Bott–Shapiro [1]. In the case of non-torsion invariants, cyclic formulas may be used to obtain more computable expressions for disorder-averaged quantities. The case of torsion indices is more complicated and we finish with some brief remarks about computing such invariants.

5.1. Symmetries and K -theory. Recall that our Hamiltonian of interest is a self-adjoint element h in the crossed-product algebra $C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d$. A Hamiltonian h with spectral gap at 0 represents an extended topological phase if h is compatible with certain symmetry involutions. The symmetries of interest to us are time-reversal symmetry (TRS), particle-hole (charge-conjugation) symmetry (PHS) and chiral (sublattice) symmetry, although we emphasise that other symmetries such as spatial involution may be considered. The following result, due in various forms to numerous authors, associates a K -theory class to a symmetry compatible Hamiltonian.

Proposition 5.1 ([9, 17, 29, 37, 58]). *Let $h \in C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}$ be self-adjoint with a spectral gap at 0. If h is compatible with time-reversal symmetry and/or particle-hole symmetry and/or chiral symmetry then we may associate a class in $K_j(C(\Omega) \rtimes \mathbb{Z}^d)$ or $KO_j(C(\Omega) \rtimes \mathbb{Z}^d)$, where j is determined by the symmetries present. The details are summarised in Table 1.*

The specific class associated to a symmetry compatible Hamiltonian in Proposition 5.1 may arise in several (largely equivalent) ways.

One may consider a subgroup G of the CT -symmetry group $\{1, T, C, CT\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with C denoting charge-conjugation (particle-hole), T time-reversal and CT sublattice/chiral symmetry. A topological phase can then be considered as a Hamiltonian with spectral gap (assumed to be at 0), whose phase $h|h|^{-1}$ acts as a grading for a projective unitary/anti-unitary (PUA) representation of $G \subset \{1, T, C, CT\}$ on a complex Hilbert space \mathcal{H} . This is the perspective first developed in [17] in the commutative setting and then extended to noncommutative algebras in [9, 58]. The even/odd nature of the time-reversal and charge-conjugation symmetries is encoded in the cocycle that arises in the projective representation. The class in $K_j(A)$ or $KO_j(A)$ for $A \subset C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d$ from Proposition 5.1 and Table 1 is determined by the subgroup G of the CT -symmetry group and its PUA representation.

One may also consider the gapped Hamiltonian h to be oddly graded by the presence of a chiral/sublattice symmetry, in which case the phase $h|h|^{-1}$ determines a class in the van Daele K -theory of the algebra, which can then be related to real or complex K -theory.¹ Other symmetries can be incorporated by considering Real structures on the observable algebra. By considering the various types of symmetries and whether they are even and odd, one can derive Table 1. See [29, 36] for further information.

The generality and flexibility of KK -theory means that we can use either the symmetry class or the van Daele class in terms of pairing or the bulk-edge correspondence. The choice of K -theory class can therefore be determined in order to model the specifics of an experimental set-up. This is important as the techniques being employed to define/measure \mathbb{Z}_2 -invariants of topological phases are still in development.

5.2. Pairings, the Clifford index and the bulk-edge correspondence. As briefly explained, a symmetry compatible gapped Hamiltonian gives rise to a class in $K_j(A_b)$ or $KO_j(A_b)$ for A_b the real or complex ungraded crossed product $C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d$. By Theorem 2.2, we can relate the K -theory groups to $KK(\mathcal{C}\ell_j, A_b)$ or $KKO(\mathcal{C}\ell_{j,0}, A_b)$. Hence we can consider the map

$$(12) \quad KKO(\mathcal{C}\ell_{j,0}, C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d) \times KKO(C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d \hat{\otimes} \mathcal{C}\ell_{0,d}, C(\Omega)) \rightarrow KKO(\mathcal{C}\ell_{j,d}, C(\Omega))$$

given by the internal Kasparov product, where the class in the group $KKO^d(C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d, C(\Omega))$ is represented by the fundamental K -cycle for the \mathbb{Z}^d -action, λ_b from Equation (10) (an analogous map occurs in the complex case).

Like the case of the quantum Hall effect, where the Hall conductance is related to a pairing of the Fermi projection with a cyclic cocycle or K -homology class, we claim that the quantities of interest in topological insulator systems arise as pairings/products of this type.

Without specifying a particular K -theory class, we can make some general comments about the pairing with the fundamental K -cycle. The unbounded product with the fundamental K -cycle has the general form

$$(\mathcal{C}\ell_{j,d}, E_{C(\Omega)}, \tilde{X}, \Gamma),$$

with $E_{C(\Omega)}$ a countably generated C^* -module with a left-action of $\mathcal{C}\ell_{j,d}$. The construction of the product is done in such a way that the left-action of $\mathcal{C}\ell_{j,d}$ graded-commutes with the unbounded operator \tilde{X} . This implies that the topological information of interest is contained in the kernel $\text{Ker}(\tilde{X})$ as a C^* -submodule of $E_{C(\Omega)}$ (when this makes sense, see [9, Appendix B]).

Let us now associate an analytic index to the product in Equation (12).

Definition 5.2. We let ${}_{r,s}\mathfrak{M}_{C(\Omega)}$ be the Grothendieck group of equivalence classes of real \mathbb{Z}_2 -graded right- $C(\Omega)$ C^* -modules carrying a graded left-representation of $\mathcal{C}\ell_{r,s}$.

Using the notation of Clifford modules, $\text{Ker}(\tilde{X})$ determines a class in the quotient group ${}_{j,d}\mathfrak{M}_{C(\Omega)}/i^*({}_{j+1,d}\mathfrak{M}_{C(\Omega)})$, where i^* comes from restricting a Clifford action of $\mathcal{C}\ell_{j+1,d}$ to $\mathcal{C}\ell_{j,d}$. Next, we use an elementary extension of the Atiyah–Bott–Shapiro isomorphism, [1] and [55, §2.3], to make the identification

$${}_{j,d}\mathfrak{M}_{C(\Omega)}/i^*({}_{j+1,d}\mathfrak{M}_{C(\Omega)}) \cong KO_{j-d}(C(\Omega)).$$

¹If there is no chiral/sublattice symmetry then for h to have odd grading we must first consider a larger graded algebra which we can then reduce to the trivially graded A_b . See [29].

Definition 5.3. The Clifford index, $\text{Index}_{j-d}(\tilde{X})$, of \tilde{X} is given by the class

$$[\text{Ker}(\tilde{X})] \in {}_{j,d}\mathfrak{M}_{C(\Omega)}/i^*_{j+1,d}\mathfrak{M}_{C(\Omega)} \cong KO_{j-d}(C(\Omega))$$

We remark that Index_k reproduces the usual (real) C^* -module Fredholm index as studied in [19, Chapter 4] if $k = 0$, see [55].

Of course we may instead wish to use an invariant probability measure \mathbf{P} on Ω to obtain invariants averaged over the disorder space rather than K -theory classes of the disorder. We will return to this question in Section 5.3.

5.2.1. The bulk-edge correspondence. Let us now apply Theorem 3.4 to our study on pairings. Denoting by $[x_b]$ the K -theory class represented by the symmetry compatible Hamiltonian, we have that

$$[x_b] \hat{\otimes}_{A_b} [\lambda_b] = (-1)^{d-1} [x_b] \hat{\otimes}_{A_b} [\text{ext}] \hat{\otimes}_{A_e} [\lambda_e]$$

with λ_e the fundamental K -cycle for the edge algebra. Using associativity of the Kasparov product to group the terms on the right in two different ways, the real index pairing will either be a pairing

$$KKO(C\ell_{j,0}, C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d) \times KKO(C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d \hat{\otimes} C\ell_{0,d}, \mathbb{R}) \rightarrow KO_{j-d}(C(\Omega)),$$

the bulk invariant, or a new (but equivalent) pairing

$$\begin{aligned} KKO(C\ell_{j,0} \hat{\otimes} C\ell_{0,1}, C(\Omega) \rtimes_{\alpha\parallel,\theta} \mathbb{Z}^{d-1}) \times KKO(C(\Omega) \rtimes_{\alpha\parallel,\theta} \mathbb{Z}^{d-1} \hat{\otimes} C\ell_{0,d-1}, \mathbb{R}) \\ \rightarrow KO_{j-d}(C(\Omega)). \end{aligned}$$

The second pairing yields an invariant that comes from $A_e = C(\Omega) \rtimes_{\alpha\parallel,\theta} \mathbb{Z}^{d-1}$, the edge algebra of a system with boundary. Theorem 3.4 ensures that regardless of our choice of pairing, the result is the same and so we obtain the bulk-edge correspondence for real and complex algebras. In particular non-trivial bulk invariants imply non-trivial edge invariants and vice versa. Furthermore, we see that the bulk-edge correspondence continues to hold under the addition of weak disorder. In complex examples, the value of the edge pairing can be interpreted as a response coefficient of the edge system like, for instance, the conductance of a current concentrated at the boundary of the sample [31, 32, 34, 52].

Remark 5.4 (Wider applications of Theorem 3.4). The bulk-edge correspondence and Theorem 3.4 are largely independent of the symmetry considerations of topological phases. Instead, it is a general property of the (real or complex) unbounded Kasparov module representing the short exact sequence

$$0 \rightarrow \mathcal{K} \otimes C(\Omega) \rtimes_{\alpha\parallel,\theta} \mathbb{Z}^{d-1} \rightarrow \mathcal{T}(\alpha_d) \rightarrow C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d \rightarrow 0$$

and the fundamental K -cycles on the ideal and quotient algebras we have constructed.

In particular, the fact that the factorisation occurs on the K -homological part of the index pairing means other K -theory classes and symmetry types can be considered without changing the result and is independent of the symmetries present. For example, if we were to consider symmetry compatible Hamiltonians of a group \tilde{G} that included spatial involution or other symmetries, then provided that the symmetry data can be associated to a class in $KKO(C^*(\tilde{G}), A_b)$ (or complex), the pairing with λ_b would still display the bulk-edge correspondence.

Separating the topological information arising from the internal symmetries of the Hamiltonian from the (non-commutative) geometry of the Brillouin zone highlights an advantage of using Kasparov theory to study topological systems with internal symmetries.

We obtain the flexibility to change the K -theoretic data without affecting the geometric information that is used to obtain the topological invariants of interest and vice versa.

5.3. Semifinite spectral triples and non-torsion invariants. Given a unital C^* -algebra (real or complex) B with possibly twisted \mathbb{Z}^d -action, we have given a general procedure to obtain an unbounded $B \rtimes_{\alpha, \theta} \mathbb{Z}^d$ - B module and class in $KKO(B \rtimes_{\alpha, \theta} \mathbb{Z}^d \hat{\otimes} C\ell_{0,d}, B)$. If the algebra B has a faithful, semifinite and norm-lower semicontinuous trace τ_B , there is a general method by which we can obtain a semifinite spectral triple as studied in [25, 39, 49]. Such a condition on B is satisfied in the physically interesting case of $B = C(\Omega, M_N)$, where the disorder space of configurations is equipped with a probability measure \mathbf{P} such that $\text{supp}(\mathbf{P}) = \Omega$. The measure is usually assumed to also be invariant and ergodic under the \mathbb{Z}^d -action.

Given the C^* -module $\ell^2(\mathbb{Z}^d, B)$ and trace τ_B , we consider the inner-product

$$\langle \lambda_1 \otimes b_1, \lambda_2 \otimes b_2 \rangle = \tau_B((\lambda_1 \otimes b_1 \mid \lambda_2 \otimes b_2)_B) = \langle \lambda_1, \lambda_2 \rangle_{\ell^2(\mathbb{Z}^d)} \tau_B(b_1^* b_2),$$

which defines the Hilbert space $\ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B)$ where $L^2(B, \tau_B)$ is the GNS space.

Lemma 5.5. *The algebra $A = B \rtimes_{\alpha, \theta} \mathbb{Z}^d$ acts on $\ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B)$.*

Proof. This follows from the identification of $\ell^2(\mathbb{Z}) \otimes L^2(B, \tau_B)$ with $\ell^2(\mathbb{Z}, B) \otimes_B L^2(B, \tau_B)$. \square

Proposition 5.6 ([39], Theorem 1.1). *Given $T \in \text{End}_B(\ell^2(\mathbb{Z}^d, B))$ with $T \geq 0$, define*

$$\text{Tr}_\tau(T) = \sup_I \sum_{\xi \in I} \tau_B[(\xi \mid T\xi)_B],$$

where the supremum is taken over all finite subsets I of $\ell^2(\mathbb{Z}, B)$ such that $\sum_{\xi \in I} \Theta_{\xi, \xi} \leq 1$. Then Tr_τ is a semifinite norm-lower semicontinuous trace with the property $\text{Tr}_\tau(\Theta_{\xi_1, \xi_2}) = \tau_B[(\xi_2 \mid \xi_1)_B]$.

Lemma 5.7. *Let $\text{End}_B^{00}(\ell^2(\mathbb{Z}^d, B))$ be the algebra of the span of rank-1 operators, Θ_{ξ_1, ξ_2} with $\xi_1, \xi_2 \in \ell^2(\mathbb{Z}^d, B)$, and \mathcal{N} be the von Neumann algebra $\text{End}_B^{00}(\ell^2(\mathbb{Z}^d, B))''$ with weak-closure taken in the bounded operators on $\ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B)$. Then the trace Tr_τ extends to a trace on the positive cone \mathcal{N}_+ .*

Proof. This is just the dual trace construction, or an explicit check can be made as in [49, Proposition 5.11]. \square

Proposition 5.8. *For $\mathcal{A} = B_{\alpha, \theta} \mathbb{Z}^d$, the tuple*

$$\left(\mathcal{A} \hat{\otimes} C\ell_{0,d}, \ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B) \otimes \bigwedge^* \mathbb{R}^d, \sum_{j=1}^d X_j \otimes \gamma^j, (\mathcal{N}, \text{Tr}_\tau) \right)$$

is a QC^∞ and d -summable semifinite spectral triple.

Proof. The commutators $[X_j, a]$ are bounded by analogous arguments to the proof of Proposition 3.2. We use the frame $\{e_m\}_{m \in \mathbb{Z}^d} \subset \ell^2(\mathbb{Z}^d, B)$ with $e_m = \delta_m \otimes 1_B$ to compute that

$$\begin{aligned} \text{Tr}_\tau((1 + |X|^2)^{-s/2}) &= \text{Tr}_\tau \left(\sum_{m \in \mathbb{Z}^d} (1 + |m|^2)^{-s/2} \Theta_{e_m, e_m} \right) \\ &= \sum_{m \in \mathbb{Z}^d} (1 + |m|^2)^{-s/2} \tau_B((e_m \mid e_m)_B) \\ &= \sum_{m \in \mathbb{Z}^d} (1 + |m|^2)^{-s/2} \tau_B(1_B) \end{aligned}$$

by the properties of Tr_τ . The sum of $(1 + |m|^2)^{-s/2}$ for $m \in \mathbb{Z}^d$ will be finite for $s > d$ and $\tau_B(1_B) = 1$. Hence $(1 + |X|^2)^{-s/2}$ is Tr_τ -trace-class for $s > d$ as required.

Our spectral triple is QC^∞ if $\mathcal{A} \hat{\otimes} C\ell_{0,d}$ preserves the domain of $(1 + D^2)^{k/2}$ for any $k \in \mathbb{N}$, which reduces to checking that \mathcal{A} preserves the domain of $(1 + |X|^2)^{k/2} \otimes 1$ on $\ell^2(\mathbb{Z}^d) \otimes L^2(B, \tau_B)$. We recall that elements of \mathcal{A} are of the form $\sum_n S^n b_n$ with $b_n \in B$ and $n \in \mathbb{Z}^d$ a multi-index such that $|n| < R$ for some R . We compute that

$$\begin{aligned} & (1 + |X|^2)^{k/2} \sum_{|n| < R} S^n b_n \cdot (\psi(x) \otimes b) \\ &= \sum_{|n| < R} (1 + |x - n|^2)^{k/2} \psi(x - n) \otimes \alpha^{-x+n}(\theta(x, n)) \alpha^{-x}(b_n) b \end{aligned}$$

for $x \in \mathbb{Z}^d$. Because n is strictly bounded and $\theta(x, n)$ is unitary for all $n, x \in \mathbb{Z}^d$, the Hilbert space norm of the sum over finite n of $(1 + |x - n|^2)^{k/2} \psi(x - n) \otimes \alpha^{-x}(\alpha^n(\theta(x, n)) b_n) b$ is well-defined and norm-convergent for $\psi \otimes b$ in the domain of $(1 + |X|^2)^{k/2}$. \square

Let us return to the case of $B = C(\Omega)$, where $C(\Omega)$ is endowed with the trace $\mathcal{T}_\mathbf{P}$ from the probability measure \mathbf{P} . The semifinite index pairing with the symmetry K -theory class $[x_b] \in KO_d(A_b)$ (or complex) is given by the composition

$$KO_d(C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d) \times KKO(C(\Omega) \rtimes_{\alpha, \theta} \mathbb{Z}^d \hat{\otimes} C\ell_{0,d}, C(\Omega)) \rightarrow KO_0(C(\Omega)) \xrightarrow{(\mathcal{T}_\mathbf{P})^*} \mathbb{R}$$

Hence the semifinite index pairing of the spectral triple of Proposition 5.8 measures disorder-averaged topological invariants. Furthermore in the case of complex algebras and modules, the semifinite local index formula [12, 13] gives us computable expressions for the index pairing in terms of traces and derivations. We expect similar results to hold for the case of real integer invariants (e.g. those that arise from $KO_0(\mathbb{R})$ or $KO_4(\mathbb{R})$), though delay a more detailed investigation to another place.

5.4. The Kane–Mele model and torsion invariants. As an example, we consider 2-dimensional systems with odd time reversal symmetry from [15, 26] and considered in [9, Section 4]. Recall the bulk Hamiltonian $H_{KM}^\omega = \begin{pmatrix} h_\omega & g \\ g^* & \mathcal{C} h_\omega \mathcal{C} \end{pmatrix}$ acting on $\mathcal{H}_b = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N}$, with h_ω a Haldane Hamiltonian, g the Rashba coupling and \mathcal{C} component-wise complex conjugation such that $g^* = -\mathcal{C} g \mathcal{C}$. We take the symmetry group $G = \{1, T\}$ whose time-reversal involution is implemented by the operator $R_T = \begin{pmatrix} 0 & \mathcal{C} \\ -\mathcal{C} & 0 \end{pmatrix}$ on \mathcal{H}_b . We follow [9, Section 4] and construct the projective symmetry class $[P_\mu^G] \in KKO(C\ell_{4,0}, C(\Omega) \rtimes \mathbb{Z}^2)$. The class $[P_\mu^G]$ is represented by the Kasparov module

$$(C\ell_{4,0}, P_\mu(A \rtimes G)_A^{\oplus 2}, 0, \Gamma),$$

where P_μ is the Fermi projection, $(A \rtimes G)_A$ is a C^* -module given by the completion of $A \rtimes G$ under the conditional expectation of the G -action on A and Γ is the self-adjoint unitary given by the phase $H_{KM}^\omega |H_{KM}^\omega|^{-1}$.

For simplicity, we will pair $[P_\mu^G]$ with the spectral triple coming from the evaluation map ev_ω . Namely we have the real spectral triple

$$\lambda_b(\omega) = \left(\mathcal{A}_b \hat{\otimes} C\ell_{0,2}, \pi_\omega(\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N}) \otimes \bigwedge^* \mathbb{R}^2, X_b, \gamma_{\bigwedge^* \mathbb{R}^2} \right)$$

with $X_b = \sum_{j=1}^2 X_j \otimes 1_{2N} \otimes \gamma^j$ and \mathcal{A}_b a dense subalgebra of $C(\Omega) \rtimes \mathbb{Z}^2$ given by matrices of elements in $C(\Omega)_\alpha \mathbb{Z}^2$.

Let us now consider a bulk-edge system. We may link bulk and edge algebras by a short exact sequence, which gives rise to a class $[\text{ext}] \in KKO(C(\Omega) \rtimes \mathbb{Z}^2 \hat{\otimes} C\ell_{0,1}, C(\Omega) \rtimes \mathbb{Z})$ by the procedure in Section 3.2. We will omit the details and instead focus on the bulk and edge pairings, where the edge spectral triple is given by

$$\lambda_e(\omega) = \left(\mathcal{A}_e \hat{\otimes} C\ell_{0,1}, \pi_\omega(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N}) \otimes \bigwedge^* \mathbb{R}, X_1 \otimes \gamma_{\text{edge}}, \gamma_{\Lambda^* \mathbb{R}} \right)$$

with \mathcal{A}_e a dense subalgebra of $C(\Omega) \rtimes \mathbb{Z}$. Our bulk pairing is the product

$$\begin{aligned} & [P_\mu^G] \hat{\otimes}_A \left[\left(\mathcal{A} \hat{\otimes} C\ell_{0,2}, \pi_\omega(\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N}) \otimes \bigwedge^* \mathbb{R}^2, X_b, \gamma_{\Lambda^* \mathbb{R}^2} \right) \right] \\ & KKO(C\ell_{4,0}, A) \times KKO(A \hat{\otimes} C\ell_{2,0}, \mathbb{R}) \rightarrow KO_2(\mathbb{R}) \cong \mathbb{Z}_2, \end{aligned}$$

which can be expressed concretely as the Clifford module valued index,

$$\begin{aligned} \langle [P_\mu^G], [\lambda_b(\omega)] \rangle &= [\text{Ker}(P_\mu X_b P_\mu)] \\ &\cong \text{Dim}_{\mathbb{C}} \text{Ker}[P_\mu((X_1 + iX_2) \otimes 1)P_\mu] \bmod 2 \end{aligned}$$

using a particular choice of Clifford generators. By Theorem 3.4 and the associativity of the Kasparov product, this is the same as the pairing

$$-([P_\mu^G] \hat{\otimes}_A [\text{ext}]) \hat{\otimes}_{A_e} \left[\left(\mathcal{A}_e \hat{\otimes} C\ell_{0,1}, \pi_\omega(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N}) \otimes \bigwedge^* \mathbb{R}, X_1 \otimes \gamma_{\text{edge}}, \gamma_{\Lambda^* \mathbb{R}} \right) \right],$$

a map

$$KKO(C\ell_{4,0} \hat{\otimes} C\ell_{0,1}, C(\Omega) \rtimes \mathbb{Z}) \times KKO((C(\Omega) \rtimes \mathbb{Z}) \hat{\otimes} C\ell_{0,1}, \mathbb{R}) \rightarrow KO_{4-1-1}(\mathbb{R}) \cong \mathbb{Z}_2,$$

which is now an invariant of the edge algebra $C(\Omega) \rtimes \mathbb{Z}$. Analytic expressions can be derived for the edge invariant by taking the Clifford index of the (edge) product module.

We would like to examine the edge pairing more closely. We first review what occurs in the complex setting as developed in [31, 32]. Let $\Delta \subset \mathbb{R}$ be an open interval of \mathbb{R} such that $\mu \in \Delta$ and Δ is in the complement of the spectrum of H_{KM}^ω . By considering states in the image of the spectral projection $P_\Delta = \chi_\Delta(\Pi_s H_{KM}^\omega \Pi_s)$ for $\Pi_s : \ell^2(\mathbb{Z}^2) \rightarrow \ell^2(\mathbb{Z} \otimes \{\dots, s-1, s\})$ the projection, we are focusing precisely on eigenstates of the Hamiltonian with edge that do not exist in the bulk system, namely edge states. We use the projection P_Δ to define the unitary

$$U(\Delta) = \exp \left(-2\pi i \frac{\Pi_s H_{KM}^\omega \Pi_s - \inf(\Delta)}{\text{Vol}(\Delta)} P_\Delta \right).$$

It is a key result of [31, 32] that $U(\Delta)$ is a unitary in $(C(\Omega) \rtimes \mathbb{Z})_{\mathbb{C}}$ and, furthermore, represents the image of the Fermi projection under the (complex) K -theory boundary map. That is, the unitary $[U(\Delta)] \in K_1((C(\Omega) \rtimes \mathbb{Z})_{\mathbb{C}})$ represents the complex Kasparov product $[P_\mu] \hat{\otimes}_{A_{\mathbb{C}}} [\text{ext}]$ for trivially graded algebras. The authors of [32] show that the pairing of $[U(\Delta)]$ with the boundary spectral triple can be expressed as

$$\begin{aligned} \sigma_e &= -\frac{e^2}{h} \hat{\mathcal{T}}(U(\Delta)^* i[X_1, U(\Delta)]) \\ (13) \quad &= -\lim_{\Delta \rightarrow \mu} \frac{1}{\text{Vol}(\Delta)} \hat{\mathcal{T}}(P_\Delta i[X_1, \Pi_s H_{KM}^\omega \Pi_s]), \end{aligned}$$

where $\hat{\mathcal{T}} = \mathcal{T} \otimes \text{Tr}$ is the trace per unit volume along the boundary and operator trace normal to the boundary. One recognises Equation (13) as measuring the conductance of an edge current (as P_Δ projects onto edge states). Unfortunately, in the Kane–Mele example, the expression $\hat{\mathcal{T}}(P_\Delta i[X_1, \Pi_s H_{KM}^\omega \Pi_s])$ is zero as there is no net current and the cyclic cocycle cannot detect the \mathbb{Z}_2 -index we associate to the edge channels. Under additional symmetries of the bulk Hamiltonian, non-zero cyclic pairings for the edge can

be computed using the spin Chern number (see [56, 57]), though the resulting indices are no-longer a property of only time-reversal symmetry. This leads us to summarise the remaining difficulties in our approach.

6. OPEN PROBLEMS

Interpreting torsion-valued invariants. A concrete representation of the torsion-valued index pairings that give rise to both bulk and edge pairings is a much more difficult task than in the complex case, where invariants can be expressed as the Fredholm index of the operators of interest. This is because we have to consider Kasparov products, which give rise to a ‘Clifford module’ index.

One advantage of unbounded Kasparov theory is that the operators we deal with and the modules we build have geometric or physical motivation and so can be linked to the underlying system. In particular it would be desirable to link the Clifford module index with a more physical expression for the edge pairing as the edge invariant is meant to be directly linked to the existence of edge channels on a system with boundary. This remains an open problem in the field and is related to the difficulty of measuring torsion-labelled states. See [30] for recent progress.

A further complication is that there are two copies of \mathbb{Z}_2 in the real K -theory of a point: in degrees 1 and 2. These different indices arise in different ways, and have different interpretations: see [11].

Disorder, localisation, and spectral gaps. Throughout this article we have required that the Hamiltonian retains a spectral gap. This is so that there is an unambiguous method of constructing K -theory classes associated to the Hamiltonian. It is well-known, however, that the complex pairing with the class of the fundamental K -cycle continues to make sense whenever the spectrum of the Hamiltonian has a *gap of extended states*.

For the quantum Hall effect Bellissard et al. showed that this phenomenon arises physically from localisation arising from disorder [6]. Mathematically this was reflected in the properties of commutators between the position operator X and the projector $\chi_{(-\infty, a]}(H)$ for a a point in such a gap of extended states of H . In turn this behaviour is seen in the expression for the Chern character, which makes sense for non-torsion invariants, but seemingly has no analogue for torsion invariants. Importantly, disorder does not affect the construction of the fundamental K -cycle.

The real local index formula. For non-torsion invariants of topological insulators, it seems reasonable to expect that the entirety of the machinery developed for complex topological insulators in [52] has an exact analogue. The use of a Chern character, the analysis of disorder and localisation, the Kubo formula relating topological invariants to linear response coefficients all make perfect sense.

While all this seems quite reasonable, there is a quite a lot of detail to check. There are many steps in the production of, for instance, the local index formula, and the heavy reliance on spectral theory means that numerous details of the proof require a careful check.

The K -theory class. Relating experiments on topological phases to details of single particle Hamiltonians with symmetries is not a simple task. The measured quantities should correspond to pairings of the fundamental K -cycle with K -theory classes of the observable algebra arising from the Hamiltonian.

Having the K -homology data (the fundamental K -cycle) fixed allows us to compare the results of future experiments with the predictions arising from pairing with different K -theory classes.

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