

Generalised time functions and finiteness of the Lorentzian distance

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Abstract

We show that finiteness of the Lorentzian distance is equivalent to the existence of generalised time functions with gradient uniformly bounded away from light cones. To derive this result we introduce new techniques to construct and manipulate achronal sets. As a consequence of these techniques we obtain a functional description of the Lorentzian distance extending the work of Franco and Moretti, [6, 12].

1 Introduction

This paper originated from asking whether Franco and Moretti's formula for the Lorentzian distance function $d : M \times M \rightarrow [0, \infty]$ could be extended to stably causal manifolds, [6, 12]. Their proofs were valid only in the globally hyperbolic case. The technical difficulties raised by this problem led to a consideration of the delicate interplay between the Lorentzian distance function, causality and time functions.

Ultimately we were led to develop new techniques for the construction of achronal sets, the manipulation of these sets and a new class of generalised time function. These new techniques allow us to prove our two main results.

Finiteness of the Lorentzian distance. *Let (M, g) be a Lorentzian manifold. The Lorentzian distance is finite if and only if there exists a function $f : M \rightarrow \mathbb{R}$, strictly monotonically increasing on timelike curves, whose gradient exists almost everywhere and is such that $\text{ess sup } g(\nabla f, \nabla f) \leq -1$.*

The Lorentzian distance formula. *Let (M, g) have finite Lorentzian distance. Then for all $p, q \in M$*

$$d(p, q) = \inf \{ \max \{ f(q) - f(p), 0 \} : f : M \rightarrow \mathbb{R}, f \text{ future directed, } \text{ess sup } g(\nabla f, \nabla f) \leq -1 \}. \quad (1)$$

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We will refer to equality (1) as the distance formula below.

Franco and Moretti had as their initial motivation the extension of Connes' formula for the Riemannian distance to Lorentzian manifolds, [5], and we note that early investigations and counter-examples appear in [14]. The tools of noncommutative geometry have thus far not been seriously extended past the globally hyperbolic setting, and we hope that our results stimulate further work on this topic.

The paper is organised as follows. Section 2 summarises those ideas from Lorentzian geometry that we require, and sets notation. In addition we review, and mildly extend, the results of Franco and Moretti. We also prove a 'reverse Lipschitz' characterisation of our generalised time functions in Proposition 2.18, which is essential for applications to Connes-type formulae for the distance.

In brief, the idea of our proof is as follows. Let $S \subset M$ be an achronal set in the Lorentzian manifold (M, g) . Then if $M = I^+(S) \cup S \cup I^-(S)$, we can try to define a function $f(x) = d(S, x) = \sup_{s \in S} d(s, x)$ when x is in the future of S , and similarly for other cases. The chief difficulty with this definition is the finiteness of f , even when the Lorentzian distance function d only takes finite values. Much of the difficulty is in finding a suitable set $S \subset M$ with which to define f .

Section 3 contains the technical advances, and is divided into three subsections. The first shows that if the Lorentzian distance is finite then it is possible to choose an achronal subset of the manifold that 'bounds' any divergent behaviour of the metric. The second proves that, under mild assumptions on M , and starting from a suitable achronal set, there exists an achronal surface which divides the manifold into the future of the set, the surface itself and the past of the surface. This is a refinement of a construction of Penrose, [15, Proposition 3.15]. The third section shows how, starting from such a 'bounding' achronal set, to construct a new achronal set. This produces a new achronal set S which separates the manifold M into the future of S , S itself and the past of S . The advantage of this new set is that we can define a generalised time function by taking the Lorentzian distance of a point to S , and this function takes finite values.

Finally, Section 4 presents the proofs of our two main results.

The Appendix provides the details on the regularity of our generalised time functions. A similar concept, also called generalised time functions, has appeared previously, [8]. Our generalised time functions have poor regularity, but in the Appendix we prove that they are continuous almost everywhere, and do have all directional derivatives, and so gradient, existing almost everywhere.

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2 Background definitions, notation and results

In the following (M, g) will always be a C^∞ , time orientable, path-connected, Lorentzian manifold M of dimension $n + 1 \geq 2$ equipped with a Lorentzian metric g with signature $(-1, 1, \dots, 1)$. We let T denote the vector field defining the time orientation. The non-time orientable case can be studied via Lorentzian covering manifolds, [9, p 181]. Here and below the measure is always the Lebesgue

measure arising from $\sqrt{-\det g}$. Throughout the rest of the paper, unless otherwise noted, we shall use the notation as given in [1]. In particular, for any $U \subset M$, $I^\pm(U) = \bigcup_{x \in U} I^\pm(x)$ and that for any $x \in M$, $I^+(x) = \{y \in M : d(x, y) > 0\}$.

A curve γ is a C^0 , piecewise C^1 , function from an interval $I \subset \mathbb{R}$ into M so that the tangent vector $\gamma' = \gamma_*(\partial_t)$ is almost everywhere (a.e.) non-zero. For $x, y \in M$ we let $\Omega_{x,y}$ denote the set of future-directed causal curves from x to y . Thus $\gamma \in \Omega_{x,y}$ satisfies $g(\gamma', \gamma') \leq 0$ (causal) everywhere it exists and $g(T, \gamma') < 0$ (future-directed).

By a standard abuse of notation, we sometimes treat γ as a set rather than a curve. Thus $x, y \in \gamma$ means $x, y \in \gamma(I)$, $\gamma \subset U$ means $\gamma(I) \subset U$, and so on. Given a causal curve $\gamma : [a, b] \rightarrow M$, the length of γ , denoted $L(\gamma)$ is defined by

$$L(\gamma) = \int_a^b \sqrt{-g(\gamma', \gamma')(t)} dt.$$

Definition 2.1 ([1, Chapter 4]). Let (M, g) be a Lorentzian manifold. The Lorentzian distance $d : M \times M \rightarrow \mathbb{R}$ is given by

$$d(p, q) := \begin{cases} \sup_{\gamma \in \Omega_{p,q}} L(\gamma) & \Omega_{p,q} \neq \emptyset \\ 0 & \Omega_{p,q} = \emptyset. \end{cases}$$

The Lorentzian distance is always lower semi-continuous, [1, Lemma 4.4].

We make use of the reverse triangle inequality for the Lorentzian distance, [1, page 140]: if $x \in M$, $y \in I^+(x)$ and $z \in I^+(y)$ then $d(x, z) \geq d(x, y) + d(y, z)$. If for all $x, y \in M$, $d(x, y) < \infty$ then we say that the Lorentzian distance is finite, or that M has finite Lorentzian distance.

Definition 2.2. Let $S \subset M$ be a subset of M . We define the functions $d(S, \cdot) : M \rightarrow \mathbb{R} \cup \{\infty\}$ and $d(\cdot, S) : M \rightarrow \mathbb{R} \cup \{\infty\}$ by $d(S, x) = \sup\{d(s, x) : s \in S\}$ and $d(x, S) = \sup\{d(x, s) : s \in S\}$.

These functions satisfy a version of the reverse triangle inequality.

Lemma 2.3. *Let $x \in M$, $y \in I^+(x)$ and $S \subset M$. Then:*

1. $x \in I^+(S)$ implies that $d(S, y) \geq d(S, x) + d(x, y)$;
2. $y \in I^-(S)$ implies that $d(x, S) \geq d(x, y) + d(y, S)$.

Proof. In each case, the reverse triangle inequality implies that:

1. $d(z, y) \geq d(z, x) + d(x, y)$ when $z \in S \cap I^-(x)$;
2. $d(x, z) \geq d(x, y) + d(y, z)$ when $z \in S \cap I^+(y)$.

Taking the supremum over these inequalities with respect to z proves the result. \square

Definition 2.4. A function $f : M \rightarrow \mathbb{R}$ such that for all timelike curves from x to y the function $f \circ \gamma$ is strictly monotonically increasing is called a future-directed generalised time function. A past-directed generalised time function f is a function so that $-f$ is a future-directed generalised time function.

It is worth noting that our definition of a generalised time function is slightly more general than that used in the literature, [1, Definition 3.23] or [11, Definition 3.48], as we do not require our generalised time functions to be strictly monotonically increasing along null curves.

We show, in Appendix A, that if f is monotonic on all timelike curves, as are generalised time functions, then all directional derivatives of f exist a.e. This is an application of the well-known differentiability a.e. of real-valued monotonic functions on an interval, [7, Theorem 9.3.1]. Moreover, [16, Chapter 5, Theorem 2], for any function $f : M \rightarrow \mathbb{R}$ and any curve $\gamma : [0, 1] \rightarrow M$ such that $f \circ \gamma$ is (not necessarily strictly) monotonically increasing, the derivative of $f \circ \gamma$ exists a.e., is integrable, and we have the inequality

$$\int_0^1 \frac{d}{dt} (f \circ \gamma)(t) dt \leq f(1) - f(0). \quad (2)$$

In what follows we will be interested in generalised time functions, $f : M \rightarrow \mathbb{R}$, so that we have the inequality $\text{ess sup}_M g(\nabla f, \nabla f) \leq -1$. This condition ensures that wherever ∇f exists, and it must exist a.e., it is timelike, as we show in Proposition 2.18. As the following example shows, this is unfortunately not enough to ensure that f is strictly monotonically increasing along all causal curves.

Example 2.5. Let $M = (-1, 1) \times (0, 1)$ considered as a sub-manifold of 2-dimensional Minkowski space. Let $S = M \cap \partial I^+((0, 0))$ be the portion of the future null cone of the origin that lies in M . By construction S is achronal and by definition of M , $M = I^+(S) \cup S \cup I^-(S)$. The Lorentzian distance of M is bounded above by 1 and hence is finite. Thus the function defined by

$$f(x) = \begin{cases} d(S, x) & \text{if } x \in I^+(S) \\ 0 & \text{if } x \in S \\ -d(x, S) & \text{if } x \in I^-(S) \end{cases}.$$

is well defined and a generalised time function. Proposition 3.13 shows that f satisfies the gradient bound $\text{ess sup}_M g(\nabla f, \nabla f) \leq -1$. However the level surface $S = f^{-1}(0)$ is not acausal, and contains null geodesics on which f is constant. The function f is therefore not strictly monotonically increasing along all causal curves.

Lorentzian manifolds, (M, g) , can be classified into a causal hierarchy. Of that hierarchy we shall need the following definitions:

- *stably causal* if there exists a continuous function $f : M \rightarrow \mathbb{R}$ that is strictly monotonically increasing on all causal curves,
- *causally simple* if it is causal and $J^\pm(x)$ is closed for all $x \in M$;
- *globally hyperbolic* if and only if it is causal and, for all $x, y \in M$, the intersection $J^+(x) \cap J^-(y)$ is compact. This is equivalent to M being isometric to the product $\mathbb{R} \times N$. See [11, Section 3.11.3 and Theorem 3.78] for a review of Bernal and Sanchez's work on this, [2, 3, 4].

Global hyperbolicity implies causal simplicity which implies stable causality. See [11] for further details and examples.

The following example of a non-continuous generalised time function with timelike gradient a.e. everywhere demonstrates that the lack of continuity in our definition of generalised time function can have a serious impact on the relationship between time functions and stable causality.

Example 2.6. Let

$$T = [-\pi, \pi] \times \mathbb{R} \setminus \left(\left\{ \left(\frac{\pi}{4}, x \right) : x \leq \frac{\pi}{4} \right\} \cup \left\{ \left(-\frac{\pi}{4}, x \right) : x \geq -\frac{\pi}{4} \right\} \right),$$

with coordinates $t \in [-\pi, \pi]$ and $s \in \mathbb{R}$. A diagram representing this manifold can be found in [11, Figure 7] and additional discussion of this example can be found in [9, Page 193] and [1, Figure 3.4]. Let $(t, s), (\tau, \sigma) \in T$ and define an equivalence relation, \sim , on T by $(t, s) \sim (\tau, \sigma)$ if and only if $s = \sigma$ and $t = -\tau = \pm\pi$. Let $M = T/\sim$. Topologically M is $S^1 \times \mathbb{R}$ with two half lines removed. Define a metric g on M by pushing the metric $g = -dt^2 + ds^2$ on T onto M via the induced map from T to M .

We claim that (M, g) is not stably causal. Indeed, consider the point $(0, 0)$. For any metric with slightly wider lightcones, there will exist $\epsilon > 0$ such that the point $(\frac{\pi}{4}, \frac{\pi}{4} + \epsilon)$ is in the future of $(0, 0)$ and $(-\frac{\pi}{4}, -\frac{\pi}{4} - \epsilon)$ is in the past of $(0, 0)$. By the definition of M , the point $(-\frac{\pi}{4}, -\frac{\pi}{4} - \epsilon)$ is in the future of $(\frac{\pi}{4}, \frac{\pi}{4} + \epsilon)$, and hence there will exist a closed timelike curve for any metric with slightly wider lightcones.

Let

$$\begin{aligned} A &= \left\{ (t, s) : t > \frac{\pi}{4}, s \in \mathbb{R} \right\} \cup \left\{ (t, s) : t > -\frac{\pi}{4}, s > t \right\} \\ B &= \left\{ (t, s) : t < \frac{\pi}{4}, s < t \right\} \cup \left\{ (t, s) : t < -\frac{\pi}{4}, s \in \mathbb{R} \right\} \end{aligned}$$

and define $f : M \rightarrow \mathbb{R}$ by

$$f(t, s) = \begin{cases} t & (t, s) \in A \\ t + 2\pi & (t, s) \in B \end{cases}$$

It can easily be checked that $\nabla f = \partial t$ wherever it exists and that f is a generalised time function despite M failing to be stably causal.

Definition 2.7. A set F is a future set (P is a past set) if $F = I^+(F)$ ($P = I^-(P)$), [15, Definition 3.1]. A set S is achronal if $S \cap I^+(S) = \emptyset$, [15, Definition 3.11], or equivalently $S \cap I^-(S) = \emptyset$. A set S is an achronal surface (or sometimes also called an achronal boundary) if $S = \partial F$ ($S = \partial P$) where F is a future set (P is a past set), [15, Definition 3.13 and Proposition 3.14]. Achronal surfaces are achronal sets, [15, Definition 3.13].

The following result, which is a paraphrase of a result by Penrose, highlights the importance of achronal surfaces.

Proposition 2.8 ([15, Proposition 3.15]). *Let (M, g) be a Lorentzian manifold. If $S \neq \emptyset$ is an achronal surface then there is a unique future set F and a unique past set P so that F, P, S are disjoint, $M = F \cup S \cup P$ and $S = \partial F = \partial P$. Furthermore, any timelike curve from P to F intersects S in a unique point.*

Proposition 2.8 will play a pivotal role below. To simplify its application we use the following two results.

Corollary 2.9. *Let (M, g) be a Lorentzian manifold and let $S \subset M$ be an achronal surface. If $S = \partial I^+(S)$ then $F = I^+(S)$ and $P = M \setminus (I^+(S) \cup S)$ are the unique future and past sets given by Proposition 2.8. In particular any timelike curve from P to F intersects S in a unique point.*

Proof. By assumption $S = \partial F = \partial P$ and $M = F \cup S \cup P$. The result now follows directly from Proposition 2.8. \square

Lemma 2.10. *Let (M, g) be a Lorentzian manifold. If $A \subset M$ then $\partial I^+(\partial I^+(A)) = \partial I^+(A)$.*

Proof. Since $\partial I^+(A)$ is achronal, $\partial I^+(A) \subset \partial I^+(\partial I^+(A))$. Let $a \in \partial I^+(\partial I^+(A))$. Then $I^+(a) \subset I^+(\partial I^+(A))$. So for all $x \in \overline{I^+(a)}$ there exists $b \in \partial I^+(A)$ so that $x \in I^+(b)$. This implies that $I^+(a) \subset I^+(A)$ and hence $a \in \overline{I^+(A)}$. Since $a \in \partial I^+(\partial I^+(A))$ we know that $a \in \partial I^+(A)$. Thus $\partial I^+(\partial I^+(A)) = \partial I^+(A)$ as required. \square

Note that in Corollary 2.9 the set P is not necessarily equal to $I^-(S)$. A core part of this paper is the construction of an achronal surface, S , so that the unique future and past sets given by Proposition 2.8 are $F = I^+(S)$ and $P = I^-(S)$. This allows us to assume that every point in M is either in S or is connected to S via a timelike curve, a fact that we will exploit to define the needed functions.

2.1 Overview of Franco's result

We briefly reprise the key arguments used by Franco in [6] to obtain his main result, stated here as Theorem 2.14, and point out that these arguments are broadly similar to those used by Moretti [12, Theorem 2.2].

The version of these results we present represents only a small generalisation, but it seems worthwhile to repeat the arguments, as they show clearly where several constraints come from.

Lemma 2.11. *Let (M, g) be a Lorentzian manifold, $x, y \in M$ and $\gamma \in \Omega_{x,y}$. If f is monotonic on every timelike curve then $|f(y) - f(x)| \geq l(\gamma) \operatorname{ess\,inf}_{\gamma} \sqrt{-g(\nabla f, \nabla f)}$.*

Proof. By Lemma A.5 the vector field ∇f exists a.e., see Definition A.6. Lemma A.7 implies that ∇f is causal. Assume that ∇f is past-directed. Let $\gamma : [0, 1] \rightarrow M \in \Omega_{x,y}$.

We calculate, using Lemmas A.3 and A.5 and Definition A.6, as well as Equation (2) that

$$f(y) - f(x) \geq \int_0^1 \frac{d}{dt} f(\gamma(t)) dt = \int_0^1 df(\gamma') dt = \int_0^1 g(\nabla f, \gamma') dt = \int_0^1 |g(\nabla f, \gamma')| dt,$$

since γ is future-directed and ∇f is past-directed. By [13, Proposition 5.30] if γ' and ∇f are both time-like the reverse Cauchy inequality holds,

$$|g(\nabla f, \gamma')| \geq \sqrt{-g(\nabla f, \nabla f)} \sqrt{-g(\gamma', \gamma')}.$$

If ∇f or γ' is null then it is clear that this inequality continues to hold. Hence

$$f(y) - f(x) \geq \int_0^1 |g(\nabla f, \gamma')| dt \geq \int_0^1 \sqrt{-g(\nabla f, \nabla f)} \sqrt{-g(\gamma', \gamma')} dt \geq l(\gamma) \operatorname{ess\,inf}_{\gamma} \sqrt{-g(\nabla f, \nabla f)}.$$

In the case that f is past-directed, ∇f is future-directed. Thus $-f$ has a past-directed gradient, whence

$$f(x) - f(y) \geq l(\gamma) \operatorname{ess\,inf}_{\gamma} \sqrt{-g(\nabla f, \nabla f)} \quad \text{and so} \quad |f(y) - f(x)| \geq l(\gamma) \operatorname{ess\,inf}_{\gamma} \sqrt{-g(\nabla f, \nabla f)},$$

as required.

If γ is causal, but neither timelike nor null, we can divide γ into null and timelike segments. Since γ is piecewise C^1 the intermediate value theorem shows that each timelike segment is an open interval. The result now follows by applying the arguments given above to each segment. \square

Corollary 2.12. *With the assumptions of Lemma 2.11 and assuming that $\text{ess inf}_M \sqrt{-g(\nabla f, \nabla f)} > 0$ and $d(x, y) < \infty$ we have*

$$|f(y) - f(x)| \geq d(x, y) \text{ess inf}_M \sqrt{-g(\nabla f, \nabla f)}.$$

Proof. Since $\text{ess inf}_\gamma \sqrt{-g(\nabla f, \nabla f)} \geq \text{ess inf}_M \sqrt{-g(\nabla f, \nabla f)}$ we have, from Lemma 2.11,

$$|f(y) - f(x)| \geq l(\gamma) \text{ess inf}_M \sqrt{-g(\nabla f, \nabla f)}.$$

Taking the supremum over curves in $\Omega_{x,y}$ gives $|f(y) - f(x)| \geq d(x, y) \text{ess inf}_M \sqrt{-g(\nabla f, \nabla f)}$. \square

In order to obtain his functional description of the Lorentzian distance, Franco proves the following, [6, Lemma 5].

Lemma 2.13. *Let (M, g) be globally hyperbolic, $x, y \in M$ and $\epsilon > 0$. Then there exists a future-directed time function f so that $\text{ess inf}_M \sqrt{-g(\nabla f, \nabla f)} \geq 1$ and $|f(y) - f(x) - d(x, y)| \leq \epsilon$.*

Global hyperbolicity is essential for Franco's construction of this time function. In particular he exploits the existence of Cauchy surfaces as well as the necessary finiteness and continuity of the Lorentzian distance. With this in hand, Franco is able to prove his main result.

Theorem 2.14. [6, Theorem 1] *Let (M, g) be a globally hyperbolic manifold then*

$$d(x, y) = \inf \{ \max \{ f(y) - f(x), 0 \} : f \in C(M, \mathbb{R}), \text{ess sup } g(\nabla f, \nabla f) \leq -1, \nabla f \text{ is past-directed} \}.$$

Since Moretti uses different differentiability conditions, his analogue of this result, [12, Theorem 2.2], is superficially different but has essentially the same conclusion and proof.

The results above imply the following about those situations where the Lorentzian distance becomes infinite.

Proposition 2.15. *Let (M, g) be a Lorentzian manifold, $x, y \in M$ and suppose that $f : M \rightarrow \mathbb{R}$ is monotonic on every timelike curve. Suppose further that there exist $\{\gamma_i : i \in \mathbb{N}\} \subset \Omega_{x,y}$ so that $l(\gamma_i) \rightarrow \infty$, as $i \rightarrow \infty$, i.e. $d(x, y) = \infty$. Then $\lim_{i \rightarrow \infty} \text{ess inf}_{\gamma_i} \sqrt{-g(\nabla f, \nabla f)} = 0$.*

Proof. This follows from Lemma 2.11, since for all i we have $|f(y) - f(x)| \geq l(\gamma_i) \text{ess inf}_{\gamma_i} \sqrt{-g(\nabla f, \nabla f)}$. \square

Corollary 2.16. *Let (M, g) be a Lorentzian manifold. If there exists a function that is monotonic on every time-like curve so that $\text{ess inf}_M \sqrt{-g(\nabla f, \nabla f)} > 0$ then M has finite Lorentzian distance.*

Proof. This is implied by the contrapositive of Proposition 2.15. \square

The behaviour described in Proposition 2.15 can occur in otherwise innocuous situations, and has to be taken into account for our construction. The following example of a causally simple non-globally hyperbolic spacetime with $x, y \in M$ so that $d(x, y) = \infty$ is taken from [11, Remark 3.66], and shows how our main construction, presented in subsection 3.3 below, fails when the Lorentzian distance is not finite. Similar examples with finite Lorentzian distance motivate the constructions of the next section.

Example 2.17. Let $M = \{(x, y) \in \mathbb{R}^2 : 2|y| > x \text{ and } x > -1\}$ with metric $ds^2 = \frac{1}{x^2+y^2} (dx^2 - dy^2)$. This is a non-globally hyperbolic, causally simple spacetime, [11, Figure 10]. As a consequence there exist analytic time functions on M . A specific example is $h(x, y) = y$ whose gradient is $\nabla h = -(y^2 + x^2)\partial_y$.

By definition, for all $(x, y) \in M$ the surface $\partial I^+((x, y))$ is an achronal surface, which further satisfies $M = I^+(\partial I^+((x, y))) \cup \partial I^+((x, y)) \cup I^-(\partial I^+((x, y)))$. Hence for any $(x, y) \in M$ and letting $S = \partial I^+((x, y))$, we can try to construct a function $f : M \rightarrow \mathbb{R}$ by the definition

$$f((u, v)) = \begin{cases} d(S, (u, v)) & \text{if } (u, v) \in I^+(S) \\ 0 & \text{if } (u, v) \in S \\ -d((u, v), S) & \text{if } (u, v) \in I^-(S) \end{cases}$$

Depending on the choice of (x, y) we have three cases. To present these cases we consider M as a submanifold of \mathbb{R}^2 and in the following statements closures are taken in \mathbb{R}^2 . The three cases are:

1. $(0, 0) \in \overline{S}$,
2. $(0, 0) \notin \overline{S}$ and $(0, 0) \in \overline{I^-(S)}$,
3. $(0, 0) \notin \overline{S}$ and $(0, 0) \in \overline{I^+(S)}$.

For the sake of this example we assume that the last case holds. Note that arguments similar to those given below will hold in the other two cases. We denote the set $\{(u, v) \in M : |u| < v\}$ by $I^+((0, 0))$. This is an abuse of notation since $(0, 0) \notin M$.

We now show that for all $(u, v) \in I^+((0, 0))$, $f(u, v) = \infty$. Let $w > 0$ and let $\gamma_w : [0, 1] \rightarrow \mathbb{R}$ be the curve given by $\gamma_w(\tau) = (0, w(1 - \tau))$. This is a past-directed timelike curve from $(0, w)$ to $(0, 0)$, and

$$g(\gamma'_w, \gamma'_w) = \frac{1}{(1 - \tau)^2}$$

A short calculation shows that $L(\gamma_w) = \infty$ for all $w > 0$. Since w was arbitrary we can choose w so that $(0, w) \in I^-(u, v)$. Since $(0, 0) \in \overline{I^+(S)}$ and $(0, 0) \notin \overline{S}$ we know that for all $\tau \in [0, 1)$,

$$(0, w(1 - \tau)) \in I^+(S).$$

Thus, from Lemma 2.3,

$$\begin{aligned} f(u, v) &= d(S, (u, v)) \geq d(S, (0, w)) + d((0, w), (u, v)) \\ &= \sup_{\tau} \{d(S, (0, w(1 - \tau))) + d((0, w(1 - \tau)), (0, w))\} + d((0, w), (u, v)) \\ &= \infty \end{aligned}$$

as claimed.

That is, despite the existence of smooth finite valued time functions, the construction we give in Proposition 3.13, for this surface, produces a function which takes infinite values. We claim that this is the case for all choices of $S = \partial I^+((x, y))$ (in the case $(0, 0) \notin \overline{S}$ and $(0, 0) \in \overline{I^-(S)}$ the function f will take the value $-\infty$).

Let $U \subset M$ be the set of points so that $f|_U \subset \mathbb{R}$. Then Lemma A.5, Definition A.6 and Proposition 2.18 imply that $\text{ess inf}_U \sqrt{-g(\nabla f, \nabla f)} \geq 1$. This is in contrast to $(x, y) \mapsto h(x, y) = y$ where $\text{ess inf}_M \sqrt{-g(\nabla h, \nabla h)} = 0$. Hence we have paid for a lower bound on the gradient of f by letting f diverge to $\pm\infty$ on M .

In order to complete our discussion of generalised time functions, we present an alternative characterisation which will play an important role later.

Proposition 2.18. *Let (M, g) be a Lorentzian manifold and $f : M \rightarrow \mathbb{R}$ a function differentiable a.e. The condition*

$$\text{for all } x \in M, \text{ for all } y \in I^+(x), \quad f(y) - f(x) \geq d(x, y) \quad (3)$$

holds if and only if $\text{ess inf}_M \sqrt{-g(\nabla f, \nabla f)} \geq 1$ and f is future-directed.

Proof. We begin by showing that condition (3) implies that ∇f is past-directed and timelike wherever it exists. This allows us to build a coordinate system using ∇f which is then used to prove the bound on the gradient. We will make extensive use of the fact that for all timelike curves $\gamma : [0, 1] \rightarrow M$, the function $f \circ \gamma$ is differentiable a.e. in $[0, 1]$, as shown in the Appendix.

So we can fix $x \in M$ where $(\nabla f)(x)$ exists. Then we take a geodesic neighbourhood U of x , and $y \in I^+(x) \cap U$. We let γ be the unique geodesic from x to y so that $d(x, y) = \int_0^1 \sqrt{-g(\gamma', \gamma')}(s) ds$. Indeed, for $0 < t \leq 1$, $d(x, \gamma(t)) = \int_0^t \sqrt{-g(\gamma', \gamma')}(s) ds$.

Now if f satisfies condition (3), then

$$f(\gamma(t)) - f(x) \geq d(x, \gamma(t)) = \int_0^t \sqrt{-g(\gamma', \gamma')}(s) ds.$$

Dividing through by t and using the mean value theorem for integrals shows that

$$\frac{f(\gamma(t)) - f(x)}{t} \geq \frac{1}{t} \int_0^t \sqrt{-g(\gamma', \gamma')}(s) ds = \sqrt{-g(\gamma', \gamma')}(t_0),$$

where $0 < t_0 < t$. Hence as $t \rightarrow 0$ we find

$$g(\nabla f, \gamma')(\gamma(0)) = \frac{d(f \circ \gamma)}{dt}(0) \geq \sqrt{-g(\gamma', \gamma')}(0). \quad (4)$$

By considering all such $y \in I^+(x) \cap U$, we see that Equation (4) holds for all timelike vectors in $T_x M$. In particular, letting T be the unit vector field defining the time orientation of M , we find that $g(\nabla f, T) \geq 1$ and hence ∇f is past-directed.

If $Z \in T_x M$ is a timelike vector, and future directed, we can write

$$Z = \alpha T + \beta V, \quad V \perp T, \quad g(V, V) = 1, \quad \alpha > 0, \quad -\alpha^2 + \beta^2 = -m^2.$$

Similarly

$$(\nabla f)(x) = \mu T + \nu W, \quad W \perp T, \quad g(W, W) = 1, \quad \mu \leq -1$$

where the value of μ follows from setting $\gamma'(0)$ equal to $T(x)$ in Equation (4). We can, and do, assume that $\beta, \nu > 0$. We set $c = g(V, W)$ and compute that

$$g((\nabla f)(x), Z) = -\mu\alpha + \nu\beta c \geq \sqrt{\alpha^2 - \beta^2} = m$$

where the inequality is from Equation (4). Now choose $V = -W$ so that $c = -1$. This yields

$$|\mu|\alpha \geq m + \nu\beta.$$

Rearranging and using the binomial series yields

$$\begin{aligned} |\mu| &\geq \frac{m}{\alpha} + \nu \sqrt{1 - \frac{m^2}{\alpha^2}} \\ &= \frac{m}{\alpha} + \nu \left(1 - \frac{m^2}{2\alpha^2} - \frac{1}{2 \times 4} \left(\frac{m^2}{\alpha^2} \right)^2 + \dots \right) \\ &= \nu + \frac{m}{\alpha} - \frac{m^2\nu}{2\alpha^2} - \frac{\nu}{2 \times 4} \frac{m^4}{\alpha^4} + \dots \end{aligned}$$

This makes sense as an infinite series, since $m^2/\alpha^2 < 1$ when $\beta \neq 0$, and for m sufficiently small it is straightforward to see that we have $|\mu| > \nu$. In short, $(\nabla f)(x)$ is timelike.

We now know that condition (3) implies that $(\nabla f)(x)$ is past-directed and timelike. Hence we may take normal coordinates $\phi : U \subset \mathbb{R}^n \rightarrow V \subset M$ about x so that $g(\partial_i, \partial_j)(x) = \delta_{ij}$, $i \neq 0$, and $g(\partial_0, \partial_j)(x) = -\delta_{0j}$ where $\partial_0(x) = \alpha \nabla f(x)$, $\alpha \neq 0$. This ensures that $\partial_i f|_x = 0$ if $i \neq 0$.

Condition (3) tells us that

$$\lim_{h \rightarrow 0^+} \frac{f \circ \phi(h, 0, \dots, 0) - f \circ \phi(0, 0, \dots, 0)}{h} \geq \lim_{h \rightarrow 0^+} \frac{d(x, \phi(h, 0, \dots, 0))}{h} \quad (5)$$

and

$$\lim_{h \rightarrow 0^-} \frac{f \circ \phi(0, 0, \dots, 0) - f \circ \phi(h, 0, \dots, 0)}{-h} \geq \lim_{h \rightarrow 0^-} \frac{d(\phi(h, 0, \dots, 0), x)}{-h}. \quad (6)$$

By construction, for h small enough, we have

$$\phi(h, 0, \dots, 0) = \exp_x(h\partial_0) = \gamma_{h\partial_0}(1) = \gamma_{\partial_0}(h),$$

where $\gamma_v : [0, a) \rightarrow M$, $a \in \mathbb{R} \cup \{\infty\}$ is the unique geodesic satisfying $\gamma_v(0) = x$ and $\gamma'_v(0) = v$ with affine parameter. Since γ_{∂_0} is a geodesic and $g(\gamma_{\partial_0}, \gamma_{\partial_0})(x) = -1$, we see that for all $0 \leq \tau \leq h$, $g(\gamma_{\partial_0}, \gamma_{\partial_0})(\tau) = -1$. Then for $h > 0$ we calculate that

$$L(\gamma_{\partial_0}|_{[0, h]}) = \int_0^h \sqrt{-g(\gamma_{\partial_0}, \gamma_{\partial_0})(\tau)} d\tau = \int_0^h d\tau = h.$$

By definition, $d(x, \phi(h, 0, \dots, 0)) = \sup_{\gamma \in \Omega_{x, \phi(h, 0, \dots, 0)}} L(\gamma)$ and as $\gamma_{\partial_0}|_{[0, h]} \in \Omega_{x, \phi(h, 0, \dots, 0)}$ we see that $d(x, \phi(h, 0, \dots, 0)) \geq h$. The same calculation when $h < 0$ gives $d(\phi(h, 0, \dots, 0), x) \geq -h$. Plugging these inequalities into Equations (5) and (6) we see that $\partial_0 f|_x \geq 1$. We may now calculate that

$$\sqrt{-g(\nabla f, \nabla f)}(x) = \sqrt{-g^{ij} \partial_i f \partial_j f}(x) = |\partial_0 f|(x) \geq 1.$$

As x was an arbitrary point where the gradient exists, we find that

$$\operatorname{ess\,inf}_M \sqrt{-g(\nabla f, \nabla f)} \geq 1.$$

For the converse statement, suppose that $\operatorname{ess\,inf}_M \sqrt{-g(\nabla f, \nabla f)} \geq 1$. Let $x, y \in M$ and γ be a timelike curve from x to y . From Lemma 2.11 we know that

$$|f(y) - f(x)| = f(y) - f(x) \geq \operatorname{ess\,inf}_\gamma \sqrt{-g(\nabla f, \nabla f)} L(\gamma) \geq L(\gamma).$$

Since $y \in I^+(x)$ we know that $d(x, y)$ is the supremum of $L(\gamma)$ over all timelike curves from x to y . Hence by taking the supremum over all timelike curves from x to y of the inequality above we get

$$f(y) - f(x) \geq d(x, y).$$

Therefore condition (3) is satisfied by f . □

3 Constructions and definitions for the proof of the main theorems

This section contains the technical details for the proofs of our main theorems. We have divided the work into three portions each of which culminates in a key result. Briefly those results are:

1. Lemma 3.7: If the Lorentzian distance is finite then there exists a special achronal subset (which we call a hatting);
2. Lemma 3.11: If there exists a future set with non-empty boundary then there exists an achronal surface S so that $M = I^+(S) \cup S \cup I^-(S)$;
3. Proposition 3.13: If the Lorentzian distance is finite then there exists a generalised time function satisfying condition (3).

3.1 Finite Lorentzian distance implies the existence of a hatting

Definition 3.1. Let M be a manifold. A sequence $(x_i)_{i \in \mathbb{N}} \subset M$ such that there exists $x \in M$ such that $d(x_i, x) \rightarrow \infty$ ($d(x, x_i) \rightarrow \infty$) as $i \rightarrow \infty$ is called future (past) divergent. Given a future (past) divergent sequence, (x_i) , let $F_{(x_i)} = I^+(\{x \in M : \lim_{i \rightarrow \infty} d(x_i, x) = \infty\})$ ($P_{(x_i)} = I^-(\{x \in M : \lim_{i \rightarrow \infty} d(x, x_i) = \infty\})$).

Lemma 3.2. Let (M, g) be a Lorentzian manifold and (x_i) a future divergent sequence. If (y_i) is a subsequence of (x_i) then (y_i) is future divergent and $F_{(x_i)} \subset F_{(y_i)}$.

Proof. Let $x \in F_{(x_i)}$. By definition $\lim_{i \rightarrow \infty} d(x_i, x) = \infty$. If $\lim_{i \rightarrow \infty} d(y_i, x) \neq \infty$ then we also have $\lim_{i \rightarrow \infty} d(x_i, x) \neq \infty$. Therefore $\lim_{i \rightarrow \infty} d(y_i, x) = \infty$. This implies that (y_i) is future divergent and that $F_{(x_i)} \subset F_{(y_i)}$. □

Lemma 3.3. Let (M, g) be a Lorentzian manifold with finite Lorentzian distance and S an achronal set. If there exists $x \in I^+(S)$ so that $d(S, x) = \infty$ then there exists a future divergent sequence in S .

Proof. Since $d(S, x) = \infty$ there exists a sequence $(x_i) \subset S$ so that $\lim_{i \rightarrow \infty} d(x_i, x) = \infty$. The sequence (x_i) is trivially a future divergent sequence in S . \square

Definition 3.4. A hatting is an achronal subset $H \subset M$ so that for every future (past) divergent sequence, $(x_i)_{i \in \mathbb{N}}$ in M , there exists $N \in \mathbb{N}$ so that for all $j \geq N$, $x_j \in I^-(H)$ ($x_j \in I^+(H)$).

Lemma 3.5. Let (M, g) be a Lorentzian manifold with finite Lorentzian distance. If $S \subset M$ is finite then for all future divergent sequences (x_i) there exists $N \in \mathbb{N}$ so that for all $j \geq N$, $x_j \notin \overline{I^+(S)}$.

Proof. For a contradiction we will assume that no such N exists. By definition and as S is finite $\overline{I^+(S)} = \bigcup_{s \in S} \overline{I^+(s)}$. As no such N exists and as the union is over a finite number of elements a pigeon hole argument shows for all divergent sequences, (x_i) , there exists a subsequence, (y_i) , of (x_i) so that $(y_i) \subset \overline{I^+(s)}$ for some $s \in S$. Lemma 3.2 implies that (y_i) is divergent. Let $y \in F_{(y_i)}$. By construction, for each i , $s \in I^-(y_i)$ and $y_i \in I^-(y)$. Thus $\lim_{i \rightarrow \infty} d(y_i, y) = \infty$ implies that $d(s, y) = \infty$. This is a contradiction, hence the required $N \in \mathbb{N}$ exists. \square

Lemma 3.6. Let (M, g) be a Lorentzian manifold. Let $S \subset M$ and let (x_i) be a future divergent sequence. If $F_{(x_i)} \cap S \neq \emptyset$ then there exists $N \in \mathbb{N}$ so that for all $j \geq N$, $x_j \in I^-(S)$.

Proof. Let $s \in F_{(x_i)} \cap S$. Since $F_{(x_i)}$ is open and non-empty, there exists $x \in F_{(x_i)} \cap I^-(s)$. Hence $x \in I^-(S)$. As $x \in F_{(x_i)}$ we know that $\lim_{i \rightarrow \infty} d(x_i, x) = \infty$. This implies that there exists $N \in \mathbb{N}$ so that for all $j \geq N$, $x_i \in I^-(x)$. Since $I^-(x) \subset I^-(S)$ we have the result. \square

Lemma 3.7. Let (M, g) be a Lorentzian manifold. If the Lorentzian distance is finite then there exists a hatting for M .

Proof. Let A be the union of all $F_{(x_i)}$ for all future divergent sequences. By construction A is an open manifold. Therefore there exists a countable dense subset F of A . Similarly, let P be a countable dense subset of the union of all $P_{(x_i)}$ where (x_i) is a past divergent sequence. Since F and P are countable, we choose an ordering so that $F = \{f_0, f_1, \dots, f_i, \dots\}$, $P = \{p_0, p_1, \dots, p_i, \dots\}$.

We build our hatting by iteration over \mathbb{N} .

Base case: Let $i = 0$ and define $S_0 = P_0 = F_0 = \{p_0\}$. Note that P_0 and F_0 are finite, $I^-(S_0) \subset I^-(P_0)$ and $I^+(S_0) \subset I^+(F_0)$. Since the Lorentzian distance is finite the sets S_0, P_0 and F_0 are achronal.

Inductive case: Assume that S_{i-1}, P_{i-1} and F_{i-1} exist and are such that S_{i-1} is achronal, P_{i-1}, F_{i-1} are finite, $I^-(S_{i-1}) \subset I^-(P_{i-1})$ and $I^+(S_{i-1}) \subset I^+(F_{i-1})$.

Assume that $i = 2k + 1$ for some $k \in \mathbb{N}$, $k \geq 0$. If there does not exist $f_k \in F$ then let $S_i = S_{i-1}$, $P_i = P_{i-1}$, $F_i = F_{i-1}$ and continue the induction. Otherwise, we have three subcases:

1. If $\{f_k\} \cup S_{i-1}$ is achronal let $S_i = \{f_k\} \cup S_{i-1}$, $P_i = \{f_k\} \cup P_{i-1}$ and $F_i = \{f_k\} \cup S_{i-1}$. It is clear that the inductive hypothesis remains true.
2. If $f_k \in I^-(S_{i-1})$ then, by construction, $f_k \in F$ hence there exists (x_i) a future divergent sequence so that $f_k \in F_{(x_i)}$. Since $I^-(S_{i-1})$ is a past set, $F_{(x_i)}$ is a future set and as $f_k \in I^-(S_{i-1}) \cap F_{(x_i)}$ there exists $\hat{f}_k \in I^+(f_k) \cap \partial I^-(S_{i-1})$. Let $S_i = \{\hat{f}_k\} \cup S_{i-1}$. By construction S_i is achronal. Let $P_i = P_{i-1}$. Since $\hat{f}_k \in \partial I^-(S_i)$, $I^-(S_i) \subset I^-(P_i)$. Let $F_i = \{f_k\} \cup F_{i-1}$. By construction $I^+(S_i) \subset I^+(F_i)$. Hence the inductive hypotheses are true.

3. Otherwise $f_k \in I^+(S_{i-1})$. Let (x_i) be a future divergent sequence so that $f_k \in F_{(x_i)}$. Since F_{i-1} is finite and as the Lorentzian distance is assumed to be finite, Lemma 3.5 implies that there exists $N \in \mathbb{N}$ so that for all $j \geq N$, $x_j \notin I^+(F_{i-1})$. Since $I^+(S_{i-1}) \subset I^+(F_{i-1})$ this implies that for all $j \geq N$ there exists $y_j^{(x_i)} \in I^-(f_k) \cap I^+(x_j) \cap \partial I^+(S_{i-1})$. Let $Y(f_k)$ be the union of all $y_j^{(x_i)}$ for all (x_i) , a future divergent sequence, so that $f_k \in F_{(x_i)}$.

By construction $Y(f_k) \subset \partial I^+(S_{i-1})$ so $S_i = Y(f_k) \cup S_{i-1}$ is achronal. Let $F_i = F_{i-1}$. Since $Y(f_k) \subset \partial I^+(S_{i-1})$, $I^+(F_i) \supset I^+(S_i)$. Let $P_i = \{f_k\} \cup P_{i-1}$. Since $Y(f_k) \subset I^-(f_k)$, $I^-(S_i) \subset I^-(P_i)$. It is clear that the inductive hypothesis are satisfied.

Assume that $i = 2k$ for some $k \geq 1$. This is the time reversed version of the three subcases above. For clarity we write them out in full. If there does not exist $p_k \in P$ then let $S_i = S_{i-1}$, $P_i = P_{i-1}$, $F_i = F_{i-1}$ and continue the induction. Otherwise, we have three subcases:

1. If $\{p_k\} \cup S_{i-1}$ is achronal then let $S_i = \{p_k\} \cup S_{i-1}$, $P_i = \{p_k\} \cup P_{i-1}$ and $F_i = \{p_k\} \cup F_{i-1}$. It is clear that the inductive hypothesis are satisfied.
2. If $p_k \in I^+(S_{i-1})$ then, by construction, there exists (x_i) a past divergent sequence so that $p_k \in P_{(x_i)}$. Since $I^+(S_{i-1})$ is a future set and $P_{(x_i)}$ is a past set and as $p_k \in I^+(S_{i-1}) \cap P_{(x_i)}$ there exists $\hat{p}_k \in I^-(p_k) \cap \partial I^+(S_{i-1})$. Let $S_i = \{\hat{p}_k\} \cup S_{i-1}$. Let $F_i = F_{i-1}$ and $P_i = \{\hat{p}_k\} \cup P_{i-1}$. Since $\hat{p}_k \in \partial I^+(S_{i-1})$, $I^+(S_i) \subset I^+(F_i)$ and it is clear that $I^-(S_i) \subset I^-(P_i)$. Hence the inductive hypotheses are satisfied.
3. Otherwise $p_k \in I^-(S_{i-1})$. Let (x_i) be a past divergent sequence so that $p_k \in P_{(x_i)}$. Since P_{i-1} is finite and as the Lorentzian distance is assumed to be finite, the time reverse of Lemma 3.5 implies there exists $N \in \mathbb{N}$ so that for all $j \geq N$, $x_j \notin I^-(P_{i-1})$. Since $I^-(S_{i-1}) \subset I^-(P_{i-1})$ this implies that for all $j \geq N$ there exists $y_j^{(x_i)} \in I^+(p_k) \cap I^-(x_j) \cap \partial I^-(S_{i-1})$. Let $Y(p_k)$ be the set of all $y_j^{(x_i)}$ for all (x_i) , a past divergent sequence, so that $p_k \in P_{(x_i)}$.
By construction $Y(p_k) \subset \partial I^-(S_{i-1})$ so $S_i = Y(p_k) \cup S_{i-1}$ is achronal. Let $P_i = P_{i-1}$. As $Y(p_k) \subset \partial I^-(S_{i-1})$ it is clear that $I^-(S_i) \subset I^-(P_i)$. Let $F_i = \{p_k\} \cup F_{i-1}$. Since $Y(p_k) \subset I^+(p_k)$ we have that $I^+(S_i) \subset I^+(F_i)$. Hence the inductive hypotheses are satisfied.

Since $S_i \subset S_{i+1}$ and each S_i is achronal the set $H = \bigcup_i S_i$ is achronal.

We now prove that for each (x_i) , a future divergent sequence, there exists $N \in \mathbb{N}$ so that for all $\ell \geq N$, $x_\ell \in I^-(H)$. By construction there exists $f_k \in F \cap F_{(x_i)}$. We have three cases ($j = 2k + 1$):

1. If $\{f_k\} \cup S_j$ is achronal then $f \in S_{j+1} \subset H$. Lemma 3.6 now gives the required N .
2. If $f_k \in I^-(S_j)$ then, by construction there exists $\hat{f}_k \in I^+(f_k) \cap S_{j+1} \subset H$. Lemma 3.6 now gives the required N .
3. Otherwise $f_k \in I^+(S_j)$. By construction there exists $N \in \mathbb{N}$ so that for all $j \geq N$ there is $y_j^{(x_i)} \in S_{j+1} \cap I^+(x_j)$. That is for all $\ell \geq N$, $x_\ell \in I^-(H)$.

The time reverse of this argument shows that for all past divergent sequences, (x_i) , there exists N so that for all $\ell \geq N$, $x_\ell \in I^+(H)$. Hence H is a hatting. \square

Every generalised time function satisfying condition (3) gives rise to a hatting.

Proposition 3.8. *Let (M, g) be a Lorentzian manifold. If $f : M \rightarrow \mathbb{R}$ is a generalised time function that satisfies condition (3) then for all $r \in \mathbb{R}$, in the range of f , $f^{-1}(r)$ is a hatting for M .*

Proof. Let $(x_i)_{i \in \mathbb{N}}$ be a future divergent sequence in M . If $(x_i) \subset \overline{I^+(f^{-1}(r))}$ then for all $y \in F_{(x_i)}$ $d(f^{-1}(r), y) = \infty$. This implies, by Lemma 2.3, that $f(y) = \infty$ which is a contradiction. Therefore there exists $N \in \mathbb{N}$ so that for all $i \geq N$, $x_i \in I^-(f^{-1}(r))$. A similar argument applied to past divergent sequences shows that $f^{-1}(r)$ is a hatting. \square

3.2 The construction of a special achronal surface

This subsection shows how to construct an achronal surface, S , so that $M = I^+(S) \cup S \cup I^-(S)$.

Lemma 3.9. *Let (M, g) be a Lorentzian manifold. Let S be an achronal surface so that $S = \partial I^+(S)$ and let $x, y \in M$ be such that $y \in I^+(x)$ and $U = I^-(y) \cap I^+(x)$. If $I^+(S) \cap U \neq \emptyset$ then $U = (I^+(S) \cap U) \cup (S \cap U) \cup (I^-(S) \cap U)$.*

Proof. It is clear that $(I^+(S) \cap U) \cup (S \cap U) \cup (I^-(S) \cap U) \subset U$. If $x \in \overline{I^+(S)}$ then the achronality of S implies that $I^+(x) \subset I^+(S)$. Hence, $U \subset I^+(S)$ and we have the result. Thus, we assume that $x \notin \overline{I^+(S)}$.

Let $w \in U$. By construction there exists a timelike curve γ from x to y through w . As $I^+(S) \cap U \neq \emptyset$ and $U \subset I^-(y)$ we know that $y \in I^+(S)$. By assumption $x \notin \overline{I^+(S)}$, hence Corollary 2.9 implies that $\gamma \cap S = \{s\}$ for a unique $s \in S$. Since γ is timelike, this implies that w lies in $I^+(s)$, $I^-(s)$, or $w = s$. Hence $w \in I^+(S)$, $w \in S$ or $w \in I^-(S)$, and $U \subset (I^+(S) \cap U) \cup (S \cap U) \cup (I^-(S) \cap U)$ as required. \square

As with many results in Lorentzian geometry a ‘time reversed’ version of this result also holds.

Lemma 3.10. *Let (M, g) be a Lorentzian manifold. Let S be an achronal surface so that $S = \partial I^-(S)$ and let $x, y \in M$ so that $y \in I^+(x)$ and $U = I^-(y) \cap I^+(x)$. If $I^-(S) \cap U \neq \emptyset$ then $U = (I^+(S) \cap U) \cup (S \cap U) \cup (I^-(S) \cap U)$.*

Lemma 3.11. *Let (M, g) be a Lorentzian manifold. If there exists a future set, F , so that $\partial F \neq \emptyset$ then there exists an achronal set S so that $\partial F \subset S$ and $M = I^+(S) \cup S \cup I^-(S)$.*

Proof. Let $S_0 = \partial F$. By assumption $S_0 \neq \emptyset$. By Lemma 2.10, $\partial I^+(S_0) = \partial I^+(\partial F) = \partial I^+(\partial I^+(F)) = \partial I^+(F) = \partial F$, since, by definition $I^+(F) = F$.

Assuming that S_i is given we construct S_{i+1} as follows.

$$S_{i+1} = \begin{cases} \partial I^-(S_i) & i+1 \text{ odd} \\ \partial I^+(S_i) & i+1 \text{ even.} \end{cases} \quad (7)$$

Assuming $i+1$ is even then, by construction and Lemma 2.10, $\partial I^+(S_{i+1}) = \partial I^+(\partial I^+(S_i)) = \partial I^+(S_i) = S_{i+1}$. Hence Corollary 2.9 can be applied to S_{i+1} . In the case that $i+1$ is odd the same argument shows that $S_{i+1} = \partial I^-(S_{i+1})$ and the time reverse of Corollary 2.9 can be applied.

Let $S = \bigcup_i S_i$. Thus $\partial F = S_0 \subset S$ as required. We now prove that, for all $i \in \mathbb{N}$, $S_i \subset S_{i+1}$. Let $y \in S_i$. There are two cases to consider.

Case one: Assume that $i + 1$ is odd. Since $y \in S_i$, $y \in \overline{I^-(S_i)}$. The achronality of S_i implies that $y \notin I^-(S_i)$ and therefore $y \in \partial I^-(S_i) = S_{i+1}$.

Case two: Assume that $i + 1$ is even. Since $y \in S_i$, $y \in \overline{I^+(S_i)}$. The achronality of S_i implies that $y \notin I^+(S_i)$ and therefore $y \in \partial I^+(S_i) = S_{i+1}$.

Thus $S_i \subset S_{i+1}$ as required.

We now show that for all $x \in M$ there exists $i \in \mathbb{N}$ so that $x \in I^+(S_i) \cup S_i \cup I^-(S_i)$. Since M is path connected there exists a curve $\gamma : [0, 1] \rightarrow M$ from $\gamma(0) \in I^+(S_0)$ to $\gamma(1) = x$. For each $y \in \gamma$ choose $z_y \in I^+(y)$ and $w_y \in I^-(y)$.

The set $\{I^-(z_y) \cap I^+(w_y) : y \in \gamma\}$ is an open cover of γ . As γ is compact there exists a finite open subcover, $\mathcal{C} = \{I^-(z_i) \cap I^+(w_i) : i = 0, \dots, m\}$.

By relabelling, if necessary, we take $U_0 = I^-(z_0) \cap I^+(w_0) \in \mathcal{C}$ so that $\gamma(0) \in U_0$. Lemma 3.9 implies that $U_0 = (I^+(S_0) \cap U_0) \cup (S_0 \cap U_0) \cup (I^-(S_0) \cap U_0)$. Let γ_0 be the connected component of γ in U_0 containing $\gamma(0)$. Define $t_1 = \sup\{t \in [0, 1] : \gamma(t) \in U_0\}$. Again by relabelling, if necessary, we take $U_1 = I^-(z_1) \cap I^+(w_1) \in \mathcal{C}$ so that $\gamma(t_1) \in U_1$.

We will show that either $U_1 = (I^+(S_0) \cap U_1) \cup (S_0 \cap U_1) \cup (I^-(S_0) \cap U_1)$ or $U_1 = (I^+(S_1) \cap U_1) \cup (S_1 \cap U_1) \cup (I^-(S_1) \cap U_1)$.

By definition of t_1 , and as U_1 is open, there exists $\epsilon > 0$ so that $\gamma(t_1 - \epsilon) \in U_1 \cap U_0$. Thus there exists $x_1 \in U_0 \cap U_1$. From above we know that $x_1 \in (I^+(S_0) \cap U_0) \cup (S_0 \cap U_0) \cup (I^-(S_0) \cap U_0)$. We have three cases to consider.

Case one: If $x_1 \in I^+(S_0)$ then Lemma 3.9 implies that $U_1 = (I^+(S_0) \cap U_0) \cup (S_0 \cap U_0) \cup (I^-(S_0) \cap U_0)$.

Case two: If $x_1 \in S_0$ then as $U_0 \cap U_1$ is open there exists $x'_1 \in U_0 \cap U_1 \cap I^+(x_1)$. Since $x'_1 \in I^+(x_1)$ and $x_1 \in S_0$ we know that $x'_1 \in I^+(S_0)$. Lemma 3.9 implies that $U_1 = (I^+(S_0) \cap U_1) \cup (S_0 \cap U_1) \cup (I^-(S_0) \cap U_1)$.

Case three: Suppose that $x_1 \in I^-(S_0)$. Then as $I^-(S_1) = I^-(\partial I^-(S_0))$, $x_1 \in I^-(S_1)$ so that $I^-(S_1) \cap U_1 \neq \emptyset$. Lemma 3.10 implies that $U_1 = (I^+(S_1) \cap U_1) \cup (S_1 \cap U_1) \cup (I^-(S_1) \cap U_1)$.

This inductive process can be repeated. The result is that, for some $i = 0, \dots, m$ with $x = \gamma(1) \in U_i \in \mathcal{C}$ we know that, for some $0 \leq j \leq i$, $U_i = (I^+(S_j) \cap U_i) \cup (S_j \cap U_i) \cup (I^-(S_j) \cap U_i)$. Hence $x \in I^+(S_j) \cup S_j \cup I^-(S_j)$ as claimed.

We now show that $M = I^+(S) \cup S \cup I^-(S)$. For all $x \in M$ there exists $j \in \mathbb{N}$ so that $x \in I^+(S_j) \cup S_j \cup I^-(S_j)$. By definition of S , $I^+(S) = \bigcup_i I^+(S_i)$ and $I^-(S) = \bigcup_i I^-(S_i)$. Thus $x \in I^+(S) \cup S \cup I^-(S)$. Since $I^+(S) \cup S \cup I^-(S) \subset M$ we have the required equality.

We now show that S is an achronal surface by showing that $S = \partial I^+(S)$. Let $x \in \partial I^+(S)$. From above there exists $j \in \mathbb{N}$ so that $x \in I^+(S_j) \cup S_j \cup I^-(S_j)$. If $x \in I^+(S_j)$ then, from the construction of S , we know that $x \in I^+(S)$. This is a contradiction as $I^+(S)$ is open. Similarly if $x \in I^-(S_j)$ we are led to a contradiction. Therefore $x \in S_j \subset S$, and so $\partial I^+(S) \subset S$.

Let $x \in S$. Since $I^+(x) \subset I^+(S)$, we know that $x \in \overline{I^+(S)}$. If $x \in \partial I^+(S)$, we are done, since then $S \subset \partial I^+(S)$ by the arbitrariness of x .

So suppose that $x \notin \partial I^+(S)$. Then $x \in I^+(S)$, by the achronality of $\partial I^+(S)$. From the definition of S this implies that there exists $i \in \mathbb{N}$ so that $x \in I^+(S_i)$. As $x \in S$ there exists $j \in \mathbb{N}$ so that $x \in S_j$.

Suppose that $i \geq j$. Since $x \in S_j$ then $x \in S_i$. Thus we have that $x \in S_i \cap I^+(S_i)$. This is a contradiction as S_i is an achronal surface. So assume that $j > i$. Since $S_i \subset S_j$ we know that $I^+(S_i) \subset I^+(S_j)$. Thus, again, we get the contradiction $x \in S_j \cap I^+(S_j)$.

Therefore, we have that $x \in \partial I^+(S)$ and hence that $S = \partial I^+(S)$, as required. \square

3.3 Combining surfaces and hattings to get generalised time functions

In this section we show how to use the hatting and the surface construction in the previous sections to construct a surface S so that $d(\cdot, S)$ and $d(S, \cdot)$ are finite valued and $M = I^+(S) \cup S \cup I^-(S)$. This allows us to construct a generalised time function that satisfies condition (3).

Lemma 3.12. *Let (M, g) be a Lorentzian manifold. If there exists a hatting, H , then there exists an achronal set S such that, $H \subset S$, $M = I^+(S) \cup S \cup I^-(S)$ and for all $x \in M$, $d(S, x) < \infty$ and $d(x, S) < \infty$.*

Proof. Since $\emptyset \neq H \subset \partial I^+(H)$, Lemma 3.11 can be used to generate an achronal surface, S , so that $H \subset S$. If there exists $x \in I^+(S)$ so that $d(S, x) = \infty$ then Lemma 3.3 implies that there exists (x_i) , a future divergent sequence, lying in S . Since H is a hatting there exists $N \in \mathbb{N}$ so that for all $j \geq N$, $x_j \in I^-(H) \subset I^-(S)$. This contradicts the achronality of S . Hence for all $x \in I^+(S)$, $d(S, x) < \infty$. The time reverse of this argument proves that for all $x \in I^-(S)$, $d(x, S) < \infty$ as required. \square

Proposition 3.13. *Let (M, g) be a Lorentzian manifold. If there exists a hatting then there exists an achronal surface S such that $M = I^+(S) \cup S \cup I^-(S)$. Moreover the function $f : M \rightarrow \mathbb{R}$ defined by*

$$f(x) = \begin{cases} d(S, x) & \text{if } x \in I^+(S) \\ 0 & \text{if } x \in S \\ -d(x, S) & \text{if } x \in I^-(S) \end{cases}.$$

is a generalised time function which satisfies condition (3).

Proof. The existence of S so that $M = I^+(S) \cup S \cup I^-(S)$ is given by Lemma 3.12.

Let $x \in M$ then as $M = I^+(S) \cup S \cup I^-(S)$ and as these sets are pairwise disjoint we know that x belongs to one of $I^+(S)$, S or $I^-(S)$. Hence f is well defined. The finiteness of f follows from Lemma 3.12. It is clear, by definition of d , that f is strictly monotonically increasing on every timelike curve.

It remains to show that f satisfies condition (3). Let $x \in M$, $y \in I^+(x)$ and for a contradiction we assume that $f(y) - f(x) < d(x, y)$. We have five cases to consider.

Case one, $f(x) > 0$ By assumption $f(y) - f(x) = d(S, y) - d(S, x) < d(x, y)$. Hence $d(S, y) < d(S, x) + d(x, y)$ which contradicts Lemma 2.3.

Case two, $f(x) = 0$ By assumption $f(y) - f(x) = d(S, y) < d(x, y)$. This contradicts the definition of d .

Case three, $f(x) < 0, f(y) > 0$ By assumption $x \in I^-(S)$ and $y \in I^+(S)$. Since S is an achronal surface so that $M = I^+(S) \cup S \cup I^-(S)$ we know that $S = \partial I^+(S) = \partial I^-(S)$, Proposition

2.8 now implies that for any $\gamma : [a, b] \rightarrow M$, $\gamma \in \Omega_{x,y}$, there exists a unique $t \in [a, b]$ so that $\gamma \cap S = \{\gamma(t)\}$. Therefore

$$f(y) - f(x) = d(S, y) + d(x, S) \geq L(\gamma|_{[a,t]}) + L(\gamma|_{[t,b]}) = L(\gamma).$$

Taking the supremum over $\Omega_{x,y}$ we see that $f(y) - f(x) = d(S, y) + d(x, S) \geq d(x, y)$. This contradicts our assumption.

Case four, $f(x) < 0, f(y) = 0$ By assumption $f(y) - f(x) = d(x, S) < d(x, y)$. This contradicts the definition of d .

Case five, $f(y) < 0$ By assumption $f(y) - f(x) = -d(y, S) + d(x, S) < d(x, y)$. Hence $d(x, S) < d(x, y) + d(y, S)$ which contradicts Lemma 2.3.

Since every case ends in a contradiction we see that f satisfies condition (3). \square

Example 3.14. Here is an example of a stably causal manifold with finite but discontinuous Lorentzian distance function. Simply take two dimensional Minkowski space, and remove the segment $\{(x, y) : y = 0, -1 \leq x \leq 1\}$.

We now refine Proposition 3.13 under the assumption that the Lorentzian distance is continuous.

Corollary 3.15. *If, in addition to the assumptions of Proposition 3.13 the Lorentzian distance is continuous then the function defined in Proposition 3.13 is continuous.*

Proof. The aim is to show that the function f we have constructed is both lower and upper semi-continuous, [10, page 101]. For lower semi-continuity, let S be our achronal surface, and observe that by the lower semi-continuity of d , which holds on all Lorentzian manifolds, for each $s \in S$ the function $p \mapsto d(s, p)$ is lower semi-continuous. Hence $p \mapsto \sup_{s \in S} d(s, p)$ is lower semi-continuous. The time symmetry of f completes the argument.

Since for each $s \in S$, $d(s, \cdot)$ is upper semi-continuous, for each $\epsilon > 0$ there exists a neighbourhood $U \subset M$, $p \in U$, so that for any $s \in S$ and $q \in U$,

$$d(s, p) \geq d(s, q) - \epsilon.$$

Taking the supremum over all $s \in S$ yields

$$f(p) \geq f(q),$$

and so if $q_i \rightarrow p$ we see that $\limsup f(q_i) \leq f(p)$, and hence f is upper semi-continuous. \square

The last corollary and the equivalence between stably causality and the existence of a continuous time function suggests the following conjecture: If (M, g) has finite and continuous Lorentzian distance then (M, g) is stably causal.

Our techniques, based on functions necessarily constant on at least some causal curves, seem not to be able to address this question, despite obtaining a continuous generalised time function when the Lorentzian distance is finite and continuous.

4 Proof of the main results

Finiteness of the Lorentzian distance. *Let (M, g) be a Lorentzian manifold. The Lorentzian distance is finite if and only if there exists a generalised time function $f : M \rightarrow \mathbb{R}$, strictly monotonically increasing on timelike curves, whose gradient exists almost everywhere and is such that $\text{ess sup } g(\nabla f, \nabla f) \leq -1$.*

Proof. Suppose that such a generalised time function exists. Then Corollary 2.16 proves that the Lorentzian distance is finite.

Conversely suppose that the Lorentzian distance is finite. Then Lemma 3.7 along with Propositions 3.13 and 2.18, imply that there exists a generalised time function $f : M \rightarrow \mathbb{R}$ so that $\text{ess sup } g(\nabla f, \nabla f) \leq -1$. \square

The Lorentzian distance formula. *Let (M, g) have finite Lorentzian distance. Then for all $p, q \in M$*

$$d(p, q) = \inf \{ \max\{f(q) - f(p), 0\} : f : M \rightarrow \mathbb{R}, f \text{ future directed, } \text{ess sup } g(\nabla f, \nabla f) \leq -1 \}.$$

Proof. We assume that either $\{p, q\}$ is achronal or $q \in I^+(p)$.

Since the Lorentzian distance is finite, Lemma 3.7 implies that there exists a hatting for M . Thus Lemma 3.12 implies that there exists an achronal surface, S_1 , so that $d(S_1, \cdot)$ and $d(\cdot, S_1)$ are finite valued. Let $S_2 = \partial(I^+(S_1) \setminus I^-(q))$. Let $x \in I^+(S_2)$. By construction $d(S_2, x) \leq \max\{d(S_1, x), d(q, x)\}$. Hence $d(S_2, \cdot)$ is finite valued. A similar argument shows that $d(\cdot, S_2)$ is finite valued. Let $S = \partial(I^-(S_2) \setminus I^+(p))$. The same arguments as above show that $d(S, \cdot)$ is finite and $d(\cdot, S)$ is finite.

Take $f : M \rightarrow \mathbb{R}$ to be as defined in Proposition 3.13 using the surface S . If $\{p, q\}$ is achronal then $p, q \in S$ so that $f(q) = 0$ and $f(p) = 0$. In this case $d(p, q) = 0 = f(q) - f(p)$. Now suppose that $q \in I^+(p)$. Let γ be a timelike curve from q to $x \in S$. By construction $x \notin S_2$. Since γ is timelike $x \notin \partial I^-(q)$. Hence $x \in \partial I^+(p)$ and therefore $d(p, x) = 0$. We thus have that $d(p, q) \geq d(p, x) + d(x, q) = d(x, q)$ with equality for $x = p$. Noting that $p \in S$, by taking the supremum over all $x \in S$ we get $d(p, q) = d(S, q) = f(q)$. Again as $p \in S$ we have that $f(p) = 0$ and hence $f(q) - f(p) = d(p, q)$.

Lastly Proposition 2.18 implies that as $f(q) - f(p) \geq d(p, q)$ then $\text{ess inf } \sqrt{-g(\nabla f, \nabla f)} \geq 1$.

Thus we have shown that

$$d(p, q) = \inf \{ \max\{f(q) - f(p), 0\} : f : M \rightarrow \mathbb{R}, f \text{ future directed, } \text{ess sup } g(\nabla f, \nabla f) \leq -1 \}.$$

as required. \square

Remark. In fact we have shown that the infimum is achieved.

A The differentiability of functions that are monotonic on timelike curves.

We begin by addressing the question of continuity.

Proposition A.1. *Let $f : M \rightarrow \mathbb{R}$ be monotonically increasing on all timelike curves and let $x \in M$. Suppose that there exists a timelike curve $\gamma : [-1, 1] \rightarrow M$ with $\gamma(0) = x$ such that $f \circ \gamma$ is continuous at 0. Then f is continuous at x .*

Proof. Let $(y_i)_{i \in \mathbb{N}} \subset M$ be a sequence of points so that $y_i \rightarrow x$. Then, for each $i \in \mathbb{N}$ there exists $k_i \in \mathbb{N}$ so that for all $j > k_i$

$$y_j \in I^- \left(\gamma \left(\frac{1}{i} \right) \right) \cap I^+ \left(\gamma \left(-\frac{1}{i} \right) \right).$$

Since f is monotonically increasing on all timelike curves this implies that, for all $j > k_i$,

$$f \left(\gamma \left(\frac{1}{i} \right) \right) \geq f(y_j) \geq f \left(\gamma \left(-\frac{1}{i} \right) \right).$$

As $f \circ \gamma$ is continuous at 0,

$$f \left(\gamma \left(\frac{1}{i} \right) \right) \rightarrow f \circ \gamma(0)$$

and

$$f \left(\gamma \left(-\frac{1}{i} \right) \right) \rightarrow f \circ \gamma(0).$$

This implies that $f(y_i) \rightarrow f(\gamma(0)) = f(x)$, as required. \square

Corollary A.2. *Let $f : M \rightarrow \mathbb{R}$ be monotonically increasing on all timelike curves. Then f is continuous a.e.*

Proof. This follows directly from Proposition A.1 and as the push forward of our measure on M is a product measure on the image of a chart $\phi : U \subset M \rightarrow V \subset \mathbb{R}^{n+1}$. \square

We now show that functions that are monotonic on timelike curves are differentiable a.e.

Lemma A.3. *Let $\gamma : I \rightarrow M$ be a timelike curve and $f : M \rightarrow \mathbb{R}$ be monotonic on any timelike curve. Then $\gamma'(f) : I \rightarrow \mathbb{R}$ exists a.e.*

Proof. By definition

$$\gamma'(f)|_{\gamma(t)} = \frac{d}{d\tau} f \circ \gamma|_t.$$

By assumption $f \circ \gamma$ is a monotonic function. Hence, standard results, e.g. [7, Theorem 9.3.1], imply that $\gamma'(f)$ exists a.e. on $\gamma(I)$. \square

Lemma A.4. *Let U be a coordinate neighbourhood of M . Let $f : M \rightarrow \mathbb{R}$ be monotonic on any timelike curve and let $v \in TU$ a vector field on U . Then $v(f)$ exists a.e. on U .*

Proof. Let $\partial_0, \dots, \partial_n$ be the coordinate vector fields on U . By using the Gram-Schmidt process we can produce an orthonormal frame field, $w_0, \dots, w_n \in TU$ over U so that for all $j = 1, \dots, n$ and $i = 0, \dots, n$ we have that $g(w_0, w_j) = -\delta_{0j}$ and $g(w_i, w_j) = \delta_{ij}$. Choose $1 > \epsilon > 0$ and let $e_0 = w_0$ and, for all $i = 1, \dots, n$ let $e_i = (1 - \epsilon)w_i + w_0$. Then for all $i = 1, \dots, n$ and $j = 0, \dots, n$ we have that $g(e_0, e_j) = -1$ and $g(e_i, e_j) = (1 - \epsilon)^2 \delta_{ij} - 1$. In particular this implies that each vector e_0, \dots, e_n is timelike.

Choose $x \in U$. Then for each $j = 0, \dots, n$ there exists an integral curve of the vector field e_j through x . Since each e_j is causal this integral curve is causal and therefore $e_j(f)$ exists a.e. on every integral curve of e_j . As these integral curves foliate U , we know that for each $j = 0, \dots, n$, $e_j(f)$ exists a.e. on U .

Since e_0, \dots, e_n are a frame over U we can express v as $v = v^j e_j$. Thus for $x \in U$ we have that $v(f) = v^j e_j(f)$, if $e_j(f)$ exists for all $j = 1, \dots, n$. Hence $v(f)$ exists a.e. on U . \square

This allows us to define the differential of f .

Lemma A.5. *Let $f : M \rightarrow \mathbb{R}$ be monotonic on any timelike curve. Then there exists a unique, a.e. defined, linear operator $df : TM \rightarrow \mathbb{R}$ so that $df(v) = v(f)$.*

Proof. We will define df locally and then show that the definitions on each coordinate patch satisfy the necessary transformation properties in order to conclude global existence.

Choose a coordinate neighbourhood U . Choose a frame e_0, \dots, e_n so that for all $i = 1, \dots, n$ and $j = 0, \dots, n$ we have that $g(e_0, e_j) = -1$ and $g(e_i, e_j) = (1 - \epsilon)^2 \delta_{ij} - 1$ with $1 > \epsilon > 0$ (see the proof of Proposition A.4 for the existence of such frames). For all $j = 0, \dots, n$ let de^j be the differential form defined by, for all $k = 0, \dots, n$, $de^j(e_k) = \delta_k^j$.

Define $df : TU \rightarrow \mathbb{R}$ by $df = e_j(f) de^j$. Since for all $j = 0, \dots, n$ the vector field $e_j(f)$ exists a.e. on U we know that df is defined a.e. on U . The linearity of df follows from the linearity of each de^j , for $j = 0, \dots, n$.

Let V be a second coordinate neighbourhood so that $V \cap U \neq \emptyset$. Let e'_0, \dots, e'_n be a frame on V so that for all $i = 1, \dots, n$ and $j = 1, \dots, n$ we have that $g(e'_0, e'_j) = -1$ and $g(e'_i, e'_j) = (1 - \epsilon)^2 \delta_{ij} - 1$. We can write $e_j = T_j^i e'_i$ on $V \cap U$. This implies that $e_j(f) = T_j^i e'_i(f)$ and that $de^j = (T^{-1})^j_i de'^i$. Hence, for $i, j, k, l = 0, \dots, n$,

$$\begin{aligned} df &= e_j(f) de^j = \delta_k^j e_j(f) de^k = \delta_k^j T_j^i e'_i(f) (T^{-1})^k_l de'^l = \delta_k^j T_j^i (T^{-1})^k_l e'_i(f) de'^l \\ &= T_k^i (T^{-1})^k_l e'_i(f) de'^l = \delta_l^i e'_i(f) de'^l = e'_l(f) de'^l. \end{aligned}$$

Which is the expression for df in the frame on V . Hence df is globally defined.

It remains to show that $df(v) = v(f)$ and that df is unique. Let $v \in TU$, then we can express v as $v = v^k e_k$. Thus $df(v) = e_j(f) de^j(v^k e_k) = v^j e_j(f) = v(f)$.

Suppose that there exists $\omega : TM \rightarrow \mathbb{R}$ a linear operator so that $\omega(v) = v(f)$. Then $\omega(e_j) = e_j(f)$ and we have $\omega = e_j(f) de^j$. This implies that $\omega = df$ as required. \square

We can now define the gradient.

Definition A.6. Let $f : M \rightarrow \mathbb{R}$ be monotonic on any timelike curve. We call df , as given in the proposition above, the differential of f . We call the a.e. defined vector field ∇f such that $df(v) = g(\nabla f, v)$, for all $v \in TM$, the gradient of f .

The gradient of a function that is monotonically increasing on all timelike curves is necessarily causal.

Lemma A.7. *If $f : M \rightarrow \mathbb{R}$ is monotonically increasing on any future-directed timelike curve, then ∇f is past-directed and causal wherever it exists.*

Proof. Let $x \in M$ be such that ∇f exists at x . Suppose that ∇f is spacelike. Then there exists $v \in T_x M$ a future-directed timelike vector so that $g(\nabla f, v) = 0$. Choose $a \in \mathbb{R}$ so that $a < 0$ and $a^2 < -\frac{g(v,v)}{g(\nabla f, \nabla f)}$ and let $w = a\nabla f + v$. Then

$$g(w, w) = a^2 g(\nabla f, \nabla f) + g(v, v) < 0$$

so that w is timelike. Let $\gamma : (-1, 1) \rightarrow M$ be a future-directed timelike curve so that $x = \gamma(0)$ and $\gamma'(0) = w$. By definition

$$\frac{d}{dt} f \circ \gamma|_{t=0} = g(\nabla f, w) = ag(\nabla f, \nabla f) < 0.$$

This contradicts the assumption that f is monotonically increasing along any future directed timelike curve. Thus ∇f is causal. A similar argument to that above can be used to show that ∇f is past-directed. \square

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