Nonunital spectral triples and metric completeness in unbounded KK-theory

Bram Mesland§*, Adam Rennie‡*

§Mathematics Institute, Zeeman Building, University of Warwick
Coventry CV4 7AL, UK

§Institut für Analysis, Leibniz Universität Hannover
Welfengarten 1, 30167 Hannover, Germany

‡School of Mathematics and Applied Statistics, University of Wollongong
Wollongong, Australia

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Abstract

We consider the general properties of bounded approximate units in non-self-adjoint operator algebras. Such algebras arise naturally from the differential structure of spectral triples and unbounded Kasparov modules. Our results allow us to present a unified approach to characterising completeness of spectral metric spaces, existence of connections on modules, self-adjointness and regularity of induced operators on tensor product C^*-modules and the lifting of Kasparov products to the unbounded category. In particular, we prove novel existence results for quasi-central approximate units in non-self-adjoint operator algebras, allowing us to strengthen Kasparov's technical theorem and extend it to this realm. Finally, we show that given any two composable KK-classes, we can find unbounded representatives whose product can be constructed to yield an unbounded representative of the Kasparov product.

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*email: `mesland@math.uni-hannover.de`, `renniea@uow.edu.au`, MSC Classification: 19K35, 46L07, 54E50

Introduction

In this paper we analyse the completeness of metric spaces associated to (nonunital) spectral triples, the existence of differentiable structures and connections on modules over algebras associated to spectral triples, and we prove that Kasparov products can be lifted to the unbounded setting in a very strong sense. The precise conditions under which such liftings exist have become important due to recent applications of the unbounded Kasparov product, [12, 13, 32, 41].

These seemingly disparate topics are in fact related by the systematic use of approximate units for differentiable algebras, introduced below. Technical advances in approximate units have often heralded conceptual advances in operator algebras and noncommutative geometry. Pertinent examples include quasicentrality [1], Higson's proof of Kasparov's technical theorem, [26], and the early approaches to summability for nonunital spectral triples, [24, 46].

In this paper we refine the notion of approximate unit further by looking at differentiable algebras of spectral triples (or unbounded Kasparov modules). Given a separable C^*-algebra A with spectral triple $(A, \mathcal{H}, \mathcal{D})$, we define a *differentiable algebra* to be a separable $*$-subalgebra $\mathcal{A}$ with

$$\mathcal{A} \subset \mathcal{A}_{\mathcal{D}} = \left\{ a \in A : [\mathcal{D}, a] \text{ is defined on } \operatorname{Dom}\mathcal{D}, \quad \|a\|_{\mathcal{D}} := \left\| \begin{pmatrix} a & 0 \\ [\mathcal{D}, a] & a \end{pmatrix} \right\|_{\infty} < \infty \right\},$$

which is closed in the norm $\|\cdot\|_{\mathcal{D}}$ and dense in the C^*-algebra A. Here $\|\cdot\|_{\infty}$ is the usual norm of operators on $\mathcal{H} \oplus \mathcal{H}$. While we can always choose an approximate unit (u_n) for A consisting of elements of the prescribed dense subalgebra $\mathcal{A}$, the requirement that (u_n) be an approximate unit for $\mathcal{A}$ is much stronger, and yields finer information.

For spectral metric spaces associated to spectral triples we obtain a characterisation of metric completeness in terms of the existence of an approximate unit (u_n) for the C^*-algebra A whose Lipschitz norm is uniformly bounded in the sense that $\sup_n \|[\mathcal{D}, u_n]\|_{\infty} < \infty$. This extends previous results of Latrémolière [39] to unbounded metrics. By addressing completeness in a way compatible with [40], we complement Latrémolière's more refined picture of unbounded spectral metric spaces.

In addition, we obtain stronger forms of metric completeness, characterised by the requirement that the 'derivatives' $[\mathcal{D}, u_n]$ of the approximate unit converge to zero in norm. This property corresponds to 'topological infinity' being at infinite distance, and reflects the behaviour of geodesically complete manifolds. We present examples illustrating this analogy in Section 2.

Beyond metric properties, we use completeness and approximate units to describe a refinement of unbounded Kasparov modules and correspondences for which the Kasparov product can be explicitly constructed. Our main results then show that any pair of composable KK-classes have representatives which can be lifted to such modules. These results rely in an essential way on the two notions of completeness we introduce in Sections 1 and 2, and generalise all previous such constructions by incorporating projective modules defined by unbounded projections. We now describe these results in more detail.

We begin Section 1 by establishing the basic concepts and notation that we will use throughout the paper, in particular (non-self-adjoint) operator algebras and their approximate units. Then we prove a range of results about bounded approximate units in operator algebras, which greatly extend known results about contractive approximate units. A key result is Theorem 1.7 which says

Let $\mathcal{A}$ be an operator algebra with bounded approximate unit (u_λ), and $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ a cb-representation. Then $\pi(u_\lambda)$ converges strongly, and hence weakly, to an idempotent $q \in \mathbb{B}(\mathcal{H})$ with the following properties:
1) for all $a \in \mathcal{A}$, $q\pi(a) = \pi(a)q = \pi(a)$;
2) $q\mathcal{H} = [\pi(\mathcal{A})\mathcal{H}]$;
3) $(1-q)\mathcal{H} = \mathrm{Nil}\,\pi(\mathcal{A})$;
4) $\|q\| \leq \|\pi\| \sup_\lambda \|u_\lambda\|$.

The close relationships between the notions of approximate unit, unbounded multiplier and strictly positive element for differentiable algebras which one would expect from the corresponding C^*-theory only hold when we have the strong form of completeness, namely $[\mathcal{D}, u_n] \to 0$ in norm. Such approximate units may be normalised by replacing them by $\tilde{u}_n := \frac{u_n}{\|u_n\|_{\mathcal{D}}}$ to obtain an approximate unit that is contractive, i.e. $\|\tilde{u}_n\|_{\mathcal{D}} \leq 1$. The result on which the rest of the paper relies is Theorem 1.25 which says

Let $\mathcal{D} : \mathrm{Dom}\,\mathcal{D} \subset E_B \to E_B$ be self-adjoint and regular and $\mathcal{A} \subset \mathrm{Lip}(\mathcal{D})$ a differentiable algebra such that $[AE_B] = E_B$. Then the following are equivalent:
1) there exists an increasing commutative approximate unit $(u_n) \subset \mathcal{A}$ with $\|[\mathcal{D}, u_n]\|_\infty \to 0$;
2) there exists a positive self-adjoint complete multiplier c for $\mathcal{A}$;
3) there is a strictly positive element $h \in \mathcal{A}$ with $\mathrm{Im}\,(\mathcal{D} \pm i)^{-1}h = \mathrm{Im}\,h(\mathcal{D} \pm i)^{-1}$, and constant $C > 0$ with $\pm i[\mathcal{D}, h] \leq Ch^2$.

Our characterisations of completeness are also essential ingredients in constructing useful modules over differentiable algebras. By considering the behaviour of approximate units for the finite rank operators on such modules, we are led to two classes of modules: projective modules, and complete projective modules. Geometric examples coming from fibre bundles necessitate the use of projections whose derivative is unbounded in the sense that $[\mathcal{D}, p]$ is unbounded, [13, 23]. This is a substantial extension of the situation considered in [32, 42].

These complete projective modules are characterised by the existence of certain types of approximate units for their compact endomorphisms, and it is in this setting that we can systematically relate frames, splittings of the Cuntz-Quillen sequence, and existence of connections. This is discussed in Section 3. In particular we gain tools for studying self-adjointness of operators arising in the context of Kasparov products, and Theorem 3.18 proves

Let $\mathcal{E}_\mathcal{B}$ be a complete projective module for the unbounded Kasparov module $(\mathcal{B}, F_C, \mathcal{D})$. Then the densely defined symmetric operator $1 \otimes_\nabla \mathcal{D}$ on $\mathcal{E}\tilde{\otimes}_\mathcal{B} F_C$ is self-adjoint and regular.

The proof of the self-adjointness of the operator $1 \otimes_\nabla \mathcal{D}$ relies on the local-global principle, due to Pierrot and Kaad-Lesch, [31, 45], and on the completeness of the module $\mathcal{E}_\mathcal{B}$. The resulting argument is in the spirit of self-adjointness proofs for Dirac operators on complete manifolds.

To prove our results about representing Kasparov classes as composable unbounded Kasparov modules, we extend the notion of quasi-central approximate units to differentiable algebras.

We obtain a novel, strong form of quasicentrality in the general context of non-self-adjoint operator algebras in Theorem 4.15. Our results concerning existence of such approximate units are new even for C^*-algebras. This study culminates in a refinement of Kasparov's technical theorem for differentiable algebras in Theorem 4.18. Both the statement and the proof are in the same spirit as Higson's version, [26].

With this tool in hand, we show that given an *arbitrary* pair of composable Kasparov classes, we can find unbounded Kasparov modules which represent these classes and whose product can be *constructed* in the unbounded setting. This is done by associating to a bounded Kasparov module a correspondence in a slightly broader sense than was used in [13, 32, 42]. Earlier forms of this lifting construction were first considered in [4] to handle external products, and later in [37], in the context of internal products. We prove successively stronger lifting results in Theorem 4.7, Proposition 4.20, Theorem 4.23 and Proposition 4.29, culminating in Theorem 4.30 which says

Let A, B, C be separable C^*-algebras, $x \in KK(A, B)$ and $y \in KK(B, C)$. There exists an unbounded Kasparov module $(\mathcal{B}, E_C, T)$ representing y and a correspondence $(\mathcal{A}, \mathcal{E}_{\mathcal{B}}, S, \nabla)$ for $(\mathcal{B}, E_C, T)$ representing x. Consequently $(\mathcal{A}, E_B\tilde{\otimes}_B E_C, \mathcal{S}\otimes 1+1\otimes_\nabla T)$ represents the Kasparov product $x \otimes_B y$.

This result can also be interpreted as an alternative proof of the existence of the Kasparov product. By proving the existence of unbounded representatives of a very special form, the product can be constructed via an explicit algebraic formula. To lift Kasparov modules to unbounded representatives, we prove the existence of, equivalently, either a frame, an approximate unit or a strictly positive element possessing certain properties. Given such a frame, connections and so products become explicitly computable.

The unbounded Kasparov modules we construct are 'complete' in the strong metric sense, so every KK-class has such a representative. On the other hand, not every unbounded representative of a Kasparov class is 'complete'. For instance Kaad's example of the half-line, [30], or Baum, Douglas and Taylor's examples, [5], from manifolds with boundary will not satisfy our completeness requirements. If we take the associated KK-class of the Dirac operator on a manifold with boundary, and then lift a bounded representative to a complete unbounded module, we will have seriously altered the geometry.

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1 Approximate units and unbounded multipliers for operator algebras

This section begins by recalling some of the basic elements of operator algebras we require. Then we address the definitions of, and relationships between, approximate units, strictly positive elements and unbounded multipliers. For C^*-algebras these notions are closely related due to the connection between the norm and the spectrum. Here we must work somewhat harder, but the outcome is a systematic way of capturing the notion of metric completeness, and this plays a significant rôle throughout the rest of the paper.

1.1 Operator algebras and differentiable algebras

By an *operator algebra* we will mean a concrete operator algebra, that is, a closed subalgebra $\mathcal{A} \subset B$ of some C^*-algebra B. By representing B isometrically on a Hilbert space $\mathcal{H}$, we can always assume that $B = \mathbb{B}(\mathcal{H})$. An *operator $*$-algebra* [32, 42] is an operator algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ with a completely bounded involution $* : \mathcal{A} \to \mathcal{A}$. This involution will in general not coincide with the involution of the ambient C^*-algebra $\mathbb{B}(\mathcal{H})$, unless $\mathcal{A}$ itself is actually a C^*-algebra.

There are two C^*-algebras canonically associated to a concrete operator $*$-algebra $\mathcal{A} \subset B$. The first is the *enveloping C^*-algebra* $C^*(\mathcal{A})$, defined to be the smallest C^*-subalgebra of B containing $\mathcal{A}$. In fact this C^*-algebra depends only on the inclusion $\mathcal{A} \subset B$, i.e. on the structure of $\mathcal{A}$ as a concrete operator algebra. The second C^*-algebra is the *C^*-closure* A, constructed from viewing $\mathcal{A}$ as a Banach $*$-algebra, and completing in the C^*-norm coming from the square root of the spectral radius of a^*a. The two C^*-algebras are almost always different, as $\mathcal{A}$ is always dense in A, but is usually not dense in $C^*(\mathcal{A})$.

The main examples of operator algebras that we consider arise in the following setting. Given an unbounded (even) (A, B) Kasparov module $(A, E_B, \mathcal{D})$, we denote the algebra of adjointable operators on the C^*-module E_B by $\mathrm{End}^*_B(E)$. The algebras A and B are (possibly trivially) $\mathbb{Z}_2$-graded, as is the module E, and all commutators are $\mathbb{Z}_2$-graded. The grading operator on E will be denoted by γ or γ_E when needed, and we observe that if B is non-trivially graded, γ_E is *not* an adjointable operator, and $\gamma_E(eb) = \gamma_E(e)\gamma_B(b)$, where γ_B is the grading on B. In addition we have the identities

$$[\mathcal{D}, \pi(a)] = \mathcal{D}\pi(a) - \pi(\gamma_A(a))\mathcal{D}, \quad [\mathcal{D}, \pi(a)]^* = -[\mathcal{D}, \pi(\gamma_A(a^*))], \quad \pi(\gamma_A(a)) = \gamma_E \pi(a)\gamma_E.$$

See [6] for more information. When no confusion can occur, we will just write γ in all cases.

We realise the full Lipschitz algebra

$$\mathcal{A}_{\mathcal{D}} = \{a \in A : \ a \operatorname{Dom} \mathcal{D} \subset \operatorname{Dom} \mathcal{D}, \ [\mathcal{D}, a] \in \mathrm{End}^*_B(E)\}$$

as an operator $*$-algebra via

$$\pi_{\mathcal{D}} : \mathcal{A}_{\mathcal{D}} \ni a \mapsto \pi_{\mathcal{D}}(a) := \begin{pmatrix} \pi(a) & 0 \\ [\mathcal{D}, \pi(a)] & \pi(\gamma(a)) \end{pmatrix} \in \mathrm{End}^*_B(E \oplus E). \tag{1.1}$$

Here $\pi : A \to \mathrm{End}^*_B(E)$ is the representation implicit in the Kasparov module $(A, E_B, \mathcal{D})$ and it will often be suppressed in the notation (as in the Introduction). We also recall that $[\mathcal{D}, \pi(a)] \in \mathrm{End}^*_B(E)$ is short hand for the densely defined commutator having an adjointable extension to all of E_B.

Throughout the paper we will denote $U = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The involution is completely isometric in this case because

$$\pi_{\mathcal{D}}(a^*) = U^* \begin{pmatrix} \pi(\gamma(a)) & 0 \\ [\mathcal{D}, \pi(\gamma(a))] & \pi(a) \end{pmatrix}^* U = U^* \pi_{\mathcal{D}}(\gamma(a))^* U,$$

cf. [42, cf. Proposition 4.1.3].

We will call $\pi_{\mathcal{D}}$ the standard Lipschitz representation of $\mathcal{A}_{\mathcal{D}}$, and always consider $\mathcal{A}_{\mathcal{D}}$ to be topologised by the operator norm $\|\pi_{\mathcal{D}}(a)\|_\infty$ in this representation. If A is not represented faithfully on E_B, the standard Lipschitz representation should be modified to

$$a \mapsto a \oplus \pi_{\mathcal{D}}(a) \in A \oplus \mathrm{End}_B^*(E \oplus E).$$

We will always suppress this modification as it is inconsequential for our computations (in fact it only obscures them).

A more general setting where this operator algebra structure can be discussed is symmetric spectral triples or symmetric Kasparov modules, [28, Section 3]. Here we have a triple $(A, E_B, \mathcal{D})$ with $\mathcal{D}$ a (closed) symmetric regular operator such that $a(1+\mathcal{D}^*\mathcal{D})^{-1/2}$ is compact for all $a \in A$. We also require that the subalgebra $\mathcal{A}$ of $a \in A$ for which $a \cdot \mathrm{Dom}\,\mathcal{D}^* \subset \mathrm{Dom}\,\mathcal{D}$ and $[\mathcal{D}^*, a]$, defined on $\mathrm{Dom}\,\mathcal{D}^*$, has an adjointable extension to all of E_B is dense in A.

Remark 1.1. In this case $[\mathcal{D}^*, a]^* = -[\mathcal{D}^*, \gamma(a^*)]$, both initially defined on $\mathrm{Dom}\,\mathcal{D}^*$. The equality on $\mathrm{Dom}\,\mathcal{D}^*$ proceeds as in [22, cf. Proposition 2.1], but to see that the equality holds on all of E_B relies on the stronger assumptions needed for unbounded Kasparov modules (as opposed to spectral triples). Specifically, the quadratic forms associated to the operators $[\mathcal{D}^*, a]^*$ and $-[\mathcal{D}^*, \gamma(a^*)]$ coincide on $\mathrm{Dom}\,\mathcal{D}^*$, and the fact that $[\mathcal{D}^*, a]$ has an adjointable (and so bounded) extension to E_B ensures that the continuous extensions to E_B coincide.

We can again use the representation

$$\pi_{\mathcal{D}} : \mathcal{A}_{\mathcal{D}} \ni a \mapsto \pi_{\mathcal{D}}(a) := \begin{pmatrix} a & 0 \\ [\mathcal{D}^*, a] & \gamma(a) \end{pmatrix} \in \mathrm{End}_B^*(E \oplus E) \tag{1.2}$$

to give $\mathcal{A}_{\mathcal{D}}$ the structure of an operator space. More generally still, associated to a closed symmetric regular operator $\mathcal{D}$ on a C^*-module E_B is the operator algebra

$$\mathrm{Lip}(\mathcal{D}) := \{T \in \mathrm{End}_B^*(E) : T\,\mathrm{Dom}\,\mathcal{D}^* \subset \mathrm{Dom}\,\mathcal{D},\ [\mathcal{D}^*, T] \in \mathrm{End}_B^*(E)\}, \tag{1.3}$$

the *Lipschitz algebra* of $\mathcal{D}$, which is an operator algebra in the representation $\pi_{\mathcal{D}}$ (cf. [42, Def. 4.1.1]). Here, as above, $[\mathcal{D}^*, T]$ initially defined on $\mathrm{Dom}\,\mathcal{D}^*$ is required to have an adjointable extension to all of E_B. By [42, Sec. 4.2], $\mathrm{Lip}(\mathcal{D})$ is spectral invariant in its C^*-closure, and hence stable under the holomorphic functional calculus. The same holds for any closed subalgebra of $\mathrm{Lip}(\mathcal{D})$.

Definition 1.2. Let $\mathcal{D} : \mathrm{Dom}\,\mathcal{D} \to E_B$ be a closed symmetric regular operator. A *differentiable algebra* is a separable operator $*$-subalgebra $\mathcal{A} \subset \mathrm{Lip}(\mathcal{D})$ which is closed in the operator space topology given by $\pi_{\mathcal{D}}$. By projecting onto the first component of $\pi_{\mathcal{D}}(\mathcal{A})$, the C^*-closure of a differentiable algebra $\mathcal{A}$ coincides with the closure of $\mathcal{A}$ viewed as a subalgebra of $\mathrm{End}_B^*(E)$, and is thus a C^*-algebra.

Remark 1.3. We will present examples at the end of Section 2 showing that for an unbounded Kasparov module $(A, E_B, \mathcal{D})$, in general one needs to choose algebras smaller than $\mathcal{A}_{\mathcal{D}}$ in order to apply the methods that we develop in the remainder of the paper. Therefore we employ the notation $(\mathcal{A}, E_B, \mathcal{D})$ for unbounded Kasparov modules, where $\mathcal{A} \subset \mathcal{A}_{\mathcal{D}}$ is a closed subalgebra in the Lipschitz topology whose C^*-closure is A.

Operator spaces and modules play a central rôle in this paper and we now introduce some concepts of operator space theory that we will need. All operator spaces we encounter are *concrete* operator spaces, that is, given explicitly as closed subspaces of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For a comprehensive treatment of operator algebras and modules, see [7].

The main feature of an operator space X is the existence of canonical *matrix norms*, that is, a norm $\|\cdot\|_n$ on $M_n(X)$ for each $n \in \mathbb{N}$. For a concrete operator space $X \subset \mathbb{B}(\mathcal{H})$ these norms are obtained from the natural representation of $M_n(X)$ on $\mathbb{B}(\mathcal{H}^n)$.

A linear map $\phi : X \to Y$ between operator spaces X and Y is *completely bounded* if

$$\|\phi\|_{\mathrm{cb}} := \sup_n \{\sup \|\phi(x_{ij})\|_{M_n(Y)} : \|(x_{ij})\|_{M_n(X)} \leq 1\} < \infty,$$

and we refer to $\|\phi\|_{\mathrm{cb}}$ as the *cb-norm* of ϕ. The map ϕ is *completely contractive* if $\|\phi\|_{\mathrm{cb}} \leq 1$.

If we are given an operator algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ and an operator space $X \subset \mathbb{B}(\mathcal{H})$, we say that X is a *concrete left operator $\mathcal{A}$-module* if $\mathcal{A} \cdot X \subset X$. Here $\cdot$ denotes the usual operator multiplication in $\mathbb{B}(\mathcal{H})$. Right modules are defined similarly.

The *Haagerup tensor product* of operator spaces X and Y is defined to be the completion of the algebraic tensor product $X \otimes Y$ over the complex numbers, in the *Haagerup norm*

$$\|z\|_h^2 := \inf\Big\{\|\sum x_i x_i^*\| \, \|\sum y_i^* y_i\| : z = \sum x_i \otimes y_i\Big\}, \tag{1.4}$$

and the completion is denoted $X\tilde{\otimes}Y$. In case X is a left and Y is a right operator $\mathcal{A}$-module, the *Haagerup module tensor product* $X\tilde{\otimes}_{\mathcal{A}}Y$ is the quotient of $X\tilde{\otimes}Y$ by the closed linear span of the expressions $x \otimes ay - xa \otimes y$. The norm on $X\tilde{\otimes}_{\mathcal{A}}Y$ is the quotient norm.

The main feature of the Haagerup tensor product is that it makes operator multiplication $X\tilde{\otimes}\mathcal{A} \to X$ continuous for operator modules. An equivalent way of defining operator modules is by requiring the multiplication to be contractive on the Haagerup tensor product, cf. [16, Cor. 3.3]. See also [7, Thm. 3.3.1] and [10].

By an *inner product operator module* [31, 42] we mean a right operator module $\mathcal{E}$ over an operator $*$-algebra $\mathcal{B}$, that comes equipped with a sesquilinear pairing

$$\mathcal{E} \times \mathcal{E} \to \mathcal{B}, \quad (e_1, e_2) \mapsto \langle e_1, e_2\rangle,$$

satisfying the usual inner product axioms,

$$\langle e_1, e_2 b\rangle = \langle e_1, e_2\rangle b, \quad \langle e_1, e_2\rangle^* = \langle e_2, e_1\rangle, \quad \langle e, e\rangle \geq 0 \text{ in } B, \quad \langle e, e\rangle = 0 \Leftrightarrow e = 0,$$

and the weak Cauchy-Schwarz inequality $\|\langle e_1, e_2\rangle\|_{\mathcal{B}} \leq C\|e_1\|_{\mathcal{E}}\|e_2\|_{\mathcal{E}}$ on the level of matrix norms, for some $C > 0$. Notice that the norm $\|\cdot\|_{\mathcal{E}}$ is part of the data and we do not require $\mathcal{E}$ to be complete in the norm $\|e\|_{\mathrm{inn}}^2 := \|\langle e, e\rangle\|_{\mathcal{B}}$, which in general will be different from $\|\cdot\|_{\mathcal{E}}$. For example, consider the operator $*$-algebra $\mathcal{A}_{\mathcal{D}}$ in the representation (1.1), viewed as an inner product module over itself via $\langle a, b\rangle := a^*b$. Thus norm of $a \in \mathcal{E} = \mathcal{A}$ is given by $\|a\|_{\mathcal{E}}^2 = \|a\|_{\mathcal{D}}^2 = \|\pi_{\mathcal{D}}(a)^*\pi_{\mathcal{D}}(a)\| \neq \|\pi_{\mathcal{D}}(a^*a)\| = \|\pi_{\mathcal{D}}(\langle a, a\rangle)\| = \|a\|_{\mathrm{inn}}^2$.

1.2 Bounded approximate units

In this section we gather some useful facts about bounded approximate units in the general setting of operator algebras. We write $\|\cdot\|$ for the norm on an operator algebra, and this can include the norm on operators on a C^*-module or Hilbert space. When we need to stress the distinction, we will write $\|\cdot\|_\infty$ for the usual norm on operators on a Hilbert space or C^*-module. For the example of a (symmetric) spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ we have $\|a\| = \|\pi_{\mathcal{D}}(a)\|_\infty$.

Definition 1.4. Let $\mathcal{A}$ be an operator algebra. A *bounded approximate unit* for $\mathcal{A}$ is a net $(u_\lambda)_{\lambda\in\Lambda} \in \mathcal{A}$ such that:
1) $\sup_\lambda \|u_\lambda\| < \infty$;
2) for all $a \in \mathcal{A}$, $\lim_{\lambda\to\infty} \|u_\lambda a - a\| = \lim_{\lambda\to\infty} \|a u_\lambda - a\| = 0$.

The bounded approximate unit is *commutative* if $u_\lambda u_\mu = u_\lambda u_\mu$ for all $\lambda, \mu \in \Lambda$ and *sequential* in case $\Lambda = \mathbb{N}$.

By a *cb-representation* of an operator algebra $\mathcal{A}$ we mean a completely bounded algebra homomorphism $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. A representation π is *essential* (also called *nondegenerate* in the literature) if $\pi(\mathcal{A})\mathcal{H}$ is dense in $\mathcal{H}$. Our first aim is to show that in any cb-representation of an operator algebra $\mathcal{A}$, a bounded approximate unit converges strongly to an idempotent.

Let $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a cb-representation. For $W \subset \mathcal{H}$, denote by $W^\perp$ its orthogonal complement, by $[W]$ the closed linear span of W, and set

$$\pi(\mathcal{A})\mathcal{H} := \{\pi(a)h : a \in \mathcal{A},\ h \in \mathcal{H}\}.$$

The Hilbert space $\mathcal{H}$ splits as a direct sum of closed orthogonal subspaces in two ways:

$$\mathcal{H} = [\pi(\mathcal{A})\mathcal{H}] \oplus [\pi(\mathcal{A})\mathcal{H}]^\perp = [\pi(\mathcal{A})^*\mathcal{H}] \oplus [\pi(\mathcal{A})^*\mathcal{H}]^\perp.$$

Another important subspace of $\mathcal{H}$ associated to π is

$$\mathrm{Nil}\ \pi(\mathcal{A}) := \{h \in \mathcal{H} : \pi(a)h = 0 \text{ for all } a \in \mathcal{A}\}.$$

Lemma 1.5. *Let $\mathcal{A}$ be an operator algebra and $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ a cb-representation. Then* $\mathrm{Nil}\ \pi(\mathcal{A}) = [\pi(\mathcal{A})^*\mathcal{H}]^\perp$.

Proof. For $h \in \mathrm{Nil}\ \pi(\mathcal{A})$ and $v \in \mathcal{H}, a \in \mathcal{A}$ we have $\langle h, \pi(a)^*v\rangle = \langle \pi(a)h, v\rangle = 0$, so $\mathrm{Nil}\ \pi(\mathcal{A}) \subset [\pi(\mathcal{A})^*\mathcal{H}]^\perp$. Now let $h \in [\pi(\mathcal{A})^*\mathcal{H}]^\perp$, $v \in \mathcal{H}$ and $a \in \mathcal{A}$. Then $\langle \pi(a)h, v\rangle = \langle h, \pi(a)^*v\rangle = 0$, so $\pi(a)h = 0$ and $h \in \mathrm{Nil}\ \pi(\mathcal{A})$. □

Lemma 1.6. *Let $\mathcal{A}$ be an operator algebra with bounded approximate unit (u_λ) and also let $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be a cb-representation. Then:*
1) $\pi(u_\lambda)h \to h$ for all $h \in [\pi(\mathcal{A})\mathcal{H}]$;
2) $\mathrm{Nil}\ \pi(\mathcal{A}) \cap [\pi(\mathcal{A})\mathcal{H}] = \{0\} = \mathrm{Nil}\ \pi(\mathcal{A})^* \cap [\pi(\mathcal{A})^*\mathcal{H}]$.

Proof. To prove 1), let $h = \pi(a)v$ and observe that $\pi(u_\lambda)h \to h$ since (u_λ) is an approximate unit. So the convergence property is satisfied by all h in a dense subset of $[\pi(\mathcal{A})\mathcal{H}]$. Uniform boundedness of (u_λ) now gives the result for all $h \in [\pi(\mathcal{A})\mathcal{H}]$.
For 2), let h be a vector in the intersection $\mathrm{Nil}\ \pi(\mathcal{A}) \cap [\pi(\mathcal{A})\mathcal{H}]$ so that $\pi(u_\lambda)h \to h$ as above, since $h \in [\pi(\mathcal{A})\mathcal{H}]$. On the other hand, since $h \in \mathrm{Nil}\ \pi(\mathcal{A})$, we have $\pi(a)h = 0$ for all $a \in \mathcal{A}$, so in particular $\pi(u_\lambda)h = 0$ for all λ, so $h = 0$. Similarly $\mathrm{Nil}\ \pi(\mathcal{A})^* \cap [\pi(\mathcal{A})^*\mathcal{H}] = \{0\}$. □

The following theorem generalises the observations in the appendix to [42]. The result has been known for contractive approximate units for a long time: see for example [7, Lemma 2.1.9] and its proof.

Theorem 1.7. *Let $\mathcal{A}$ be an operator algebra with bounded approximate unit (u_λ), and $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ a cb-representation. Then the net $(\pi(u_\lambda))$ converges strongly, and hence weakly, to an idempotent $q \in \mathbb{B}(\mathcal{H})$ with the following properties:*
1) for all $a \in \mathcal{A}$, $q\pi(a) = \pi(a)q = \pi(a)$;
2) $q\mathcal{H} = [\pi(\mathcal{A})\mathcal{H}]$;
3) $(1-q)\mathcal{H} = \mathrm{Nil}\ \pi(\mathcal{A})$;
4) $\|q\| \leq \|\pi\| \sup_\lambda \|u_\lambda\|$.

Proof. Denote by p the projection onto $[\pi(\mathcal{A})\mathcal{H}]$ and by p_* the projection onto $[\pi(\mathcal{A})^*\mathcal{H}]$. The bounded and self-adjoint operator

$$x \mapsto (p + (1-p_*))x,$$

is injective, for if $(p+(1-p_*))x = 0$ then $px = -(1-p_*)x$ so

$$px \in [\pi(\mathcal{A})\mathcal{H}] \cap [\pi(\mathcal{A})^*\mathcal{H}]^\perp = [\pi(\mathcal{A})\mathcal{H}] \cap \mathrm{Nil}\ \pi(\mathcal{A}) = \{0\}.$$

Therefore $px = (1-p_*)x = 0$ and $x = (1-p)x = p_*x$, so

$$x \in [\pi(\mathcal{A})\mathcal{H}]^\perp \cap [\pi(\mathcal{A})^*\mathcal{H}] = \mathrm{Nil}\ \pi(\mathcal{A})^* \cap [\pi(\mathcal{A})^*\mathcal{H}] = \{0\}.$$

Since $p + (1-p_*)$ is self-adjoint,

$$[\mathrm{Im}(p+(1-p_*))] = \ker(p+(1-p_*))^\perp = \mathcal{H},$$

and therefore $\mathrm{Im}(p+(1-p_*))$ is dense in $\mathcal{H}$. In particular, the subspace $[\pi(\mathcal{A})\mathcal{H}] + \mathrm{Nil}\ \pi(\mathcal{A})$ is dense in $\mathcal{H}$. Now let $\xi \in \mathcal{H}$ and $\varepsilon > 0$. Choose $x \in [\pi(\mathcal{A})\mathcal{H}]$ and $y \in \mathrm{Nil}\ \pi(\mathcal{A})$ such that $\|\xi - x - y\| < \frac{\varepsilon}{4C}$, with $C := \sup \|\pi(u_\lambda)\|$. Now choose $\lambda < \mu$ large enough such that $\|\pi(u_\lambda - u_\mu)x\| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\|\pi(u_\lambda - u_\mu)\xi\| &\leq \|\pi(u_\lambda - u_\mu)(x+y)\| + \|\pi(u_\lambda - u_\mu)(\xi - x - y)\| \\ &\leq \|\pi(u_\lambda - u_\mu)x\| + \|\pi(u_\lambda - u_\mu)\|\|(\xi - x - y)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

which shows that $(\pi(u_\lambda))$ is a strong Cauchy net. Since the strong operator topology is complete on bounded sets, the sequence has a limit q. By definition of q

$$q\pi(a) = \pi(a) = \pi(a)q, \tag{1.5}$$

which proves 1). From this it follows that

$$q^2\xi = \lim_\lambda \pi(u_\lambda)q\xi = \lim_\lambda \pi(u_\lambda)\xi = q\xi,$$

so q is idempotent and in particular has closed range. It is immediate from the definiton of q and Equation (1.5) that

$$\pi(\mathcal{A})\mathcal{H} \subset \mathrm{Im}\, q \subset [\pi(\mathcal{A})\mathcal{H}],$$

and so $\operatorname{Im} q = [\pi(A)\mathcal{H}]$, proving 2). For 3), observe that

$$\pi(a)(1-q) = (1-q)\pi(a) = 0,$$

so we have $\operatorname{Im}(1-q) \subset \operatorname{Nil} \pi(\mathcal{A})$. If $h \in \operatorname{Nil} \pi(\mathcal{A})$, then $qh = 0$, so $h = (1-q)h \in \operatorname{Im}(1-q)$. Finally, 4) follows from

$$\|q\| = \sup_{\|h\|\leq 1} \|qh\| \leq \sup_{\|h\|\leq 1} \|\lim_{\lambda} \pi(u_\lambda)h\| \leq \|\pi\| \sup_{\lambda} \|u_\lambda\| \qquad \square$$

Corollary 1.8. *The Hilbert space $\mathcal{H}$ splits as a non-orthogonal direct sum $\mathcal{H} \cong [\pi(\mathcal{A})\mathcal{H}] \oplus [\pi(\mathcal{A})^*\mathcal{H}]^\perp$.*

Such splittings for C^*-modules need to be handled with more care, and we only treat the case of symmetric Kasparov modules with some additional convergence hypotheses. Recall that the *strict topology* on $\operatorname{End}^*_B(E)$ is defined by the seminorms $\|T\|_e := \max\{\|Te\|, \|T^*e\|\}$, and thus models pointwise convergence on E_B.

Proposition 1.9. *Let $(\mathcal{A}, E_B, \mathcal{D})$ be a symmetric Kasparov module for which $[\pi(A)E_B]$ is a complemented submodule of E_B and $p \in \operatorname{End}^*_B(E_B)$ the corresponding projection. Let (u_n) be an even sequential bounded approximate unit for the differentiable algebra $\mathcal{A}$. Then:*
1) p is the strict limit of (u_n);
2) $p[\mathcal{D}^*, u_n]p \to 0$ strictly;
3) if $(\mathcal{D}u_n e)$ converges for all $e \in \operatorname{Dom} \mathcal{D}^*$ then $p \in \operatorname{Lip}(\mathcal{D}^*)$ and $[\mathcal{D}^*, p]$ is the strict limit of the sequence $([\mathcal{D}^*, u_n])$.

Proof. Let p be the projection onto $[\pi(A)E_B]$, which exists because this submodule is complemented. For $e \in [\pi(A)E_B]$ we have $u_n e \to e$, since $u_n a \to a$ in the C^*-norm. Moreover $pa = ap = a$ for all $a \in A$, and thus $(1-p)a = a(1-p) = 0$. Therefore

$$\lim_n u_n e = \lim_n u_n pe + u_n(1-p)e = \lim_n u_n pe = pe,$$

and $u_n \to p$ strictly, proving 1). Since (u_n) is a bounded approximate unit for $\mathcal{A}$, the sequence of operators $[\mathcal{D}^*, u_n]$ is uniformly bounded. For $a \in \mathcal{A}$ and $e \in \operatorname{Dom} \mathcal{D}^*$ we have

$$[\mathcal{D}^*, u_n]ae = [\mathcal{D}^*, u_n a]e - u_n[\mathcal{D}^*, a]e.$$

Since $ae = pae = ape$, multiplying on the left by p yields

$$p[\mathcal{D}^*, u_n]pae = p[\mathcal{D}^*, u_n a]pe - pu_n[\mathcal{D}^*, a]pe.$$

Both terms on the right hand side converge to $p[\mathcal{D}^*, a]pe$, and so the right hand side converges to zero. Hence the left hand side also converges to zero. As vectors of the form ae are dense in pE_B, we see that $p[\mathcal{D}^*, u_n]p$ converges pointwise to zero. Since we have a symmetric Kasparov module and u_n is even, it holds that $(p[\mathcal{D}^*, u_n]p)^* = -p[\mathcal{D}^*, u_n^*]p$, and so (u_n^*) is also a bounded approximate unit for $\mathcal{A}$. Repeating the previous arguments for u_n^* shows that $p[\mathcal{D}, u_n^*]p$ converges pointwise to zero, and so we see that $p[\mathcal{D}^*, u_n]p$ converges strictly to 0, which proves 2).

To prove 3) we first show that p maps $\operatorname{Dom} \mathcal{D}^*$ into $\operatorname{Dom} \mathcal{D}$. As $(\mathcal{A}, E_B, \mathcal{D})$ is a symmetric Kasparov module, each u_n maps the domain of $\mathcal{D}^*$ into the domain of $\mathcal{D}$. Since, by assumption,

$$\pi_{\mathcal{D}}(u_n)\begin{pmatrix} e \\ \mathcal{D}^*e \end{pmatrix} = \begin{pmatrix} u_n e \\ \mathcal{D}u_n e \end{pmatrix},$$

is convergent and by 1) the projection p is the strict limit of the u_n, we find that

$$\lim_{n\to\infty} u_n \begin{pmatrix} e \\ \mathcal{D}^* e \end{pmatrix} = \begin{pmatrix} pe \\ x \end{pmatrix}.$$

Now the graph of $\mathcal{D}$ is closed so it follows that $pe \in \operatorname{Dom} \mathcal{D}$ and

$$x = \lim_{n\to\infty} \mathcal{D} u_n e = \mathcal{D} pe. \tag{1.6}$$

Now observe that, since $p \operatorname{Dom} \mathcal{D}^* \subset \operatorname{Dom} \mathcal{D}$ we can write for $e \in \operatorname{Dom} \mathcal{D}^*$

$$\begin{aligned}
[\mathcal{D}^*, u_n]e &= [\mathcal{D}^*, u_n]pe + [\mathcal{D}^*, u_n](1-p)e \\
&= \mathcal{D} u_n pe - u_n \mathcal{D} pe + \mathcal{D} u_n (1-p)e - u_n \mathcal{D}^*(1-p)e \\
&= \mathcal{D} u_n e - u_n \mathcal{D} pe - u_n \mathcal{D}^*(1-p)e \\
&\to \mathcal{D} pe - p\mathcal{D} pe - p\mathcal{D}^*(1-p)e \qquad\qquad \text{by Equation (1.6)} \\
&= [\mathcal{D}^*, p]e,
\end{aligned}$$

which tells us that $[\mathcal{D}^*, u_n]$ converges to $[\mathcal{D}^*, p]$ on $\operatorname{Dom} \mathcal{D}^*$. Since the sequence $[\mathcal{D}^*, u_n]$ is bounded, it converges strictly on all of E_B, and the operator $[\mathcal{D}^*, p]$ is thus bounded on $\operatorname{Dom} \mathcal{D}^*$. This proves 3). □

Remark 1.10. It would be desirable to remove the convergence hypothesis in 3). At present it seems unlikely to be possible without further assumptions.

In fact our seeming flexibility in allowing symmetric operators is redundant in the presence of a bounded approximate unit.

Corollary 1.11. *Let $(\mathcal{A}, E_B, \mathcal{D})$ be a symmetric Kasparov module with $A \cdot E_B$ dense in E_B. If $\mathcal{A}$ has a sequential bounded approximate unit then $\mathcal{D}$ is self-adjoint.*

Proof. Suppose we have an approximate identity (u_n) with $[\mathcal{D}^*, u_n]$ uniformly bounded in n. Then by Lemma 1.9, $[\mathcal{D}^*, u_n] \to 0$ strictly, since $p = 1$. Thus for $e \in \operatorname{Dom} \mathcal{D}^*$ we find

$$\mathcal{D}^* u_n e = [\mathcal{D}^*, u_n]e + u_n \mathcal{D}^* e \to \mathcal{D}^* e.$$

As we also have $u_n e \to e$, and $u_n e \in \operatorname{Dom} \mathcal{D}$, we see that e is in the graph norm completion of $\operatorname{Dom} \mathcal{D}$. As $e \in \operatorname{Dom} \mathcal{D}^*$ was arbitrary, $\operatorname{Dom} \mathcal{D}^* \subset \operatorname{Dom} \mathcal{D}$ and so $\mathcal{D}$ is self-adjoint. □

For the differential algebras appearing in unbounded (symmetric) Kasparov modules, we can always assume that approximate units are self-adjoint and even and we do so from here on.

1.3 Bounded and unbounded multipliers of differentiable algebras

It is a well-known fact that for a Banach algebra $\mathcal{A}$ with a bounded approximate unit, the multiplier algebra $\mathbb{M}(\mathcal{A})$ is isomorphic to the strict closure of $\mathcal{A}$, and contains $\mathcal{A}$ as an essential ideal [44, Ch 5]. Similarly, a representation $A \to \mathbb{B}(\mathcal{H})$ of a C^*-algebra A on a Hilbert space $\mathcal{H}$ extends to a representation of the multiplier algebra $\mathbb{M}(A)$ on that same Hilbert space. We discuss these notions here for operator algebras with bounded approximate unit.

For an operator algebra $\mathcal{A}$, $\mathbb{M}(\mathcal{A})$ inherits matrix norms by viewing elements of $\mathbb{M}(\mathcal{A})$ as operators on $\mathcal{A}$. The next lemma shows that the presence of a bounded approximate unit ensures that this norm, when restricted to $\mathcal{A}$, is cb-equivalent to the original norm on $\mathcal{A}$, ensuring that the inclusion is a cb-equivalence.

Lemma 1.12 (cf. Chapter 5 of [44])**.** *Let $\mathcal{A}$ be an operator algebra with bounded approximate unit. Then the norm on $M_n(\mathcal{A})$ is equivalent to the norm*

$$\|a\|_{\mathrm{op},n} := \sup_{\|b\|_n \leq 1} \|ab\|_n, \quad \|a\|_{\mathrm{op},n} \leq \|a\|_n \leq C\|a\|_{\mathrm{op},n}, \tag{1.7}$$

with C a constant independent of n.

Proof. Obviously, it holds that $\|a\|_{\mathrm{op}} \leq \|a\|$. If u_λ is a bounded approximate unit, then $\frac{1}{C}\|u_\lambda\| \leq 1$ for some fixed constant C and all λ. For any $\varepsilon > 0$ there exists λ such that $\|b - bu_\lambda\| < \varepsilon$ and thus

$$\frac{1}{C}(\|b\| - \varepsilon) < \frac{1}{C}(\|b\| - \|b - bu_\lambda\|) \leq \frac{1}{C}\|bu_\lambda\| \leq \|b\|_{\mathrm{op}},$$

which proves the assertion. The argument for the matrix norms $\|\cdot\|_n$ is verbatim the same using the bounded approximate unit $(u_\lambda \cdot \mathrm{Id}_n)$. □

Definition 1.13. Let $\mathcal{A}$ be an operator algebra with bounded approximate unit. We define the *multiplier algebra* $\mathbb{M}(\mathcal{A})$ to be the *strict closure* of $\mathcal{A}$. That is

$$\mathbb{M}(\mathcal{A}) := \left\{T : \mathcal{A} \to \mathcal{A} : \exists \text{ a net } (b_\lambda) \subset \mathcal{A} \text{ such that } \forall a \in \mathcal{A} \ \lim \|b_\lambda a - Ta\| = \lim \|ab_\lambda - aT\| = 0\right\},$$

with norm $\|T\| := \|T\|_{\mathrm{op}}$ cf. Lemma 1.12.

It is worth noting that the strict topology on $\mathrm{End}^*_B(E)$ as defined before Proposition 1.9 coincides with the strict topology in the sense of Definition 1.13 defined by the ideal $\mathbb{K}(E_B)$.

Lemma 1.14. *Let $\mathcal{A}$ be an operator algebra with bounded approximate unit (u_λ) and $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be an essential cb-representation. Then π extends uniquely to a cb-representation $\pi : \mathbb{M}(\mathcal{A}) \to \mathbb{B}(\mathcal{H})$ such that $\pi(1) = 1$.*

Proof. By assumption $\mathcal{H} = [\pi(\mathcal{A})\mathcal{H}]$, so for all $h \in \mathcal{H}$ we have $\pi(u_\lambda)h \to h$ by Lemma 1.6. Since $\mathbb{M}(\mathcal{A})$ is the strict closure of $\mathcal{A}$, for all $b \in \mathbb{M}(\mathcal{A})$ it holds that $\sup_\lambda \|bu_\lambda\| < \infty$ and for all $a \in \mathcal{A}$, $(bu_\lambda a)$ is a Cauchy net in $\mathcal{A}$. Therefore

$$\pi(b)\pi(a)h := \lim_\lambda \pi(bu_\lambda a)h,$$

is a Cauchy net for all $a \in \mathcal{A}$ and $h \in \mathcal{H}$. Thus $\pi(bu_\lambda)h$ converges for $h \in \pi(\mathcal{A})\mathcal{H}$. Since this subspace is dense in $\mathcal{H}$ and the net $(\pi(bu_\lambda))$ is uniformly bounded, the net is strongly Cauchy on $\mathcal{H}$. This proves that the assigment $h \mapsto \lim_\lambda \pi(bu_\lambda)h$ defines a bounded operator on $\mathcal{H}$. For $a \in \mathcal{A}$ and $b \in \mathbb{M}(\mathcal{A})$, it is immediate from the definition that $\pi(ab) = \pi(a)\pi(b)$. Then for $a, b \in \mathbb{M}(\mathcal{A})$ we have

$$\pi(a)\pi(b)h = \pi(a)\big(\lim_\lambda \pi(bu_\lambda)h\big) = \lim_\lambda \pi(abu_\lambda)h = \pi(ab)h,$$

proving that the extension of π is a homomorphism. Since for all $a \in \mathcal{A}$ and $b \in \mathbb{M}(\mathcal{A})$ we have $bu_\lambda a \to ba$ in $\mathcal{A}$, it is immediate that any other cb-extension of π must coincide with the one given, proving uniqueness. □

Lemma 1.15. *Let $\mathcal{A}$ be an operator algebra with bounded approximate unit $(u_\lambda) \subset \mathcal{A}$ and $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ an essential cb-isomorphic representation. Then:*
1) the strict closure $\mathbb{M}(\mathcal{A})$ of $\pi(\mathcal{A})$ is cb-isomorphic to the idealiser of $\pi(\mathcal{A}) \subset \mathbb{B}(\mathcal{H})$;
2) every element $T \in \mathbb{M}(\mathcal{A})$ is the strict limit of a bounded net in $\mathcal{A}$;
3) if $\mathcal{J} \subset \mathcal{A}$ is a closed ideal, then $\mathcal{J}$ is a closed ideal in $\mathbb{M}(\mathcal{A})$.

Proof. By Lemma 1.14, π extends to a representation of $\mathbb{M}(\mathcal{A})$. Let T be an element of $\pi(\mathbb{M}(\mathcal{A}))$, so that there is a net $(b_\lambda) \subset \mathcal{A}$ with the property that for all $a \in \mathcal{A}$

$$\|b_\lambda a - Ta\|,\ \|ab_\lambda - aT\| \to 0.$$

Since $\mathcal{A}$ is norm closed and π is cb-isomorphic, it follows that $\pi(Ta)$, $\pi(aT) \in \pi(\mathcal{A})$ for all $a \in \mathcal{A}$, so $\pi(T)$ idealises $\pi(\mathcal{A})$. Now let $T \in \mathbb{B}(\mathcal{H})$ be such that $T\pi(a)$, $\pi(a)T \in \pi(\mathcal{A})$ for all $a \in \mathcal{A}$. Consider the net $T\pi(u_\lambda) \in \pi(\mathcal{A})$. For $a \in \mathcal{A}$ we have

$$\|T\pi(u_\lambda a) - T\pi(a)\| \leq \|T\|\|\pi(u_\lambda a - a)\| \to 0, \quad \|\pi(a)T\pi(u_\lambda) - \pi(a)T\| \to 0,$$

so since π is cb-isomorphic and essential, T is the image of an element in $\mathbb{M}(\mathcal{A})$. For the second statement, observe that T is the strict limit of the bounded net (Tu_λ), as in Lemma 1.12. For the third assertion, let $T \in \mathbb{M}(\pi(\mathcal{A}))$ and $j \in \mathcal{J}$. Since $Tj \in \mathcal{A}$, the net $(u_\lambda Tj)$ converges to Tj in norm. But $u_\lambda T \in \mathcal{A}$ so this net actually lies in $\mathcal{J}$. Since $\mathcal{J}$ is closed, $Tj \in \mathcal{J}$, and similarly for jT. □

Theorem 1.16. *Any cb-representation $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ of an operator algebra with bounded approximate unit extends uniquely to a representation $\pi : \mathbb{M}(\mathcal{A}) \to \mathbb{B}(\mathcal{H})$ of the multiplier algebra $\mathbb{M}(\mathcal{A})$, such that $\pi(1)$ is an idempotent satisfying $\pi(1)\mathcal{H} = [\pi(\mathcal{A})\mathcal{H}]$ and $(1-\pi(1))\mathcal{H} = \mathrm{Nil}\,\pi(\mathcal{A})$.*

Proof. The Hilbert space $\mathcal{H}$ is cb-isomorphic to the nonorthogonal direct sum $q\mathcal{H} \oplus (1-q)\mathcal{H}$, with q as in Proposition 1.7. The representation π is essential on $q\mathcal{H}$ and 0 on $(1-q)\mathcal{H}$. Thus, Lemma 1.14 gives a representation $\mathbb{M}(\mathcal{A}) \to \mathbb{B}(q\mathcal{H})$, which extends to 0 on $(1-q)\mathcal{H}$, thus giving the desired representation $\pi : \mathbb{M}(\mathcal{A}) \to \mathbb{B}(\mathcal{H})$. By construction $\pi(1) = q$. □

We now consider multiplier algebras for closed subalgebras of $\mathrm{Lip}(\mathcal{D})$, and in particular for differentiable algebras of spectral triples.

Proposition 1.17. *Let $\mathcal{D} : \mathrm{Dom}\,\mathcal{D} \to E_B$ be a self-adjoint regular operator and $\mathcal{A} \subset \mathrm{Lip}(\mathcal{D})$ a closed subalgebra with bounded approximate unit and assume $[AE_B] = E_B$. The multiplier algebra $\mathbb{M}(\mathcal{A})$ is cb-isomorphic to the algebra*

$$\Big\{T \in \mathbb{M}(A) : T\,\mathrm{Dom}\,\mathcal{D} \subset \mathrm{Dom}\,\mathcal{D},\quad T\mathcal{A}, \mathcal{A}T \subset \mathcal{A},\quad [\mathcal{D}, T] \in \mathrm{End}^*_B(E_B)\Big\}, \tag{1.8}$$

topologised by the representation given in Equation (1.2). *The inclusion $\mathbb{M}(\mathcal{A}) \to \mathbb{M}(A)$ is spectral invariant.*

Proof. The algebra defined in (1.8) is clearly a subalgebra of $\mathbb{M}(A)$ contained in the idealiser of $\pi_{\mathcal{D}}(\mathcal{A})$ inside $\mathrm{End}_B(E \oplus E)$. The other inclusion can be seen by writing

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} a & 0 \\ [\mathcal{D}, a] & a \end{pmatrix} = \begin{pmatrix} T_{11}a + T_{12}[\mathcal{D}, a] & T_{12}a \\ T_{21}a + T_{22}[\mathcal{D}, a] & T_{22}a \end{pmatrix},$$

and observing that for this to be an element of $\mathcal{A}$ for all $a \in \mathcal{A}$, $T_{12}a = 0$ for all $a \in \mathcal{A}$ and hence $T_{12} = 0$ since A is essential. It then follows that $T_{11}a = T_{22}a$ for all $a \in \mathcal{A}$ which implies $T_{11} = T_{22}$, again because A is essential. Writing $T_{11} = T$, one again derives from essentiality of $\mathcal{A}$ that T must preserve the domain of $\mathcal{D}$. Finally we get the equation $[\mathcal{D}, Ta] = T_{21}a + T[\mathcal{D}, a]$, which implies that $T_{21} = [\mathcal{D}, T]$ which is therefore bounded. Thus the algebra (1.8) contains the idealiser of $\pi_{\mathcal{D}}(\mathcal{A})$, and is therefore equal to it.

Since $[AE_B] = E_B$, we have $\pi_{\mathcal{D}}(1) = 1$, using Lemma 1.14, and the representation $\pi_{\mathcal{D}}$ is essential. An argument similar to that given in the proof of Lemma 1.14 shows that the strict closure $\mathbb{M}(\mathcal{A})$ maps into $\mathrm{End}^*_B(E \oplus E)$, whereas the argument given in Lemma 1.15 shows that the idealiser of $\pi_{\mathcal{D}}(\mathcal{A})$ in $\mathrm{End}^*_B(E \oplus E)$ coincides with the image of this strict closure. The norm on the strict closure is given by Equation (1.7), and the equivalence of norms given there proves that $\mathbb{M}(\mathcal{A})$ is cb-isomorphic to the idealiser (1.8). Spectral invariance of the inclusion $\mathbb{M}(\mathcal{A}) \subset \mathbb{M}(A)$ now follows from spectral invariance of the inclusion $\mathrm{Lip}(\mathcal{D}) \subset \mathrm{End}^*_B(E_B)$, cf. [42, Thm B.3]. □

In [43] it was shown that any operator algebra $\mathcal{A}$ admits a canonical unitisation. In this paper, our main examples are closed subalgebras $\mathcal{A} \subset \mathrm{Lip}(\mathcal{D})$, where $\mathcal{D}$ is a self-adjoint regular operator on a C^*-module E_B with essential A representation. In this setting we can construct unitisations concretely.

Definition 1.18. Let $\mathcal{D} : \mathrm{Dom}\,\mathcal{D} \to E_B$ be a self-adjoint regular operator and $\mathcal{A} \subset \mathrm{Lip}(\mathcal{D})$ a differentiable algebra with bounded approximate unit and C^*-closure A. If $[AE_B] = E_B$, the *unitisation* $\mathcal{A}^+ \subset \mathbb{M}(\mathcal{A}) \subset \mathrm{Lip}(\mathcal{D})$ is the algebra generated by $\mathcal{A}$ and $\pi_{\mathcal{D}}(1) = 1$.

Remark 1.19. The requirement that $[AE_B] = E_B$ is not a severe restriction. By [34, Lemma 2.8] every class in $KK(A, B)$ can be represented by a bounded Kasparov module (A, E_B, F) with $[AE_B] = E_B$. Combining this with Kucerovsky's lifting results [37, Lemma 1.4, Lemma 2.2], every class in $KK(A, B)$ can be represented by an unbounded module $(A, E_B, \mathcal{D})$ with $[AE_B] = E_B$. Thus, the only serious hypothesis in Definition 1.18 is that $\mathcal{A}$ have a bounded approximate unit. Unless otherwise stated,

from now on we assume that all unbounded Kasparov modules are essential.

Unbounded multipliers of C^*-algebras were introduced by Baaj ([3]) and Woronowicz ([51]). In the differentiable setting, the definition of unbounded multiplier requires a bit more care, because of the absence of the strong relation between norm and spectrum.

Definition 1.20. Let $\mathcal{A}$ be a differentiable algebra. A linear map $c : \mathrm{Dom}\,c \subset \mathcal{A} \to \mathcal{A}$, defined on the dense right ideal $\mathrm{Dom}\,c \subset \mathcal{A}$ is a *multiplier* if $c(ab) = (ca)b$ for all $a \in \mathrm{Dom}\,c$ and $b \in \mathcal{A}$. The operator c is a *symmetric unbounded multiplier* if:
1) c is closed;
2) for all $a, b \in \mathrm{Dom}\,c$ we have $(ca)^*b = a^*(cb)$;
and c is *self-adjoint* if
3) $c \pm i$ are surjective and $(c \pm i)^{-1} \in \mathbb{M}(\mathcal{A})$.
The multiplier c is *positive* if for all $a \in \mathrm{Dom}\,c$ we have $(ca)^*a \geq 0$ in A, the C^*-closure of $\mathcal{A}$.

The spectral invariance of the inclusion $\mathbb{M}(\mathcal{A}) \to \mathbb{M}(A)$ (cf. Proposition 1.17) ensures that some of the usual properties of positive and self-adjoint multipliers remain valid in the differentiable context. As a first consequence of the inclusion $\mathbb{M}(\mathcal{A}) \to \mathbb{M}(A)$, the resolvents $(c \pm i)^{-1} \in \mathbb{M}(\mathcal{A})$

define elements in the C^*-multiplier algebra $\mathbb{M}(A)$. Hence c defines a self-adjoint multiplier on the C^*-algebra A in the usual sense, with $\operatorname{Dom} c = \operatorname{Im}(c \pm i)^{-1} \subset A$.

Lemma 1.21. *Let $c : \operatorname{Dom} c \to \mathcal{A}$ be a self-adjoint multiplier. For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the operators $(c \pm \lambda) : \operatorname{Dom} c \to \mathcal{A}$ are bijective and $(c \pm \lambda)^{-1} \in \mathbb{M}(\mathcal{A})$. Moreover if c is positive then for all $\lambda \in \mathbb{C} \setminus [0, \infty)$, the operators $c - \lambda : \operatorname{Dom} c \to \mathcal{A}$ are bijective and $(c - \lambda)^{-1} \in \mathbb{M}(\mathcal{A})$.*

Proof. The operators $c \pm \lambda$ are bijective in the C^*-closure A, and thus $(c \pm \lambda)(c \pm i)^{-1} \in \mathbb{M}(A)$ are invertible. Spectral invariance then tells us that $g = (c \pm \lambda)(c \pm i)^{-1}$ is invertible in $\mathbb{M}(\mathcal{A})$, whence $c \pm \lambda : \operatorname{Dom} c \to \mathcal{A}$ is bijective. The inverse satisfies the equation

$$g^{-1} = (c \pm i)(c \pm \lambda)^{-1} = 1 \mp (\lambda - i)(c \pm \lambda)^{-1},$$

in $\mathbb{M}(A)$, and since both $1, g^{-1} \in \mathbb{M}(\mathcal{A})$, it follows that $(c \pm \lambda)^{-1} \in \mathbb{M}(\mathcal{A})$. The positive case is proved similarly. $\square$

If there is an orthogonal decomposition $E_B = [\pi(A)E_B] \oplus [\pi(A)E_B]^{\perp}$ (which is always the case for Hilbert spaces) we can extend the self-adjoint multiplier to a self-adjoint operator on E_B by defining $c([\pi(A)E_B]^{\perp}) = 0$. We denote this extension to E_B by c as well. It is the *affiliated operator* from [3, 51]. In case $\mathcal{A}$ has a bounded approximate unit, condition 3) of Definition 1.20 can be weakened to the requirement that $c \pm i$ have dense range and $(c \pm i)^{-1}$ are norm bounded, as we now show.

Lemma 1.22. *Let $\mathcal{A}$ be a differentiable algebra with bounded approximate unit. Then any multiplier $c : \operatorname{Dom} c \to \mathcal{A}$ satisfying $(ca)^*b = a^*cb$ for all $a, b \in \operatorname{Dom} c$ is closable.*

Proof. Since $\mathcal{A}$ has a bounded approximate unit, the norm on $\mathcal{A}$ is equivalent to the norm $\|a\|_{\mathrm{op}} := \sup_{\|a\| \leq 1} \|ab\|$ by Lemma 1.12 . Let (a_n) be a sequence in $\operatorname{Dom} c$ with $a_n \to 0$ and $ca_n \to b$. Since $\|(ca_n)\| = \|(ca_n)^*\|$ and c is symmetric, for arbitrary $a \in \operatorname{Dom} c$, we get

$$b^*a = \lim_{n \to \infty} (ca_n)^*a = \lim_{n \to \infty} a_n^*(ca) = 0.$$

From this it follows that $\|b^*\|_{\mathrm{op}} = 0$, and therefore $\|b^*\| = 0$ so $\|b\| = 0$. $\square$

Corollary 1.23. *Let $\mathcal{A}$ be a differentiable algebra with bounded approximate unit and let c be a symmetric multiplier such that $(c \pm i)^{-1}$ are densely defined and bounded. Then the closure of c is a self-adjoint unbounded multiplier with $\operatorname{Dom} c = \operatorname{Im}(c \pm i)^{-1}$.*

In the context of separable C^*-algebras, the notion of unbounded multiplier, approximate unit, and strictly positive element are closely related. For a differentiable algebra $\mathcal{A}$, an element $h \in \mathcal{A}$ is *strictly positive* if it has positive spectrum and $h\mathcal{A}$ is dense in $\mathcal{A}$ (for the topology coming from $\pi_{\mathcal{D}}$). Note that this implies that h is strictly positive in the C^*-algebra A.

A more refined notion of unbounded multiplier for differentiable algebras which is compatible with strict positivity is given in the next definition. The core idea is abstracted from [27, Definition 10.2.8], which gives a commutator approach to properness of the metric. Examples illustrating the connection are presented in Section 2.

Definition 1.24. Let $(\mathcal{A}, E_B, \mathcal{D})$ be an unbounded Kasparov module, and c a self-adjoint multiplier of $\mathcal{A}$. Then c is a *complete multiplier* if:
1) $(c \pm i)^{-1} \in \mathcal{A}$;
2) $\operatorname{Im}(\mathcal{D} \pm i)^{-1}(c \pm i)^{-1} = \operatorname{Im}(c \pm i)^{-1}(\mathcal{D} \pm i)^{-1} \subset E_B$;
3) $[\mathcal{D}, c]$ is bounded on the set $\operatorname{Im}(\mathcal{D} \pm i)^{-1}(c \pm i)^{-1}$.

It should be noted that the condition in 2) is natural when dealing with commutators of unbounded operators. The sets mentioned are the natural domain for the operators $\mathcal{D}c$ and $c\mathcal{D}$, as c maps $\operatorname{Im}(c \pm i)^{-1}(\mathcal{D} \pm i)^{-1}$ into $\operatorname{Dom} \mathcal{D}$ and similarly for $\mathcal{D}$.

The following theorem provides the relationships between unbounded complete multipliers, approximate units, and strictly positive elements for differentiable algebras, and gives us our strong notion of completeness. This strong completeness is analogous to that of a geodesically complete Riemannian manifold, and is much stronger than completeness of a general complete metric space. We exemplify these statements in Section 2.

Theorem 1.25. *Let $\mathcal{D} : \operatorname{Dom} \mathcal{D} \subset E_B \to E_B$ be self-adjoint and regular and $\mathcal{A} \subset \operatorname{Lip}(\mathcal{D})$ a differentiable algebra such that $[AE_B] = E_B$. Then the following are equivalent:*
1) there exists an increasing commutative approximate unit $(u_n) \subset \mathcal{A}$ with $\|[\mathcal{D}, u_n]\|_\infty \to 0$;
2) there exists a positive self-adjoint complete multiplier c for $\mathcal{A}$;
3) there is a strictly positive element $h \in \mathcal{A}$ with $\operatorname{Im}(\mathcal{D} \pm i)^{-1}h = \operatorname{Im} h(\mathcal{D} \pm i)^{-1}$, and constant $C > 0$ with $\pm i[\mathcal{D}, h] \leq Ch^2$.

Proof. We show that 1) $\Leftrightarrow$ 2) and 2) $\Leftrightarrow$ 3).

We assume 1), so that there is an increasing commutative approximate unit $(u_n) \subset \mathcal{A}$ with $[\mathcal{D}, u_n] \to 0$ in norm. Suppose without loss of generality that there exists $0 < \varepsilon < 1$ such that $\|[\mathcal{D}, u_n]\|_\infty < \varepsilon^{2n}$. Moreover, let $\{a_i\}_{i \in \mathbb{N}}$ be a subset of $\mathcal{A}$ whose linear span is dense, and assume without loss of generality that for $1 \leq i \leq n$ we have $\|(u_{n+1} - u_n)a_i\| < \varepsilon^{2n}$. Write $d_n := u_{n+1} - u_n \geq 0$ and define

$$c = \sum_{n=1}^{\infty} \varepsilon^{-n} d_n,$$

which is a sum of positive elements of $\mathcal{A}$. Then c is densely defined, since for fixed a_i and $i < k < \ell$ we have

$$\|\sum_{n=k}^{\ell} \varepsilon^{-n} d_n a_i\| \leq \sum_{n=k}^{\ell} \varepsilon^{-n} \|(u_{n+1} - u_n)a_i\| \leq \sum_{n=k}^{\ell} \varepsilon^{n},$$

which goes to zero as $k \to \infty$ and therefore $ca_i \in \mathcal{A}$. Moreover, c is obviously symmetric, so by Corollary 1.23 it suffices to show that the resolvents $(c \pm i)^{-1}$ are densely defined and bounded. Consider the truncations $c_k := \sum_{n=1}^{k} \varepsilon^{-n} d_n \in \mathcal{A}$. By Proposition 1.17, $\mathbb{M}(\mathcal{A})$ is spectral invariant in $\mathbb{M}(A)$, so as the operators $c_k \pm i \in \mathbb{M}(\mathcal{A})$ are invertible in $\mathbb{M}(A)$, the resolvents $(c_k \pm i)^{-1}$ are elements of $\mathbb{M}(\mathcal{A})$. Subsequently estimate

$$\begin{aligned}\|[\mathcal{D}, c_k]\|_\infty = \Big\| \sum_{n=1}^{k} \varepsilon^{-n}([\mathcal{D}, u_{n+1}] - [\mathcal{D}, u_n]) \Big\|_\infty &\leq \sum_{n=1}^{k} \varepsilon^{-n}(\|[\mathcal{D}, u_{n+1}]\|_\infty + \|[\mathcal{D}, u_n]\|_\infty) \\ &\leq 2\sum_{n=1}^{k} \varepsilon^{n},\end{aligned}$$

from which we deduce that $\sup_k \|[\mathcal{D}, c_k]\|_\infty < \infty$. Therefore

$$\sup_k \|[\mathcal{D}, (c_k \pm i)^{-1}]\|_\infty = \sup_k \|(c_k \pm i)^{-1}[\mathcal{D}, c_k](c_k \pm i)^{-1}\|_\infty \leq \sup_k \|[\mathcal{D}, c_k]\|_\infty < \infty,$$

so $(c_k \pm i)^{-1}$ is a bounded sequence in $\mathbb{M}(\mathcal{A})$. Moreover, for the elements a_i we have

$$((c_\ell \pm i)^{-1} - (c_m \pm i)^{-1})a_i = (c_\ell \pm i)^{-1}(c_m \pm i)^{-1} \sum_{n=\ell}^{m} \varepsilon^{-n} d_n a_i,$$

so the sequence is strictly Cauchy, with limit $(c \pm i)^{-1} \in \mathbb{M}(\mathcal{A})$, whence these operators are densely defined and bounded. Hence the closure of c is a positive, self-adjoint unbounded multiplier on $\mathcal{A}$.

Now we show that properties 1)-3) of Definition 1.24 hold true for c, starting with points 2) and 3). For 2), we need to show that the domain equality $\mathrm{Im}\,(c\pm i)^{-1}(\mathcal{D}\pm i)^{-1} = \mathrm{Im}\,(\mathcal{D}\pm i)^{-1}(c\pm i)^{-1}$ is true. Observe that for each $y \in E_B$, the vector $(c \pm i)^{-1}(\mathcal{D} \pm i)^{-1}y$ is a limit

$$\lim_{k\to\infty} (c_k \pm i)^{-1}(\mathcal{D} \pm i)^{-1}y.$$

Writing

$$(c_k \pm i)^{-1}(\mathcal{D} \pm i)^{-1} = (\mathcal{D} \pm i)^{-1}(c_k \pm i)^{-1} + (\mathcal{D} \pm i)^{-1}(c_k \pm i)^{-1}[\mathcal{D}, c_k](c_k \pm i)^{-1}(\mathcal{D} \pm i)^{-1},$$

and recalling that the sequence $[\mathcal{D}, c_k](c_k \pm i)^{-1}(\mathcal{D} \pm i)^{-1}$ is uniformly bounded in operator norm, it follows that

$$\lim_{k\to\infty} (c_k \pm i)^{-1}[\mathcal{D}, c_k](c_k \pm i)^{-1}(\mathcal{D} \pm i)^{-1}y = (c \pm i)^{-1}[\mathcal{D}, c](c \pm i)^{-1}(\mathcal{D} \pm i)^{-1}y \in \mathrm{Im}\,(c \pm i)^{-1}.$$

Thus $\mathrm{Im}\,(c \pm i)^{-1}(\mathcal{D} \pm i)^{-1} \subset \mathrm{Im}\,(\mathcal{D} \pm i)^{-1}(c \pm i)^{-1}$. The other inclusion is proved in the same way by writing

$$(\mathcal{D} \pm i)^{-1}(c_k \pm i)^{-1} = (c_k \pm i)^{-1}(\mathcal{D} \pm i)^{-1} + (c_k \pm i)^{-1}(\mathcal{D} \pm i)^{-1}[\mathcal{D}, c_k](c_k \pm i)^{-1}(\mathcal{D} \pm i)^{-1}.$$

To prove that point 3) of Definition 1.24 holds, observe that the commutator $[\mathcal{D}, c]$, defined on $\mathrm{Im}\,(c \pm i)^{-1}(\mathcal{D} \pm i)^{-1}$, is bounded because it is the strong limit of the operators $[\mathcal{D}, c_k]$ on this subset, and $\sup_k \|[\mathcal{D}, c_k]\|_\infty$ is bounded.

Lastly, for 1), we need to show that $(c \pm i)^{-1} \in \mathcal{A}$. Since these are elements of $\mathbb{M}(\mathcal{A})$, we have $a(c \pm i)^{-1}, (c \pm i)^{-1}a \in \mathcal{A}$ for $a \in \mathcal{A}$ and $(c \pm i)^{-1} \in \mathrm{Lip}(\mathcal{D})$ by Proposition 1.17. We claim that it suffices to show that $(c \pm i)^{-1} \in A$. For then $u_n(c \pm i)^{-1} \to (c \pm i)^{-1}$ in C^*-norm and and since $[\mathcal{D}, c]$ is bounded and $[\mathcal{D}, u_n] \to 0$ we find

$$\begin{aligned}[\mathcal{D}, u_n(c \pm i)^{-1}] &= u_n[\mathcal{D}, (c \pm i)^{-1}] + [\mathcal{D}, u_n](c \pm i)^{-1} \\ &= -u_n(c \pm i)^{-1}[\mathcal{D}, c](c \pm i)^{-1} + [\mathcal{D}, u_n](c \pm i)^{-1} \to [\mathcal{D}, (c \pm i)^{-1}].\end{aligned}$$

Hence $\pi_{\mathcal{D}}(u_n(c \pm i)^{-1}) \to \pi_{\mathcal{D}}((c \pm i)^{-1})$ and since $u_n(c \pm i)^{-1} \in \mathcal{A}$, it follows that $(c \pm i)^{-1} \in \mathcal{A}$.

To prove that $(c \pm i)^{-1} \in A$, we restrict to the commutative C^*-subalgebra $B \subset A$ generated by the u_n, so that by Gelfand theory there is a locally compact Hausdorff space X with $B = C^*(\{u_n\}) \cong C_0(X)$ via the Gelfand transform. Every closed unbounded multiplier is determined by its Gelfand transform [50, Thm 2.1,2.3]. Under this identification, we wish to show that $(c \pm i)^{-1} \in C_0(X) \subset A$. To this end, fix $t \in (0,1)$ and consider the sets

$$X_n := \{x \in X : u_n(x) \geq t\}.$$

The X_n form an increasing sequence of compact sets such that $X = \cup X_n$. We claim that

$$\sum \varepsilon^{-n} d_n(x) \geq (1-t)\varepsilon^{-k}, \quad \text{for } x \in X \setminus X_k,$$

which implies that $(c \pm i)^{-1} \in C_0(X)$. For such $x \in X \setminus X_k$, and any $m \geq k$ it holds that

$$\begin{aligned}
\sum_{n=0}^{\infty} \varepsilon^{-n} d_n(x) \geq \sum_{n=k}^{\infty} \varepsilon^{-n} d_n(x) &= \sum_{n=k}^{m} \varepsilon^{-n} d_n(x) + \sum_{n>m} \varepsilon^{-n} d_n(x) \\
&\geq \sum_{n=k}^{m} \varepsilon^{-k} d_n(x) + \sum_{n>m} \varepsilon^{-n} d_n(x) \\
&= \varepsilon^{-k}(u_{m+1} - u_k)(x) + \sum_{n>m} \varepsilon^{-n} d_n(x) \\
&\geq \varepsilon^{-k}(u_{m+1}(x) - t) + \sum_{n>m} \varepsilon^{-n} d_n(x),
\end{aligned}$$

and since $u_{m+1}(x) \to 1$ and $\sum_{n>m} \varepsilon^{-n} d_n(x) \to 0$, the estimate follows. This proves 1) $\Rightarrow$ 2).

Conversely, let c be an unbounded positive complete multiplier for $\mathcal{A}$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = e^{-x/n}$. For $y \in \operatorname{Dom} \mathcal{D}$

$$[\mathcal{D}, f_n(c)]y = \int_0^1 \frac{d}{ds}\left(e^{-c(1-s)/n}\mathcal{D}e^{-cs/n}y\right) ds = -\frac{1}{n}\int_0^1 e^{-c(1-s)/n}[\mathcal{D}, c]e^{-cs/n}y\, ds,$$

and since both sides are bounded, this equality extends to all of E_B. Moreover

$$\|[\mathcal{D}, f_n(c)]\|_\infty \leq \frac{1}{n}\|[\mathcal{D}, c]\|_\infty,$$

so that $[\mathcal{D}, f_n(c)] \to 0$ in norm as $n \to \infty$. Finally we need to see that the $f_n(c)$ define an approximate unit. The density of $(c \pm i)^{-1}\mathcal{A}$ in $\mathcal{A}$ says that the inclusion of the commutative subalgebra $\mathcal{C}$ generated by $(c \pm i)^{-1}$ in $\mathcal{A}$ is essential. Since $(f_n(c))$ is obviously an approximate unit for $\mathcal{C}$, we are done.

To see that 2) and 3) are equivalent, let c be an unbounded multiplier on $\mathcal{A}$ and set $h := (1 + c)^{-1}$, which is positive with dense range in $\mathcal{A}$. On the other hand, if $h \in \mathcal{A}$ is positive with dense range, then $c := h^{-1}$ is densely defined on $\operatorname{Dom} h^{-1} = \operatorname{Im} h$. The domain condition follows from the fact that $(h^{-1} \pm i)^{-1} = h(1 \pm ih)^{-1}$, and $1 \pm ih \in \mathbb{M}(\mathcal{A})$ is invertible, so that $(1 \pm ih)^{-1}$ maps $\operatorname{Dom} \mathcal{D} = \operatorname{Im} (\mathcal{D} \pm i)^{-1}$ bijectively onto itself. Then

$$\operatorname{Im} h(1 \pm ih)^{-1}(\mathcal{D} \pm i)^{-1} = \operatorname{Im} h(\mathcal{D} \pm i)^{-1} = \operatorname{Im} (\mathcal{D} \pm i)^{-1}h = \operatorname{Im} (\mathcal{D} \pm i)^{-1}h(1 \pm ih)^{-1}.$$

From the further assumption that $i[\mathcal{D}, h] \leq Ch^2$, it follows that for $e \in \operatorname{Im} h(\mathcal{D} \pm i)^{-1}h$

$$\langle i[\mathcal{D}, h^{-1}]e, e\rangle = -\langle h^{-1}i[\mathcal{D}, h]h^{-1}e, e\rangle = \langle i[\mathcal{D}, h]h^{-1}e, h^{-1}e\rangle \leq C\langle he, h^{-1}e\rangle = C\langle e, e\rangle.$$

Taking a sequence $hy_n \to y \in E_B$, boundedness on the whole of $\operatorname{Im} h(\mathcal{D} \pm i)^{-1}$ follows. □

From now on, in addition to being self-adjoint and even, in view of Theorem 1.25 we assume all approximate units for differentiable algebras to be commutative.

2 Metric completeness via approximate units

We recall that if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a unital spectral triple, then the formula

$$d(\phi, \psi) := \sup\{|\phi(a) - \psi(a)| : \|[\mathcal{D}, a]\| \leq 1\}, \quad \phi, \psi \in \mathcal{S}(A) \tag{2.1}$$

defines a metric on the state space $\mathcal{S}(A)$ of A provided that the set

$$B := \{[a] \in A/\mathbb{C}1 : \|[\mathcal{D}, a]\| \leq 1\} \tag{2.2}$$

is bounded. In the non-unital case we do not need to consider the quotient Banach space $A/\mathbb{C}1$ in Equation (2.2), just A, and again the same conditions guarantee that we obtain a *bounded* metric. It is known that in the unital case the resulting metric topology agrees with the weak* topology provided that B is pre-compact, [39, 47]. We refer to the formula in Equation (2.1) as Connes' formula.

One would like to define unbounded metrics so that they restrict to bounded metrics on each weak* compact subset of $\mathcal{S}(A)$, but it turns out that this is too strong. Latrémolière identifies a class of tame compact subsets for which this is possible, [40, Definition 2.28], and shows by example that not all compact subset of $\mathcal{S}(A)$ are tame. As well as the difficulty in discussing the weak*-topology, examples show that there is also the need to consider extended metrics, so that points can be at infinite distance.

Our initial results concerning completeness of metric spaces rely on a weaker notion of approximate unit than we needed earlier, though we will see below how these various notions are related. For now, given a (symmetric) spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we say that $(u_n) \subset \mathcal{A} \subset A$ is an *adequate approximate unit* if (u_n) is a sequential approximate unit for A (in its C^*-norm topology) and $\sup \|[\mathcal{D}^*, u_n]\|_\infty < \infty$. This is a weaker notion than a bounded approximate unit for $\mathcal{A}$.

Proposition 2.1. *Let (X, d) be a metric space, $\mathcal{A} = \mathrm{Lip}_0(X)$ be the algebra of Lipschitz functions vanishing at infinity and $A = C_0(X)$. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a symmetric spectral triple such that for all $a \in \mathrm{Lip}_0(X)$*

$$C_1\|a\|_{\mathrm{Lip},d} \leq \|[\mathcal{D}^*, a]\|_\infty \leq C_2\|a\|_{\mathrm{Lip},d}$$

where $0 < C_1 \leq C_2 < \infty$ are constants and $\|a\|_{\mathrm{Lip},d}$ is the Lipschitz seminorm of $a \in \mathrm{Lip}_0(X)$. If $\mathcal{A}$ has an adequate approximate unit, then (X, d) is metrically complete.

Remark 2.2. 1) The condition of the theorem implies that Connes' formula, Equation (2.1), defines a metric d^C which is bi-Lipschitz equivalent to d.
2) The algebra $\mathrm{Lip}_0(X)$ is typically not separable in the Lipschitz norm, [49], but our results also apply to closed separable subalgebras of $\mathrm{Lip}_0(X)$, such as our differentiable algebras, cf. Definition 1.2. More examples are presented below.

Proof. We give the proof in the self-adjoint case, as the symmetric case is the same. We will prove that if (X, d) is not complete then for any sequential approximate unit $(u_k) \subset \mathrm{Lip}_0(X)$ for $C_0(X)$, the sequence $\|[\mathcal{D}, u_k]\|_\infty$ is unbounded, and so (u_k) can not be an adequate approximate unit. Since the metric d is bi-Lipschitz equivalent to Connes' metric, for any $y, z \in X$ we have

$$|u_k(y) - u_k(z)| \leq C\|[\mathcal{D}, u_k]\|_\infty \, d(y, z)$$

for a constant $C > 0$. Now let x be in the metric completion $\overline{X}$ of X, and $x \notin X$. Let $1/2 > \varepsilon > 0$, fix $y \in B_{1/n}(x) \cap X$ and let k be large enough so that $u_k(y) > 1 - \varepsilon$. This is possible since u_k is an approximate unit. Now let $z \in B_{1/n}(x) \cap X$ be such that $u_k(z) < \varepsilon$, possible since u_k vanishes at infinity. Then for this choice of k and $y, z \in B_{1/n}(x) \cap X$

$$1 - 2\varepsilon < |u_k(y) - u_k(z)| \leq C\|[\mathcal{D}, u_k]\|_\infty \, d(y, z) < \frac{2C}{n}\|[\mathcal{D}, u_k]\|_\infty.$$

Hence we see that for any n there is a $k = k(n)$ such that

$$\frac{n(1 - 2\varepsilon)}{2C} < \|[\mathcal{D}, u_k]\|_\infty.$$

Since this is true for any approximate unit $(u_k) \subset \mathrm{Lip}_0(X)$, we are done. $\square$

Corollary 2.3. *Let (M, g) be a Riemannian spin^c manifold, $A = C_0(M)$, $\mathcal{A} = \mathrm{Lip}_0(M)$, and $(A, L^2(M, S), \mathcal{D})$ the Dirac spectral triple of the spin^c structure. If $\mathcal{A}$ has an adequate approximate unit then the Riemannian manifold (M, g) is geodesically complete, and $\mathcal{D}$ is self-adjoint.*

Proof. The point here is that (M, g) need not, a priori, be complete, in particular it may be the interior of a manifold with boundary. First we recall that by [17], the norm $\|[\mathcal{D}^*, f]\|_\infty$ is equal to the Lipschitz norm of f (with respect to the geodesic distance) for all $f \in \mathrm{Lip}_0(M)$. Thus we can apply Proposition 2.1 to obtain the first statement. In particular if such an approximate unit exists, (M, g) is metrically complete, and so geodesically complete by the Hopf-Rinow theorem.

The self-adjointness of the Dirac operator now follows as in [27, Prop 10.2.10]. $\square$

In this last result we managed to deduce self-adjointness of a (potentially) symmetric operator using just an adequate approximate unit, whereas Corollary 1.11 requires the existence of an honest bounded approximate unit for the Lipschitz topology. This is essentially due to the special form of the geodesic metric on a Riemannian manifold. The Hopf-Rinow theorem says that completeness implies that 'topological infinity' is at infinite distance.

The issues are perhaps best seen as follows. For any metric space (X, d), we obtain a new metric space of bounded diameter by taking the new metric $\tilde{d} = d/(1+d)$. Then one can check that (X, d) is complete if and only if $(X, \tilde{d})$ is complete. The identity map on X is typically not a bi-Lipschitz map between these metric spaces, and the property of having an adequate approximate unit whose Lipschitz constants go to zero is not preserved by this operation.

We collect a few examples from the world of metric spaces about approximate units for Lipschitz algebras and differentiable algebras. The first result is rather negative.

Lemma 2.4. *Let (X, d) be a finite-diameter, noncompact, complete metric space. Then there is no adequate approximate unit in $\mathrm{Lip}_0(X)$ whose Lipschitz constants go to zero.*

Proof. Let (u_n) be an approximate unit in $\mathrm{Lip}_0(X)$. Since (u_n) is a norm approximate unit, for any $x \in X$ and $1/2 > \delta > 0$ we can find N such that $u_N(x) > 1 - \delta$. Since u_N vanishes at infinity we can find $y \in X$ such that $u_N(y) < \delta$. Then $u_N(x) - u_N(y) > 1 - 2\delta$, and as $d(x,y) \leq \mathrm{diam}(X)$ we find that

$$\frac{u_N(x) - u_N(y)}{d(x,y)} > \frac{1-2\delta}{d(x,y)} \geq \frac{1-2\delta}{\mathrm{diam}(X)}.$$

Hence the Lipschitz norm of the u_N's is bounded below. □

Hence finite diameter complete spaces do not have spectral triples which both recover the metric and satisfy the conditions of Theorem 1.25. Also observe that we did not ask for an approximate unit (u_n) for $\mathrm{Lip}_0(X)$ in the Lipschitz topology with $\|u_n\|_{\mathrm{Lip},d} \to 0$. These typically do not exist.

Lemma 2.5. *Let $(u_n) \subset C_c^\infty(\mathbb{R})$ be a differentiable approximate unit for the supremum norm topology on $C_0(\mathbb{R})$ such that the Lipschitz constants go to zero as $n \to \infty$ (these exist). Then (u_n) is not an approximate unit for the Lipschitz topology on* $\mathrm{Lip}_0(\mathbb{R})$.

Proof. Let $f(x) = \sin(x^3)/(1+x^2) \in \mathrm{Lip}_0(\mathbb{R})$. The mean value theorem says that given $x, y \in \mathbb{R}$ there is some w between x and y such that

$$|(f - u_nf)(x) - (f - u_nf)(y)| = |(f - u_nf)'(w)|d(x,y) = |(1 - u_n(w))f'(w) - u_n'(w)f(w)|d(x,y).$$

Since the Lipschitz constants of the u_n converge to zero, and $u_n' \to 0$ uniformly, we see that $u_n'f \to 0$ uniformly. As the derivative of f is $f'(x) = 3x^2\cos(x^3)/(1+x^2) - 2x\sin(x^3)/(1+x^2)^2$, and u_n vanishes at infinity for each n, we see that $|(1 - u_n(w))f'(w)|$ does not go to zero uniformly. □

Remark 2.6. The function $f(x) = \sin(x^3)/(1+x^2)$ also appears in [14, p 43], to demonstrate that derivatives must be controlled to handle summability in the nonunital setting.

Despite this lack of success, even with our strongest completeness condition, there are positive results, and these demonstrate the need to take smaller algebras than $\mathrm{Lip}_0(X)$. Recall, [20], the pointwise Lipschitz constant of a function f at a non-isolated point $x \in X$ defined by

$$\mathrm{Lip}(f)(x) := \limsup_{y \to x,\, y \neq x} \frac{|f(x) - f(y)|}{d(x,y)}.$$

If x is isolated we set $\mathrm{Lip}(f)(x) = 0$. Then we set

$$\mathrm{L}_{00}(X) = \{f \in \mathrm{Lip}(X) :\ f \text{ and } \mathrm{Lip}(f) \text{ vanish at infinity}\}.$$

The function $\mathrm{Lip}(f)$ need not be continuous, but we can still ask for it to be small outside a compact set. The space $\mathrm{L}_{00}(X)$ is not always a Banach space in its natural norm $\|f\|_\infty + \|\mathrm{Lip}(f)\|_\infty$, but we can take its completion, which is a subspace of $C_0(X)$. We denote this Banach space by $\mathrm{Lip}_{00}(X)$.

Lemma 2.7. *Let (X,d) be a metric space and $(u_n) \subset \mathrm{Lip}_{00}(X)$ an adequate approximate unit such that $\|u_n\|_{\mathrm{Lip}} \to 0$ as $n \to \infty$. Then (u_n) is an approximate identity for* $\mathrm{Lip}_{00}(X)$.

Proof. We just need to show that for $f \in \mathrm{Lip}_{00}(X)$ we have $\|f - u_n f\|_{\mathrm{Lip}} \to 0$. That is, we need to show that

$$\sup_{x \neq y} \left| \frac{(f - u_n f)(x) - (f - u_n f)(y)}{d(x,y)} \right| \to 0 \text{ as } n \to \infty,$$

which is to say, we need to show that

$$\sup_{x \neq y} \left| \frac{(u_n(x) - u_n(y))f(x) - (f(y) - f(x))(u_n(y) - 1)}{d(x,y)} \right| \\ \leq \|u_n\|_{\mathrm{Lip}} \|f\|_\infty + \sup_{y \in X} |\mathrm{Lip}(f)(y)(u_n(y) - 1)| \to 0 \text{ as } n \to \infty,$$

the second term going to zero since $\mathrm{Lip}(f)$ vanishes at infinity. □

The last two lemmas show why we need to be able to restrict to closed subalgebras of $\mathrm{Lip}(\mathcal{D})$ which may be smaller than $\mathrm{Lip}(\mathcal{D}) \cap A$, but which are still norm dense in A. For general metric spaces it is not clear that one can always find suitable algebras which have adequate approximate units. When the metric is suitably infinite and the metric space nice enough, we can find approximate units for $\mathrm{Lip}_{00}(X)$. This result resembles the equivalences of Theorem 1.25, and captures the idea that topological infinity is at infinite distance.

Proposition 2.8. *Let (X, d) be a metric space and $x_0 \in X$ such that the function $x \mapsto d(x_0, x)$ is proper. Then we obtain an approximate unit for $\mathrm{Lip}_{00}(X)$ whose Lipschitz constants go to zero. Hence (X, d) is complete.*

Proof. Fix $x_0 \in X$ as in the statement, and let

$$K_N = \{x \in X : d(x, x_0) \leq N\}.$$

Then the K_N form an increasing sequence of compact sets whose union is X. Define functions on X by

$$u_N(x) = \begin{cases} 1 & x \in K_N \\ \frac{N}{N-1}\left(1 - \frac{d(x_0,x)}{N^2}\right) & x \in K_{N^2} \setminus K_N \\ 0 & x \notin K_{N^2} \end{cases}.$$

Checking the various cases shows that each $u_N \in \mathrm{Lip}_{00}(X)$ is a bounded Lipschitz function whose Lipschitz constant is bounded by $1/N(N-1)$, and so $\|u_N\|_{\mathrm{Lip}} \to 0$ as $N \to \infty$. Moreover it is clear that (u_N) is a sup norm approximate unit for $C_0(X)$, and so by Proposition 2.1 and Lemma 2.7 we are done. □

For $\mathbb{R}^n$, and more generally geodesically complete manifolds M, we can always construct an approximate unit as in Proposition 2.8. As a consequence, we can construct a bounded approximate unit for $\mathrm{Lip}_{00}(M)$, and then Corollary 1.11 tells us directly that Dirac-type operators on M are self-adjoint.

Given a spectral metric space, we can still deduce the completeness of the state space $\mathcal{S}(A)$ from the existence of an adequate approximate unit, as was first shown by Latrémolière [39] for the case of bounded metrics. As the context is somewhat different, we give the argument.

Proposition 2.9. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a symmetric spectral triple for which Connes' formula*

$$d(\sigma, \tau) := \sup\{|\sigma(a) - \tau(a)| : \|[\mathcal{D}^*, a]\|_\infty \leq 1\},$$

defines an extended metric on the state space $\mathcal{S}(A)$ (so d may take the value ∞). If $\mathcal{A}$ has an adequate approximate unit then $(\mathcal{S}(A), d)$ is complete.

Proof. Let (u_n) be an adequate approximate unit. Let σ_k be a sequence of states that is Cauchy for the Connes metric, i.e. for $k < \ell$

$$\sup\{|\sigma_k(a) - \sigma_\ell(a)| : \|[\mathcal{D}^*, a]\|_\infty \leq 1\} \to 0,$$

as $k \to \infty$. Then $\sigma(a) := \lim_k \sigma_k(a)$, for $a \in \mathcal{A}$, is a well defined map $\mathcal{A} \to \mathbb{C}$. It is positive since for positive a, $\sigma(a)$ is a limit of positive numbers. It remains to show that σ has norm 1. To this end, let $a \in \mathcal{A}$ be in the unit ball for the C^*-norm. Then $|\sigma_k(a)| \leq 1$, so $|\sigma(a)| \leq 1$, showing that $\|\sigma\| \leq 1$, and thus σ extends to all of A. Now since u_n is an approximate unit, we have $\sigma_k(u_n) \to 1$ for fixed k and $n \to \infty$. Since $[\mathcal{D}^*, u_n]$ is bounded, we may assume that $\|[\mathcal{D}^*, u_n]\|_\infty \leq C$ for all n and some positive constant C. This means that for $k < \ell$

$$\sup_n |\sigma_k(u_n) - \sigma_\ell(u_n)| \to 0,$$

as $k \to \infty$. Hence there exist $\varepsilon > 0$ and k sufficiently large such that for all n

$$|\sigma(u_n) - \sigma_k(u_n)| < \varepsilon/2.$$

Now choose n large enough such that $\|\sigma_k(u_n) - 1\| < \varepsilon/2$. Then

$$|\sigma(u_n) - 1| \leq |\sigma(u_n) - \sigma_k(u_n)| + |\sigma_k(u_n) - 1| \leq \varepsilon.$$

This shows that $\sigma(u_n) \to 1$, and in particular that $\|\sigma\| = 1$ and $\sigma \in \mathcal{S}(A)$. $\square$

In particular, the presence of an adequate approximate unit ensures that the metric topology limit of states is a state, and so such a sequence is a tight set, [40, Definition 2.2]. It is likely that our approach to completeness can further complement Latrémolière's approach to locally compact quantum metric spaces.

Finally, let us consider what can be said about closed subalgebras of $\mathrm{Lip}(\mathcal{D})$ for a general (symmetric) spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Proposition 2.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a symmetric spectral triple. Suppose that A has an adequate approximate unit $(u_n) \subset \mathcal{A}$ such that $[\mathcal{D}^*, u_n] \to 0$ in operator norm. Then (u_n) is an approximate unit for $\mathcal{A}$ if and only if (u_n) is an operator norm topology approximate unit for the C^*-algebra generated by A and the commutators $[\mathcal{D}^*, a]$, $a \in \mathcal{A}$.*

Proof. This just boils down to asking when $[\mathcal{D}^*, au_n - a] \to 0$ in operator norm. Using the Leibniz rule,

$$[\mathcal{D}^*, au_n - a] = a[\mathcal{D}^*, u_n] + [\mathcal{D}^*, a](u_n - 1),$$

we obtain the result immediately. $\square$

3 Approximate units and connections on operator modules

Having demonstrated the usefulness of approximate identities in differentiable algebras, we now refine our concepts to address the existence of connections on modules and the unbounded Kasparov product. Using connections to identify explicit representatives of Kasparov products has been used in several contexts, [13, 32, 41, 42], but doing this in a naive algebraic way leads to problems, as shown in [30, 48].

3.1 Projective modules

For an operator algebra $\mathcal{B}$ with bounded approximate unit v_λ, the right $\mathcal{B}$-module $\mathcal{H}_{\mathcal{B}} := \mathcal{H}\tilde{\otimes}\mathcal{B}$ is called the *standard rigged module*, [8]. For notational convenience we write $\hat{\mathbb{Z}} := \mathbb{Z} \setminus \{0\}$. The module $\mathcal{H}_{\mathcal{B}}$ can be concretely defined using an isometric representation $\pi : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ as the space of column vectors

$$\Big\{(b_i)_{i\in\hat{\mathbb{Z}}} :\ b_i \in \mathcal{B},\ \sum_{i\in\hat{\mathbb{Z}}} \pi(b_i)^*\pi(b_i) < \infty\Big\},$$

where the sum converges in norm. From now on we fix a $\mathbb{Z}_2$-graded C^*-algebra B and an essential unbounded (B, C) Kasparov module $(\mathcal{B}, F_C, \mathcal{D})$ with γ the $\mathbb{Z}_2$-grading operator[1]. We fix the representation

$$\pi_{\mathcal{D}}(b) := \begin{pmatrix} b & 0 \\ [\mathcal{D}, b] & \gamma(b) \end{pmatrix} \in \mathrm{End}^*_C(F \oplus F), \quad b \in \mathcal{B},$$

and we assume $\mathcal{B}$ to have a bounded approximate unit. The graded operator $\mathcal{B}^+$-module $\mathcal{H}_{\mathcal{B}^+}$ is the graded Haagerup tensor product of the graded Hilbert space $\ell^2(\hat{\mathbb{Z}})$ and the graded algebra $\mathcal{B}^+$. Thus the module $\mathcal{H}_{\mathcal{B}^+}$ is naturally $\mathbb{Z}_2$-graded via

$$\Gamma(b_i)_{i\in\hat{\mathbb{Z}}} := (\mathrm{sign}(i)\gamma(b_i))_{i\in\hat{\mathbb{Z}}}, \tag{3.1}$$

and defining the self-adjoint unitary

$$\varepsilon : \mathcal{H}_{\mathcal{B}^+} \to \mathcal{H}_{\mathcal{B}^+}, \quad \varepsilon(b_i)_{i\in\hat{\mathbb{Z}}} = (\mathrm{sign}(i)b_i)_{i\in\hat{\mathbb{Z}}},$$

the grading operator (3.1) on $\mathcal{H}_{\mathcal{B}^+}$ decomposes as $\Gamma := \varepsilon\mathrm{diag}(\gamma_{\mathcal{B}^+}) = \mathrm{diag}(\gamma_{\mathcal{B}^+})\varepsilon$. This allows us to write the representation presenting $\mathcal{H}_{\mathcal{B}^+}$ as a concrete operator $\mathcal{B}^+$-module as

$$(b_i)_{i\in\hat{\mathbb{Z}}} \mapsto \begin{pmatrix} b_i & 0 \\ \mathrm{sign}(i)[\mathcal{D}, b_i]_{\mathcal{B}^+} & \mathrm{sign}(i)\gamma_{\mathcal{B}^+}(b_i) \end{pmatrix}_{i\in\hat{\mathbb{Z}}} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} b_i & 0 \\ [\mathcal{D}, b_i]_{\mathcal{B}^+} & \gamma_{\mathcal{B}^+}(b_i) \end{pmatrix}_{i\in\hat{\mathbb{Z}}}.$$

We will always consider $\mathcal{H}_{\mathcal{B}^+}$ where $\mathcal{B}^+$ is the unitisation of the differentiable algebra $\mathcal{B}$ (cf. Definition 1.18).

The *compact operators* $\mathbb{K}(\mathcal{H}_{\mathcal{B}^+})$ on $\mathcal{H}_{\mathcal{B}^+}$ are defined to be the Haagerup tensor product $\mathbb{K}\tilde{\otimes}\mathcal{B}^+$, as defined in Equation (1.4). The algebra $\mathbb{K}(\mathcal{H}_{\mathcal{B}^+})$ has a bounded approximate unit

$$\chi_n = \sum_{1\leq|i|\leq n} |e_i\rangle\langle e_i|,$$

[1] We recall that if C is non-trivially $\mathbb{Z}_2$-graded then γ is not adjointable as an operator on F_C

where e_i is the *standard basis* of $\mathcal{H}_{\mathcal{B}^+}$. In [32, 42] it was shown that the standard B-valued inner product on the module $\mathcal{H}_{\mathcal{B}^+}$ actually takes values in $\mathcal{B}^+$. Then one defines the *adjointable operators* $\mathrm{End}^*_{\mathcal{B}^+}(\mathcal{H}_{\mathcal{B}^+})$ as the algebra of completely bounded maps $T : \mathcal{H}_{\mathcal{B}^+} \to \mathcal{H}_{\mathcal{B}^+}$ that admit an adjoint with respect to the standard inner product, so that

$$\langle Te, f\rangle = \langle e, T^* f\rangle. \tag{3.2}$$

In [32] the class of submodules of $\mathcal{H}_{\mathcal{B}^+}$ defined by projection operators in $\mathrm{End}^*_{\mathcal{B}^+}(\mathcal{H}_{\mathcal{B}^+})$ are called *operator $*$-modules*, and were classified by Kaad in [29] for the case of commutative $\mathcal{B}$. The class of *stably rigged modules* discussed in [42] is essentially the same. In [13], this class is enlarged by incorporating countable direct sums of projections in $\mathrm{End}^*_{\mathcal{B}^+}(\mathcal{H}_{\mathcal{B}^+})$. The present paper further broadens the class of modules that can be used to construct the Kasparov product, refining the approximate unit techniques of [42].

In [13] the notion of unbounded projection operator was introduced, in order to deal with the differential structure on the C^*-module arising from the Hopf fibration. In this section we develop the theory of such modules beyond the case of direct sums of bounded projections. This will be put to use to demonstrate existence of connections on projective operator modules.

Definition 3.1 (cf. [13])**.** Let $\mathcal{B}$ be an operator $*$-algebra. A *projective operator module* is an inner product operator module $\mathcal{E}$ over $\mathcal{B}$ that is isometrically unitarily isomorphic to $p\,\mathrm{Dom}\,p$ for some possibly unbounded even projection in $\mathcal{H}_{\mathcal{B}^+}$, such that the canonical basis vectors $\{e_i\}_{i\in\hat{\mathbb{Z}}}$ are contained in $\mathrm{Dom}\,p$. Here $\mathcal{E}$ is regarded as a $\mathcal{B}^+$-module in the usual way.

Note that a projective operator module $\mathcal{E}$ over $\mathcal{B}$ admits a canonical C^*-completion, coming from the inner product. Equivalently, this completion can be obtained as the Haagerup tensor product $\mathcal{E}\tilde{\otimes}_{\mathcal{B}}B$ over the completely contractive inclusion $\mathcal{B}\to B$ [13, Corollary 2.18]. We now characterise when a given C^*-module E over B admits a projective $\mathcal{B}$-submodule. The algebra of finite rank operators on E is denoted $\mathrm{Fin}_B(E)$. By a *(homogenous) frame for E* we mean a sequence $(x_i)_{i\in\hat{\mathbb{Z}}}$ with the property that

$$\gamma_E(x_i) = \begin{cases} x_i & \text{if } i>0 \\ -x_i & \text{if } i<0 \end{cases}, \qquad \text{and that} \qquad \chi_n = \sum_{1\leq |i|\leq n} |x_i\rangle\langle x_i| \in \mathrm{Fin}_B(E) \tag{3.3}$$

is an approximate unit for $\mathrm{Fin}_B(E)$ with $\|\chi_n\|_{\mathrm{End}^*_B(E)} \leq 1$ (that is (χ_n) is contractive). We refer to χ_n as the *frame approximate unit for* (x_i). All frames will be homogenous unless stated otherwise, so that $\gamma_E(x_i) = \mathrm{sign}(i)x_i$.

Proposition 3.2. *Let $\mathcal{B}$ be a differentiable algebra and E_B a graded C^*-module over the C^*-closure B. Then E_B is the completion of a projective operator $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}\subset E_B$ if and only if there is a frame $(x_i)_{i\in\hat{\mathbb{Z}}}$ such that each of the column vectors $v_j = (\langle x_i, x_j\rangle)_{i\in\hat{\mathbb{Z}}}$ has finite norm in $\mathcal{H}_{\mathcal{B}^+}$. We call such a frame (x_i), and the associated approximate unit χ_n,* column finite*.*

Proof. $\Rightarrow$ When $\mathcal{E}_{\mathcal{B}}$ is projective we may assume that $\mathcal{E}_{\mathcal{B}}\subset\mathcal{H}_{\mathcal{B}^+}$ and $\{e_i\}$ is the canonical basis of $\mathcal{H}_{\mathcal{B}^+}$, then setting $x_i = pe_i$ we observe that

$$\lim_{k\to\infty}\Big\langle pe_i, \sum_{1\leq |j|\leq k} pe_k\langle pe_k, pe_j\rangle\Big\rangle = \lim_{k\to\infty}\Big\langle pe_i, \sum_{1\leq |j|\leq k} e_k\langle e_k, pe_j\rangle\Big\rangle,$$

is norm convergent since pe_i and pe_j are in $\mathcal{H}_{\mathcal{B}^+}$. So $\chi_n := \sum_{1\leq|i|\leq n} |pe_i\rangle\langle pe_i|$ is a column finite approximate unit for $\mathrm{Fin}_B(E)$.
$\Leftarrow$ We show that the matrix $p = (\langle x_i, x_j\rangle)_{ij}$ is an even projection in $\mathcal{H}_{\mathcal{B}^+}$ with domain

$$\mathrm{Dom}\, p := \Big\{(b_i)_{i\in\hat{\mathbb{Z}}} \in \mathcal{H}_{\mathcal{B}^+} : \forall i \in \hat{\mathbb{Z}} \lim_{k\to\infty}\Big(\sum_{1\leq|j|\leq k} \langle x_i, x_j\rangle b_j\Big) \in \mathcal{B}\Big\}.$$

It is clear that p is densely defined, as the canonical basis vectors $e_i \in \mathcal{H}_{\mathcal{B}^+}$ lie in the domain of p by column finiteness. Moreover p is closed. To see this, first denote by q_i the projection onto the submodule spanned by the basis vector e_i. By column finiteness, $q_i p \in \mathrm{End}^*_{\mathcal{B}^+}(\mathcal{H}_{\mathcal{B}^+})$. Now if $\mathcal{H}_{\mathcal{B}^+} \ni z_n \to z$ and $pz_n \to h$, then $q_i p z_n \to q_i h$ and $q_i p z_n \to q_i p z$. Thus $q_i p z = q_i h$ for all i and $pz = h \in \mathcal{H}_{\mathcal{B}^+}$ so p is closed on its domain.

Next we show that the symmetric operator p is self-adjoint. Let $z \in \mathrm{Dom}\, p^*$, i.e. there is $x \in \mathcal{H}_{\mathcal{B}^+}$ such that for all $w \in \mathrm{Dom}\, p$ we have $\langle pw, z\rangle = \langle w, x\rangle$. Since the basis vectors e_i are in the domain, we can compute

$$\lim_n \sum_{1\leq|i|\leq n} q_i p z = \lim_n \sum_{1\leq|i|\leq n} e_i\langle e_i, q_i p z\rangle = \lim_n \sum_{1\leq|i|\leq n} e_i\langle pe_i, z\rangle = \lim_n \sum_{1\leq|i|\leq n} e_i\langle e_i, x\rangle = x.$$

This means that $pz = x$, so $z \in \mathrm{Dom}\, p$ and p is self-adjoint. Now define

$$\mathcal{E}_{\mathcal{B}} := \{e \in E_B : (\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}} \in \mathrm{Dom}\, p\}, \tag{3.4}$$

and observe that $x_i \in \mathcal{E}_{\mathcal{B}}$ by definition, so $\mathcal{E}_{\mathcal{B}}$ is dense in E_B. The module $\mathcal{E}_{\mathcal{B}}$ is closed in $\mathcal{H}_{\mathcal{B}^+}$ because a convergent net $e_\lambda \in \mathcal{E}_{\mathcal{B}}$ in particular converges in E_B and therefore the limit must be of the form $(\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}}$. $\square$

Note that a column finite approximate unit is row finite as well, because of the relation between the internal and external adjoint: $(\pi_{\mathcal{D}}(\langle x_i, x_j\rangle))^* = U\pi_{\mathcal{D}}(\langle x_j, x_i\rangle)U^*$.

3.2 Connections and splittings

We refine the notion of a connection on a projective module defined in [13] by employing approximate units. The main improvement over [13] is a new operator space topology on a projective module $\mathcal{E}_{\mathcal{B}}$, which is precisely the (weak) topology making the natural Grassmann connection continuous. Completing in this topology yields a possibly larger module $\mathcal{E}^\nabla$. We will prove that for bounded projections, the modules $\mathcal{E}_{\mathcal{B}}$ and $\mathcal{E}^\nabla$ are cb-isomorphic.

Recall from [13] the definition of the universal 1-forms $\Omega^1(B, \mathcal{B})$ associated to a unital differentiable algebra $\mathcal{B}$, defined to be the kernel of the graded multiplication map

$$m : B\tilde{\otimes}\mathcal{B} \to B, \quad b_1 \otimes b_2 \mapsto \gamma(b_1)b_2.$$

In the nonunital case we use $\Omega^1(B^+, \mathcal{B}^+)$, so that the universal derivation

$$\mathrm{d} : \mathcal{B} \mapsto \Omega^1(B^+, \mathcal{B}^+), \quad b \mapsto 1 \otimes b - \gamma(b) \otimes 1, \tag{3.5}$$

is well defined. We will look at splittings of the *universal exact sequence*

$$0 \to E\tilde{\otimes}_{B^+}\Omega^1(B^+, \mathcal{B}^+) \to E\tilde{\otimes}_{\mathbb{C}}\mathcal{B}^+ \xrightarrow{m} E \to 0,$$

that are compatible with the projective submodule $\mathcal{E}_{\mathcal{B}} \subset E_B$. Here $m : E\tilde{\otimes}_{\mathbb{C}}\mathcal{B}^+ \to E$, $m(e\tilde{\otimes}b) = \gamma(e)b$ is the graded multiplication map. We adapt the algebraic notion of splitting to our setting.

Definition 3.3. A completely bounded, graded, $\mathcal{B}^+$-module map $s : \mathcal{E}_{\mathcal{B}} \to E\tilde{\otimes}_{\mathbb{C}}\mathcal{B}^+$ is a *splitting* if $m \circ s$ coincides with the inclusion map $\mathcal{E}_{\mathcal{B}} \to E_B$.

We can now prove the analogue of the Cuntz-Quillen characterisation of algebraic projectivity [19, Proposition 8.1, Corollary 8.2] in the present analytic setting.

Proposition 3.4. *Let $(x_i)_{i\in\hat{\mathbb{Z}}}$ be a column finite frame as in Equation* (3.3)*, defining a projective $\mathcal{B}$-submodule $\mathcal{E}_{\mathcal{B}} \subset E_B$. The map*

$$s : \mathcal{E}_{\mathcal{B}} \to E\tilde{\otimes}_{\mathbb{C}}\mathcal{B}^+, \qquad e \mapsto \sum_{i\in\hat{\mathbb{Z}}} \gamma(x_i) \otimes \langle x_i, e\rangle, \tag{3.6}$$

defines a contractive $\mathcal{B}^+$-linear splitting of the universal exact sequence.

Proof. First we show that for $e \in \mathcal{E}_{\mathcal{B}}$, $s(e)$ actually defines an element of $E\tilde{\otimes}_{\mathbb{C}}\mathcal{B}^+$. To this end let $\varepsilon > 0$ and choose n, m such that

$$\Big\| \sum_{n\le|i|\le m} \pi_{\mathcal{D}}(\langle x_i, e\rangle)^* \pi_{\mathcal{D}}(\langle x_i, e\rangle)\Big\| < \varepsilon,$$

which is possible because $e \in \mathcal{E}_{\mathcal{B}}$. Now estimate

$$\begin{aligned} \Big\| \sum_{n\le|i|\le m} \gamma(x_i) \otimes \langle x_i, e\rangle\Big\|_h^2 &\le \Big\| \sum_{n\le|i|\le m} |x_i\rangle\langle x_i| \Big\|_{\mathbb{K}(E)} \Big\| \sum_{n\le|i|\le m} \pi_{\mathcal{D}}(\langle x_i, e\rangle)^* \pi_{\mathcal{D}}(\langle x_i, e\rangle)\Big\|_{\mathcal{B}^+} \\ &\le \Big\| \sum_{n\le|i|\le m} \pi_{\mathcal{D}}(\langle x_i, e\rangle)^* \pi_{\mathcal{D}}(\langle x_i, e\rangle)\Big\|_{\mathcal{B}^+} < \varepsilon, \end{aligned}$$

which shows that the partial sums of the series defining s form a Cauchy sequence in the Haagerup norm, cf. Equation (1.4). To show continuity of s we again estimate with the Haagerup norm to see that

$$\|s(e)\|_h^2 \le \lim_{k\to\infty} \Big\| \sum_{1\le|i|\le k} |x_i\rangle\langle x_i| \Big\|_{\mathbb{K}(E)} \Big\| \sum_{1\le|i|\le k} \pi_{\mathcal{D}}(\langle x_i, e\rangle)^* \pi_{\mathcal{D}}(\langle x_i, e\rangle)\Big\|_{\mathcal{B}} \le \|e\|_{\mathcal{E}}^2$$

and we are done. □

Recall that a *connection* on a (graded, projective) operator module $\mathcal{E}_{\mathcal{B}}$ is a completely bounded linear operator $\nabla : \mathcal{E}_{\mathcal{B}} \to E\tilde{\otimes}_{B^+}\Omega^1(B^+, \mathcal{B}^+)$ satisfying the Leibniz rule

$$\nabla(eb) = \nabla(e)b + \gamma(e)\tilde{\otimes}\mathrm{d}b,$$

where d is defined in Equation (3.5). Associated to a splitting $s : \mathcal{E}_{\mathcal{B}} \to E\tilde{\otimes}_{\mathbb{C}}\mathcal{B}^+$ is a universal connection

$$\nabla_s : \mathcal{E}_{\mathcal{B}} \to E\tilde{\otimes}_{B^+}\Omega^1(B^+, \mathcal{B}^+), \qquad e \mapsto s(e) - \gamma(e) \otimes 1,$$

which in the case of a column finite frame as in (3.3) takes the form

$$\begin{aligned}\nabla_s(e) = s(e) - \gamma(e)\otimes 1 &= \sum \gamma(x_i)\otimes\langle x_i, e\rangle - \gamma(e)\otimes 1 = \sum \gamma(x_i)\otimes\langle x_i, e\rangle - \gamma(x_i\langle x_i, e\rangle)\otimes 1\\ &= \sum \gamma(x_i)(1\otimes\langle x_i, e\rangle - \gamma(\langle x_i, e\rangle)\otimes 1) = \sum \gamma(x_i)\otimes d\langle x_i, e\rangle.\end{aligned}$$

On the other hand, the frame (3.3) induces a stabilisation map

$$v : \mathcal{E}_{\mathcal{B}} \to \mathcal{H}_{\mathcal{B}^+} \qquad e \mapsto (\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}},$$

with adjoint

$$v^* : \mathcal{H}_{\mathcal{B}^+} \to \mathcal{E}_{\mathcal{B}} \qquad (b_i)_{i\in\hat{\mathbb{Z}}} \mapsto \sum_{i\in\hat{\mathbb{Z}}} x_i b_i,$$

and $v^*v = \mathrm{Id}_{\mathcal{E}}$. The associated projection $p = vv^*$, is given by the matrix $p = (\langle x_i, x_j\rangle)$.

Recalling that the grading operator (3.1) on $\mathcal{H}_{\mathcal{B}^+}$ decomposes as $\Gamma := \varepsilon\mathrm{diag}(\gamma_{\mathcal{B}^+})$, the module $\mathcal{H}_{\mathcal{B}^+}$ admits a canonical connection $\varepsilon\mathrm{d} : (b_i) \mapsto \varepsilon(\mathrm{d}b_i)$. The isometry v induces a connection $v^*\varepsilon\mathrm{d}v : \mathcal{E} \to E\tilde{\otimes}_{B^+}\Omega^1(B^+, \mathcal{B}^+)$, which we call the *Grassmann connection.* These considerations prove the following lemma.

Lemma 3.5. *Let $(x_i)_{i\in\hat{\mathbb{Z}}}$ be a (homogenous) frame and $v : E \to \mathcal{H}_{B^+}$ the associated isometry. Then v is even, that is, $v(\gamma(e)) = \Gamma(v(e))$. If $(x_i)_{i\in\hat{\mathbb{Z}}}$ is column finite the connection $\nabla_s : \mathcal{E}_{\mathcal{B}} \to E\tilde{\otimes}_{B^+}\Omega^1(B^+, \mathcal{B}^+)$ associated to the splitting* (3.6) *equals $v^*\varepsilon\mathrm{d}v$.*

In order to deal with unbounded projections we need to extend the techniques developed in [13] and introduce a slightly different operator space structure. To this end we need to pass from the universal derivation d to the represented derivation $\delta_{\mathcal{D}} : b \mapsto [\mathcal{D}, b]$ coming from the defining Kasparov module $(\mathcal{B}, F_C, \mathcal{D})$ for $\mathcal{B}$. Recall, Remark 1.19, that we assume $[BF_C] = F_C$ and define the unitisation $\mathcal{B}^+$ according to Definition 1.18. The closed linear span of *represented 1-forms* is the operator space

$$\Omega^1_{\mathcal{D}} := \overline{\Big\{\sum_i \pi(b_i)[\mathcal{D}, \pi(b_i')] : b_i, b_i' \in \mathcal{B}\Big\}} \subset \mathrm{End}^*_C(F),$$

which is a $(B, \mathcal{B})$-bimodule. Using the universality of the derivation d, there is a completely bounded $(B, \mathcal{B})$-bimodule map

$$j_{\mathcal{D}} : \Omega^1(B^+, \mathcal{B}^+) \to \Omega^1_{\mathcal{D}},$$

uniquely determined by $\mathrm{d}b \mapsto [\mathcal{D}, \pi(b)]$ since $\pi(1) = 1$, cf. [13, Prop 2.22]. In this way we obtain a connection

$$\nabla_{\mathcal{D}} : \mathcal{E}_{\mathcal{B}} \xrightarrow{\nabla_s} E\tilde{\otimes}_B\Omega^1(B^+, \mathcal{B}^+) \xrightarrow{1\otimes j_{\mathcal{D}}} E\tilde{\otimes}_B\Omega^1_{\mathcal{D}},$$

sometimes referred to as a *represented connection.* In the case of a free module, the tensor product $\mathcal{H}_{\mathcal{B}^+}\tilde{\otimes}_{B^+}\Omega^1_{\mathcal{D}}$ can be identified with the space $\mathcal{H}_{\Omega^1_{\mathcal{D}}} := \mathcal{H}\tilde{\otimes}\Omega^1_{\mathcal{D}}$ of square summable sequences of forms (ω_j), where $\sum_j \omega_j^*\omega_j$ converges in $\mathrm{End}^*_C(F)$. Let (x_i) be a column finite frame for $\mathcal{E}_{\mathcal{B}}$, and consider the space

$$\mathcal{E}^{\nabla} := \Big\{e \in E_B : \lim_{n\to\infty}\Big(\sum_{1\leq|k|\leq n}\langle x_i, \gamma(x_k)\rangle[\mathcal{D}, \langle x_k, e\rangle]\Big)_{i\in\hat{\mathbb{Z}}} \in \mathcal{H}_{\Omega^1_{\mathcal{D}}}\Big\}. \tag{3.7}$$

This space $\mathcal{E}^\nabla$ may be strictly larger than $\mathcal{E}_\mathcal{B}$, and is an operator module in the representation

$$\pi_\nabla(e) := \begin{pmatrix} v(e) & 0 \\ vv^*\varepsilon[\mathcal{D}, v(e)] & v(\gamma(e)) \end{pmatrix} \in \bigoplus_{i\in\hat{\mathbb{Z}}} \mathrm{End}^*_C(F \oplus F), \quad \|e\|_{\mathcal{E}^\nabla} := \|\pi_\nabla(e)\| \tag{3.8}$$

where we have used slightly abusive notation for the entrywise graded commutator with $\mathcal{D}$ in the indicated column vector. We will write γ for $\mathrm{diag}(\gamma_{\mathcal{B}^+})$ and $\mathcal{D}_\varepsilon$ for the self-adjoint regular operator $\varepsilon\mathrm{diag}(\mathcal{D})$ on $\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F$. There is an equality of domains $\mathrm{Dom}\,\mathrm{diag}(\mathcal{D}) = \mathrm{Dom}\,\mathcal{D}_\varepsilon$, and the closed graded derivations

$$[\mathrm{diag}(\mathcal{D}), T]_\gamma := \mathrm{diag}(\mathcal{D})T - \gamma T\gamma\mathrm{diag}(\mathcal{D}), \quad [\mathcal{D}_\varepsilon, T]_\Gamma := \mathcal{D}_\varepsilon T - \Gamma T\Gamma\mathcal{D}_\varepsilon, \tag{3.9}$$

are related via $[\mathcal{D}_\varepsilon, T]_\Gamma = [\mathrm{diag}(\mathcal{D}), \varepsilon T]_\gamma = \varepsilon[\mathrm{diag}(\mathcal{D}), T]_\gamma$. Therefore these derivations have the same domain inside $\mathrm{End}^*_C(\mathcal{H}_{\mathcal{B}^+}\tilde{\otimes}_{B^+}F)$. With regards to gradings on the module $\mathcal{E}^\nabla$ defined in (3.7), observe that there is an identity

$$\begin{aligned}\Big(\sum_{1\leq|k|\leq n} \langle x_i, \gamma(x_k)\rangle[\mathcal{D}, \langle x_k, e\rangle]\Big)^* &= \sum_{1\leq|k|\leq n} -[\mathcal{D}, \gamma\langle e, x_k\rangle]\langle\gamma(x_k), x_i)\rangle \\ &= \sum_{1\leq|k|\leq n} \gamma\big(\,[\mathcal{D}, \langle e, x_k\rangle]\langle x_k, \gamma(x_i)\rangle\,\big),\end{aligned}$$

and thus for $e \in \mathcal{E}^\nabla$, the series of row vectors

$$\left(\sum_{1\leq|k|\leq n} [\mathcal{D}, \langle e, x_k\rangle]\langle x_k, \gamma(x_i)\rangle\right)^t_{i\in\hat{\mathbb{Z}}} \tag{3.10}$$

is convergent.

Lemma 3.6. *Let $p : \mathrm{Dom}\, p \to \mathcal{H}_{\mathcal{B}^+}$ be a projection such that $\mathcal{E}_\mathcal{B} = p\,\mathrm{Dom}\, p$ is a projective operator module, and consider the operator module $\mathcal{E}^\nabla$ defined in Equation* (3.7)*.*
1) There is a completely contractive dense inclusion $\iota : \mathcal{E}_\mathcal{B} \to \mathcal{E}^\nabla$.
2) If $p \in \mathrm{End}^*_{\mathcal{B}^+}(\mathcal{H}_{\mathcal{B}^+})$ then ι is a cb-isomorphism.

Proof. The estimate

$$\left\|\begin{pmatrix} v(e) & 0 \\ vv^*\varepsilon[\mathcal{D}, v(e)] & \gamma(v(e)) \end{pmatrix}\right\| = \left\|\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} v(e) & 0 \\ \varepsilon[\mathcal{D}, v(e)] & v(\gamma(e)) \end{pmatrix}\right\| \leq \left\|\begin{pmatrix} v(e) & 0 \\ \varepsilon[\mathcal{D}, v(e)] & \gamma(v(e)) \end{pmatrix}\right\|,$$

proves 1). For 2), observe first that $\Gamma p = p\Gamma$, so $p\varepsilon = p\gamma\varepsilon\gamma = p\Gamma\gamma = \Gamma p\gamma = \varepsilon\gamma p\gamma = \varepsilon\gamma(p)$ and

$$[\mathcal{D}_\varepsilon, p]_\Gamma v(e) = [\mathrm{diag}(\mathcal{D}), \varepsilon p]_\gamma v(e) = [\mathcal{D}, \varepsilon v(e)] - \varepsilon\gamma p\gamma[\mathcal{D}, v(e)] = [\mathcal{D}, \varepsilon v(e)] - p\varepsilon[\mathcal{D}, v(e)].$$

Thus we can write

$$\begin{pmatrix} v(e) & 0 \\ vv^*\varepsilon[\mathcal{D}, v(e)] & \gamma(v(e)) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -[\mathcal{D}_\varepsilon, p]_\Gamma & 1 \end{pmatrix}\begin{pmatrix} v(e) & 0 \\ \varepsilon[\mathcal{D}, v(e)] & v(\gamma(e)) \end{pmatrix},$$

and as $[\mathcal{D}_\varepsilon, p]_\Gamma$ is bounded, the matrix $\begin{pmatrix} 1 & 0 \\ -[\mathcal{D}_\varepsilon, p]_\Gamma & 1 \end{pmatrix}$ is invertible. The assertion follows. □

Proposition 3.7. *The module $\mathcal{E}^\nabla$ has the following properties.*
1) The inner product $\mathcal{E}_\mathcal{B} \times \mathcal{E}_\mathcal{B} \to \mathcal{B}$ extends to an inner product $\mathcal{E}^\nabla \times \mathcal{E}^\nabla \to \mathcal{B}$.
2) For each $e \in \mathcal{E}^\nabla$ the operator $e^* : \mathcal{E}^\nabla \to \mathcal{B}$ defined by $f \mapsto \langle e, f\rangle$ is completely bounded and adjointable, with adjoint $b \mapsto eb$, and satisfies the estimate $\|e^*\|_{cb} \leq 2\|e\|_{\mathcal{E}^\nabla}$.

Proof. For 1) we must show that for $e, f \in \mathcal{E}^\nabla$ the inner product $\langle e, f\rangle \in \mathcal{B}$. Let $(x_i)_{i\in\hat{\mathbb{Z}}}$ be a defining column finite frame for $\mathcal{E}_\mathcal{B}$. By definition the series of column vectors

$$\sum_{j\in\hat{\mathbb{Z}}}(\langle x_i, \gamma(x_j)\rangle[\mathcal{D}, \langle x_j, e\rangle])_{i\in\hat{\mathbb{Z}}},$$

is norm convergent for $e \in \mathcal{E}^\nabla$ by (3.7). Consider the partial sums

$$\Big[\mathcal{D}, \sum_{1\leq|j|\leq n}\langle e, x_j\rangle\langle x_j, f\rangle\Big] = \sum_{1\leq|j|\leq n}\gamma(\langle e, x_j\rangle)[\mathcal{D}, \langle x_j, f\rangle] + [\mathcal{D}, \langle e, x_j\rangle]\langle x_j, f\rangle.$$

The two terms on the right hand side are convergent sums, since (using the pairing of row and column vectors)

$$\begin{aligned}\Big\|\sum_{1\leq|j|\leq n}\gamma(\langle e, x_j\rangle)[\mathcal{D}, \langle x_j, f\rangle]\Big\| &= \Big\|\sum_{1\leq|j|\leq n}\sum_{i\in\hat{\mathbb{Z}}}\langle\gamma(e), x_i\rangle\langle x_i, \gamma(x_j)\rangle[\mathcal{D}, \langle x_j, f\rangle]\Big\| \\ &= \Big\|\sum_{1\leq|j|\leq n}(\langle x_i, \gamma(e)\rangle)^*_{i\in\hat{\mathbb{Z}}}\cdot(\langle x_i, \gamma(x_j)\rangle[\mathcal{D}, \langle x_j, f\rangle])_{i\in\hat{\mathbb{Z}}}\Big\| \\ &\leq \|e\|_E\,\Big\|\sum_{1\leq|j|\leq n}(\langle x_i, \gamma(x_j)\rangle[\mathcal{D}, \langle x_j, f\rangle])_{i\in\hat{\mathbb{Z}}}\Big\|,\end{aligned}$$

and similarly for the other term. Since $\sum_{i\in\hat{\mathbb{Z}}}\langle e, x_i\rangle\langle x_i, f\rangle$ converges to $\langle e, f\rangle$ and $[\mathcal{D}, \cdot]$ is a closed derivation, it follows that $\langle e, f\rangle \in \mathcal{B}$. Therefore we can write

$$\begin{aligned}\begin{pmatrix}\langle e, f\rangle & 0\\ [\mathcal{D}, \langle e, f\rangle] & \gamma\langle e, f\rangle\end{pmatrix} &= \sum_{i\in\hat{\mathbb{Z}}}\begin{pmatrix}\langle e, x_i\rangle & 0\\ 0 & \langle\gamma(e), x_i\rangle\end{pmatrix}\begin{pmatrix}\langle x_i, f\rangle & 0\\ \sum_{j\in\hat{\mathbb{Z}}}\langle x_i, \gamma(x_j)\rangle[\mathcal{D}, \langle x_i, f\rangle] & \langle x_i, \gamma(f)\rangle\end{pmatrix} \\ &\quad + \begin{pmatrix}0 & 0\\ \sum_{j\in\hat{\mathbb{Z}}}[\mathcal{D}, \langle e, x_j\rangle]\langle x_j, \gamma(x_i)\rangle & 0\end{pmatrix}\begin{pmatrix}\langle\gamma(x_i), f\rangle & 0\\ 0 & \langle x_i, f\rangle\end{pmatrix}.\end{aligned}$$

These series are convergent in view of (3.7) and (3.10) and the equalities also hold when f is a matrix of elements of $\mathcal{E}^\nabla$ yielding the estimate

$$\|e^*(f)\|_\mathcal{B} \leq (\|e\|_E\|\|f\|_{\mathcal{E}^\nabla} + \|e\|_{\mathcal{E}^\nabla}\|f\|_E) \leq 2\|e\|_{\mathcal{E}^\nabla}\|f\|_{\mathcal{E}^\nabla},$$

whence $\|e^*\|_{cb} \leq 2\|e\|_{\mathcal{E}^\nabla}$. □

3.3 Complete projective modules

We now define several algebras of operators on the modules $\mathcal{E}_\mathcal{B}$ and $\mathcal{E}^\nabla$. Recall that $\mathcal{E}_\mathcal{B} \subset \mathcal{E}^\nabla$ is a proper submodule in general by Lemma 3.6. As for C^*-modules, we denote the space of finite rank operators by $\mathrm{Fin}_\mathcal{B}(\mathcal{E})$. We give $\mathrm{Fin}_\mathcal{B}(\mathcal{E})$ the operator algebra structure determined by regarding $\mathrm{Fin}_\mathcal{B}(\mathcal{E})$ as an algebra of operators on $\mathcal{E}^\nabla$:

$$\pi_\nabla(K) := \begin{pmatrix}vKv^* & 0\\ p[\mathcal{D}_\varepsilon, vKv^*]p & vKv^*\end{pmatrix}, \tag{3.11}$$

where $\mathcal{D}$ comes from the defining Kasparov module $(\mathcal{B}, F_C, \mathcal{D})$ for $\mathcal{B}$ and p comes from a defining column finite frame $(x_i)_{i\in\hat{\mathbb{Z}}}$. For $e, f \in \mathcal{E}_{\mathcal{B}}$, consider the column, respectively row, vectors

$$v|e\rangle = v(e) = (\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}}, \quad \langle f|v^* = v(f)^* = (\langle f, x_i\rangle)^t_{i\in\hat{\mathbb{Z}}}, \tag{3.12}$$

which are elements of $\mathcal{H}_{\mathcal{B}+}$ and $\mathcal{H}^t_{\mathcal{B}+}$ respectively. Thus the rank one operator $|e\rangle\langle f|$ is such that $[\mathcal{D}_\varepsilon, v(|e\rangle\langle f|)v^*]$ is a bounded matrix. Therefore the representation (3.11) is well-defined on $\mathrm{Fin}_{\mathcal{B}}(\mathcal{E})$. We emphasise that we do *not* consider finite rank operators associated to vectors coming from $\mathcal{E}^\nabla$ here. The ideal of *compact operators* $\mathbb{K}(\mathcal{E}^\nabla)$ is defined to be the closure of $\mathrm{Fin}_{\mathcal{B}}(\mathcal{E})$ in the operator space norm $\|\pi_\nabla(\cdot)\|_\infty$. We now address the issue of approximate units for $\mathbb{K}(\mathcal{E}^\nabla)$.

Lemma 3.8. *Let $(\mathcal{B}, F_C, \mathcal{D})$ be the defining Kasparov module for $\mathcal{B}$, and $\mathcal{E}_{\mathcal{B}}$ a projective operator module. For a column finite frame approximate unit χ_n associated to the defining frame $(x_i)_{i\in\hat{\mathbb{Z}}}$, any $K \in \mathrm{Fin}_{\mathcal{B}}(\mathcal{E})$ satisfies*

$$vKv^* \operatorname{Dom} \mathcal{D}_\varepsilon \subset \operatorname{Dom} \mathcal{D}_\varepsilon,$$

and $[\mathcal{D}_\varepsilon, vKv^*]$ extends to a bounded adjointable operator in $\mathrm{End}^*_C(\mathcal{H}_B\tilde{\otimes}_B F)$. Moreover

$$\begin{aligned}\lim_{n\to\infty} v\chi_n v^*[\mathcal{D}_\varepsilon, vKv^*] &= vv^*[\mathcal{D}_\varepsilon, vKv^*],\\ \lim_{n\to\infty} [\mathcal{D}_\varepsilon, vKv^*]v\chi_n v^* &= [\mathcal{D}_\varepsilon, vKv^*]vv^*,\end{aligned} \tag{3.13}$$

in operator norm.

Proof. It suffices to prove this for rank one operators $K = |e\rangle\langle f|$. In that case, Equation (3.12) shows that vKv^* is given by the infinite matrix

$$(\langle x_i, e\rangle\langle f, x_j\rangle)_{ij} \in \mathbb{K}\tilde{\otimes}\mathcal{B},$$

and thus is in the domain of the derivation $[\mathcal{D}_\varepsilon, \cdot]$. The norm limits (3.13) are given by

$$\begin{aligned}\lim_{n\to\infty} v\chi_n v^*[\mathcal{D}_\varepsilon, vKv^*] &= \lim_{n\to\infty}\left(\sum_{1\le|k|\le n}\langle x_i, \gamma(x_k)\rangle[\mathcal{D}, \langle x_k, e\rangle\langle f, x_j\rangle]\right)_{ij}\\ &= \lim_{n\to\infty}\left(\sum_{1\le|k|\le n}\langle x_i, \gamma(x_k)\rangle\gamma\langle x_k, e\rangle[\mathcal{D}, \langle f, x_j\rangle] + \langle x_i, \gamma(x_k)\rangle[\mathcal{D}, \langle x_k, e\rangle]\langle f, x_j\rangle\right)_{ij}\\ &= \big(\langle x_i, \gamma(e)\rangle[\mathcal{D}, \langle f, x_j\rangle]\big)_{ij} + \lim_{n\to\infty}\left(\sum_{1\le|k|\le n}\langle x_i, \gamma(x_k)\rangle[\mathcal{D}, \langle x_k, e\rangle]\langle f, x_j\rangle\right)_{ij},\end{aligned}$$

where the first term is a well-defined infinite matrix because $f \in \mathcal{E}$ and the second term is a norm convergent limit because $e \in \mathcal{E} \subset \mathcal{E}^\nabla$. The other limit is handled verbatim. $\square$

Given a frame approximate unit (χ_n) for $\mathbb{K}(E_B)$, denote by $\mathscr{C}(\chi_n)$ the convex hull of (χ_n). This is the algebraic convex hull, and *not* the closed convex hull.

Definition 3.9. Let $(\mathcal{B}, F_C, \mathcal{D})$ be the defining Kasparov module for $\mathcal{B}$, and $\mathcal{E}_\mathcal{B}$ a projective operator module with column finite frame approximate unit χ_n. The module $\mathcal{E}_\mathcal{B}$ is a *complete projective operator module* if there is an approximate unit $(u_n) \subset \mathscr{C}(\chi_n)$ for $\mathbb{K}(E_B)$ such that the sequence of operators $p[\mathcal{D}_\varepsilon, vu_nv^*]p : \mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F \to \mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F$ converges to 0 strictly.

This definition should be viewed in the light of property 2) of Proposition 1.9 as well as Corollary 1.11.

Proposition 3.10. *Let $\mathcal{E}_\mathcal{B}$ be a complete projective module over $\mathcal{B}$. Then $\mathbb{K}(\mathcal{E}^\nabla)$ has a bounded approximate unit consisting of elements of $\mathrm{Fin}_\mathcal{B}(\mathcal{E})$.*

Proof. Let $\chi_n = \sum_{1\leq|i|\leq n} |x_i\rangle\langle x_i|$ be the defining column finite frame approximate unit. Consider an approximate unit $(u_n) \in \mathscr{C}(\chi_n)$ as in Definition 3.9. It follows from the uniform boundedness principle that $\sup_n \|\pi_\nabla(u_n)\| < \infty$: this follows because for each $x \in \mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F$ the sequence $p[\mathcal{D}, vu_nv^*]px$ converges, so that

$$\sup_n \|p[\mathcal{D}_\varepsilon, vu_nv^*]px\| < \infty, \quad \text{and therefore} \quad \sup_n \|p[\mathcal{D}_\varepsilon, vu_nv^*]p\| < \infty.$$

For $K \in \mathrm{Fin}_\mathcal{B}(\mathcal{E})$ it then follows that

$$p[\mathcal{D}_\varepsilon, vu_nKv^*]p = p[\mathcal{D}_\varepsilon, vu_nv^*]vKv^* + vu_nv^*[\mathcal{D}_\varepsilon, vKv^*]p \to p[\mathcal{D}_\varepsilon, vKv^*]p,$$

by Lemma 3.8. Now since $\mathrm{Fin}_\mathcal{B}(\mathcal{E}) \subset \mathbb{K}(\mathcal{E}^\nabla)$ is dense and $\pi_\nabla(u_n)$ is uniformly bounded, it follows that $\pi_\nabla(u_nK) \to \pi_\nabla(K)$ and $\pi_\nabla(Ku_n) \to \pi_\nabla(K)$ for all $K \in \mathbb{K}(\mathcal{E}^\nabla)$. □

Remark 3.11. Proposition 3.10 can be made into an if and only if statement when we restrict to bounded approximate units contained in $\mathscr{C}(\chi_n)$. Uniform boundedness of such an approximate unit u_k gives a uniformly bounded sequence $p[\mathcal{D}_\varepsilon, vu_kv^*]p$, which converges pointwise on the algebraic tensor product $\mathcal{H}_{\mathcal{B}^+} \otimes \mathrm{Dom}\,\mathcal{D}_\varepsilon$, and hence everywhere.

We now present some sufficient conditions for a projective operator module to be complete.

Proposition 3.12. *For a projective operator module $\mathcal{E}_\mathcal{B} = p\,\mathrm{Dom}\,p$ with defining column finite frame (x_i) and corresponding approximate unit (χ_n), each of the following conditions imply completeness of the module $\mathcal{E}_\mathcal{B}$:*
1) there is an approximate unit $(u_n) \in \mathscr{C}(\chi_n)$ for $\mathbb{K}(E_B)$ such that the operators $p[\mathcal{D}_\varepsilon, u_n]p$ converge to 0 in norm on the C^*-module $\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F_C$;
2) the projection p is a countable direct sum of finite even projections $p_k \in M_{2m_k}(\mathcal{B}^+)$;
3) the projection p is an element of $\mathrm{End}^*_{\mathcal{B}^+}(\mathcal{H}_{\mathcal{B}^+})$;

Proof. Because norm convergence implies strict convergence, 1) implies that the sequence $p[\mathcal{D}_\varepsilon, u_n]p$ converges strictly on $\mathcal{H}_{B^+}$, hence the module $\mathcal{E}_\mathcal{B}$ is complete in the sense of Definition 3.9. Thus, to prove 2), it is enough to show that 2) $\Rightarrow$ 1).

2) $\Rightarrow$ 1) For a countable family of finite projections p_i with $[\mathcal{D}_\varepsilon, p_i]$ bounded, for each i we have $p_i[\mathcal{D}_\varepsilon, p_i]p_i = 0$ and

$$p_k = \sum_{1\leq|i|\leq m_k} |pe_i^k\rangle\langle pe_i^k|.$$

By identifying the direct sum $\bigoplus_{k=0}^\infty (\mathcal{B}^+)^{2m_k}$ with $\mathcal{H}_{\mathcal{B}^+}$ and setting $p = \oplus_i^\infty p_i$, we can define an approximate unit $u_n = \oplus_{i=1}^n p_i$. The explicit form of p_i given above shows that u_n is a subsequence of the approximate unit associated to the frame (pe_k^i), and so in the convex hull. Then $p[\mathcal{D}, u_n]p = \sum_{i=1}^n p[\mathcal{D}, p_i]p = \sum_{i=1}^n p_i[\mathcal{D}, p_i]p_i = 0$.

To show that 3) implies completeness, observe that $p \in \operatorname{End}^*_{\mathcal{B}^+}(\mathcal{H}_{\mathcal{B}^+})$ if and only if $p \otimes \operatorname{Id}_F$ preserves the domain of $\mathcal{D}_\varepsilon$ and $[\mathcal{D}_\varepsilon, p \otimes \operatorname{Id}_F]$ is a bounded operator. Let q_n, $i \in \hat{\mathbb{Z}}$ denote the projection onto the submodule generated by the basis vectors e_i, $1 \leq |i| \leq n$, let $x_i = pe_i$, and $\chi_n = \sum_{i=1}^n |x_i\rangle\langle x_i|$.

Now *on the image of* p, a short calculation shows that $\chi_n y = \chi_n py = pq_n py$. Then

$$p[\mathcal{D}_\varepsilon, \chi_n]p = p[\mathcal{D}_\varepsilon, pq_n]p = p[\mathcal{D}_\varepsilon, p]q_n p.$$

The projections q_n converge strongly to the identity on $\mathcal{H}_{B^+}$, and $[\mathcal{D}_\varepsilon, p]$ is bounded. Therefore it follows that for any $x \in \mathcal{H}_{B^+}$, $p[\mathcal{D}_\varepsilon, \chi_n]px = p[\mathcal{D}_\varepsilon, p]q_n px \to p[\mathcal{D}_\varepsilon, p]px = 0$, and so Definition 3.9 is satisfied. □

3.4 Self-adjointness and regularity

We now come to the study of self-adjointness and regularity of induced operators $1 \otimes_\nabla \mathcal{D}$ on tensor product modules. The setting for this construction is as follows. Let $(\mathcal{B}, F_C, \mathcal{D})$ be the unbounded Kasparov module defining $\mathcal{B}$, which we recall, Remark 1.19, is essential so that $[BF_C] = F_C$. Given a projective module $\mathcal{E}_\mathcal{B} \subset \mathcal{E}^\nabla$ with grading γ one obtains an odd symmetric operator

$$1 \otimes_\nabla \mathcal{D} : \mathcal{E} \otimes_\mathcal{B} \operatorname{Dom} \mathcal{D} \to E \tilde{\otimes}_B F, \tag{3.14}$$

via the usual formula $1 \otimes_\nabla \mathcal{D}(e \otimes f) := \gamma(e) \otimes \mathcal{D}f + \nabla_\mathcal{D}(e)f$. We extend $1 \otimes_\nabla \mathcal{D}$ to its minimal closure.

In [13] it was shown that this operator is self-adjoint and regular in the case where p is a direct sum of bounded projection operators. In [30] it was shown that there exist unbounded projections for which the resulting operator is not self-adjoint. The counterexample uses the half-line, a noncomplete metric space. In this section we show that for complete projective modules the induced operator is self-adjoint and regular, by an argument similar to that for the Dirac operator on a complete manifold.

Write $\partial := v(1 \otimes_\nabla \mathcal{D})v^*$ with domain and definition

$$\operatorname{Dom} \partial = v \operatorname{Dom}(1 \otimes_\nabla \mathcal{D}) \oplus (1-p)\mathcal{H}_{B^+} \tilde{\otimes}_{B^+} F_C, \quad \partial(vy + (1-p)z) = v^*(1 \otimes_\nabla \mathcal{D})y. \tag{3.15}$$

We have $\mathrm{G}(\partial) \subset (\mathcal{H}_{B^+} \tilde{\otimes}_{B^+} F_C)^{\oplus 2}$, and the graph of the adjoint operator ∂^* is given by $\mathrm{G}(\partial^*) := U\mathrm{G}(\partial)^\perp$, where we recall that $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Lemma 3.13. *The operator* $1 \otimes_\nabla \mathcal{D}$ *is self-adjoint and regular on* $\operatorname{Dom}(1 \otimes_\nabla \mathcal{D})$ *if and only if the operator* ∂ *is self-adjoint and regular on* $\operatorname{Dom} \partial$.

Proof. Recall that a closed, densely-defined symmetric operator $T : \operatorname{Dom} T \to E$ is self-adjoint and regular if and only if the operators $T \pm i : \operatorname{Dom} T \to E$ have dense range, cf. [38, Lemma 9.7, 9.8] and [31, Proposition 4.1].

Suppose that $1 \otimes_\nabla \mathcal{D} \pm i$ have dense range. Then, for $x = vy + (1-p)z$ with $y \in \operatorname{Dom}(1 \otimes_\nabla \mathcal{D})$ we have

$$(\partial \pm i)x = v(1 \otimes_\nabla \mathcal{D} \pm i)y + i(1-p)z, \quad (1 \otimes_\nabla \mathcal{D} \pm i)y = v^*(\partial \pm i)x.$$

Since $\operatorname{Im} v$ and $\operatorname{Im}(1-p)$ are orthogonal, it follows that $\partial \pm i$ has dense range in $\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F_C$ if and only if $(1\otimes_\nabla \mathcal{D})\pm i$ has dense range in $E\tilde{\otimes}_B F_C$. □

We now prove that ∂ is self-adjoint and regular. Since the representation of B on F_C is assumed to be essential, we have the identification

$$\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F_C \xrightarrow{\sim} \bigoplus_{i\in\hat{\mathbb{Z}}} F_C, \tag{3.16}$$

and the self-adjoint regular operator $\mathcal{D}_\varepsilon$ coincides with the operator $1\otimes_d \mathcal{D}$ on $\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F_C$ and $\varepsilon\mathrm{d}$ the trivial connection.

Lemma 3.14. *Let $(\mathcal{B}, F_C, \mathcal{D})$ be the defining unbounded Kasparov module for $\mathcal{B}$, with $[BF_C] = F_C$, and let $\mathcal{E}_\mathcal{B} \subset \mathcal{E}^\nabla$ be a complete projective module over $\mathcal{B}$ with grading γ and defining frame $(x_i)_{i\in\hat{\mathbb{Z}}}$. For an elementary tensor $e\otimes f \in \mathcal{E}\otimes_{\mathcal{B}^+}\operatorname{Dom}\mathcal{D}$, $(1\otimes_\nabla\mathcal{D})(e\otimes f)$ is given by the formula*

$$\begin{aligned}
\gamma(e)\otimes\mathcal{D}f + \nabla(e)f &= \sum_{i\in\hat{\mathbb{Z}}} x_i\otimes\langle x_i,\gamma(e)\rangle\mathcal{D}f + \gamma(x_i)\otimes[\mathcal{D},\langle x_i,e\rangle]f \\
&= \sum_{i\in\hat{\mathbb{Z}}} x_i\otimes\langle x_i,\gamma(e)\rangle\mathcal{D}f + \sum_{i,j\in\hat{\mathbb{Z}}} x_i\otimes\langle x_i,\gamma(x_j)\rangle[\mathcal{D},\langle x_j,e\rangle]f && (3.17)\\
&= \sum_{i\in\hat{\mathbb{Z}}} x_i\otimes\mathcal{D}\langle\gamma(x_i),e\rangle f = \sum_{i\in\hat{\mathbb{Z}}}\gamma(x_i)\otimes\mathcal{D}\langle x_i,e\rangle f. && (3.18)
\end{aligned}$$

More symbolically, $1\otimes_\nabla\mathcal{D} = v^*\partial v = v^*\mathcal{D}_\varepsilon v$ on $\mathcal{E}\otimes_\mathcal{B}\operatorname{Dom}\mathcal{D}$ and $\partial = p\mathcal{D}_\varepsilon p$ on $v\mathcal{E}\otimes_{\mathcal{B}^+}\operatorname{Dom}\mathcal{D}$. The map

$$g:\mathcal{E}^\nabla\tilde{\otimes}_\mathcal{B}\mathrm{G}(\mathcal{D})\to\mathrm{G}(1\otimes_\nabla\mathcal{D}),\quad e\otimes\begin{pmatrix} f\\ \mathcal{D}f\end{pmatrix}\mapsto\begin{pmatrix} e\otimes f\\ (1\otimes_\nabla\mathcal{D})(e\otimes f)\end{pmatrix}, \tag{3.19}$$

is a completely contractive operator with dense range.

Proof. First we show that the sum (3.17) is convergent, so that the map (3.19) is well-defined. The first term on the right hand side of (3.17) converges trivially. For the second term we prove slightly more, estimating for finite sums $\sum_k e_k\otimes f_k\in\mathcal{E}^\nabla\otimes_\mathcal{B}\operatorname{Dom}\mathcal{D}$

$$\begin{aligned}
&\Big\|\sum_{i,j,k} x_i\otimes\langle x_i,\gamma(x_j)\rangle[\mathcal{D},\langle x_j,e_k\rangle]f_k\Big\|_h^2\\
&\qquad\le\Big\|\sum_{i\in\hat{\mathbb{Z}}}|x_i\rangle\langle x_i|\Big\|_{\mathbb{K}(E)}\Big\|\Big(\sum_{j,k\in\hat{\mathbb{Z}}}\langle x_i,\gamma(x_j)\rangle[\mathcal{D},\langle x_i,e_k\rangle]f_k\rangle\Big)_{i\in\hat{\mathbb{Z}}}\Big\|^2\\
&\qquad\le\|\sum_k\pi_\nabla(e_k)\pi_\nabla(e_k)^*\|\,\|\sum_k\langle f_k,f_k\rangle\| && (3.20)\\
&\qquad\le\|\sum_k\pi_\nabla(e_k)\pi_\nabla(e_k)^*\|\,\Big\|\sum\Big\langle\begin{pmatrix} f_k\\ \mathcal{D}f_k\end{pmatrix},\begin{pmatrix} f_k\\ \mathcal{D}f_k\end{pmatrix}\Big\rangle\Big\|,
\end{aligned}$$

proving that both (3.17) and (3.18) are well-defined. The estimate (3.20) also provides half of the estimates needed to prove continuity of the map g. The other half is proving continuity of

$e\otimes f\mapsto \gamma(e)\otimes \mathcal{D}f$. So, again, consider a finite sum $\sum_k e_k\otimes f_k\in \mathcal{E}^{\nabla}\otimes_{\mathcal{B}}\operatorname{Dom}\mathcal{D}$. We have the estimate

$$\Big\|\sum_k \gamma(e_k)\otimes \mathcal{D}f_k\Big\|^2 \leq \Big\|\sum |\gamma(e_k)\rangle\langle\gamma(e_k)|\Big\|_{\mathbb{K}(E)}\Big\|\sum\langle \mathcal{D}f_k,\mathcal{D}f_k\rangle\Big\|$$
$$\leq \|\sum_k \pi_{\nabla}(e_k)\pi_{\nabla}(e_k)^*\|\,\Big\|\sum\Big\langle\begin{pmatrix} f_k\\ \mathcal{D}f_k\end{pmatrix},\begin{pmatrix} f_k\\ \mathcal{D}f_k\end{pmatrix}\Big\rangle\Big\|,$$

by using the fact that the C^*-module tensor product $E\otimes_B F$ is isometrically isomorphic to the Haagerup tensor product $E\tilde{\otimes}_B F$ cf. [9, Thm. 4.3].

Combining the two norm estimates above and taking the infimum over all representations in the tensor product shows that the map g satisfies

$$\Big\|g\Big(\sum_k e_k\otimes\begin{pmatrix} f_k\\ \mathcal{D}f_k\end{pmatrix}\Big)\Big\|\leq 2\Big\|\sum_k \pi_{\nabla}(e_k)\otimes\begin{pmatrix} f_k\\ \mathcal{D}f_k\end{pmatrix}\Big\|_h,$$

and we are done. □

Lemma 3.15. *Let $\mathcal{E}_{\mathcal{B}}$ be a projective operator module with column finite frame (x_i) and $R\in\mathscr{C}(\chi_n)$. Then*
1) *the operator vRv^* maps $\operatorname{Dom}\mathcal{D}_{\varepsilon}$ into $\operatorname{Dom}\partial$;*
2) *the operator vRv^* maps $\operatorname{Dom}\partial^*$ into $\operatorname{Dom}\mathcal{D}_{\varepsilon}$;*
3) *if $\mathcal{E}_{\mathcal{B}}$ is complete, then vRv^* maps $\operatorname{Dom}\partial^*$ into $\operatorname{Dom}\mathcal{D}_{\varepsilon}\cap\operatorname{Dom}\partial\subset\operatorname{Dom}\partial$.*

Proof. It suffices to show that the frame approximate unit χ_n of (x_i) has the properties $1), 2)$ and 3), for then any finite convex combination R of χ_n's also has these properties. For 1), consider the adjointable operators

$$\pi^p_{\mathcal{D}}(\chi_k):=\begin{pmatrix} v\chi_k v^* & 0\\ p[\mathcal{D}_{\varepsilon},v\chi_k v^*] & v\chi_k v^*\end{pmatrix}:(\mathcal{H}_B\tilde{\otimes}_B F)^{\oplus 2}\to (vE\tilde{\otimes}_B F)^{\oplus 2}.$$

For $x=h\otimes f\in\mathcal{H}_{\mathcal{B}}\otimes_{\mathcal{B}}\operatorname{Dom}\mathcal{D}\subset\operatorname{Dom}\mathcal{D}_{\varepsilon}$, we have that

$$v\chi_k v^*(x)=\sum_{1\leq|i|\leq k} vx_i\otimes\langle x_i,v^*(h)\rangle f=\sum_{1\leq|i|\leq k} vx_i\otimes\langle vx_i,h\rangle f, \tag{3.21}$$

and since $v(x_i),h\in\mathcal{H}_{\mathcal{B}^+}$, we have $\langle vx_i,h\rangle\in\mathcal{B}$ and thus $\langle vx_i,h\rangle f\in\operatorname{Dom}\mathcal{D}$. Hence the finite sum (3.21) is an element of $v\mathcal{E}\otimes_{\mathcal{B}}\operatorname{Dom}\mathcal{D}\subset\operatorname{Dom}\partial$. By Lemma 3.14 we get that

$$\pi^p_{\mathcal{D}}(\chi_k)\begin{pmatrix} x\\ \mathcal{D}_{\varepsilon}x\end{pmatrix}=\begin{pmatrix} v\chi_k v^*x\\ p(\mathcal{D}_{\varepsilon})v\chi_k v^*x\end{pmatrix}=\begin{pmatrix} v\chi_k v^*x\\ \partial v\chi_k v^*x\end{pmatrix}.$$

It follows from (3.16) that $\mathcal{H}_{\mathcal{B}}\otimes_{\mathcal{B}}\operatorname{Dom}\mathcal{D}$ is a core for $\mathcal{D}_{\varepsilon}$, for it contains the algebraic direct sum $\oplus_{i\in\hat{\mathbb{Z}}}\operatorname{Dom}\mathcal{D}$. Thus the bounded operators $\pi^p_{\mathcal{D}}(\chi_k)$ map a dense subspace of $\mathrm{G}(\mathcal{D}_{\varepsilon})$ into $\mathrm{G}(\partial)$, and therefore they map all of $\mathrm{G}(\mathcal{D}_{\varepsilon})$ into $\mathrm{G}(\partial)$. This proves 1).

For 2), consider the adjoint $\pi^p_{\mathcal{D}}(\chi_k)^*$, which by 1) maps $\mathrm{G}(\partial)^\perp$ into $\mathrm{G}(\mathcal{D}_\varepsilon)^\perp$. The equalities $\mathrm{G}(\partial)^\perp = U\mathrm{G}(\partial^*)$ and $\mathrm{G}(\mathcal{D}_\varepsilon)^\perp = U\mathrm{G}(\mathcal{D}_\varepsilon)$ allow us to compute, for $x \in \mathrm{Dom}\,\partial^*$

$$\begin{aligned}\pi^p_{\mathcal{D}}(\chi_k)^*\begin{pmatrix}-\partial^* x\\ x\end{pmatrix} &= \begin{pmatrix}v\chi_k v^* & -[\mathcal{D}_\varepsilon, v\chi_k v^*]p\\ 0 & v\chi_k v^*\end{pmatrix}\begin{pmatrix}-\partial^* x\\ x\end{pmatrix}\\ &= \begin{pmatrix}-v\chi_k^*\partial^* x - [\mathcal{D}_\varepsilon, v\chi_k v^*]x\\ v\chi_k v^* x\end{pmatrix} \in U\mathrm{G}(\mathcal{D}_\varepsilon).\end{aligned}$$

Hence $v\chi_k v^* x \in \mathrm{Dom}\,\mathcal{D}_\varepsilon$ whenever $x \in \mathrm{Dom}\,\partial^*$ which proves 2).

For 3) it suffices to show that $v\chi_n v^*$ maps $\mathrm{Dom}\,\partial^*$ into $\mathrm{Dom}\,\partial$ and then use 2). Let $x \in \mathrm{Dom}\,\partial^*$ and, since $\mathcal{E}_\mathcal{B}$ is complete, let $u_k \in \mathscr{C}(\chi_n)$ be the approximate unit from Definition 3.9. By 2) $v\chi_n v^* x \in \mathrm{Dom}\,\mathcal{D}_\varepsilon$ and by 1) $vu_k v^* v\chi_n v^* x \in \mathrm{Dom}\,\partial$. We have $\lim_k vu_n v^* v\chi_n v^* x = v\chi_n v^* x$ in norm in $\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F_C$. Now the operator $p[\mathcal{D}_\varepsilon, vu_k v^*]p$ is defined on the dense subspace $v\mathcal{E}\otimes_{\mathcal{B}^+}\mathrm{Dom}\,\mathcal{D}$, and bounded there. Hence it extends to a bounded operator on the whole module $\mathcal{H}_{\mathcal{B}^+}\tilde{\otimes}_{\mathcal{B}^+}F_C$. The relation

$$(\partial vu_k v^* - vu_k v^*\mathcal{D}_\varepsilon)p = p[\mathcal{D}_\varepsilon, vu_k v^*]p, \tag{3.22}$$

which is valid on the subspace $\mathrm{Dom}\,\partial \cap \mathrm{Dom}\,\mathcal{D}_\varepsilon \cap p\mathcal{H}_{\mathcal{B}^+}\tilde{\otimes}_{\mathcal{B}^+}F_C$, along with the boundedness of $p[\mathcal{D}_\varepsilon, vu_k v^*]p$, imply that the left hand side of Equation (3.22) is bounded. Combining all these facts with the strict convergence $p[\mathcal{D}_\varepsilon, vu_k v^*]p \to 0$, we find that

$$\begin{aligned}\lim_k \partial vu_k v^* v\chi_n v^* x &= \lim_k vu_k v^*\mathcal{D}_\varepsilon v\chi_n v^* x + \partial vu_k v^* v\chi_n v^* - vu_k v^*\mathcal{D}_\varepsilon v\chi_n v^* x\\ &= \lim_k vu_k v^* p\mathcal{D}_\varepsilon p v\chi_n v^* x + p[\mathcal{D}_\varepsilon, vu_k v^*]pv\chi_n v^* x\end{aligned}$$

which since $vu_k v^* \to p$ strictly, shows that the sequence converges. Since ∂ is closed, $v\chi_n v^* x \in \mathrm{Dom}\,\partial$. □

The paper [31] introduces a local-global principle for regular operators on C^*-modules, though this had been independently developed by Pierrot in [45]. The main technical tool developed is the following. Let E_B be a C^*-module and $\sigma : B \to \mathbb{C}$ a state and $\mathcal{H}_\sigma = L^2(B, \sigma)$ the associated GNS representation. The *localisation* E^σ is the Hilbert space completion of E_B in the inner product $\langle e, f\rangle_\sigma := \sigma(\langle e, f\rangle)$, and there is a dense inclusion $\iota_\sigma : E_B \to E^\sigma$ and a $*$-representation $\pi_\sigma : \mathrm{End}^*_B(E_B) \to \mathbb{B}(E^\sigma)$. Equivalently, $E^\sigma = E\otimes_B L^2(B, \sigma)$, where $L^2(B, \sigma)$ denotes the GNS representation space of B defined by the state σ. A closed, densely defined symmetric operator T on E induces a closed densely defined symmetric operator T^σ in E^σ, by defining it on the dense subspace $\iota_\sigma(\mathrm{Dom}\,T) \subset E^\sigma$ and taking the closure. It then holds that $\iota_\sigma(\mathrm{Dom}\,T^*) \subset \mathrm{Dom}(T^\sigma)^*$, cf. [31, Lemma 2.5].

Theorem 3.16 (Theorem 4.2, [31], Théorème 1.18 [45])**.** *Let T be a closed densely defined symmetric operator in the C^*-module E_B. Then T is self-adjoint and regular if and only if all localisations T^σ are self-adjoint.*

For an unbounded Kasparov module $(\mathcal{B}, F_C, \mathcal{D})$ and a state $\sigma : C \to \mathbb{C}$ we obtain a contractive map $\mathcal{B} \to \mathrm{Lip}(\mathcal{D}^\sigma)$. This follows because by definition $\iota_\sigma(\mathrm{Dom}\,\mathcal{D})$ is a core for $\mathcal{D}^\sigma$ and for all $b \in \mathcal{B}$ and $f \in \mathrm{Dom}\,\mathcal{D}$ we have $\pi_\sigma(b)\iota_\sigma(f) = \iota_\sigma(bf) \in \iota_\sigma(\mathrm{Dom}\,\mathcal{D})$. Thus $\pi_\sigma(b)$ preserves the core $\iota_\sigma(\mathcal{D})$ for $\mathcal{D}^\sigma$. The commutator satisfies

$$\begin{aligned}\|[\mathcal{D}^\sigma, \pi_\sigma(b)]\iota_\sigma(f)\|^2 = \|\iota_\sigma([\mathcal{D}, b]f)\|^2 = \sigma(\langle[\mathcal{D}, b]f, [\mathcal{D}, b]f\rangle) &\leq \|[\mathcal{D}, b]\|^2\sigma(\langle f, f\rangle)\\ &= \|[\mathcal{D}, b]\|^2\|\iota_\sigma(f)\|^2,\end{aligned} \tag{3.23}$$

and is thus bounded on this core. Thus $[\mathcal{D}^\sigma, \pi_\sigma(b)] = \pi_\sigma([\mathcal{D}, b])$ and we can write $\pi_{\mathcal{D}^\sigma}(b) = \pi_\sigma(\pi_{\mathcal{D}}(b))$ and hence the map $\pi_{\mathcal{D}}(b) \mapsto \pi_{\mathcal{D}^\sigma}(b)$ is completely bounded. We let $\mathcal{B}^\sigma$ be the completion of $\mathcal{B}$ is the norm induced by $\pi_{\mathcal{D}^\sigma}$, and define the localised module $\mathcal{E}_{\mathcal{B}^\sigma}$ over $\mathcal{B}^\sigma$ via the map $\mathcal{H}_{\mathcal{B}+} \to \mathcal{H}_{\mathcal{B}^\sigma+}$.

Lemma 3.17. *Let $\mathcal{E}_{\mathcal{B}}$ be a complete projective operator module for $(\mathcal{B}, F_C, \mathcal{D})$ with column finite frame (x_i) and frame approximate unit χ_n. Then for all states $\sigma : C \to C$, the localised module $\mathcal{E}_{\mathcal{B}^\sigma}$ is a complete projective module for $(\mathcal{B}^\sigma, F^\sigma, \mathcal{D}^\sigma)$. Moreover, under the identification $E\tilde{\otimes}_B F^\sigma \cong (E\tilde{\otimes}_B F)^\sigma$ we have $1 \otimes_\nabla \mathcal{D}^\sigma = (1 \otimes_\nabla \mathcal{D})^\sigma$ as unbounded operators. Therefore*
1) for each n, the operator $v\pi_\sigma(\chi_n)v^*$ maps $\mathrm{Dom}(\partial^\sigma)^*$ into $\mathrm{Dom}\,\partial^\sigma$;
2) there is an approximate unit $(u_n) \subset \mathscr{C}(\chi_n)$ such that $p[\mathcal{D}^\sigma_\varepsilon, v\pi_\sigma(u_n)v^*]p$ converges to 0 $*$-strongly on $\mathcal{H}_{B+}\tilde{\otimes}_{B+}F^\sigma$.

Proof. To check that $\mathcal{E}_{\mathcal{B}^\sigma}$ is a complete projective module, it suffices to show that the defining frame (x_i) of $\mathcal{E}_{\mathcal{B}}$ is column finite for $\mathcal{B}^\sigma$, and that there exists an approximate unit $(u_n) \in \mathscr{C}(\chi_n)$ such that $p[D^\sigma_\varepsilon, vu_nv^*]p \to 0$ $*$-strongly on $\mathcal{H}_{B+}\tilde{\otimes}_{B+}F^\sigma$. Column finiteness follows from complete boundedness of the map $\pi_{\mathcal{D}}(b) \mapsto \pi_{\mathcal{D}^\sigma}(b)$, proved after (3.23) above. This is because complete boundedness shows that for all $e \in \mathcal{E}_{\mathcal{B}}$

$$\|\pi_{\mathcal{D}^\sigma}(\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}}\| \leq \|\|\pi_{\mathcal{D}}(\langle x_i, e\rangle)_{i\in\hat{\mathbb{Z}}}\|,$$

and so in particular for all the vectors x_j. Definition 3.9 gives an approximate unit (u_n) in the convex set $\mathscr{C}(\chi_n) \subset \mathrm{Fin}_{\mathcal{B}}(\mathcal{E})$ for which the sequence $p[\mathcal{D}^\sigma_\varepsilon, vu_nv^*]p$ converges to 0 strictly on $\mathcal{H}_{B+}\tilde{\otimes}_{B+}F$ and is therefore uniformly bounded in n. Thus the localised sequence $p[\mathcal{D}^\sigma_\varepsilon, \pi_\sigma(u_n)]p = \pi_\sigma(p[\mathcal{D}_\varepsilon, u_n]p)$ is bounded as well and converges strongly to 0 on the dense subspace $\mathcal{H}_{B+}\tilde{\otimes}\iota_\sigma(F)$ and thus on all of $\mathcal{H}_{B+}\tilde{\otimes}F^\sigma$. Hence $\mathcal{E}_{\mathcal{B}^\sigma}$ is a complete projective module for $(\mathcal{B}^\sigma, F^\sigma, \mathcal{D}^\sigma)$, which in particular proves 2).

The operator $1 \otimes_\nabla \mathcal{D}^\sigma$ is defined on its core $\mathcal{E} \otimes_{\mathcal{B}+} \mathrm{Dom}\,\mathcal{D}^\sigma$ while $(1 \otimes_\nabla \mathcal{D})^\sigma$ is defined on $\iota_\sigma(\mathrm{Dom}\,1 \otimes_\nabla D)$. We claim that the subspace

$$X := \iota_\sigma(\mathcal{E} \otimes_{\mathcal{B}} \mathrm{Dom}\,\mathcal{D}) = \mathcal{E} \otimes_{\mathcal{B}} \iota_\sigma(\mathrm{Dom}\,\mathcal{D}) \subset \iota_\sigma(\mathrm{Dom}\,1 \otimes_\nabla D),$$

is a common core for $(1 \otimes_\nabla \mathcal{D})^\sigma$ and $1 \otimes_\nabla \mathcal{D}^\sigma$. Since $\mathcal{E} \otimes_{\mathcal{B}} \mathrm{Dom}\,\mathcal{D}$ is a core for $1 \otimes_\nabla \mathcal{D}$, its image under ι_σ is a core for $(1 \otimes_\nabla \mathcal{D})^\sigma$. To see that it is also a core for $1 \otimes_\nabla \mathcal{D}^\sigma$, we use the definition

$$1 \otimes_\nabla \mathcal{D}^\sigma(e \otimes f) = \gamma(e) \otimes \mathcal{D}f + \nabla_{\mathcal{D}}(e)f = \gamma(e) \otimes \mathcal{D}^\sigma f + \sum \gamma(x_i) \otimes [\mathcal{D}^\sigma, \langle x_i, e\rangle]f,$$

and take a sequence $f_k \in \mathrm{Dom}\,\mathcal{D}$ converging to $f \in \mathrm{Dom}\,\mathcal{D}^\sigma$ in the graph norm. The term $\gamma(e) \otimes \mathcal{D}^\sigma f_k$ will then converge to $\gamma(e) \otimes \mathcal{D}^\sigma f$. The other term can be estimated using the Haagerup norm

$$\begin{aligned}\Big\|\sum \gamma(x_i) \otimes [\mathcal{D}, \langle x_i, e\rangle](f_k - f_\ell)\Big\|_h^2 &\leq \Big\|\sum |x_i\rangle\langle x_i|\Big\|_{\mathbb{K}(E)} \big\|([\mathcal{D}, \langle x_i, e\rangle](f_k - f_\ell))_{i\in\hat{\mathbb{Z}}}\big\|^2 \\ &\leq \big\|([\mathcal{D}, \langle x_i, e\rangle])_{i\in\hat{\mathbb{Z}}}\big\|^2 \|f_k - f_\ell\|^2,\end{aligned}$$

and the norm of the column $([\mathcal{D}, \langle x_i, e\rangle])_{i\in\hat{\mathbb{Z}}}$ is finite because $e \in \mathcal{E}_{\mathcal{B}}$. Therefore we can approximate any $y \in \mathcal{E} \otimes_{\mathcal{B}} \mathrm{Dom}\,\mathcal{D}^\sigma$ by elements of X in the graph norm of $1 \otimes_\nabla \mathcal{D}^\sigma$. Thus the closure of $1 \otimes_\nabla \mathcal{D}^\sigma$ on X contains $\mathcal{E} \otimes_{\mathcal{B}} \mathrm{Dom}\,\mathcal{D}^\sigma$ which is the defining core for $1 \otimes_\nabla \mathcal{D}^\sigma$. Therefore X

is a common core and since the operators $(1\otimes_\nabla \mathcal{D})^\sigma$ and $1\otimes_\nabla \mathcal{D}^\sigma$ coincide on X, it follows that $(1\otimes_\nabla \mathcal{D})^\sigma = 1\otimes_\nabla \mathcal{D}^\sigma$.

Statement 1) now follows by applying Lemma 3.15 to the frame (x_i) of the complete projective module $\mathcal{E}_{\mathcal{B}^\sigma}$. $\square$

We now come to the main application of complete projective modules: self-adjointness of the connection operator $1\otimes_\nabla \mathcal{D}$. A further application of the domain mapping properties of approximate units then allows us to show that $\mathbb{K}(\mathcal{E}^\nabla)$ is a differentiable algebra.

Theorem 3.18. *Let $\mathcal{E}_\mathcal{B}$ be a complete projective module for $(\mathcal{B}, F_C, \mathcal{D})$. Then the operator $1\otimes_\nabla \mathcal{D}$ is self-adjoint and regular.*

Proof. We must show that for all states $\sigma : C \to \mathbb{C}$ the operator ∂^σ on the Hilbert space $(E\tilde{\otimes}_B F)^\sigma \cong E\tilde{\otimes}_B F^\sigma$ is self-adjoint. Let $(u_n)\subset \mathscr{C}(\chi_n)$ be an approximate unit as in Definition 3.9. By Lemma 3.17, $v\pi_\sigma(u_n)v^*$ maps $\mathrm{Dom}(\partial^\sigma)^*$ into $\mathrm{Dom}\,\partial^\sigma$ and $p[\mathcal{D}_\varepsilon^\sigma, v\pi_\sigma(u_n)v^*]p$ converges to 0 $*$-strongly on $\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F^\sigma$. Using Lemma 3.14 we have that $\partial^\sigma x = p\mathcal{D}_\varepsilon p x$ for x in the dense subspace $\mathcal{H}_{\mathcal{B}^+}\otimes \mathrm{Dom}\,\mathcal{D}^\sigma$, which is a core for ∂^σ. Thus, suppressing π_σ in the notation, we have

$$[\partial^\sigma, vu_kv^*]x = \partial^\sigma vu_kv^*x - vu_kv^*\partial^\sigma x = p\mathcal{D}_\varepsilon^\sigma p vu_kv^*x - vu_kv^*p\mathcal{D}_\varepsilon^\sigma p x = p[\mathcal{D}_\varepsilon^\sigma, vu_kv^*]px \to 0, \tag{3.24}$$

in norm, and by uniform boundedness of $p[\mathcal{D}_\varepsilon, vu_kv^*]p$, the convergence holds for all $x\in \mathcal{H}_{\mathcal{B}^+}\tilde{\otimes} F^\sigma$. Since there is an equality of closures

$$\overline{[(\partial^\sigma)^*, vu_kv^*]} = \overline{[\partial^\sigma, vu_kv^*]},$$

and the latter operator is bounded, it follows that $[(\partial^\sigma)^*, vu_kv^*]\to 0$ strictly on $\mathcal{H}_{\mathcal{B}^+}\tilde{\otimes}_{\mathcal{B}^+}F$. Therefore if $y\in \mathrm{Dom}(\partial^\sigma)^*$ then $vu_kv^*y\in\mathrm{Dom}\,\partial^\sigma$ by Lemma 3.17, and $vu_kv^*y\to y$. Whence by (3.24) we can compute

$$(\partial^\sigma)^*y = \lim vu_kv^*(\partial^\sigma)^*y = \lim \partial^\sigma vu_kv^*y - [(\partial^\sigma)^*, vu_kv^*]y = \lim \partial^\sigma vu_kv^*y,$$

and it follows that $\partial^\sigma vu_kv^*y$ is convergent to $(\partial^\sigma)^*y$ since $vu_kv^*(\partial^\sigma)^*y$ is. So $\mathrm{Dom}\,\partial^\sigma$ is a core for $(\partial^\sigma)^*$, and as ∂^σ is a closed symmetric operator, it is self-adjoint. The local-global principle of [31, 45] now says that ∂ is self-adjoint and regular. $\square$

We now describe the algebra of adjointable operators on a complete projective module. Let the isometry of C^*-modules $v: E\to\mathcal{H}_{B^+}$ be such that it induces a column finite frame $(x_i)_{i\in\hat{\mathbb{Z}}}$, which in turn determines a complete projective submodule $\mathcal{E}_\mathcal{B}\subset\mathcal{E}^\nabla\subset E_B$. The defining representation

$$\pi_\nabla(K) = \begin{pmatrix} vKv^* & 0\\ p[\mathcal{D}_\varepsilon, vKv^*]p & vKv^*\end{pmatrix},$$

preserves the submodule $(p\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F)\oplus(p\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F)$, and annihilates the orthogonal complement.

Definition 3.19. The *algebra of adjointable operators* on the module $\mathcal{E}^\nabla$ is the idealiser of $\pi_\nabla(\mathbb{K}(\mathcal{E}^\nabla))$ in $\mathrm{End}^*_C((p\mathcal{H}_{B^+}\tilde{\otimes}_{B^+}F)^{\oplus 2})$. It is denoted $\mathrm{End}^*_\mathcal{B}(\mathcal{E}^\nabla)$.

Proposition 3.20. *If $\mathcal{E}_{\mathcal{B}}$ is a complete projective module then* $\mathrm{End}^*_{\mathcal{B}}(\mathcal{E}^{\nabla})$ *is an operator $*$-algebra, isometrically isomorphic to* $\mathbb{M}(\mathbb{K}(\mathcal{E}^{\nabla}))$ *and coinciding with a closed subalgebra of* $\mathrm{Lip}(1 \otimes_{\nabla} \mathcal{D})$*. Hence* $\mathbb{K}(\mathcal{E}^{\nabla})$ *is a differentiable algebra.*

Proof. This essentially follows from Propositions 1.9 and 1.17. Since $1 \otimes_{\nabla} \mathcal{D}$ is self-adjoint and regular the commutators $[1 \otimes_{\nabla} \mathcal{D}, vKv^*]$ coincide with the operators $p[\mathcal{D}_{\varepsilon}, vKv^*]p$ when K is finite rank. Suppose now that $T \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{E}^{\nabla})$, so there is a sequence of operators T_n such that for all $K \in \mathrm{Fin}_{\mathcal{B}}(\mathcal{E})$ we have $T_nK \in \mathrm{Fin}_{\mathcal{B}}(\mathcal{E})$ and both T_nK and $p[\mathcal{D}_{\varepsilon}, vT_nKv^*]p$ are convergent. Then since

$$p[\mathcal{D}_{\varepsilon}, vT_nKv^*]p = [\partial, vT_nKv^*],$$

it follows that

$$\partial(vT_nKv^*x) = [\partial, vT_nKv^*]x + vT_nKv^*\partial(x),$$

is convergent for all $x \in \mathrm{Dom}\,\partial$. Thus TK preserves $\mathrm{Dom}\,\partial$ for all $K \in \mathrm{Fin}_{\mathcal{B}}(\mathcal{E})$ and $v\mathrm{Fin}_B(\mathcal{E})v^* \cdot \mathrm{Dom}\,\partial$ is dense in $p\,\mathrm{Dom}\,\partial$ in the graph norm by definition of $1 \otimes_{\nabla} \mathcal{D}$. Thus T preserves a core, and on this core the commutator

$$[\partial, vTv^*]vKv^*x = [\partial, vTKv^*]x - v\gamma(T)v^*[\partial, vKv^*]x,$$

is a bounded operator. Thus $vTv^* \in \mathrm{Lip}(\partial)$, which is equivalent to $T \in \mathrm{Lip}(1 \otimes_{\nabla} \mathcal{D})$ as desired. The argument now proceeds as in Proposition 1.17. □

4 Completeness and the Kasparov product

The constructive approach to the Kasparov product has appeared in several slightly different versions in recent years, [13, 32, 42]. The variations have come from the assumptions imposed on the correspondences $(\mathcal{A}, \mathcal{E}_{\mathcal{B}}, S, \nabla)$ which refine the notion of unbounded Kasparov module. The most recent refinement in [13] was the inclusion of a class of unbounded projections into the theory, required to deal with examples arising from the Hopf fibration. Unbounded projections also appear in the construction of products for Cuntz-Krieger algebras [25], the natural Kasparov module for $SU_q(2)$ [35, 48] and the differential approach to the stabilisation theorem [30]. In the previous section of the present paper, the notion of complete projective module enlarges the class of unbounded projections we can work with.

4.1 Constructing the unbounded Kasparov product

In this section we will show that the lifting constructions of [4, 37] can be refined in such a way that we can lift a pair of cycles (A, E_B, F_1) and (B, F_C, F_2) to an unbounded Kasparov module $(\mathcal{B}, F_C, T)$ and a correspondence $(\mathcal{A}, E_{\mathcal{B}}, S, \nabla)$ for $(\mathcal{B}, F_C, T)$. This has the advantage that their Kasparov product as constructed through Theorem 4.4 is then well-defined. Since we only have to lift two classes, we provide a significant improvement over the results of [37], where it was shown that any three KK-classes, with one the product of the other two, can be lifted to unbounded classes in a way compatible with Kucerovsky's conditions for representing products [36].

Our first task is to assemble the results in the literature and blend them with the present work in order to give sufficient conditions under which the unbounded Kasparov product can be constructed. These conditions will allow us to show in Section 4.4 that any Kasparov product can be realised as the composition of a correspondence and an unbounded Kasparov module.

Definition 4.1. Given $(\mathcal{B}, F_C, T)$ an unbounded Kasparov module with bounded approximate unit for $\mathcal{B}$, an $\mathcal{A}$-$\mathcal{B}$ *correspondence* for $(\mathcal{B}, F_C, T)$ is a quadruple $(\mathcal{A}, \mathcal{E}_{\mathcal{B}}, S, \nabla)$ such that:
1) $\mathcal{E}_{\mathcal{B}}$ is a complete projective operator module over the algebra $\mathcal{B}$;
2) $\mathcal{A}$ is a $*$-algebra and $\mathcal{A} \subset \mathrm{End}^*_{\mathcal{B}}(\mathcal{E}^\nabla) \cap \mathrm{Lip}(S)$;
3) $S : \mathrm{Dom}\, S \to E$ is a self-adjoint regular operator such that $(S \pm i)^{-1} \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{E}^\nabla)$ and $a(S \pm i)^{-1} \in \mathbb{K}_{\mathcal{B}}(\mathcal{E}^\nabla)$ for $a \in \mathcal{A}$;
4) $\nabla : \mathcal{E}^\nabla \to E\tilde{\otimes}_B \Omega^1_{\mathcal{D}}$ is a connection such that $\nabla((S \pm i)^{-1}\mathcal{E}^\nabla) \subset \mathrm{Dom}\, S \otimes 1$ and the operator $[\nabla, S](S \pm i)^{-1} : \mathcal{E}^\nabla \to E\tilde{\otimes}_{\mathcal{B}} \Omega^1_T$ is completely bounded.

The correspondence is called *strongly complete* if there is an approximate unit (u_n) for the C^*-closure A of $\mathcal{A}$ such that both $[S, u_n] \to 0$ and $[1 \otimes_\nabla T, u_n \otimes \mathrm{Id}_F] \to 0$ in C^*-norm.

In condition 4), we regard ∇ as an odd operator so the commutator is the graded commutator $[\nabla, S] = \nabla S - \gamma(S)\nabla$. One of the key points in the construction of the Kasparov product is the self-adjointness of the product operator, and this is deduced from the general framework of weakly anti-commuting operators described in the appendix.

Lemma 4.2. *Let $(\mathcal{A}, \mathcal{E}_{\mathcal{B}}, S, \nabla)$ be an $\mathcal{A}$-$\mathcal{B}$ correspondence for $(\mathcal{B}, F_C, T)$. The self-adjoint regular operators $s := S \otimes 1$ and $t := 1 \otimes_\nabla T$ weakly anticommute in $E\tilde{\otimes}_B F$.*

Proof. We will show that the conditions of Definition 4.1 imply those of Definition A.1. By Lemma 3.14 the map

$$g : \mathcal{E}^\nabla \tilde{\otimes}_{\mathcal{B}} \mathrm{G}(T) \to \mathrm{G}(1 \otimes_\nabla T), \quad e \otimes \begin{pmatrix} f \\ Tf \end{pmatrix} \mapsto \begin{pmatrix} e \otimes f \\ 1 \otimes_\nabla T(e \otimes f) \end{pmatrix},$$

has dense range. This means that the submodule $X := \mathcal{E}^\nabla \otimes_{\mathcal{B}} \mathrm{Dom}\, T$, is a core for $1 \otimes_\nabla T$. Since $(S \pm i)^{-1} : \mathcal{E}^\nabla \to \mathcal{E}^\nabla$, the resolvents $(s \pm i)^{-1}$ preserve the core X, so 1) of Definition A.1 is satisfied. By condition 4) of Definition 4.1 it follows that $t(s \pm i)^{-1} X \subset \mathrm{Dom}\, s$, so 2) of Definition A.1 is satisfied as well. On the core X the graded commutator can be computed as

$$\begin{aligned} [t, (s \pm i)^{-1}] = t(s \pm i)^{-1} + (s \mp i)^{-1} t &= (s \mp i)^{-1}(t(s \pm i) + (s \mp i)t)(s \pm i)^{-1} \\ &= (s \mp i)^{-1}[s, t](s \pm i)^{-1}, \end{aligned}$$

and this is a bounded operator because

$$[s, t](e \otimes f) = (S \otimes 1)(\gamma(e) \otimes Tf + \nabla(e)f) + \gamma(Se) \otimes Tf + \nabla(Se) \otimes f = [\nabla, S](e)f.$$

Thus, $(s \pm i)^{-1}$ preserve the domain of t and $[s, t](s \pm i)^{-1}$ are bounded there, proving condition 3) of Definition A.1. □

Lemma 4.3. *Let $(\mathcal{A}, \mathcal{E}_{\mathcal{B}}, S, \nabla)$ be an $\mathcal{A}$-$\mathcal{B}$ correspondence for $(\mathcal{B}, F_C, T)$. For any $K \in \mathbb{K}(E) \otimes 1$ and $t := 1 \otimes_\nabla T$, the operators $(t \pm i)^{-1} K$ and $K(t \pm i)^{-1}$ are compact in $E\tilde{\otimes}_B F$.*

Proof. For any $e \in \mathcal{E}$ we have the norm convergent series $\sum_{i \in \hat{\mathbb{Z}}} [T_\varepsilon, \langle x_i, e\rangle]^* [T_\varepsilon, \langle x_i, e\rangle]$, which implies that the operator

$$\begin{aligned} t|e\rangle - |\gamma(e)\rangle T : f \to \sum_{i \in \hat{\mathbb{Z}}} x_i \otimes T\langle x_i, e\rangle - \gamma(e) \otimes Tf &= \sum x_i \otimes (T\langle x_i, e\rangle - \langle x_i, \gamma(e)\rangle T) f \qquad (4.1) \\ &= \sum_i x_i \otimes [T_\varepsilon, \langle x_i, e\rangle] f, \end{aligned}$$

is bounded. Moreover, for any $e \in E$ the operator $|e\rangle(T \pm i)^{-1} : F \to E\tilde{\otimes}_B F$, is compact for if u_n is an approximate unit for the C^*-algebra B then $|eu_n\rangle \to |e\rangle$ in norm and thus

$$|e\rangle(T \pm i)^{-1} = \lim_{n\to\infty} |eu_n\rangle(T \pm i)^{-1} = |e\rangle u_n(T \pm i)^{-1},$$

is a norm limit of compact operators, whence compact. Therefore, by (4.1)

$$(t \pm i)^{-1}|e_1\rangle\langle e_2| = |\gamma(e_1)\rangle(T \pm i)^{-1}\langle e_2| + (t \pm i)^{-1}(|e_1\rangle T - t|\gamma(e_1)\rangle \pm i|e_1 - \gamma(e_1)\rangle)(T \pm i)^{-1}\langle e_2|,$$

is a compact operator. Hence $(t \pm i)^{-1}K$ is compact for all $K \in \mathbb{K}(E) \otimes 1$. □

The following theorem encompasses and generalises the constructions of the unbounded Kasparov product that have appeared in [13, 32, 42].

Theorem 4.4. *Let $(\mathcal{B}, F_C, T)$ be an unbounded Kasparov module and let $(\mathcal{A}, \mathcal{E}_{\mathcal{B}}, S, \nabla)$ be an A-B correspondence for $(\mathcal{B}, F_C, T)$. Then $(\mathcal{A}, (E\tilde{\otimes}_B F)_C, S \otimes 1 + 1 \otimes_\nabla T)$ is an unbounded Kasparov module representing the Kasparov product of $(\mathcal{A}, E_B, S)$ and $(\mathcal{B}, F_C, T)$.*

Proof. The operator $1 \otimes_\nabla T$ is self-adjoint and regular in $E\tilde{\otimes}_B F$ by Theorem 3.18, as is the operator $S \otimes 1$. Now $S \otimes 1$ and $1 \otimes_\nabla T$ weakly anticommute by Lemma 4.2, and hence their sum is self-adjoint and regular in $E\tilde{\otimes}_B F$ by Theorem A.4. Lemma 4.3 replaces [32, Proposition 6.6] so that the argument of [32, Theorem 6.7] shows that $S \otimes 1 + 1 \otimes_\nabla T$ has locally compact resolvent. Thus $(\mathcal{A}, (E\tilde{\otimes}_B F)_C, S \otimes 1 + 1 \otimes_\nabla T)$ is an unbounded Kasparov module. One then shows, exactly as in [32, Theorem 7.2] and [42, Theorem 6.3.4], that the hypotheses of Kucerovsky's theorem, [36, Theorem 13], are satisfied. Hence this cycle represents the Kasparov product. □

Now we embark on a series of lifting results of increasing sophistication, whose ultimate aim is to show that any pair of composable KK-classes can be represented by unbounded Kasparov modules satisfying the hypotheses of Theorem 4.4. Recall that for a bounded (A, B)-Kasparov module (A, E_B, F) the associated ideal of A-*locally compact operators* is

$$J_A(E_B) := \{T \in \mathrm{End}^*_B(E_B) : aT,\, Ta \in \mathbb{K}(E_B) \text{ for all } a \in A\}.$$

The operator F is in the idealiser of $J_A(E_B)$, for if $T \in J_A(E_B)$ then $FTa \in \mathbb{K}(E_B)$ since $Ta \in \mathbb{K}(E_B)$ and $aFT = FaT - [F, a]T \in \mathbb{K}(E_B)$ as well. Moreover, $1 - F^2$ and hence $(1 - F^2)^{\frac{1}{2}}$ are both elements of $J_A(E_B)$. The C^*-algebra $J_A(E_B)$ is not σ-unital in general. The following counterexample to σ-unitality arose from discussions of the first author with J. Kaad: Let I be an ideal in a unital C^*-algebra B. Take $E := C_0(\mathbb{N}, I)$ viewed as a C^*-module over $C_0(\mathbb{N}, I)$ and let $A := C_0(\mathbb{N}, B)$. Then $J_A(E) = C_b(\mathbb{N}, I)$ which is not σ-unital.

Lemma 4.5. *For a Kasparov module (A, E_B, F) with $F^* = F$, define*

$$J_F := \mathbb{K}(E_B) + C^*(1 - F^2) + FC^*(1 - F^2).$$

Then J_F is a separable C^*-subalgebra of $J_A(E_B)$ containing $\mathbb{K}(E_B)$ as an ideal and with the property that FJ_F, J_FF, AJ_F, $J_FA \subset J_F$.

Proof. The space $C^*(1-F^2)+FC^*(1-F^2)$ is a commutative separable C^*-algebra. First, it is a separable linear space. Second, because the operator F commutes with $C^*(1-F^2)$, it is $*$-closed. Finally, for a, b, c, $d \in C^*(1-F^2)$, we have

$$(a+Fb)(c+Fd) = ac + F(ad+bc) + F^2 bd = ac + F(ad+bc) + (F^2-1)bd + bd,$$

which shows that $C^*(1-F^2)+FC^*(1-F^2)$ is C^*-algebra. It thus follows by [33, Section 3, Lemma 2] that J_F is a separable C^*-algebra containing $\mathbb{K}(E_B)$ as an ideal. Moreover, since $F\mathbb{K}(E_B) \subset \mathbb{K}(E_B)$ and $F^2C^*(1-F^2) \subset C^*(1-F^2)$, just as for F above, it follows that FJ_F and hence also $J_F F$ are in J_F. That AJ_F, $J_F A \subset J_F$ is immediate. □

In [4] it was shown that any bounded Kasparov module (A, E_B, F) can be represented by an unbounded Kasparov module $(A, E_B, \mathcal{D})$. The operator $\mathcal{D}$ is obtained from F by constructing a suitable strictly positive ℓ element in the ideal $J_A(E_B)$ and then setting $\mathcal{D} := F\ell^{-1}$. The element ℓ is constructed from an approximate unit for J_F with certain quasicentrality properties.

Definition 4.6. Let (A, E_B, F) be a Kasparov module with $[AE_B] = E_B$, $F = F^*$ and $1-F^2 \geq 0$. A strictly positive element $\ell \in J_F$ is *admissible* if:
1) $F : \ell E \to \ell E$ and there exists $C > 0$ with $\pm i[F,\ell] \leq C\ell^2$;
2) $(1-F^2)^{\frac{1}{2}}\ell^{-1}$ is bounded on the range of ℓ and has norm $c < 1$;
3) there is a total subset $\{a_i\} \subset A$ for which $a_i : \ell E \to \ell E$, the commutators $[\ell^{-1}, a_i]$ and $[F, a_i]\ell^{-1}$ are bounded on the range of ℓ, and so extend to operators in $\mathrm{End}^*_B(E)$.

Theorem 4.7. *If $\ell \in J_F$ is admissible then $\mathcal{D} := \frac{1}{2}(F\ell^{-1} + \ell^{-1}F)$ is a self-adjoint regular operator with the property that $(\mathcal{A}, E_B, \mathcal{D})$ is an unbounded Kasparov module defining the same class as (A, E_B, F).*

Proof. Because F preserves the image of ℓ, the operator $\ell^{-1}F$ is defined on $\mathrm{Im}\,\ell$, and in particular $F\ell^{-1}$ has a densely defined adjoint. Moreover,

$$\langle \pm i[F,\ell^{-1}]\ell e, \ell e\rangle = \langle \pm i\ell^{-1}[F,\ell]e, \ell e\rangle = \langle \pm i[F,\ell^{-1}]e, e\rangle \leq C\langle \ell e, \ell e\rangle,$$

so $[F,\ell^{-1}]$ is bounded on $\mathrm{Im}\,\ell$. It is shown in [37, Lemmas 1.4, 2.2] that $F\ell^{-1}$ defines an almost self-adjoint regular operator on E_B with resolvent in J_F and that it has bounded commutators with all the a_i. Thus $(\mathcal{A}, E_B, \mathcal{D})$ is an unbounded Kasparov module and it suffices to show the equivalence of the Kasparov modules defined by F and $\tilde{\mathcal{D}}(1+\tilde{\mathcal{D}}^*\tilde{\mathcal{D}})^{-1/2}$, where $\tilde{\mathcal{D}} = F\ell^{-1}$. By [6, Proposition 17.2.7], it suffices to show that

$$a\big(F\tilde{\mathcal{D}}(1+\tilde{\mathcal{D}}^*\tilde{\mathcal{D}})^{-1/2} + (\tilde{\mathcal{D}}(1+\tilde{\mathcal{D}}^*\tilde{\mathcal{D}})^{-1/2}F\big)a^*$$

is positive modulo compacts for all $a \in A$. Simplifying yields

$$\begin{aligned}
&F\tilde{\mathcal{D}}(1+\tilde{\mathcal{D}}^*\tilde{\mathcal{D}})^{-1/2} + \tilde{\mathcal{D}}(1+\tilde{\mathcal{D}}^*\tilde{\mathcal{D}})^{-1/2}F \\
&= F^2\ell^{-1}(1+(F\ell^{-1})^*(F\ell^{-1}))^{-1/2} + F\ell^{-1}(1+(F\ell^{-1})^*(F\ell^{-1}))^{-1/2}F \\
&= F[F,\ell^{-1}](1+(F\ell^{-1})^*(F\ell^{-1}))^{-1/2} + F\ell^{-1}[F,(1+(F\ell^{-1})^*(F\ell^{-1}))^{-1/2}] \\
&\qquad + 2F\ell^{-1/2}[\ell^{-1/2},(1+(F\ell^{-1})^*F\ell^{-1})^{-1/2}]F + 2F\ell^{-1/2}(1+(F\ell^{-1})^*F\ell^{-1})^{-1/2}\ell^{-1/2}F
\end{aligned}$$

The first term on the right hand side of the last equality is compact when multiplied by any $a \in A$ on the right and the last term is positive, so we are left with the second and third terms. Now we compute the commutator in the second term using the integral formula for fractional powers, [14]. So

$$
\begin{aligned}
&\ell^{-1}[F,(1+(F\ell^{-1})^*(F\ell^{-1}))^{-1/2}] = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\ell^{-1}[F,(\lambda+1+(F\ell^{-1})^*(F\ell^{-1}))^{-1}]\,d\lambda \\
&= -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\ell^{-1}(\lambda+1+\ell^{-1}F^2\ell^{-1})^{-1}[F,(F\ell^{-1})^*(F\ell^{-1})](\lambda+1+\ell^{-1}F^2\ell^{-1})^{-1}d\lambda \\
&= -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\ell^{-1}(\lambda+1+\ell^{-1}F^2\ell^{-1})^{-1}[F,\ell^{-1}]F^2\ell^{-1}(\lambda+1+\ell^{-1}F^2\ell^{-1})^{-1}d\lambda \\
&-\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\ell^{-1}(\lambda+1+\ell^{-1}F^2\ell^{-1})^{-1}\ell^{-1}F^2[F,\ell^{-1}](\lambda+1+\ell^{-1}F^2\ell^{-1})^{-1}d\lambda\,.
\end{aligned}
$$

We observe that $\ell^{-1}(1-F^2)^{1/2}$ is bounded and of norm $c<1$ by Definition 4.6. It follows that $\ell^{-1}F^2\ell^{-1} = \ell^{-2}-\ell^{-1}(1-F^2)\ell^{-1} \geq \ell^{-2}-c^2 1$. The functional calculus then yields the estimates [15, Appendix A] (for the norm of endomorphisms on E_B)

$$
\|\ell^{-1}(1+\lambda+\ell^{-2}-c^2)^{-1}\| \leq \frac{1}{2\sqrt{1+\lambda-c^2}}, \quad \|(1+\lambda+\ell^{-2}-c^2)^{-1}\| \leq \frac{1}{1+\lambda-c^2}.
$$

Thus the integral converges in norm. Since multiplying the integrand on right and left by an element of A yields a compact endomorphism, the same is true of the integral. For the third term the integral formula yields

$$
\begin{aligned}
&\ell^{-1/2}[\ell^{-1/2},(1+\ell^{-1}F^2\ell^{-1})^{-1/2}] \\
&= -\ell^{-1/2}\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda+\ell^{-1}F^2\ell^{-1})^{-1}\ell^{-1}[\ell^{-1/2},F^2]\ell^{-1}(1+\ell^{-1}F^2\ell^{-1}+\lambda)^{-1}\,d\lambda.
\end{aligned}
$$

In order to obtain the norm convergence of this integral, we write

$$
\ell^{-1}[\ell^{-1/2},F^2]\ell^{-1} = \ell^{-1/2}(\ell^{-1}(F^2-1)\ell^{-1}) - (\ell^{-1}(F^2-1)\ell^{-1})\ell^{-1/2}
$$

and

$$
\ell^{-1/2}(1+\lambda+\ell^{-1}F^2\ell^{-1})^{-1} = \ell^{-1/2}(1+\lambda+\ell^{-1}F^2\ell^{-1})^{-1}\ell^{-1/2}\ell^{1/2}.
$$

Since $\ell^{-1/2}(1+\lambda+\ell^{-1}F^2\ell^{-1})^{-1}\ell^{-1/2} \leq \ell^{-1}(1+\lambda+\ell^{-2}-c^2)^{-1}$, the same norm estimates we used for the second term give us

$$
\begin{aligned}
&\|\ell^{-1/2}[\ell^{-1/2},(1+\ell^{-1}F^2\ell^{-})^{-1/2}]\| \\
&\leq \frac{c^2}{2\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda-c^2)^{-3/2}d\lambda + \frac{c^2\|\ell^{1/2}\|}{4\pi}\int_0^\infty \lambda^{-1/2}(1+\lambda-c^2)^{-1}d\lambda < \infty
\end{aligned}
$$

and so the integral converges in norm. As multiplying the integrand on both sides by an element of A yields a compact endomorphism, the same is true of the operator defined by the integral. This completes the proof. □

4.2 Quasicentral approximate units

The construction of an admissible multiplier ℓ needed in Theorem 4.7 uses quasicentral approximate units as in [4]. In order to lift Kasparov modules to correspondences, this notion needs to be refined. The existence of quasicentral approximate units in C^*-algebras has been crucial for the development of KK-theory, notably in Higson's proof of the Kasparov technical theorem.

In this section $\mathcal{B}$ will always denote a unital operator algebra, and $\mathcal{J} \subset \mathcal{B}$ a closed ideal with bounded approximate unit.

For such $\mathcal{J}$ and $\mathcal{B}$, we wish to prove the existence of quasicentral approximate units. We will do this by using the argument of Akemann and Pedersen [1]. This method was employed in [2, Theorem 3.1] to construct quasicentral approximate units for closed ideals in operator algebras with *contractive* approximate units. By virtue of Proposition 1.7, the technique works for operator algebras with bounded approximate unit.

In Theorem 4.15 below, we prove a strong form of quasicentrality, unknown even in the case of C^*-algebras. Namely, we will view the ideal $\mathcal{J} \subset \mathcal{B}$ as sitting inside $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ as 'scalar' matrices $\mathcal{J} \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}}$. Although $\mathcal{J}$ is not an ideal inside $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$, we will see that $\mathcal{J}$ admits approximate units that are quasicentral inside $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$. That is $[u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}}, T] \to 0$ in norm for all $T \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$.

By an *ideal* in an operator algebra we will always mean a closed, two sided ideal. For an operator algebra $\mathcal{B}$ its *amplification* is

$$\mathcal{B}^\infty := \{(b_i)_{i\in\hat{\mathbb{Z}}} \in \prod_{i\in\hat{\mathbb{Z}}} \mathcal{B} : \sup_{i\in\hat{\mathbb{Z}}} \|b_i\| < \infty\},$$

which is canonically an operator algebra in the indicated norm.

For a general operator algebra $\mathcal{B}$, the module of infinite columns $\mathcal{H}_{\mathcal{B}}$ is paired with the module of infinite rows $\mathcal{H}^t_{\mathcal{B}}$ via

$$((b_i)^t_{i\in\hat{\mathbb{Z}}}, (c_i)_{i\in\hat{\mathbb{Z}}}) := \sum b_i c_i,$$

and $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ is defined to be the algebra of completely bounded operators $T : \mathcal{H}_{\mathcal{B}} \to \mathcal{H}_{\mathcal{B}}$ for which there exists $\tilde{T} : \mathcal{H}^t_{\mathcal{B}} \to \mathcal{H}^t_{\mathcal{B}}$ such that $(x, Ty) = (\tilde{T}x, y)$ for all $x \in \mathcal{H}^t_{\mathcal{B}}$, $y \in \mathcal{H}_{\mathcal{B}}$, cf. [8, Section 3]. For operator $*$-algebras, the spaces $\mathcal{H}^t_{\mathcal{B}}$ and $\mathcal{H}_{\mathcal{B}}$ are anti-isomorphic ([42, Lemma 4.4.1]) and this definition of $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ is equivalent to the one given earlier in (3.2).

We wish to describe $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ as an algebra of infinite matrices. Since $\mathcal{B}$ is unital, to an element $T \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ we can associate matrix coefficients (T_{ij}) by using the canonical basis e_i of $\mathcal{H}_{\mathcal{B}}$. For an infinite matrix $T := (T_{ij})_{i,j\in\hat{\mathbb{Z}}}$ the *N-truncation* is the finite $2N \times 2N$-matrix

$$T_N = (T_{ij})_N := (T_{ij})_{1\leq|i|,|j|\leq N}.$$

Lemma 4.8. *Let $T \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ and $\pi : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ be a completely isometric representation. Then the matrix coefficients $T_{ij} \in \mathcal{B}$ satisfy*

$$\sum_i \pi(T_{ij})^*\pi(T_{ij}) < \infty, \qquad \sum_j \pi(T_{ij})\pi(T_{ij})^* < \infty,$$

where these series are norm convergent in $\mathbb{B}(\mathcal{H})$*. For any* $(b_i)_{i\in\hat{\mathbb{Z}}} \in \mathcal{H}_{\mathcal{B}}$ *the series*

$$\sum_{j\in\hat{\mathbb{Z}}} \pi(T_{ij})\pi(b_j), \quad \sum_j \pi(T_{ij})^*\pi(b_j),$$

are norm convergent in $\mathbb{B}(\mathcal{H})$*.*

Proof. Using the basis vectors e_i and considering

$$T(e_j) = (T_{ij})_{i\in\hat{\mathbb{Z}}} \in \mathcal{H}_{\mathcal{B}}, \quad \tilde{T}(e_j)^t = (T_{ji})_{i\in\hat{\mathbb{Z}}} \in \mathcal{H}^t_{\mathcal{B}},$$

we obtain the stated conditions on the rows and columns of T. Considering the series $\sum_j T_{ij}b_j$, estimate the tails by

$$\begin{aligned}\Big\|\sum_{|j|\geq n} \pi(T_{ij}b_j)\Big\|^2 &= \Big\|\sum_{|j|,|k|\geq n} \pi(x_k)^*\pi(T_{ik})^*\pi(T_{ij})\pi(b_j)\Big\| \\ &= \big\|(\pi(b_k))^*_{|k|\geq n}\cdot(\pi(T_{ik})^*)_{|k|\geq n}\cdot(\pi(T_{ij}))^t_{|j|\geq n}\cdot(\pi(b_j))_{|j|\geq n}\big\| \\ &\leq \Big\|\sum \pi(x_j)^*\pi(x_j)\Big\|\,\big\|(\pi(T_{ik})^*)_{|k|\geq n}\cdot(\pi(T_{ij}))^t_{|j|\geq n}\big\| \\ &= \|x\|^2_{\mathcal{H}_{\mathcal{B}}}\,\Big\|\sum_{|j|\geq n}\pi(T_{ij})\pi(T_{ij})^*\Big\| \to 0,\end{aligned}$$

as $n\to\infty$ because the the rows $(T_{ij})_{j\in\hat{\mathbb{Z}}}$ are elements of $\mathcal{H}_{\mathcal{B}}$. The argument for the series $\sum_i \pi(T_{ij})^*\pi(T_{ij})$ is similar, now using the condition on the columns of (T_{ij}). $\square$

Given a closed, two-sided ideal $\mathcal{J}\subset\mathcal{B}$ there is an embedding $\mathcal{H}_{\mathcal{J}}\to\mathcal{H}_{\mathcal{B}}$ and we define the subalgebra

$$\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}},\mathcal{J}) = \{T\in\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}) : T\mathcal{H}_{\mathcal{B}}\subset\mathcal{H}_{\mathcal{J}}\}. \tag{4.2}$$

Lemma 4.9. *Every* $T\in\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ *has the property that* $T\mathcal{H}_{\mathcal{J}}\subset\mathcal{H}_{\mathcal{J}}$*. Consequently the subalgebra* $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}},\mathcal{J})$ *is a closed two-sided ideal in* $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$*. The algebra* $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}},\mathcal{J})$ *can be equivalently defined as the subalgebra of those* $T\in\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ *all of whose matrix coefficients* $T_{ij}\in\mathcal{J}$*.*

Proof. Let $x=(b_j)_{j\in\hat{\mathbb{Z}}}\in\mathcal{H}_{\mathcal{J}}$ and e_i the standard basis elements of $\mathcal{H}_{\mathcal{B}}$. We need to show that for $T\in\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$, the coordinates $\langle e_i, Tx\rangle$ are elements of $\mathcal{J}$. For an isometric representation π we have

$$\pi(\langle e_i, Tx\rangle) = \sum_{j\in\hat{\mathbb{Z}}}\pi(T_{ij}b_j)\in\mathcal{J},$$

which is a convergent series by Lemma 4.8. The elements of the series lie in $\mathcal{J}$ since $b_j\in\mathcal{J}$, and $\mathcal{J}$ is closed, so it follows that $\langle e_i, Tx\rangle\in\mathcal{J}$.

To see that $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}},\mathcal{J})$ is closed, we use the fact that $\mathcal{H}_{\mathcal{J}}\subset\mathcal{H}_{\mathcal{B}}$ is closed. For then if T_n is a sequence in $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}},\mathcal{J})$ which is Cauchy for the norm on $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ then for each $x\in\mathcal{H}_{\mathcal{B}}$ the sequence $T_nx\in\mathcal{H}_{\mathcal{J}}$ is Cauchy. Hence the limit $Tx\in\mathcal{H}_{\mathcal{B}}$ is actually an element of $\mathcal{H}_{\mathcal{J}}$. For $S\in\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$, $T\in\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}},\mathcal{J})$ and $x\in\mathcal{H}_{\mathcal{B}}$ we have $STx\in\mathcal{H}_{\mathcal{J}}$ because $Tx\in\mathcal{H}_{\mathcal{J}}$ and $TSx\in\mathcal{H}_{\mathcal{J}}$ by definition of $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}},\mathcal{J})$.

It is immediate that for $T\in\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}},\mathcal{J})$ all the T_{ij} are in $\mathcal{J}$. Conversely, if $T_{ij}\in\mathcal{J}$ for all i,j then by Lemma 4.8 for any $(b_i)_{i\in\hat{\mathbb{Z}}}\in\mathcal{H}_{\mathcal{B}}$ we have $T(b_i)_{i\in\hat{\mathbb{Z}}} = (\sum T_{ij}b_j)_{i\in\hat{\mathbb{Z}}}$ which is an element of $\mathcal{H}_{\mathcal{B}}$ all of whose coordinates are in $\mathcal{J}$ and hence an element of $\mathcal{H}_{\mathcal{J}}$. $\square$

For a directed set Λ, the set

$$\Lambda^\infty := \{(\lambda_i)_{i\in\hat{\mathbb{Z}}} : \lambda_i \in \Lambda\},$$

is a directed set with the partial order

$$(\lambda_i) \le (\mu_i) \Leftrightarrow \lambda_i \le \mu_i, \quad \text{for all } i.$$

If $\{u_\lambda\}_{\lambda\in\Lambda}$ is a bounded approximate unit for an ideal $\mathcal{J} \subset \mathcal{B}$, then

$$u_{(\lambda_i)} := \{\text{diag}\,(u_{\lambda_i})_{i\in\hat{\mathbb{Z}}}\}_{(\lambda_i)\in\Lambda^\infty},$$

is a net in $\text{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J})$ indexed by Λ^∞. The diagonal matrices $v_{n,\lambda}$ defined by

$$(v_{n,\lambda})_{ii} := \begin{cases} u_\lambda & |i| \le n \\ 0, & |i| > n \end{cases}$$

constitute a subnet. The algebra $\text{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J})$ admits an approximate unit whenever $\mathcal{J}$ does.

Lemma 4.10. *If $(u_\lambda)_{\lambda\in\Lambda}$ is a bounded approximate unit for $\mathcal{J}$, then:*
1) the net $(u_{(\lambda_i)})_{(\lambda_i)\in\Lambda^\infty}$, is a bounded approximate unit for $\text{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J})$;
2) the subnet $(v_{n,\lambda})$ is a bounded approximate unit for $\mathbb{K}\tilde{\otimes}\mathcal{J}$. In particular $\mathbb{K}\tilde{\otimes}\mathcal{J}$ has a sequential approximate unit whenever $\mathcal{J}$ does.

Proof. Given an operator $T := (b_{ij})$ and $\varepsilon > 0$, by Lemma 4.8 we can choose λ_j, $j \in \hat{\mathbb{Z}}$, such that the columns $(b_{ij})_i$ satisfy

$$\|(b_{ij} - b_{ij}u_{\lambda_j})_i\| < \varepsilon 2^{-(|j|+1)}.$$

Since $(\lambda_j)_{j\in\hat{\mathbb{Z}}} \in \Lambda^\infty$, the matrix $u_{(\lambda_j)} = \text{diag}(u_{\lambda_j})$ is an element of the directed set in 1). We can estimate

$$\begin{aligned}\|T\text{diag}(u_{\lambda_j}) - T\| = \|(b_{ij})\text{diag}(u_{\lambda_j}) - (b_{ij})\| &= \|\sum_j (b_{ij}u_{\lambda_j} - b_{ij})_i\| \\ &\le \sum_j \|(b_{ij}u_{\lambda_j} - b_{ij})_i\| < \sum_j \varepsilon 2^{-(|j|+1)} \le \varepsilon,\end{aligned}$$

showing we have a right approximate unit. In a similar way one shows that the directed set is a left approximate unit. The proof of 2) is similar but easier. □

We define two representations $\pi : \mathcal{B} \to \mathbb{B}(\mathcal{H}_\pi)$ and $\rho : \mathcal{B} \to \mathbb{B}(\mathcal{H}_\rho)$ to be *cb-equivalent* if there exists a cb-isomorphism $g : \mathcal{H}_\pi \to \mathcal{H}_\rho$ such that $\pi = g^{-1}\rho g$.

Lemma 4.11. *Let $\pi : \text{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}) \to \mathbb{B}(\mathcal{H})$ be a cb-representation. There exist idempotents $q_i \in \pi(\text{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})) \subset \mathbb{B}(\mathcal{H})$ such that $\sum_i q_i = \pi(1)$ and $\sup_i \|q_i\| < \infty$. Consequently, π is cb-equivalent to the representation*

$$\text{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}) \to \mathbb{B}\Big(\bigoplus_{i\in\hat{\mathbb{Z}}} q_i\mathcal{H}\Big) \qquad (b_{ij}) \mapsto (q_i\pi(b_{ij})q_j).$$

Proof. Using matrix coefficients and embedding $\mathcal{B}$ in the (i,i)-diagonal slot of $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ we obtain from π a family of representations $\pi_i : \mathcal{B} \to \mathbb{B}(\mathcal{H})$ satisfying $\pi_i\pi_j = 0$. The elements $q_i := \pi_i(1)$ are the corresponding idempotents in $\mathbb{B}(\mathcal{H})$. We also write $q = \pi(1)$. If we write $\mathcal{H}_i := q_i\mathcal{H}$, then since $q_iq_j = \delta_{ij}$, and $\|q_i\| \leq \|\pi\|$, we have a cb-isomorphism $q\mathcal{H} \cong \bigoplus_{i\in\hat{\mathbb{Z}}} \mathcal{H}_i$ and so a cb-isomorphism $g : \mathcal{H} \to (1-q)\mathcal{H} \oplus \bigoplus_{i\in\hat{\mathbb{Z}}} \mathcal{H}_i$. Using the identifications $[\pi(\mathcal{B})\mathcal{H}] = q\mathcal{H}$ and $\mathrm{Nil}\,\pi(\mathcal{B}) = (1-q)\mathcal{H}$, this clearly gives a cb-equivalence between π and the matrix representation $(q_i\pi(a_{ij})q_j)$. □

Lemma 4.12. *Let $\mathcal{J} \subset \mathcal{B}$ be an ideal in a unital operator algebra $\mathcal{B}$ and let (u_λ) be a bounded approximate unit for $\mathcal{J}$. Let $\pi : \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}) \to \mathbb{B}(\mathcal{H})$ be a cb-representation. Consider the subalgebras $\mathcal{J} \cong \mathcal{J} \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}} \subset \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J}) \subset \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ and the representations $\pi : \mathcal{J} \to \mathbb{B}(\mathcal{H})$ and $\pi : \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J}) \to \mathbb{B}(\mathcal{H})$, defined by restriction. Then:*
1) there is an equality of essential subspaces $[\pi(\mathcal{J})\mathcal{H}] = [\pi(\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J}))\mathcal{H}]$;
2) the idempotent $q = w\text{-}\lim u_\lambda \in \mathbb{B}(\mathcal{H})$ from Proposition 1.7 commutes with $\pi(T)$ for all $T \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$.

Proof. First we show that $[\pi(\mathcal{J})\mathcal{H}] = [\pi(\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J}))\mathcal{H}]$. It is clear that

$$[\pi(\mathcal{J})\mathcal{H}] \subset [\pi(\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J}))\mathcal{H}],$$

so we proceed to show the reverse inclusion. By Lemma 4.11 we may assume that there are idempotents $q_i : \mathcal{H} \to \mathcal{H}$ such that $\mathcal{H} = \bigoplus_{i\in\hat{\mathbb{Z}}} \mathcal{H}_i$ and $\pi(a_{ij}) = (q_i\pi(a_{ij})q_j)$. We wish to show that for $(h_i) \in [\pi(\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J}))\mathcal{H}]$, it holds that $\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}})(h_i) \to (h_i)$, so that $(h_i) \in [\pi(\mathcal{J} \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}})\mathcal{H}]$. Thus we must show that for every $\varepsilon > 0$ there exists $\mu \in \Lambda$ such that for all $\lambda \geq \mu$ it holds that $\|\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}})(h_i) - (h_i)\| < \varepsilon$. So let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$\Big\| \sum_{|i|>N} \langle h_i, h_i \rangle \Big\|^{\frac{1}{2}} < \frac{\varepsilon}{2(C\|\pi\| + 1)},$$

where $C := \sup_\lambda \|u_\lambda\|$. We claim we can choose μ such that for all $\lambda \geq \mu$ and $1 \leq |i| \leq N-1$

$$\|\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}})(h_i)_{|i|<N} - (h_i)_{|i|<N}\| < \frac{\varepsilon}{2N}. \tag{4.3}$$

To see this, first observe that $(h_i)_{1\leq|i|\leq N} \in [\pi(\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J}))\mathcal{H}_i]$. This is the case because

$$q_{[N]} := \sum_{1\leq|i|\leq N} q_i \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}),$$

and $(h_i)_{1\leq|i|\leq N} = q_{[N]}(h_i)$. Then, by Lemma 1.6 and Lemma 4.10

$$\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}})(h_i)_{|i|<N} \to (h_i)_{|i|<N},$$

and since we only deal with finitely many entries (at most $2N$), this means we can choose μ

as in Equation (4.3). Thus we have for $\lambda \geq \mu$ that

$$\begin{aligned}
\|\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}})(h_i) - (h_i)\| &\leq \|\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}})(h_i)_{1\leq |i| \leq N} - (h_i)_{1\leq |i|\leq N}\| \\
&\qquad + \|\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}})(h_i)_{|i|>N} - (h_i)_{|i|>N}\| \\
&\leq \sum_{1\leq |i|\leq N} \|\pi_i(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}})h_i - h_i\| \\
&\qquad + \Big\| \sum_{|i|>N} \langle (\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}} - \mathrm{Id}_{\mathcal{H}_\mathcal{B}})h_i, (\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}} - \mathrm{Id}_{\mathcal{H}_\mathcal{B}})h_i\rangle \Big\|^{\frac{1}{2}} \\
&< \frac{\varepsilon}{2} + (\|\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}})\| + 1) \Big\| \sum_{|i|>N} \langle h_i, h_i\rangle \Big\|^{\frac{1}{2}} \\
&< \frac{\varepsilon}{2} + \frac{(\|\pi(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}})\| + 1)\varepsilon}{2(C\|\pi\| + 1)} < \varepsilon,
\end{aligned}$$

showing that $\pi(u_\lambda)(h_i) \to (h_i)$.

In the same vein $[\pi(\mathcal{J})^*\mathcal{H}] = [\pi(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}, \mathcal{J}))^*\mathcal{H}]$. As $\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}, \mathcal{J}))$ is an ideal in $\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B})$, the subspace $[\pi(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}, \mathcal{J}))\mathcal{H}]$ is $\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B})$-invariant. The topological complement, given by $[\pi(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}, \mathcal{J}))^*\mathcal{H}]^\perp$, is $\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B})$ invariant as well. For if $v \in [\pi(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}, \mathcal{J}))^*\mathcal{H}]^\perp$ and $h \in [\pi(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}, \mathcal{J}))^*\mathcal{H}]$, then $\pi(T)^*h \in [\pi(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}, \mathcal{J}))^*\mathcal{H}]$ because $\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}, \mathcal{J})$ is an ideal and thus

$$\langle \pi(T)v, h\rangle = \langle v, \pi(T)^*h\rangle = 0.$$

That is $\pi(T)v \in [\pi(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}, \mathcal{J}))^*\mathcal{H}]^\perp$. From 3), 4) of Proposition 1.7, we see that

$$q\pi(T)q = \pi(T)q, \quad (1-q)\pi(T)(1-q) = \pi(T)(1-q),$$

from which $q\pi(T) = \pi(T)q$ follows readily. □

Lemma 4.13. *Let (u_λ) be a bounded approximate unit for an ideal $\mathcal{I}$ in an operator algebra $\mathcal{B}$. Then for all $T \in \mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B})$, $[u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}}, T] \xrightarrow{w} 0$. That is, $u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}}$ commutes with $\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B})$ weakly asymptotically.*

Proof. The argument we give is modelled on the proof of [1, Lemma 3.1]. We assume that $\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B})$, and hence $\mathcal{J}$ and $\mathcal{B}$ are completely isometrically embedded in $\mathbb{B}(\mathcal{H})$. Let the linear functional $\phi : \mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}) \to \mathbb{C}$ be continuous. By the Hahn-Banach theorem we may extend ϕ to the enveloping C^*-algebra $C^*(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}))$, the C^*-algebra generated by $\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}) \subset \mathbb{B}(\mathcal{H})$. Since every element in the dual of a C^*-algebra is a linear combination of four states, it suffices to prove weak convergence with respect to all states of $C^*(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}))$. If we denote by $\pi_u : C^*(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B})) \to \mathbb{B}(\mathcal{H}_u)$ the universal GNS-representation of $C^*(\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B}))$, the state ϕ has the form $b \mapsto \langle v, \pi_u(b)v\rangle$, where v is a vector in the GNS-space $\mathcal{H}_u$. Since (u_λ) is an approximate unit for $\mathcal{J}$, $\pi(u_\lambda)$ converges strongly to an idempotent q onto $[\pi(\mathcal{J})\mathcal{H}_u]$. By Lemma 4.12, q commutes with $\pi(a_{ij})$. Hence

$$\lim_\lambda \phi([u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}}, T]) = \langle v, [q, \pi_u(T)]v\rangle = 0.$$ □

Definition 4.14. Let $\mathcal{J} \subset \mathcal{B}$ be an ideal and u_λ a bounded approximate unit for $\mathcal{J}$. The bounded approximate unit u_λ is said to be $\mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B})$-*quasicentral* if for all $T \in \mathrm{End}^*_\mathcal{B}(\mathcal{H}_\mathcal{B})$ we have $\lim_\lambda \|[T, u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}}]\| = 0$.

We are now ready to establish the existence of $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$-quasicentral approximate units. It should be noted that this result is new even for C^*-algebras.

Theorem 4.15. *Let $(u_\lambda)_{\lambda\in\Lambda}$ be a bounded approximate unit for a closed ideal $\mathcal{J}$ in a unital operator algebra $\mathcal{B}$. Then there is an $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$-quasicentral approximate unit $(v_\mu)_{\mu\in M}$ for $\mathcal{J}$, contained in the convex hull of (u_λ).*

Proof. The proof is formally identical to that of [1, Theorem 3.2]. We assume $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ is isometrically isomorphically embedded in $\mathbb{B}(\mathcal{H})$. Denote by $\mathscr{C}(u_\lambda)$ the convex hull of $(u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}})$. Choose elements $b_1, \ldots, b_n \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$, $v \in \mathscr{C}(u_\lambda)$ and $\varepsilon > 0$. Consider

$$\mathscr{C}(u_\lambda) \subset \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})^n = \bigoplus_{i=1}^{n} \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}),$$

by diagonal embedding and set $b = \mathrm{diag}(b_1, \ldots, b_n)$. The set $\mathscr{C}_b := \{[u, b] : u \in \mathscr{C}(u_\lambda)\}$ is convex and hence its norm and weak closures in $\mathbb{B}(\mathcal{H}^n)$ coincide. By Lemma 4.13, $[u_\lambda \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}}, b] \xrightarrow{w} 0$, and hence 0 must be a norm limit of elements of $\mathscr{C}_b$. That is, there exists $v \in \mathscr{C}_b$ with $\|[v, b]\| < \varepsilon$. Letting Ω denote the set of finite subsets of $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$, the argument shows that for each pair $(\lambda, \omega) \in \Lambda \times \Omega$ there is a $v_{\lambda,\omega} \in \mathscr{C}(u_\mu : \lambda \leq \mu)$ for which

$$\|[v_{\lambda\omega}, b]\| < \frac{1}{|\omega|},$$

for all $b \in \omega$. The relation

$$(\lambda, \omega) \leq (\lambda', \omega') \Leftrightarrow \lambda \leq \lambda' \text{ and } \omega \subset \omega',$$

defines a partial order on $\Lambda \times \Omega$, with respect to which $(v_{\lambda\omega})$ is a bounded approximate unit. $\square$

The next theorem considers quasicentral approximate units for algebras of multipliers, relative to a second ideal. This result is not as general as the above theorem, but provides the statement we need for our refinement of the Kasparov technical theorem in the next section.

Theorem 4.16. *Let $\mathcal{B}$ be a unital operator algebra and $\mathcal{K}$ an ideal with bounded approximate unit (v_n). Assume that $\mathcal{J}$, $\mathcal{A} \subset \mathcal{B}$ are subalgebras such that $\mathcal{J}$ is an ideal in $\mathcal{B}$, $\mathcal{J}\mathcal{A}$, $\mathcal{A}\mathcal{J} \subset \mathcal{K}$ and $\mathcal{K}\mathcal{A} = \mathcal{A}\mathcal{K} = \mathcal{K}$. If $\mathcal{A}$ has a bounded approximate unit (u_k) then there exists an $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J})$ quasicentral approximate unit for $\mathcal{A}$ contained in the convex hull of (u_k).*

Proof. Assume without loss of generality that $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ is completely isometrically embedded in $\mathbb{B}(\mathcal{H})$ and let $C^*(\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})) \subset \mathbb{B}(\mathcal{H})$ denote its enveloping C^*-algebra in this representation. We wish to prove that for all functionals $\phi : \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{K}) \to \mathbb{C}$, and all $T \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J})$ we have $\phi([u_k, T]) \to 0$.

As in the proof of Lemma 4.13, it will suffice to prove this for vector states $\phi = \langle v, \cdot v\rangle$ on $C^*(\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}))$ coming from the universal GNS representation $\pi_u : C^*(\mathrm{End}_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})) \to \mathbb{B}(\mathcal{H}_u)$. Since both $\mathcal{K}$ and $\mathcal{A}$ have bounded approximate units, Proposition 1.7 gives two idempotents: p mapping onto

$$[\pi_u(\mathcal{K} \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}})\mathcal{H}_u] = [\pi_u(\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{K}))\mathcal{H}_u],$$

(cf. Lemma 4.12) and q mapping onto $[\pi_u(\mathcal{A} \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{B}}})\mathcal{H}_u]$ as the strong limits of (v_n) and (u_k) respectively. Since $\mathcal{A}\mathcal{K} = \mathcal{K}$, we have $[\pi_u(\mathcal{A})[\pi_u(\mathcal{K})\mathcal{H}_u]] = [\pi_u(\mathcal{K})\mathcal{H}_u]$ and thus $u_k \to 1$ strongly on $[\pi_u(\mathcal{K})\mathcal{H}_u] = p\mathcal{H}_u$, again by Proposition 1.7. We claim that $pq = qp = p$. To see this, first observe that $pq = qp$, since by Lemma 4.13, p commutes with all elements of $\mathcal{B}$ and hence in particular with $\mathcal{A}$. Therefore, for any $h \in \mathcal{H}_u$ we have

$$qph = \lim u_k ph = \lim pu_k h = pqh.$$

Then since (u_k) converges strongly to 1 on $[\pi_u(\mathcal{K})\mathcal{H}_u]$ it follows that

$$qph = \lim \pi(u_k)ph = ph,$$

which proves our claim. Now let $T \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J})$ and consider $[u_k, T] \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{K})$. The operator T commutes with p and since $[u_k, T] \in \mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{K})$ this operator equals $p[u_k, T]p$, and

$$\begin{aligned}\lim \phi([u_k, T]) = \lim\langle v, \pi_u([u_k, T])v\rangle &= \lim\langle v, p\pi_u([u_k, T])pv\rangle \\ &\to \langle v, p[q, \pi_u(T)]pv\rangle = \langle v, [p, \pi_u(T)]v\rangle = 0.\end{aligned}$$

Thus, the commutators $[u_k, T]$ converge to 0 in the weak topology of $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{K})$. The same argument as in the proof of Theorem 4.15 now shows that the convex hull of (u_k) contains an approximate unit that is quasicentral for $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}, \mathcal{J})$. □

4.3 Completeness and the technical theorem

Having established the existence of quasi-central approximate units in operator algebras with bounded approximate unit, we can formulate an extension of Kasparov's technical theorem in the spirit of Higson, [26]. For practical purposes we state the following corollary of Theorem 4.15 as a Lemma. When (u_n) is a sequential approximate unit, we say that $(v_n) \in \mathscr{C}(u_k)$ is a sequence of *far out* convex combinations if $(v_n) \in \mathscr{C}(u_k : k \geq n)$.

Lemma 4.17. *Let $\mathcal{J}$ be an ideal in a separable unital operator algebra $\mathcal{B}$, (u_n) a sequential bounded approximate unit for $\mathcal{J}$. Let $(z_i)_{i\in\mathbb{N}} \subset \mathcal{B}$ a countable norm bounded subset of $\mathcal{B}$ and $1 > \varepsilon > 0$. There exists a $\mathcal{B}$ quasicentral, sequential bounded approximate unit (v_n) for $\mathcal{J}$, contained in the convex hull of (u_n), such that*

$$\sup_{i\in\mathbb{N}} \|[v_{n+1} - v_n, z_i]\| < \varepsilon^n.$$

Proof. Assuming $\mathcal{B}$ separable and (u_n) countable, the new approximate unit can be chosen in such a way as to satisfy the asserted properties. This is done by choosing a countable dense subset $\{b_1, b_2, \dots\}$ of $\mathcal{B}$, embedding $\mathcal{B}$ in $\mathcal{B}^\infty$ diagonally as usual, and considering $z := \mathrm{diag}(z_1, \dots, z_i, \dots) \in \mathcal{B}^\infty$. From the proof of Theorem 4.15 we obtain an $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$ quasicentral approximate unit $v_{n,\omega}$ indexed by $\mathbb{N} \times \Omega$, with Ω the set of finite subsets of $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$. Now choose $v_0 := v_{0\{z\}}$ and inductively assume $v_k := v_{n_k,\omega_k} \in \mathscr{C}(u_n)$ was chosen from $v_{n,\omega}$ in such a way that

$$v_k = \sum_{i=n_k}^{m_k} \theta_i u_i, \quad \theta_i \in [0, 1], \quad \sum_{i=n_k}^{m_k} \theta_i = 1,$$

for some $n_k, m_k \in \mathbb{N}$. Now choose

$$v_{k+1} := v_{n_{k+1},\{z,b_1,\dots,b_{n_{k+1}}\}},$$

where $n_{k+1} > m_k$, yielding a sequential approximate unit for $\mathcal{J}$, which is quasicentral for the (b_i) and hence for $\mathcal{B}$, as well as for the element $z \in \mathcal{B}^\infty$. It should be noted that quasicentrality does not necessarily hold for all of $\mathcal{B}^\infty$. By choosing a subsequence, we can realise that

$$\|[(v_{n+1}-v_n)\cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}}, x]\| = \|[(v_{n+1}-v_n)\cdot \mathrm{Id}_{\mathcal{H}_\mathcal{B}}, \mathrm{diag}(z_1,\dots,z_i,\dots)]\| = \sup_{i\in\mathbb{N}} \|[v_{n+1}-v_n, z_i]\| < \varepsilon^n,$$

for the given sequence z_i as desired. □

Theorem 4.18. *Let $\mathcal{B}$ be a unital operator algebra, $\mathcal{K} \subset \mathcal{B}$ an ideal with countable bounded commutative approximate unit and $\mathcal{A}, \mathcal{J} \subset \mathcal{B}$ closed separable subalgebras with $\mathcal{A}\mathcal{J}$, $\mathcal{J}\mathcal{A} \subset \mathcal{K}$, $\mathcal{K} \subset \mathcal{J}$ and $\mathcal{A}\mathcal{K} = \mathcal{K}\mathcal{A} = \mathcal{K}$. Suppose we are given*
1) a bounded total subset $\{a_i\} \subset \mathcal{A}$ and countable bounded approximate units $(u'_k) \subset \mathcal{A}$, $(v'_k) \in \mathcal{J}$, and
2) $F \in \mathcal{B}$ such that $1 - F^2 \in \mathcal{J}, F\mathcal{J}, \mathcal{J}F \subset \mathcal{J}$, $[F, \mathcal{A}] \subset \mathcal{K}$ and $\lim_{k\to\infty} \|[F, u'_k]\| = 0$.

For any $0 < \varepsilon < 1$, there exist countable bounded approximate units $(v_n), (u_k)$ contained in the convex hull $\mathscr{C}(v'_k)$ and $\mathscr{C}(u'_k)$ respectively such that for $d_n := v_{n+1} - v_n$ the following convergence properties hold:
1) $\|[d_n, F]\| < \varepsilon^{2n}$;
2) $\|[d_n, a_i]\| < \varepsilon^{2n}$ for all i;
3) $\|[d_n, u_k]\| < \varepsilon^{2n}$ for all k;
4) $\|[d_n, u_k]\| < \varepsilon^{2k}$ for all n;
5) $\|d_n[F, a_i]\| < \varepsilon^{2n}$ for $n \geq i$;
6) $\|d_n[F, u_k]\| < \varepsilon^{2k}$ for all n;
7) $\|d_n[F, u_k]\| < \varepsilon^{2n}$ for $n \geq k$;
8) $\|u_k a_i - a_i\| < \varepsilon^{2k}$ for $k > i$;
9) $\|d_n[F, u_k a_i - a_i]\| < \varepsilon^{k+2n}$ for $n > k > i$;
10)$\|[d_n, u_k a_i - a_i]\| < \varepsilon^{k+2n}$ for $n > k > i$.

In fact these properties continue to hold for any subsequence $(\tilde{v}_n) := (v_{k_n})$ and the conclusion holds for any finite number of subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ satisfying the hypotheses on $\mathcal{A}$.

Proof. The case of a finite number of algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ is reduced to that of a single algebra by setting $\mathcal{A} := \oplus_{i=1}^n \mathcal{A}_i \subset \oplus_{i=1}^n \mathcal{B}$ and $\oplus_{i=1}^n \mathcal{J}$. We thus prove the theorem for a single algebra $\mathcal{A}$.

The hypotheses imply that $\tilde{\mathcal{J}} := 1 + F + \mathcal{J} \subset \mathcal{B}$ is an algebra and that $\mathcal{J}$ is an ideal in $\tilde{\mathcal{J}}$. Theorem 4.15 gives us an F-quasicentral approximate unit for $\mathcal{J}$. Using this approximate unit, consider $\tilde{\mathcal{B}} := 1 + \mathcal{A} + \mathcal{J} \subset \mathcal{B}$, in which $\mathcal{J}$ is an ideal as well. Embed $\tilde{\mathcal{B}}$, and hence $\mathcal{K}$, $\mathcal{J}$ and $\mathcal{A}$ into $\mathrm{End}^*_{\tilde{\mathcal{B}}}(\mathcal{H}_{\tilde{\mathcal{B}}})$ as multiples of the identity operator $\mathrm{Id}_{\mathcal{H}_{\tilde{\mathcal{B}}}}$. The elements

$$a := \mathrm{diag}(a_1, \dots, a_i, \dots) \quad \text{and} \quad u := \mathrm{diag}(u'_1, \dots, u'_k, \dots),$$

are elements of $\mathrm{End}^*_{\tilde{\mathcal{B}}}(\mathcal{H}_{\tilde{\mathcal{B}}})$ as well. Thus by Theorem 4.15, there exists a countable, commutative, approximate unit $(v_n) \subset \mathscr{C}(v'_n)$, which is quasicentral for a, u, whilst retaining quasicentrality for F.

Write $d_n := v_{n+1} - v_n$. By quasicentrality for F, a and u, we can re-index the (v_n) if necessary, and we may assume that

$$\|[d_n, F]\|,\ \|[d_n, a]\|,\ \|[d_n, u]\| < \varepsilon^{2n}, \tag{4.4}$$

which in particular means that

$$\|[d_n, u'_k]\|,\ \|[d_n, a_i]\| < \varepsilon^{2n}, \quad \text{for all } i, k.$$

This proves the estimates 1), 2), as well as 3) for for the approximate unit u'_k.

Next we construct an approximate unit $(u_k) \subset \mathscr{C}(u'_k)$ satisfying 3) and 4). We again consider the algebra $\tilde{\mathcal{B}} := 1 + \mathcal{A} + \mathcal{J} \subset \mathcal{B}$, in which $\mathcal{J}$ is an ideal. The uniformly bounded sequence $d_n \in \mathcal{J}$ defines an element $d := \operatorname{diag}(d_1, \dots, d_n, \dots) \in \operatorname{End}^*_{\tilde{\mathcal{B}}}(\mathcal{H}_{\tilde{\mathcal{B}}}, \mathcal{J})$. Apply Theorem 4.16 to the approximate unit (u'_k) to obtain a countable approximate unit $(u_k) \subset \mathscr{C}(u'_k)$ which is quasicentral for d. Thus we may assume that

$$\|[d_n, u_k]\| < \varepsilon^{2k}, \quad \text{for all } n.$$

Property 3) is preserved under convex combinations, and is thus valid for u_k.

For 5), since $[F, a_i] \in \mathcal{K} \subset \mathcal{J}$, we may reindex v_n and assume that

$$\|d_n[F, a_i]\| < \varepsilon^{2n}, \quad \text{whenever } n \geq i, \tag{4.5}$$

as desired.

To prove 6), observe that because by assumption $\lim_{k\to\infty} \|[F, u'_k]\| = 0$, we may assume that

$$\|[F, u_k]\| < \frac{\varepsilon^{2k}}{2C},$$

with $C := \sup_k \|v'_k\|$. Then since $\|d_n\| \leq 2C$ the claim follows.

For 7) we use that $[F, u_k] \in \mathcal{K} \subset \mathcal{J}$ so $d_n[F, u_k] \xrightarrow{n} 0$ for each k, which, by passing to a subsequence of the d_n if necessary, amounts to

$$\|d_n[F, u_k]\| < \varepsilon^{2n}, \quad \text{whenever } n \geq k.$$

For each a_i we have the norm convergence $u_k a_i \xrightarrow{k} a_i$. Therefore, given $0 < \varepsilon < 1$, we may re-index the approximate unit u_k for $\mathcal{A}$ such that

$$\|u_k a_i - a_i\| < \varepsilon^k, \text{ whenever } k > i.$$

Note that such a re-indexing does not affect the norm convergence $[F, u_k] \xrightarrow{k} 0$ nor the properties $3), 4)$ and 6). To preserve property 7) we may need to pass to a reindexing of v_n, which can be done without affecting properties $1) - 6)$. This means that we may assume that for all $k > i$ we have $\|u_k a_i - a_i\| < \varepsilon^{2k}$, which proves 8).

For 9) and 10), we can, since 8) is true, assume that for all $i \in \hat{\mathbb{Z}}$ we have that

$$z_i := \operatorname{diag}\left(\frac{u_k a_i - a_i}{\varepsilon^k}\right)_k \in \mathcal{A}^\infty \subset \operatorname{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}}).$$

Apply Lemma 4.17 to obtain an approximate unit (v_n) for $\mathcal{J}$ which is quasicentral for all the z_i, and we may achieve

$$\|[d_n, z_i]\| < \varepsilon^{2n}, \quad \text{whenever } n \geq i, \tag{4.6}$$

as well. Note that the estimates (4.4), (4.5), (4.6) remain valid when (v_n) is replaced by a subsequence or a sequence of far out convex combinations. The same is true when (u_k) is replaced by a subsequence $(\tilde{u}_k) := (u_{n_k})$, so that u is replaced by $\tilde{u} := \mathrm{diag}(\tilde{u}_k)_k$ and z_i by $\tilde{z}_i := (\frac{\tilde{u}_k a_i - a_i}{\varepsilon^k})_k$:

$$\|[d_n, \tilde{u}]\| = \sup_k \|[d_n, \tilde{u}_k]\| = \|[d_n, \tilde{u}_{n_k}]\| \leq \sup_k \|[d_n, u_k]\| < \varepsilon^{2n},$$

and

$$\|[d_n, \tilde{z}_i]\| = \sup_k \|[d_n, \frac{\tilde{u}_k a_i - a_i}{\varepsilon^k}]\| \leq \sup_k \|[d_n, \frac{u_{n_k} a_i - a_i}{\varepsilon^{n_k}}]\| \leq \sup_k \|[d_n, \frac{u_k a_i - a_i}{\varepsilon^k}]\| < \varepsilon^{2n}.$$

Lastly, since for fixed i, k we have $[F, \frac{u_k a_i - a_i}{\varepsilon^k}] \in \mathfrak{J}$, the sequence

$$d_n[F, \frac{u_k a_i - a_i}{\varepsilon^k}] \xrightarrow{n} 0,$$

which means that another re-indexing achieves

$$\|d_n[F, \frac{u_k a_i - a_i}{\varepsilon^k}]\| < \varepsilon^{2n},$$

for $n > k > i$ and thus

$$\|d_n[F, u_k a_i - a_i]\| < \varepsilon^{k+2n}, \quad \text{whenever } n > k > i.$$

This completes the proof of 9) and 10). $\square$

For our first lifting result, we need the following elementary result concerning the strict topology on a C^*-module.

Lemma 4.19. *Let $(T_n) \subset \mathbb{K}(E_B)$ be a sequence converging strictly to $T \in \mathbb{K}(E_B)$. Then there exists a sequence $(S_n) \subset \mathscr{C}(T_n)$ such that $S_n \to T$ in norm. Hence for any essential Kasparov module (A, E_B, F), A has an approximate unit (u_n) such that $[F, u_n] \to 0$ in norm.*

Proof. We need to show that $T_n \to T$ in the weak topology of $\mathbb{K}(E_B)$, for then there exists a sequence in the convex hull of the T_n that converges to S_n in norm. Since every linear functional on a C^*-algebra is a linear combination of four states, it suffices to show that for all states $\sigma : \mathbb{K}(E_B) \to \mathbb{C}$ we have $\sigma(T_n) \to \sigma(T)$. Any such σ can be realised as a vector state for a vector v_σ in the universal GNS-representation $\mathcal{H}_u$ of $\mathbb{K}(E)$. The representation $\pi_u : \mathbb{K}(E_B) \to \mathbb{B}(\mathcal{H}_u)$ is essential, so $\pi_u(T_n)$ converges to $\pi_u(T)$ weakly in $\mathbb{B}(\mathcal{H}_u)$. Thus $\sigma(T_n) = \langle v_\sigma, \pi_u(T_n) v_\sigma \rangle_u \to \langle v_\sigma, \pi_u(T) v_\sigma \rangle_u = \sigma(T)$ and we are done.

Let (A, E_B, F) be a Kasparov module for which the A representation is essential. Any approximate unit (u_n) for A will converge strictly to the identity operator on E_B and $[F, u_n] \to 0$ strictly in E_B. Therefore the previous argument yields a contractive approximate unit (u'_n) for A such that $[F, u'_n] \to 0$ in norm. $\square$

Proposition 4.20. *Let A, B be separable C^*-algebras. Any class in $KK(A, B)$ can be represented by an unbounded Kasparov module $(\mathcal{A}, E_B, S)$ such that $\mathcal{A}$ admits an approximate unit (u_n) with $[S, u_n] \to 0$ in norm.*

Proof. Let (A, E_B, F) be a Kasparov module for which the A representation is essential. Let (u'_k) be an approximate unit for A such that $[F, u'_k] \to 0$ in norm, as in Lemma 4.19. Let v'_n be a contractive approximate unit for J_F, where J_F is defined in Lemma 4.5.

Applying Theorem 4.18 with $\mathcal{B} = \mathrm{End}^*_B(E)$, $\mathcal{K} = \mathbb{K}(E)$, $\mathcal{J} = J_F$, $\mathcal{A} = A$, we obtain approximate units (v_n) for J_F and (u_n) for A with the properties 1)-10) of Theorem 4.18. Recalling that $d_n := v_{n+1} - v_n$, we define

$$\ell^{-1} = c := \sum_{n=0}^{\infty} \varepsilon^{-n} d_n,$$

which is an unbounded multiplier of the ideal J_F. By 2) of Theorem 4.18, $[\ell^{-1}, a]$ is bounded on $\mathrm{Im}\, \ell$ for a dense set of $a \in A$, and by 5), $[F, a]\ell^{-1}$ is also bounded on $\mathrm{Im}\, \ell$ for the same dense set of $a \in A$. By 1) $[F, \ell^{-1}]$ is bounded on $\mathrm{Im}\, \ell$ and since $(1 - F^2)^{\frac{1}{2}} \in J_F$ we may also assume $(1 - F^2)^{\frac{1}{2}} \ell^{-1}$ to be bounded of norm < 1. Thus ℓ is admissible and by Theorem 4.7 the operator $S := \frac{1}{2}(F\ell^{-1} + \ell^{-1}F)$ lifts (E_B, F) to an unbounded Kasparov module.

Thus it only remains to check that $[S, u_k] \to 0$ and that $[S, u_k a_i - a_i] \xrightarrow{k} 0$ for all i. The properties $3), 4), 6)$ and 7) now imply that

$$\begin{aligned}\|[F\ell^{-1}, u_k]\| &= \|F[\ell^{-1}, u_k] + [F, u_k]\ell^{-1}\| = \|\sum_{n=0}^{\infty} F\varepsilon^{-n}[d_n, u_k] + \varepsilon^{-n}[F, u_k]d_n\| \\ &\leq \|\sum_{n\leq k} F\varepsilon^{-n}[d_n, u_k] + \varepsilon^{-n}[F, u_k]d_n\| + \|\sum_{n>k} F\varepsilon^{-n}[d_n, u_k] + \varepsilon^{-n}[F, u_k]d_n\|.\end{aligned}$$

Now applying properties 4) and 6) to the first term and properties 3) and 7) to the second we estimate

$$\|[F\ell^{-1}, u_k]\| \leq \sum_{n\leq k} 2\varepsilon^{2k-n} + \sum_{k>n} 2\varepsilon^{n} = \varepsilon^k (\sum_{n\leq k} 2\varepsilon^n + \sum_{n=1}^{\infty} 2\varepsilon^n) \leq C\varepsilon^k,$$

and thus $\lim_{k\to\infty}[S, u_k] \to 0$. Lastly, observe that by 8) we have $\|u_k a_i - a_i\| < \varepsilon^{2k}$ whenever $k > i$. Then for $k > i$ we can estimate

$$\begin{aligned}\|[F\ell^{-1}, u_k a_i - a_i]\| &\leq \|[F, u_k a_i - a_i]\ell^{-1}\| + \|F[\ell^{-1}, u_k a_i - a_i]\| \\ &\leq \sum \varepsilon^{-n}\|F[d_n, u_k a_i - a_i]\| + \varepsilon^{-n}\|[F, u_k a_i - a_i]d_n\| \\ &\leq \sum_{n\leq k} \varepsilon^{-n}\|F[d_n, u_k a_i - a_i]\| + \varepsilon^{-n}\|[F, u_k a_i - a_i]d_n\| \\ &\qquad + \sum_{n>k} \varepsilon^{-n}\|F[d_n, u_k a_i - a_i]\| + \varepsilon^{-n}\|[F, u_k a_i - a_i]d_n\| \\ &\leq \sum_{n\leq k} C\varepsilon^{2k-n} + \sum_{n>k} C\varepsilon^{n+k} \qquad \text{by 8), 9) and 10)} \\ &\leq 2C\varepsilon^k,\end{aligned}$$

and thus $[S, u_k a_i - a_i] \xrightarrow{k} 0$ for all i. □

In the case of Fredholm modules, the absence of technicalities related to complementability of submodules allow for a better version of the above proposition. For an A-Fredholm module

$(A, \mathcal{H}, F)$, we write p for the projection onto $[\pi(A)\mathcal{H}]$, also note that $[F, p] \in J$ since

$$[F, p]\pi(a) = [F, p\pi(a)] - p[F, \pi(a)] = [F, \pi(a)] - p[F, \pi(a)].$$

Lemma 4.21. *Let A be a separable C^*-algebra, $(A, \mathcal{H}, F)$ be a Fredholm module with $F = F^*$ and $F^2 \leq 1$. Let p be the projection onto $[\pi(A)\mathcal{H}]$. There exists $h \in J$ such that:*
1) h has dense range;
2) $[pFp, h] = 0$.

Proof. Since the module $(\mathcal{H}, pFp)$ is a compact perturbation of $(\mathcal{H}, F)$, and $1-p \in J$, it suffices to construct $h \in \mathbb{B}(p\mathcal{H})$, since $h + (1-p) \in \mathbb{B}(\mathcal{H})$ then has the desired properties. Thus, we replace $\mathcal{H}$ with $p\mathcal{H} = [\pi(A)\mathcal{H}]$ and F with pFp, so we may assume $p = 1$. The operator $1 - F^2$ is in J and $\mathcal{H}$ splits as

$$\mathcal{H} = [\mathrm{Im}\,(1 - F^2)] \oplus \ker(1 - F^2).$$

The operator F respects this decomposition since F is self-adjoint. Choose a strictly positive $k \in \mathbb{K}(\ker(1-F^2))$, and consider $k + FkF : \ker(1-F^2) \to \ker(1-F^2)$. This element commutes with $F|_{\ker(1-F^2)}$ since $F^2 = 1$ on this subspace. Now define

$$h := (1 - F^2) + k + FkF \in J,$$

which commutes with F. Moreover $h \geq 0$ since $1 - F^2 \geq 0$ and h has dense range because it is an orthogonal sum of elements with dense range in their respective subspaces. □

Using this particular h, one can lift the Fredholm module directly to a self-adjoint element, and one does not need to assume that the representation is essential.

Definition 4.22. Let A, B, C be separable C^*-algebras (A, E_B, F) an essential (A, B) Kasparov module and $(\mathcal{B}, F_C, T)$ an essential unbounded Kasparov module such that $\mathcal{B}$ has bounded approximate unit.

Suppose we are given a complete projective $\mathcal{B}$-submodule $\mathcal{E}_{\mathcal{B}} \subset E_B$ with Grassmann connection $\nabla : \mathcal{E}_{\mathcal{B}}^{\nabla} \to E\tilde{\otimes}_B\Omega^1_T$ so that $1 \otimes_\nabla T$ is the associated self-adjoint regular operator on $E\tilde{\otimes}_B F_C$. The pair $(\mathcal{E}_{\mathcal{B}}, \nabla)$ is *compatible* with (A, E_B, F) if
1) $F \otimes 1, (1 - F^2)^{\frac{1}{2}} \otimes 1 \in \mathrm{Lip}(1 \otimes_\nabla T)$;
2) there are bounded total subsets $\{a_i\} \subset A$ and $\{c_j\} \subset J_F$ consisting of self-adjoint elements such that for all i, j, the elements $a_i, c_j \in \mathrm{Lip}(1 \otimes_\nabla T)$ and the closed subalgebras $\mathcal{A}, \mathcal{J} \subset \mathrm{Lip}(1 \otimes_\nabla T)$ generated by $\{a_i\}$ and $\{c_j\}$ satisfy $\mathcal{A}\mathcal{J}$, $\mathcal{J}\mathcal{A}$, $F\mathcal{J}$, $\mathcal{J}F \subset \mathcal{J}$;
3) there is an approximate unit $(u_k) \subset A$ for which

$$\lim_{k\to\infty} \|[1 \otimes_\nabla T, u_k]\| \to 0, \quad \lim_{k\to\infty} \|[F, u_k]\| \to 0, \text{ and for all } i, \lim_{k\to\infty} \|[1 \otimes_\nabla T, u_k a_i - a_i]\| = 0;$$

4) there is an approximate unit (w_n) for J_F such that $\lim_{n\to\infty} \|[1 \otimes_\nabla T, w_n]\| = 0$ and for all j, $\lim_{n\to\infty} \|[1 \otimes_\nabla T, w_n c_j - c_j]\| \to 0$.

The proof of the next result brings together our various technical innovations. First, the characterisation of our strongest form of completeness from Theorem 1.25 is present to ensure that the operator $[s, t](s \pm i)^{-1}$ is bounded. Our version of Kasparov's technical theorem, Theorem 4.18, is used only for C^*-algebras, but we use quasicentrality for a differentiable algebra $\mathcal{J}$ in precisely one place, to ensure that $[F, \ell^{-1}]$ is not just bounded, but even in $\mathrm{Lip}(1 \otimes_\nabla T)$.

Theorem 4.23. *Let A, B be separable C^*-algebras, $\mathcal{B}$ a differentiable algebra of an unbounded Kasparov module $(\mathcal{B}, F_C, T)$ and (A, E_B, F) a bounded (A, B) Kasparov module. If E_B admits a compatible complete projective submodule $\mathcal{E}_{\mathcal{B}} \subset E_B$, then (A, E_B, F) can be lifted to a correspondence $(\mathcal{A}, \mathcal{E}_{\mathcal{B}}, S, \nabla)$ for $(\mathcal{B}, F_C, T)$.*

Proof. Since we are given a compatible complete projective submodule $\mathcal{E}_{\mathcal{B}} \subset E_B$ we use the notation and data of Definition 4.22. By Proposition 3.20 the elements a_i and c_j are in $\mathrm{End}^*_{\mathcal{B}}(\mathcal{E}^{\nabla})$. Denote by $\mathcal{A}$ and $\mathcal{J}$ the closed subalgebras of $\mathrm{End}^*_{\mathcal{B}}(\mathcal{E}^{\nabla})$ defined under 2) and let

$$\nabla : \mathcal{E}^{\nabla}_{\mathcal{B}} \to E\tilde{\otimes}_B \Omega^1_T,$$

be the Grassmann connection.

We proceed in two steps. By hypothesis, there is a countable approximate unit (w_n) for $\mathcal{J}$ satsifying condition 4) of Definition 4.22 and $F\mathcal{J}$, $\mathcal{J}F \subset \mathcal{J}$. By Theorem 4.15 there is an F quasi-central approximate unit v'_n for $\mathcal{J}$ contained in the convex hull of w_n, and we set $d'_n = v'_{n+1} - v'_n$, and we can assume $\|[F, d'_n]\|_{\mathcal{J}} = \|[F, d'_n]\|_{1\otimes_\nabla T} < \varepsilon^{2n}$, cf. Theorem 4.18. Since $(v'_n) = \sum_{i=k_n}^{m_n} \theta_i w_i$ is built from far out convex combinations of w_n, this approximate unit will in particular again satisfy

$$[1 \otimes_\nabla T, v'_n] = \sum_{i=k_n}^{m_n} \theta_i [1 \otimes_\nabla T, w_i] \to 0,$$

in norm. We may without loss of generality assume that $\|[1 \otimes_\nabla T, v'_n]\| < \varepsilon^{2n}$. Thus we have

$$\|[F, v'_{n+1} - v'_n]\|_{1\otimes_\nabla T} < \varepsilon^{2n}, \quad \|[1 \otimes_\nabla T, v'_n]\| < \varepsilon^{2n}. \tag{4.7}$$

In particular (v'_n) is an approximate unit for the C^*-algebra J_F. By applying Theorem 4.18 with $\mathcal{B} = \mathrm{End}^*_B(E)$, $\mathcal{J} = J_F$, $\mathcal{A} = A$, and $\mathcal{K} = \mathbb{K}(E_B)$ we obtain the approximate unit (v_n) for J_F with the properties 1)-10) of Theorem 4.18.

It is important to notice that this can be achieved without losing the properties (4.7), because these are stable under far out convex combinations. Thus, by Theorem 1.25 we obtain the positive unbounded multiplier

$$\ell^{-1} = c := \sum \varepsilon^{-n} d_n, \qquad d_n = v_{n+1} - v_n$$

for the algebra $\mathcal{J}$, with the property that $[1 \otimes_\nabla T, \ell^{-1}]$ is bounded. Because the (v_n), as an approximate unit for the C^*-algebra J_F, have properties 1)-10) from Theorem 4.18, the argument from the proof of Proposition 4.20 can be repeated to see that the self-adjoint lift $S := \frac{1}{2}(F\ell^{-1} + \ell^{-1}F)$ makes $(\mathcal{A}, E_B, S)$ into an unbounded Kasparov module such that $\mathcal{A}$ has an approximate unit with $[S, u_n] \to 0$ in norm. So 1) of Definition 4.1 is satisfied, and 2) is satisfied by the assumption that $\mathcal{E}_{\mathcal{B}}$ is complete and 3) by assumptions 2) and 3) of Definition 4.22. We now turn to proving 4) of Definition 4.1.

It follows from the properties (4.7) that $[F, \ell^{-1}] \in \mathrm{Lip}(1 \otimes_\nabla T)$. To see this, observe that the finite sums $\sum_{n=0}^{k} \varepsilon^{-n}[F, d_n]$ preserve the domain of $1 \otimes_\nabla T$, and using the properties (4.7) we find that

$$\|[1 \otimes_\nabla T, [F, \ell^{-1}]]\| = \|\sum_n \varepsilon^{-n}[1 \otimes_\nabla T, [F, d_n]]\| \leq \sum_n \varepsilon^{-n}\|[1 \otimes_\nabla T, [F, d_n]]\| \leq \sum \varepsilon^n < \infty.$$

Now compute on $\mathrm{Dom}\, 1\otimes_\nabla T$

$$\begin{aligned}[1\otimes_\nabla T, S](S\pm i)^{-1} &= [1\otimes_\nabla T, S](S\pm i)^{-1} = [1\otimes_\nabla T, F\ell^{-1} - [F,\ell^{-1}]](S\pm i)^{-1}\\ &= [1\otimes_\nabla T, F\ell^{-1}](S\pm i)^{-1} - [1\otimes_\nabla T, [F,\ell^{-1}]](S\pm i)^{-1},\end{aligned}$$

by which it suffices to show that $[1\otimes_\nabla T, F\ell^{-1}](S\pm i)^{-1}$ is bounded. Note that we may assume that $\|[F,\ell^{-1}]\|<1$ and thus that $F\ell^{-1}\pm i : \mathrm{Im}\,\ell \to E_B$ is surjective and has adjointable inverse $(F\ell^{-1}\pm i)^{-1}$. Then by the resolvent equation

$$(S\pm i)^{-1} = (F\ell^{-1} - \frac{1}{2}[F,\ell^{-1}]\pm i)^{-1} = (F\ell^{-1}\pm i)^{-1} + \frac{1}{2}(F\ell^{-1}\pm i)^{-1}[F,\ell^{-1}](S\pm i)^{-1},$$

and it suffices to show that $[1\otimes_\nabla T, F\ell^{-1}](F\ell^{-1}\pm i)^{-1}$ is bounded on $\mathrm{Dom}\, 1\otimes_\nabla T$. Then

$$\begin{aligned}[1\otimes_\nabla T, F\ell^{-1}](F\ell^{-1}\pm i)^{-1} &= [1\otimes_\nabla T, F]\ell^{-1}(F\ell^{-1}+i)^{-1} + F[1\otimes_\nabla T,\ell^{-1}](F\ell^{-1}\pm i)^{-1}\\ &= [1\otimes_\nabla T, F](F\pm i\ell)^{-1} + F[1\otimes_\nabla T,\ell^{-1}](F\ell^{-1}\pm i)^{-1},\end{aligned}$$

which is bounded on $\mathrm{Dom}\, 1\otimes_\nabla T$. Therefore $(\mathcal{A}, \mathcal{E}_\mathcal{B}, S, \nabla)$ has the required properties. $\square$

4.4 Lifting Kasparov modules to correspondences

In view of Theorem 4.23, in order to lift a pair of Kasparov modules (A, E_B, G_1) and (B, F_C, G_2) so that we can construct their Kasparov product, we need to find an unbounded representative $(\mathcal{B}, F_C, T)$ such that (A, E_B, G_1) admits a compatible pair $(\mathcal{E}_\mathcal{B}, \nabla)$ of a complete projective submodule and connection in the sense of Definition 4.22. We begin with a Lemma about a special subalgebra of the multipliers of the linking algebra. Recall that the linking algebra $\mathcal{L}(E_B)$ of the C^*-module E_B is the algebra of compact endomorphisms of $E_B\oplus B$.

Lemma 4.24. *Let E_B be a C^*-module and $A\subset \mathrm{End}^*_B(E)$ an essential C^*-subalgebra with self-adjoint contractive approximate unit (u_n^A), and let (u_n^B) be an approximate unit for B. Then the collection of operators*

$$\mathcal{L}_A(E_B) := \left\{\begin{pmatrix} a+K & |e\rangle \\ \langle f| & b\end{pmatrix} : a\in A,\ K\in\mathbb{K}(E_B),\ b\in B,\ e,\ f\in E_B\right\}\subset \mathrm{End}^*_B(E\oplus B), \quad (4.8)$$

is a C^*-subalgebra containing the linking algebra $\mathcal{L}(E_B)$ as an ideal and $u_n := \begin{pmatrix} u_n^A & 0\\ 0 & u_n^B\end{pmatrix}$ is an approximate unit for $\mathcal{L}_A(E_B)$. The algebra $\mathcal{L}_{A^+}(E_{B^+})$ coincides with the unitisation $\mathcal{L}_A(E_B)^+$.

Proof. A quick computation shows that $\mathcal{L}_A(E_B)$ $*$-algebra. Since $\mathbb{K}(E_B)$ is an ideal in $\mathrm{End}^*_B(E)$ it follows that $A+\mathbb{K}(E_B)$ is a C^*-algebra. Moreover, because A is essential, $[AE_B]=E_B$, it follows that $u_n^A\langle e| = \langle u_n^A e| \to \langle e|$ and $|e\rangle u_n^A = |u_n^A e\rangle \to |e\rangle$ for all $e\in E_B$. Thus $u_n^A K\to K$ for all $K\in\mathbb{K}(E_B)$, and u_n^A is approximate unit for $A+\mathbb{K}(E_B)$. Using the corresponding properties for u_n^B, the same argument shows that $\begin{pmatrix} u_n^A & 0\\ 0 & u_n^B\end{pmatrix}$ is an approximate unit for $\mathcal{L}_A(E)$. The statement on the unitisation is immediate. $\square$

We now recall a definition from [18], where the notion of connection was introduced in the bounded picture of KK-theory.

Definition 4.25. Let (B, F_C, G) be a (B, C) Kasparov module and E_B a C^*-module over B. An operator $G \in \mathrm{End}^*_C(E \otimes_B F_C)$ is a G_2-*connection* if for each $e \in E$ the operator

$$\left[\begin{pmatrix} G & 0 \\ 0 & G_2 \end{pmatrix}, \begin{pmatrix} 0 & |e\rangle \\ \langle e| & 0 \end{pmatrix}\right] \in \mathrm{End}^*_C(E \tilde{\otimes}_B F_C \oplus F_C),$$

is compact.

It is well known that G_2-connections always exist, by realising E_B as a complemented submodule of $\mathcal{H}_{B^+}$ via $v : E_B \to \mathcal{H}_{B^+}$ and defining $G := v^* \varepsilon \mathrm{diag}(G_2) v =: v^* G_{2,\varepsilon} v$. It is useful to observe that in $\mathrm{End}^*_C(E \tilde{\otimes}_B F \oplus F)$ we have the identity

$$\begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} = \sum_{i \in \hat{\mathbb{Z}}} \begin{pmatrix} 0 & |\gamma(x_i)\rangle \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \langle x_i| & 0 \end{pmatrix},$$

which can also be written as a matrix product

$$\begin{pmatrix} 0 & |x_i\rangle \\ 0 & 0 \end{pmatrix}^t_{i \in \hat{\mathbb{Z}}} \cdot \left(\varepsilon \mathrm{diag} \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & 0 \\ \langle x_i| & 0 \end{pmatrix}_{i \in \hat{\mathbb{Z}}},$$

of a row, a diagonal matrix and a column.

Lemma 4.26. *Let $A \subset \mathrm{End}^*_B(E_B)$ be an essential subalgebra with approximate unit $(\tilde{u}_n)$ and (B, F_C, G_2) a Kasparov module. Let $(x_i)_{i \in \hat{\mathbb{Z}}}$ be a homogenous frame for E_B with stabilisation isometry $v : E_B \to \mathcal{H}_{B^+}$ and associated G_2-connection $G = v^* \varepsilon \mathrm{diag}(G_2) v$. Then there is a G-quasicentral approximate unit (u_n) for A contained in the convex hull $\mathscr{C}(\tilde{u}_n)$: that is $[G, u_n] \to 0$ in norm.*

Proof. This will follow from a direct application of Theorem 4.15. Consider the algebra $\mathcal{L}_{A^+}(E_{B^+})$ and the ideal $\mathcal{L}_A(E_{B^+})$ as defined in (4.8), with its approximate unit $u'_n = \begin{pmatrix} \tilde{u}_n & 0 \\ 0 & 1 \end{pmatrix}$. We view the row, respectively column,

$$|x\rangle := \begin{pmatrix} 0 & |x_i\rangle \\ 0 & 0 \end{pmatrix}^t_{i \in \hat{\mathbb{Z}}}, \quad \langle x| := \begin{pmatrix} 0 & 0 \\ \langle x_i| & 0 \end{pmatrix}_{i \in \hat{\mathbb{Z}}},$$

as elements in $\mathrm{End}^*_{\mathcal{L}_{A^+}(E_{B^+})}(\mathcal{H}_{\mathcal{L}_{A^+}(E_{B^+})})$. By Theorem 4.15, the convex hull $\mathscr{C}(\tilde{u}_n)$ contains an $\langle x|, |x\rangle$-quasicentral approximate unit $u''_n = \begin{pmatrix} u_n & 0 \\ 0 & 1 \end{pmatrix}$. Observe that

$$\left[u''_n \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{L}_{A^+}(E_{B^+})}}, \varepsilon \cdot \mathrm{diag} \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix}\right] = 0.$$

Writing u''_n for $u''_n \cdot \mathrm{Id}_{\mathcal{H}_{\mathcal{L}_{A^+}(E_{B^+})}}$ when necessary, we can compute

$$\begin{aligned}
\begin{pmatrix} [G, u_n] & 0 \\ 0 & 0 \end{pmatrix} &= \left[\begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_n & 0 \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 0 & |x\rangle \\ 0 & 0 \end{pmatrix} \cdot \left(\varepsilon \cdot \mathrm{diag} \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & 0 \\ \langle x| & 0 \end{pmatrix}, \begin{pmatrix} u_n & 0 \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} 0 & |x\rangle \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} u_n & 0 \\ 0 & 1 \end{pmatrix}\right] \left(\varepsilon \cdot \mathrm{diag} \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & 0 \\ \langle x| & 0 \end{pmatrix} \\
&\qquad + \begin{pmatrix} 0 & |x\rangle \\ 0 & 0 \end{pmatrix} \cdot \left(\varepsilon \cdot \mathrm{diag} \begin{pmatrix} 0 & 0 \\ 0 & G_2 \end{pmatrix}\right) \cdot \left[\begin{pmatrix} 0 & 0 \\ \langle x| & 0 \end{pmatrix}, \begin{pmatrix} u_n & 0 \\ 0 & 1 \end{pmatrix}\right]
\end{aligned}$$

which converges to 0 in norm by Theorem 4.15. Therefore $[G, u_n]$ converges to 0 in norm. $\square$

Given a $*$-homomorphism $B \to \mathrm{End}^*_C(F)$, the algebra $\mathrm{End}^*_B(E_B \oplus B)$ is naturally represented on the C^*-module $E\tilde{\otimes}_B F \oplus F$. In particular, the linking-type algebras defined in Lemma 4.24 act on $E\tilde{\otimes}_B F \oplus F$. We prepare the setting for the proof of our final lifting result, which is yet another application of Theorem 4.18. Recall that given a bounded Kasparov module (A, E_B, G_1), we define J_{G_1} to be the C^*-algebra generated by $\mathbb{K}(E_B)$ and $\mathrm{Id}_{E_B} - G_1^2$.

Lemma 4.27. *Let (A, E_B, G_1) and (B, F_C, G_2) be Kasparov modules and $(x_i)_{i\in\hat{\mathbb{Z}}}$ a homogenous frame for E_B with stabilisation isometry $v : E_B \to \mathcal{H}_{B^+}$ and associated G_2-connection $G = v^*G_{2,\varepsilon}v$. Write $\tilde{G} := \left(\begin{smallmatrix} G & 0 \\ 0 & G_2 \end{smallmatrix}\right)$ and $\tilde{G}_1 := \left(\begin{smallmatrix} G_1\otimes 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. As in Lemma 4.24 consider the algebras*

$$A_0 := \mathcal{L}_A(E_B), \quad A_1 := \mathcal{L}_{J_{G_1}}(E_B).$$

For $p = 0, 1$ define the C^*-algebras $\tilde{J}_p$ generated by $\mathbb{K}(E_B) \otimes 1 \oplus \mathbb{K}(F_C)$, $[\tilde{G}, A_p], [\tilde{G}, \tilde{G}_1]$ and $1 - \tilde{G}^2$ on $E\tilde{\otimes}F_C \oplus F_C$. Let $\tilde{B}_p$ be the C^*-algebras generated by $\mathrm{Id}_{E\tilde{\otimes}_B F\oplus F}$, A_p and $\tilde{J}_p$. Let K_p be the C^*-subalgebra of $\tilde{B}_p$ generated by $A_p\tilde{J}_p$ and $\tilde{J}_pA_p$. Finally, define $J_p = K_p + \tilde{J}_p$. Then:
1) A_p admits $\tilde{G}, G_1 \otimes 1$-quasicentral approximate units u_n^p of the form

$$(u_n^0) = \begin{pmatrix} \tilde{u}_n^A & 0 \\ 0 & \tilde{u}_n^B \end{pmatrix}, \quad (u_n^1) = \begin{pmatrix} \tilde{w}_n^1 & 0 \\ 0 & \tilde{u}_n^B \end{pmatrix},$$

where $(\tilde{u}_n^A)$, $(\tilde{u}_n^B)$, $(\tilde{w}_n^1)$ are approximate units for A, B and J_{G_1} respectively;
2) J_p admits an approximate unit $(v_n^p) = \left(\begin{smallmatrix} \tilde{v}_n^p & 0 \\ 0 & \tilde{w}_n^2 \end{smallmatrix}\right)$ where $(\tilde{v}_n^p)$ is an approximate unit for the algebra generated by $[G, A_p]$, $[G, G_1 \otimes 1]$, $1 - G^2$ and $\mathbb{K}(E_B) \otimes 1$ and $(\tilde{w}_n^2)$ is an approximate unit for J_{G_2};
3) K_p is an ideal in $\tilde{B}_p$;
4) $A_pK_p = K_pA_p = K_p$, that is K_p is A_p-essential;
5) A_pJ_p, $J_pA_p \subset K_p \subset J_p$;
6) $[G, A_p] \subset K_p$.

Proof. To prove 1), we show that both A_0 and A_1 admit approximate units (u_n^0), (u_n^1) that are quasicentral for $\tilde{G}$ and $\tilde{G}_1 \otimes 1$. Since both A and J_{G_1} are essential on E_B, by Lemma 4.26 there exist approximate units $(\tilde{u}_n^A)$, $(\tilde{w}_n^1)$ for A and J_{G_1}, respectively, that are quasicentral for G. Since $[G_1, A]$, $[G_1, J_{G_1}] \subset \mathbb{K}(E_B)$, the approximate units $(\tilde{u}_n^A)$, $(\tilde{w}_n^1)$ can be chosen G_1-quasicentral as well by Lemma 4.19. For the same reason B admits a G_2 quasicentral approximate unit $(\tilde{u}_n^B)$. By Lemma 4.24 setting

$$(u_n^0) := \begin{pmatrix} \tilde{u}_n^A & 0 \\ 0 & \tilde{u}_n^B \end{pmatrix}, \quad (u_n^1) := \begin{pmatrix} \tilde{w}_n^1 & 0 \\ 0 & \tilde{u}_n^B \end{pmatrix},$$

yields $\tilde{G}$, $G_1 \otimes 1$-quasicentral approximate units for A_0 and A_1.

To prove 2) we first show that any $\tilde{G}$-quasicentral approximate unit (v_n^p) for $\tilde{J}_p$ is an approximate unit for J_p. Since $\tilde{J}_p$ is essential (it contains $\mathcal{L}(E_B)$), existence of such (v_n^p) is guaranteed by Lemma 4.26. It is clear that (v_n^p) is an approximate unit for $\tilde{J}_p$ and $\tilde{J}_pA_p$. For $A_p\tilde{J}_p$ it suffices to show that

$$v_n^p a[\tilde{G}, b] = v_n^p[\tilde{G}, ab] - v_n^p[\tilde{G}, a]b \to 0,$$
$$v_n^p a[\tilde{G}, \tilde{G}_1] = [\tilde{G}, v_n^p a\tilde{G}_1] - [\tilde{G}, v_n^p]a\tilde{G}_1 - v_n^p[\tilde{G}, a]\tilde{G}_1 \to 0,$$

$$v_n^p a(1-\tilde{G}^2) = v_n^p(1-\tilde{G}^2)a + [\tilde{G}, v_n^p][\tilde{G}, a] - \tilde{G} v_n[\tilde{G}, a] - v_n^p[\tilde{G}, a]\tilde{G} \to 0,$$

which all follow from $\tilde{G}$-quasicentrality of (v_n^p).

Now we proceed to the statement of 2). Consider the subalgebra of $\mathrm{End}^*_C(E\tilde{\otimes}F)$ generated by $[G, A_p]$, $[G, G_1 \otimes 1]$, $1-G^2$ and $\mathbb{K}(E_B)\otimes 1$, and choose a G-quasicentral approximate unit $(\tilde{v}_n^p)$ for this algebra, as well as a G_2-quasicentral approximate unit $(\tilde{w}_n^2)$ for the algebra J_{G_2}. Then $(v_n^p) := \mathrm{diag}(\tilde{v}_n^p, \tilde{w}_n^2)$ is clearly an approximate unit for the diagonal entries of J_p. For the off-diagonal entries, it suffices to show that for all $e \in E_B$, (v_n^p) is a two-sided approximate unit for the operators

$$\left[\begin{pmatrix} G & 0 \\ 0 & G_2 \end{pmatrix}, \begin{pmatrix} 0 & |e\rangle \\ 0 & 0 \end{pmatrix}\right] = \sum_{i\in\hat{\mathbb{Z}}} \begin{pmatrix} 0 & |x_i\rangle[G_2, \langle x_i, e\rangle] \\ 0 & 0 \end{pmatrix}. \tag{4.9}$$

Since $e \in E_B$, the series $\sum_{i\in\hat{\mathbb{Z}}}[G_2, \langle x_i, e\rangle]^*[G_2, \langle x_i, e\rangle]$ is norm convergent in $\mathbb{K}(F_C) \subset J_{G_2}$. Therefore (v_n^p) is a right approximate unit for the operator in Equation (4.9). On the other hand, since the series $\sum_{i\in\hat{\mathbb{Z}}}|x_i\rangle[G_2, \langle x_i, e\rangle]$ is norm convergent and $(\tilde{v}_n^p)$ is an approximate for $\mathbb{K}(E_B)\otimes 1$, it follows that (v_n^p) is a left approximate unit for operators of the form (4.9).

For 3) it is sufficient to observe that A_p^2 is dense in A_p and $\tilde{J}_p^2$ is dense in $\tilde{J}_p$. Thus the subalgebra generated by $A_p\tilde{J}_p$, $\tilde{J}_pA_p$ is indeed a two-sided ideal in $\tilde{B}_p$.

For 4), to show that $A_pK_p = K_pA_p = K_p$ it suffices to show that $\tilde{J}_pA_p \subset A_pK_p$ and $A_p\tilde{J}_p \subset K_pA_p$. To show that $\tilde{J}_pA_p \subset A_pK_p$, it suffices to show that for all $a, b \in A_p$ we have

$$u_n^p[\tilde{G}, a]b \to [\tilde{G}, a]b, \quad u_n^p[\tilde{G}, G_1\otimes 1]a \to [\tilde{G}, G_1 \otimes 1]a, \quad u_n^p(1-\tilde{G}^2)a \to (1-\tilde{G}^2)a.$$

These limits all follow directly from $\tilde{G}, G_1\otimes 1$ quasicentrality of (u_n^p).

Property 5) is immediate from the definition of K_p and J_p.

Property 6) follows from the convergence $u_n^p[G, a] = [G, u_n^p a] - [G, u_n^p]a \to [G, a]$ for all $a \in A_p$, since $u_n^p[G, a] \in A_pJ_p \subset K_p$. □

Lemma 4.28. *Let (A, E_B, G) be a Kasparov module, $\{a_i\} \subset A$ a bounded self-adjoint total subset for A, and $\mathscr{A} := \mathrm{alg}\{a_i\}$, the (non-closed) subalgebra generated by $\{a_i\}$. There exists a bounded self-adjoint total subset $\{c_j\}$ for J_G, as defined in Lemma 4.5, such that $\mathscr{J}$, the (non-closed) linear span of the $\{c_j\}$, satisfies*

$$\mathscr{A}\mathscr{J},\ \mathscr{J}\mathscr{A},\ G\mathscr{J},\ \mathscr{J}G \subset \mathscr{J}.$$

Proof. Choose some total subset $\{c_j'\}$ for J_G and let $\mathscr{J}_0$ be the (non-closed) algebra generated by $(1-G^2)$, $\{c_j'\}$ and $[G, \mathscr{A}]$. Now consider the (non-closed) algebra $\mathscr{J}_1$ and linear subspace $\mathscr{J}$ given by

$$\mathscr{J}_1 := \mathrm{alg}\{\mathscr{J}_0, \mathscr{A}\mathscr{J}_0, \mathscr{J}_0\mathscr{A}, \mathscr{A}\mathscr{J}_0\mathscr{A}\} \subset J_G, \qquad \mathscr{J} := \mathrm{span}\{G\mathscr{J}_1, \mathscr{J}_1G, G\mathscr{J}_1G\} \subset J_G.$$

Since $\mathscr{A}$ is an algebra, $\mathscr{A}^2 \subset \mathscr{A}$ and thus $\mathscr{A}_0\mathscr{J}_1 \subset \mathscr{J}_1$ and similarly for the reversed product set. Since $\mathscr{J}_1$ is an algebra and thus

$$G^2\mathscr{J}_1 = (G^2-1)\mathscr{J}_1 + \mathscr{J}_1 \subset \mathscr{J}_1$$

we have $G\mathscr{J} \subset \mathscr{J}$ and since $[G, \mathscr{A}] \subset \mathscr{J}_0 \subset \mathscr{J}_1$ we have $\mathscr{A}\mathscr{J} \subset \mathscr{J}$. Since $\mathscr{J}$ is countable, $\{c_j'\}$ can be extended to a bounded self-adjoint set $\{c_j\}$ such that $\mathrm{span}(c_j) = \mathscr{J}$. □

Proposition 4.29. *Let A, B, C be separable C^*-algebras, and let (A, E_B, G_1), (B, F_C, G_2) be essential Kasparov modules. Then (B, F_C, G_2) can be lifted to an unbounded Kasparov module $(\mathcal{B}, F_C, T)$ such that $\mathcal{B}$ has an approximate unit (u_n) with $[T, u_n] \to 0$ in norm, and moreover E_B admits a compatible complete projective $\mathcal{B}$-submodule $\mathcal{E}_{\mathcal{B}} \subset E_B$ in the sense of Definition 4.22.*

Proof. To prove the theorem, we have to lift G_2 to an unbounded representative T with resolvent in J_{G_2}, such that B admits a differentiable subalgebra with an approximate unit (u_n^B) such that $[T, u_n^B] \to 0$ in norm. Moreover, the lift T should also satisfy Definition 4.22. This means we have to provide a column finite frame $(x_i)_{i\in\hat{\mathbb{Z}}}$ for E_B such that the resulting submodule $\mathcal{E}_{\mathcal{B}}$ is complete, as well as satisfying properties 1)-4). These properties imply that, for ∇ the Grassmann connection associated to the frame $(x_i)_{i\in\hat{\mathbb{Z}}}$, the operators G_1 and $(1-G_1^2)^{\frac{1}{2}}$ are elements of $\mathrm{Lip}(1\otimes_\nabla T)$ and the algebras A and J_{G_1} admit differentiable subalgebras $\mathcal{A}$ and $\mathcal{J}_{G_1}$, generated by $(1-G_1^2)$ and $\mathbb{K}(E_B)$, with approximate units $(u_n^{\mathcal{A}})$ and $(w_n^{\mathcal{J}})$ such that $\lim_{n\to\infty}[1\otimes_\nabla T, u_n^{\mathcal{A}}] = \lim_{n\to\infty}[1\otimes_\nabla T, w_n^{\mathcal{J}}] = 0$. In order to achieve this, we once again employ linking-type algebras.

Choose a stabilisation isometry $v : E_B \to \mathcal{H}_{B^+}$. Consider the frame $(x_i)_{i\in\hat{\mathbb{Z}}} := (v^*e_i)_{i\in\hat{\mathbb{Z}}}$ and the corresponding frame approximate unit $(\chi_n) := \sum |x_i\rangle\langle x_i|$. By Lemma 4.26 there exists an approxmate unit $(w_n) \in \mathscr{C}(\chi_n)$ with $[G, w_n] \to 0$ in norm. We proceed with the notation introduced in Lemma 4.27, and apply Theorem 4.18 with

$$\mathcal{B} = \tilde{B}_0 \oplus \tilde{B}_1, \quad \mathcal{J} = J_0 \oplus J_1, \quad F = \tilde{G} = \begin{pmatrix} G & 0 \\ 0 & G_{2,\varepsilon} \end{pmatrix}, \quad \mathcal{A} = A_0 \oplus A_1, \quad \mathcal{K} := K_0 \oplus K_1,$$

which by Lemma 4.27 satisfy $\mathcal{A}\mathcal{J}$, $\mathcal{J}\mathcal{A} \subset \mathcal{K}$, $\mathcal{A}\mathcal{K} = \mathcal{K}\mathcal{A} = \mathcal{K}$ and the algebra $\mathcal{B}$ is unital. Since the approximate units (u_n^0) for A_0 and (w_n) for $\mathbb{K}(E_B)$ are $\tilde{G}$-quasicentral, and (v_n^p) are approximate units for J_p, we can apply Theorem 4.18 by setting $(u_n') := (u_n^0) \oplus (u_n^1)$, $(v_n') = (v_n^0) \oplus (v_n^1)$. In doing so we obtain approximate units (u_n), (v_n) satisfying properties $1)-10)$ from Theorem 4.18. In addition to these properties, with $d_n^p = v_{n+1}^p - v_n^p$, we may also assume that:

11) $\|d_n^1[\tilde{G}, \mathrm{diag}(w_k, 0)]\| < \varepsilon^{2k}$ for all n;
12) $\|d_n^1[\tilde{G}, \mathrm{diag}(w_k, 0)]\| < \varepsilon^{2k}$ for $n \geq k$;
13) $\|[d_n^1, \mathrm{diag}(0, \langle x_i, x_k\rangle)_{i\in\hat{\mathbb{Z}}}]\| < \varepsilon^{2n}$ for $n \geq k$;
14) $\|d_n^1[\tilde{G}, \mathrm{diag}(0, \langle x_i, x_k\rangle)_{i\in\hat{\mathbb{Z}}}]\| < \varepsilon^{2n}$ for $n \geq k$;
15) $\|[d_n^p, \langle x|]\| < \varepsilon^{2n}$.

Properties 11) and 12) can be achieved because $\mathbb{K}(E_B) \otimes 1 \oplus 0 \subset J_1$, so this only requires a further convexity argument when running the proof of Theorem 4.18. Properties 13), 14) and 15) are a direct application of Theorem 4.15 by viewing the columns $\langle x|$ and $(\langle x_i, x_k\rangle)_{i\in\hat{\mathbb{Z}}}$ as elements in $\mathrm{End}^*_{\mathcal{L}(E_B)^+}(\mathcal{H}_{\mathcal{L}(E_B)^+})$.

The (v_n^p) so obtained from Theorem 4.18 define two unbounded multipliers on $E\tilde{\otimes}F_C \oplus F_C$

$$h_p^{-1} = \sum_n c^n d_n^p = \begin{pmatrix} k_p^{-1} & 0 \\ 0 & \ell^{-1} \end{pmatrix}, \quad v_n^p = \begin{pmatrix} \tilde{v}_n^p & 0 \\ 0 & w_n^2 \end{pmatrix}, \quad d_n^p = v_{n+1}^p - v_n^p,$$

which we use to lift $\tilde{G}$ in two ways. That we obtain the same ℓ^{-1} for $p = 0, 1$ follows from the form of (v_n^p), that is, the approximate unit (w_n^2) for J_{G_2} occurs in both lower right corners.

From the specific form of the (u_n^p) and (v_n^p), cf. Lemma 4.27 2) and 3), it follows that we obtain new approximate units (u_n^A) for A, (w_n^1) for J_{G_1} and (v_n^p) for the algebras generated by $[G, A_p]$, $[G, G_1 \otimes 1]$, $1 - G^2$ and $\mathbb{K}(E_B) \otimes 1$.

This allows us to define unbounded lifts $\tilde{T}_0 := Gk_0^{-1}$ and $\tilde{T}_1 := Gk_1^{-1}$. Choosing total subsets $\{a_i\}$ and $\{c_j\}$ as in Lemma 4.28 one proves, as in Proposition 4.20, that A admits a differentiable subalgebra $\mathcal{A}$ for which (u_n^A) is an approximate unit with $\lim[\tilde{T}_0, u_n^A] \to 0$ in norm. For J_{G_1} the same statement holds with respect to $\tilde{T}_1$. Moreover, 11) and 12) ensure that $[\tilde{T}_1, w_n] \to 0$ in norm as well, again with the same proof as Proposition 4.20. Furthermore properties 13) and 14) guarantee that the columns $[T, \langle x_i, x_k\rangle]_{i\in\hat{\mathbb{Z}}}$ are elements of $\mathcal{H}_{\mathrm{End}_C^*(F)}$. That is, the frame (x_i) is column finite for T.

It must be noted that our method does not allow us to obtain a uniform bound on the norms of these columns, and thus we are not able to produce a projection operator in $\mathrm{End}^*_{\mathcal{B}}(\mathcal{H}_{\mathcal{B}})$. Later, we will see that we do obtain a complete projective module.

We now compare the connection operator $1 \otimes_\nabla T$ of the frame $(x_i)_{i\in\hat{\mathbb{Z}}}$ to the operators $\tilde{T}_p$, $p = 0, 1$, which we have used to lift the bounded connection G. Condition 15) guarantees that $[h_p^{-1}, \langle x|]$ is a bounded operator. Since

$$\left[h_p^{-1}, \langle x|\right] = \begin{pmatrix} 0 & 0 \\ \langle x_i|k_p^{-1} - \ell^{-1}\langle x_i| & 0 \end{pmatrix}_{i\in\hat{\mathbb{Z}}}, \tag{4.10}$$

it follows that $v \operatorname{Im} k_p \subset \operatorname{Im} \ell$ and $v^* \operatorname{Im} \ell \subset \operatorname{Im} k_p$. We wish to show that the difference $\tilde{T}_p - 1 \otimes_\nabla T$ is bounded. To this end we compute

$$v^* G_{2,\varepsilon}\ell^{-1} v - Gk_p^{-1} = |x\rangle G_{2,\varepsilon}\ell^{-1}\langle x| - |x\rangle G_{2,\varepsilon}\langle x|k_p^{-1} = |x\rangle G_{2,\varepsilon}(\ell^{-1}\langle x| - \langle x|k_p^{-1}), \tag{4.11}$$

which is bounded by construction. Note that this implies the self-adjointness of $1 \otimes_\nabla T$, and also that $\tilde{T}_0$ and $\tilde{T}_1$ have the same domain.

It now follows that $\mathcal{A}$ and $\mathcal{J}$ are the closures of $\{a_i\}$ and $\{c_j\}$ inside $\mathrm{Lip}(1 \otimes_\nabla T)$ respectively, and thus by Lemma 4.28 it follows that $\mathcal{A}\mathcal{J}, \mathcal{J}\mathcal{A}, F\mathcal{J}, \mathcal{J}F \subset \mathcal{J}$.

Since $[\tilde{T}_1, u_n] \to 0$ in norm and $u_n \to 1$ strictly on $E\tilde{\otimes}_B F$, it follows that $[1 \otimes_\nabla T, u_n] \to 0$ strictly, and so is a bounded sequence. Hence $p[T_\varepsilon, vu_nv^*]p = v[v^*T_\varepsilon v, u_n]v^* \to 0$ on a dense subspace of $\mathcal{H}_{B^+}\tilde{\otimes}F$, and by boundedness of the sequence, strictly on all of $\mathcal{H}_{B^+}\tilde{\otimes}F$. Hence the frame (x_i) defines a complete projective submodule $\mathcal{E}_{\mathcal{B}} \subset E_B$, which also (and independently) proves the self-adjointness of $1 \otimes_\nabla T$.

Lastly, we must show that there are approximate units $(u_n^{\mathcal{A}}) \in \mathscr{C}(u_n^A)$ for $\mathcal{A}$ and $(w_n)^{\mathcal{J}} \in \mathscr{C}(w_n^1)$ for J_{G_1} that satisfy

$$\lim[1 \otimes_\nabla T, u_n^{\mathcal{A}}] = \lim[1 \otimes_\nabla T, w_n^{\mathcal{J}}] \to 0, \tag{4.12}$$

in norm. Observe that we can obtain the strict convergences of Equation (4.12), for both (u_n^A) and (w_n^1) converge strictly to 1 on $E\tilde{\otimes}_B F$ and the lifts T_0 and T_1 are bounded perturbations of $1 \otimes_\nabla T$.

The column in Equation (4.10) is an element of $\mathrm{End}^*_{\tilde{B}_p}(\mathcal{H}_{\tilde{B}_p}, J_p)$, because for each i we have $\left[h_p^{-1}, \langle x_i|\right] \in J_p$. Thus by Theorem 4.16 there exist $[h_p^{-1}, \langle x|]$-quasicentral approximate units in the convex hull $\mathscr{C}(u_n^p)$, which at the same time remain $\tilde{G}$-quasicentral. The resulting approximate units will again be of the form indicated in Lemma 4.27, 1). Now compute

$$\begin{pmatrix}[Gk_p^{-1}-v^*G_{2,\varepsilon}\ell^{-1}v,u_n^{\mathcal{A}}] & 0\\ 0 & 0\end{pmatrix}=\left[|x\rangle\begin{pmatrix}0 & 0\\ 0 & G_{2,\varepsilon}\end{pmatrix}[h_p^{-1},\langle x|],\begin{pmatrix}u_n^{\mathcal{A}} & 0\\ 0 & u_n^B\end{pmatrix}\right]$$
$$=\left[|x\rangle\tilde{G}[h_p^{-1},\langle x|],\begin{pmatrix}u_n^{\mathcal{A}} & 0\\ 0 & u_n^B\end{pmatrix}\right]\to 0,$$

and the same computation works for $(w_n^{\mathcal{J}})$. □

Theorem 4.30. *Let A,B,C be separable C^*-algebras, $x\in KK(A,B)$ and $y\in KK(B,C)$. There exists an unbounded Kasparov module $(\mathcal{B},F_C,T)$ representing y and a correspondence $(\mathcal{A},\mathcal{E}_{\mathcal{B}},S,\nabla)$ for $(\mathcal{B},F_C,T)$ representing x. Consequently $(\mathcal{A},E\tilde{\otimes}_BF_C,S\otimes 1+1\otimes_\nabla T)$ represents the Kasparov product $x\otimes_B y$.*

Proof. First represent x and y by essential Kasparov modules. By Proposition 4.29, for any pair of essential Kasparov modules, the second module can be lifted such that the first module admits a compatible complete projective submodule. Now apply Theorem 4.23. □

By the same method, and lifting simultaneously with $n+1$ Kasparov modules instead of 2, one can prove that for classes $x_n,\dots,x_0$ with $x_j\in KK(A_{j+1},A_j)$, one can find an unbounded Kasparov module $(\mathcal{A}_1,E_{A_0},T_0)$ representing x_0 and correspondences $(\mathcal{A}_{j+1},\mathcal{E}_{\mathcal{A}_j},T_j,\nabla_j)$ representing x_j such that for each $1\le j\le n$, $(\mathcal{A}_{j+1},\mathcal{E}_{\mathcal{A}_j},T_j,\nabla_j)$ is compatible with

$$\left(\bigotimes_{i=1}^{j}(\mathcal{A}_{i+1},\mathcal{E}_{\mathcal{A}_i},T_i,\nabla_i)\right)\otimes(\mathcal{A}_1,E_{A_0},T_0).$$

A Weakly anticommuting operators

Definition A.1 (cf. [31])**.** Let E_B be a graded C^*-module and s,t odd self-adjoint regular operators such that for all $\lambda,\mu\in\mathbb{R}\setminus\{0\}$:
1) there is a core X for t such that $(s\pm\lambda i)^{-1}X\subset\operatorname{Dom}t$;
2) $t(s\pm\lambda i)^{-1}X\subset\operatorname{Dom}s$;
3) $[s,t](s\pm\lambda i)^{-1}$ is bounded on X.

Then we say that the pair (s,t) *weakly anticommutes*, or that t *anticommutes weakly with* s. Note that this relation is not symmetric in s and t, and that the graded commutator is defined on $\operatorname{Im}(s+\lambda i)^{-1}X$.

It was proved in [31] that the sum of weakly anticommuting operators occurring in odd Kasparov products is self-adjoint and regular on $\operatorname{Dom}s\cap\operatorname{Dom}t$. Since we are concerned here with the general graded case, a few words are in order.

Lemma A.2. *If (s,t) is a weakly anticommuting pair then the operators $(s\pm\lambda i)^{-1}$ preserve the domain of t and $[s,t](s\pm\lambda i)^{-1}$ is bounded on* $\operatorname{Dom}t$*. Consequently*

$$s\left((s-\lambda i)^{-1}\operatorname{Dom}t\right)\subset\operatorname{Dom}t,\quad t\left(\operatorname{Im}(s-\lambda i)^{-1}\operatorname{Dom}t\right)\subset\operatorname{Dom}s.\tag{A.1}$$

Therefore $[s,t]$ is defined on $(s-\lambda i)^{-1}\operatorname{Dom}t=\operatorname{Im}(s-\lambda i)^{-1}(t-\mu i)^{-1}$.

Proof. For $x \in X$, the commutator $[t, (s \pm \lambda i)^{-1}]$ can be expanded as

$$\begin{aligned}[t, (s \pm \lambda i)^{-1}]x &= (t(s \pm \lambda i)^{-1}x + (s \mp \lambda i)^{-1}tx = (s \mp \lambda i)^{-1}((s \mp \lambda i)t + t(s \pm \lambda i))(s \pm \lambda i)^{-1}x \\ &= (s \mp \lambda i)^{-1}[s, t](s \pm \lambda i)^{-1}x,\end{aligned}$$

and by 2) of Definition A.1 this operator is bounded. Thus by [22, Proposition 2.1] $(s \pm \lambda i)^{-1}$ preserves the domain of t. Since X is a core for t, for every $x \in \operatorname{Dom} t$ we can choose a sequence $x_n \in X$ such that $x_n \to x$ and $tx_n \to tx$. Then

$$\begin{aligned}t(s - \lambda i)^{-1}x_n &= -(s + \lambda i)^{-1}tx_n + (s + \lambda i)^{-1}[s, t](s - \lambda i)^{-1}x_n \\ &\to -(s + \lambda i)^{-1}tx + (s + \lambda i)^{-1}[s, t](s - \lambda i)^{-1}x \in \operatorname{Dom} s,\end{aligned}$$

which is a Cauchy sequence, so the limit equals $t(s - \lambda i)^{-1}x \in \operatorname{Dom} s$. The statement that $s\left((s - \lambda i)^{-1} \operatorname{Dom} t\right) \subset \operatorname{Dom} t$ follows directly from the equality

$$s(s - \lambda i)^{-1}(t - \mu i)^{-1} = (t - \mu i)^{-1} + \lambda i(s - \lambda i)^{-1}(t - \mu i)^{-1}$$

This proves (A.1). □

Lemma A.3. *For $|\lambda| > 1$ there is a constant C such that $\|[s, t](s \pm \lambda i)^{-1}\| < C$. Thus, for $|\lambda|$ sufficiently large, we may assume $\varepsilon := \|(s \mp \lambda i)^{-1}[s, t](s \pm \lambda i)^{-1}\| < 1$.*

Proof. Let $C := 2\|[s, t](s + i)^{-1}\|$ and write

$$(s + \lambda i)^{-1} = (s + i)^{-1} - (s + i)^{-1}(\lambda - 1)i(s + \lambda i)^{-1}.$$

Using this, estimate

$$\begin{aligned}\|[s, t](s + \lambda i)^{-1}\| &\leq \|[s, t](s + i)^{-1} - [s, t](s + i)^{-1}(\lambda - 1)i(s + \lambda i)^{-1}\| \\ &\leq \|[s, t](s + i)^{-1}\| + \|[s, t](s + i)^{-1}\|\|(\lambda - 1)i(s + \lambda i)^{-1}\| \\ &\leq \frac{C}{2}\left(1 + \frac{|\lambda - 1|}{|\lambda|}\right) < C,\end{aligned}$$

since $|\lambda| > 1$. Consequently, using that $\|(s \pm \lambda i)^{-1}\| \leq \frac{1}{|\lambda|}$, we find that

$$\|(s \mp \lambda i)^{-1}[s, t](s \pm \lambda i)^{-1}\| \leq \frac{C}{|\lambda|} < 1,$$

for $|\lambda|$ sufficiently large. □

Theorem A.4 (cf.[31])**.** *If (s, t) weakly anitcommutes, then $s + t$ is closed, self-adjoint and regular on $\operatorname{Dom} s \cap \operatorname{Dom} t$, and $\operatorname{Im}(s \pm i)^{-1}(t \pm i)^{-1}$ is a core for $s + t$. The same holds for $s - t$.*

Proof. It was shown in [31] that the sum $s + t$ of such operators is closed, self-adjoint and regular on $\operatorname{Dom} s \cap \operatorname{Dom} t$ by a localisation argument. However, to get the statement on the core, we proceed by adapting the spectral argument given in [42]. Choose λ large enough as in Lemma A.3. The operators

$$x := (t + \mu i)^{-1} - (s - \lambda i)^{-1}, \quad y := x^* = (t - \mu i)^{-1} - (s + \lambda i)^{-1},$$

have dense range. This follows because xx^* is strictly positive by esitmating

$$\begin{aligned}
xx^* &= (\mu^2+t^2)^{-1} + (\lambda^2+s^2)^{-1} - (t+\mu i)^{-1}(s+\lambda i)^{-1} - (s-\lambda i)^{-1}(t-\mu i)^{-1} \\
&= (\mu^2+t^2)^{-1} + (\lambda^2+s^2)^{-1} - (t+\mu i)^{-1}(s+\lambda i)^{-1}([s,t]-2\lambda\mu)(s-\lambda i)^{-1}(t-\mu i)^{-1} \\
&= (\mu^2+t^2)^{-1} + (\lambda^2+s^2)^{-1} + 2\lambda\mu(t+\mu i)^{-1}(\lambda^2+s^2)^{-1}(t-\mu i)^{-1} \\
&\qquad\qquad - (t+\mu i)^{-1}(s+\lambda i)^{-1}[s,t](s-\lambda i)^{-1}(t-\mu i)^{-1} \\
&\geq (1-\varepsilon)(\mu^2+t^2)^{-1} + (\lambda^2+s^2)^{-1} + 2\lambda\mu(t+\mu i)^{-1}(\lambda^2+s^2)^{-1}(t-\mu i)^{-1},
\end{aligned}$$

using A.1 and Lemma A.3. Since the latter operator is strictly positive, so is xx^* by [38, Corollary 10.2]. The same holds for x^*x. The rest of the proof now follows by using the factorisations

$$x = (s+t+(\mu-\lambda)i - (s+\lambda i)^{-1}([s,t]-2\lambda\mu))(s-\lambda i)^{-1}(t-\mu i)^{-1},$$

$$y = (s+t+(\lambda-\mu)i - (s-\lambda i)^{-1}([s,t]-2\lambda\mu))(s+\lambda i)^{-1}(t+\mu i)^{-1},$$

as in [42, Theorem 6.18], and applying [42, Lemma 6.1.7]. The statement for $s-t$ is obtained by observing that $(s,-t)$ weakly anticommutes. □

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