

PSEUDO-RIEMANNIAN SPECTRAL TRIPLES AND THE HARMONIC OSCILLATOR

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Abstract

We define pseudo-Riemannian spectral triples, an analytic context broad enough to encompass a spectral description of a wide class of pseudo-Riemannian manifolds, as well as their noncommutative generalisations. Our main theorem shows that to each pseudo-Riemannian spectral triple we can associate a genuine spectral triple, and so a K -homology class. With some additional assumptions we can then apply the local index theorem. We give a range of examples and some applications. The example of the harmonic oscillator in particular shows that our main theorem applies to much more than just classical pseudo-Riemannian manifolds.

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1. INTRODUCTION

Spectral triples provide a way to extend Riemannian geometry to noncommutative spaces, retaining the connection to the underlying topology via K -homology. In this paper we provide a definition of pseudo-Riemannian spectral triple, enabling a noncommutative analogue of pseudo-Riemannian geometry, and show that we can still maintain contact with the underlying topology. We do this by ‘Wick rotating’ to a spectral triple analogue of our pseudo-Riemannian spectral triple. Below we discuss the class of pseudo-Riemannian manifolds which give examples of our construction: a key point is that we work with Hilbert spaces not Krein spaces, and so the construction relies on a global splitting of the tangent bundle into timelike and spacelike sub-bundles.

Our main theorem, Theorem 5.1, states that one can associate a spectral triple to a pseudo-Riemannian spectral triple via Wick rotation. Under additional assumptions, the process of Wick rotating is shown to preserve spectral dimension, smoothness and integrability, as we define them. Thus one obtains a K -homology class and the tools to compute index pairings using the local index formula. Since the most important Lorentzian manifolds are noncompact, we have taken care to ensure that our definitions are consistent with the nonunital version of the local index formula, as proved in [CGRS2].

Section 2 recalls what we need from the theory of nonunital spectral triples, while Section 3 recalls some pseudo-Riemannian geometry, in order to set notation and provide motivation. Here we also show how certain pseudo-Riemannian spin manifolds provide examples for our theory.

In Section 4 we discuss some technicalities about unbounded operators before presenting our definition of pseudo-Riemannian spectral triples. We also provide definitions of smoothness and summability, and a range of examples. An unexpected example is provided by the harmonic oscillator.

Section 5 begins with our main theorem, which shows that we can obtain a spectral triple from a pseudo-Riemannian spectral triple. We also give a sufficient condition on a smoothly summable pseudo-Riemannian spectral triple ensuring that the resulting spectral triple is smoothly summable, so that we can employ the local index formula. This sufficient condition is enough for all our examples, except the harmonic oscillator. The remainder of the section looks at the examples, in particular the oscillator, as well as one simple non-existence result for certain kinds of harmonic one forms on compact manifolds.

The classical examples we have presented all arise by taking a pseudo-Riemannian spin manifold and generating a Riemannian metric: for this to work, we require a global splitting of the tangent bundle into timelike and spacelike sub-bundles, and that the resulting Riemannian metric be complete and of bounded geometry. This can always be achieved in the globally hyperbolic case when $M_n = \mathbb{R} \times M_{n-1}$ provided that the induced metric on M_{n-1} is complete and of bounded geometry.

It would be desirable to have a method of producing a spectral triple, and so K -homology class, associated to a more general pseudo-Riemannian metric, or at least Lorentzian metric. The reason we can not is that, in the Lorentzian case, the weakest physically reasonable causality condition is stable causality, and the Lorentzian metrics of such manifolds need not have complete Riemannian manifolds associated to them by our Wick rotation procedure.

Hence to deal with general Lorentzian manifolds, one would have to be able to deal with Riemannian manifolds with boundary, and even more general objects. This requires careful consideration of appropriate boundary conditions, and we refer to [BDT] for a comprehensive discussion of boundary conditions in K -homology and [IL] for a definition of ‘spectral triple with boundary’. We will return to this question in a future work.

As a final comment, we remark that the purpose of this paper is to develop a framework for accessing the topological data associated to a pseudo-Riemannian manifold, commutative or noncommutative. The details of the geometry and/or physics of such a manifold should be accessed using the pseudo-Riemannian structure and not the associated Riemannian one.

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2. NONUNITAL SPECTRAL TRIPLES

In this section we will summarise the definitions and results concerning nonunital spectral triples. Much of this material is from [CGRS2] where a more detailed account can be found.

Definition 2.1. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by

- 1) A representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ of a $*$ -algebra \mathcal{A} on the Hilbert space \mathcal{H} .
- 2) A self-adjoint (unbounded, densely defined) operator $\mathcal{D} : \text{dom } \mathcal{D} \rightarrow \mathcal{H}$ such that $[\mathcal{D}, \pi(a)]$ extends to a bounded operator on \mathcal{H} for all $a \in \mathcal{A}$ and $\pi(a)(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ is compact for all $a \in \mathcal{A}$. The triple is said to be even if there is an operator $\Gamma = \Gamma^*$ such that $\Gamma^2 = 1$, $[\Gamma, \pi(a)] = 0$ for all $a \in \mathcal{A}$ and $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$ (i.e. Γ is a \mathbb{Z}_2 -grading such that \mathcal{D} is odd and $\pi(\mathcal{A})$ is even.) Otherwise the triple is called odd.

Remark 2.2. We will systematically omit the representation π in future.

Our aim is to define and study generalisations of spectral triples which model pseudo-Riemannian manifolds. The interesting situation in this setting is when the underlying space is noncompact. In order to discuss summability in this context, we recall a few definitions and results from [CGRS2], where a general definition of summability in the nonunital/noncompact context was developed.

Definition 2.3. Let \mathcal{D} be a densely defined self-adjoint operator on the Hilbert space \mathcal{H} . Then for each $p \geq 1$ and $s > p$ we define a weight φ_s on $\mathcal{B}(\mathcal{H})$ by

$$\varphi_s(T) := \text{Trace}((1 + \mathcal{D}^2)^{-s/4} T (1 + \mathcal{D}^2)^{-s/4}), \quad 0 \leq T \in \mathcal{B}(\mathcal{H}),$$

and the subspace $\mathcal{B}_2(\mathcal{D}, p)$ of $\mathcal{B}(\mathcal{H})$ by

$$\mathcal{B}_2(\mathcal{D}, p) := \bigcap_{s > p} \left(\text{dom}(\varphi_s)^{1/2} \bigcap (\text{dom}(\varphi_s)^{1/2})^* \right).$$

The norms

$$(1) \quad \mathcal{B}_2(\mathcal{D}, p) \ni T \mapsto \mathcal{Q}_n(T) := (\|T\|^2 + \varphi_{p+1/n}(|T|^2) + \varphi_{p+1/n}(|T^*|^2))^{1/2}, \quad n = 1, 2, 3, \dots,$$

take finite values on $\mathcal{B}_2(\mathcal{D}, p)$ and provide a topology on $\mathcal{B}_2(\mathcal{D}, p)$ stronger than the norm topology. We will always suppose that $\mathcal{B}_2(\mathcal{D}, p)$ has the topology defined by these norms.

The space $\mathcal{B}_2(\mathcal{D}, p)$ is in fact a Fréchet algebra, [CGRS2, Proposition 2.6], and plays the role of bounded square integrable operators. Next we introduce the bounded integrable operators. On $\mathcal{B}_2(\mathcal{D}, p)^2$, the span of products TS , with $T, S \in \mathcal{B}_2(\mathcal{D}, p)$, define norms

$$(2) \quad \mathcal{P}_n(T) := \inf \left\{ \sum_{i=1}^k \mathcal{Q}_n(T_{1,i}) \mathcal{Q}_n(T_{2,i}) : T = \sum_{i=1}^k T_{1,i} T_{2,i}, \quad T_{1,i}, T_{2,i} \in \mathcal{B}_2(\mathcal{D}, p) \right\}.$$

Here the sums are finite and the infimum runs over all possible such representations of T . It is shown in [CGRS2, page 13] that the \mathcal{P}_n are norms on $\mathcal{B}_2(\mathcal{D}, p)^2$.

Definition 2.4. Let \mathcal{D} be a densely defined self-adjoint operator on \mathcal{H} and $p \geq 1$. Define $\mathcal{B}_1(\mathcal{D}, p)$ to be the completion of $\mathcal{B}_2(\mathcal{D}, p)^2$ with respect to the family of norms $\{\mathcal{P}_n : n = 1, 2, \dots\}$.

Definition 2.5. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, is said to be finitely summable if there exists $s > 0$ such that for all $a \in \mathcal{A}$, $a(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{H})$. In such a case, we let

$$p := \inf \{s > 0 : \forall a \in \mathcal{A}, \text{Trace}(|a|(1 + \mathcal{D}^2)^{-s/2}) < \infty\},$$

and call p the spectral dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Remarks 2.6. For the definition of the spectral dimension above to be meaningful, one needs two facts. First, if \mathcal{A} is the algebra of a finitely summable spectral triple, we have $|a|(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{H})$ for all $a \in \mathcal{A}$, which follows by using the polar decomposition $a = v|a|$ and writing

$$|a|(1 + \mathcal{D}^2)^{-s/2} = v^* a (1 + \mathcal{D}^2)^{-s/2}.$$

Observe that we are *not* asserting that $|a| \in \mathcal{A}$, which is typically not true in examples. The second fact we require is that $\text{Trace}(a(1 + \mathcal{D}^2)^{-s/2}) \geq 0$ for $a \geq 0$, which follows from [Bik, Theorem 3].

In contrast to the unital case, checking the finite summability condition for a nonunital spectral triple can be difficult. This is because our definition relies on control of the trace norm of the non-self-adjoint operators $a(1 + \mathcal{D}^2)^{-s/2}$, $a \in \mathcal{A}$. It is shown in [CGRS2, Propositions 3.16, 3.17] that for a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to be finitely summable with spectral dimension p , it is necessary that $\mathcal{A} \subset \mathcal{B}_1(\mathcal{D}, p)$ and this condition is *almost* sufficient as well.

Anticipating the pseudodifferential calculus, we introduce subalgebras of $\mathcal{B}_1(\mathcal{D}, p)$ which ‘see’ smoothness as well as summability. There are several operators naturally associated to our notions of smoothness.

Definition 2.7. Let \mathcal{D} be a densely defined self-adjoint operator on the Hilbert space \mathcal{H} . Set $\mathcal{H}_\infty = \bigcap_{k \geq 0} \text{dom } \mathcal{D}^k$. For an operator $T \in \mathcal{B}(\mathcal{H})$ such that $T : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ we set

$$\delta(T) := [(1 + \mathcal{D}^2)^{1/2}, T].$$

In addition, we recursively set

$$T^{(n)} := [\mathcal{D}^2, T^{(n-1)}], \quad n = 1, 2, \dots \quad \text{and} \quad T^{(0)} := T.$$

Finally, let

$$(3) \quad L(T) := (1 + \mathcal{D}^2)^{-1/2} [\mathcal{D}^2, T], \quad R(T) := [\mathcal{D}^2, T] (1 + \mathcal{D}^2)^{-1/2}.$$

It is proven in [CoM, Co5] and [CPRS2, Proposition 6.5] that

$$(4) \quad \bigcap_{n \geq 0} \operatorname{dom} L^n = \bigcap_{n \geq 0} \operatorname{dom} R^n = \bigcap_{k, l \geq 0} \operatorname{dom} L^k \circ R^l = \bigcap_{n \in \mathbb{N}} \operatorname{dom} \delta^n.$$

To define $\delta^k(T)$ for $T \in \mathcal{B}(\mathcal{H})$ we need $T : \mathcal{H}_l \rightarrow \mathcal{H}_l$ for each $l = 1, \dots, k$. This is necessary before discussing boundedness of $\delta^k(T)$.

Definition 2.8. Let \mathcal{D} be a densely defined self-adjoint operator on the Hilbert space \mathcal{H} , and $p \geq 1$. Then define for $k = 0, 1, 2, \dots$

$$\mathcal{B}_1^k(\mathcal{D}, p) := \{T \in \mathcal{B}(\mathcal{H}) : T : \mathcal{H}_l \rightarrow \mathcal{H}_l, \text{ and } \delta^l(T) \in \mathcal{B}_1(\mathcal{D}, p) \ \forall l = 0, \dots, k\}.$$

Also set

$$\mathcal{B}_1^\infty(\mathcal{D}, p) := \bigcap_{k=0}^{\infty} \mathcal{B}_1^k(\mathcal{D}, p).$$

We equip $\mathcal{B}_1^k(\mathcal{D}, p)$, $k = 0, 1, 2, \dots, \infty$, with the topology determined by the seminorms $\mathcal{P}_{n,l}$ defined by

$$\mathcal{B}(\mathcal{H}) \ni T \mapsto \mathcal{P}_{n,l}(T) := \sum_{j=0}^l \mathcal{P}_n(\delta^j(T)), \quad n = 1, 2, \dots, \quad l = 0, \dots, k.$$

Definition 2.9. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. Then we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k summable if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finitely summable with spectral dimension p and

$$\mathcal{A} \cup [\mathcal{D}, \mathcal{A}] \subset \mathcal{B}_1^k(\mathcal{D}, p).$$

We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is smoothly summable if it is QC^k summable for all $k \in \mathbb{N}$ or, equivalently, if

$$\mathcal{A} \cup [\mathcal{D}, \mathcal{A}] \subset \mathcal{B}_1^\infty(\mathcal{D}, p).$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is smoothly summable with spectral dimension p , the δ - φ -topology on \mathcal{A} is determined by the family of norms

$$\mathcal{A} \ni a \mapsto \mathcal{P}_{n,k}(a) + \mathcal{P}_{n,k}([\mathcal{D}, a]), \quad n = 1, 2, \dots, \quad k = 0, 1, 2, \dots,$$

where the norms $\mathcal{P}_{n,k}$ are those of Definition 2.8.

In [CGRS2] it was shown that the local index formula holds for smoothly summable spectral triples. Examples show, [CGRS2, page 45], that we can have $\mathcal{A} \subset \mathcal{B}_1^\infty(\mathcal{D}, p)$, while $[\mathcal{D}, \mathcal{A}] \notin \mathcal{B}_1^\infty(\mathcal{D}, p)$. These examples show that we need assumptions to control the summability of $[\mathcal{D}, a]$ as well as $a \in \mathcal{A}$.

Next we recall the version of pseudodifferential calculus for nonunital spectral triples developed in [CGRS2].

Definition 2.10. Let \mathcal{D} be a densely defined self-adjoint operator on the Hilbert space \mathcal{H} and $p \geq 1$. The set of **order- r tame pseudodifferential operators** associated with $(\mathcal{H}, \mathcal{D})$ and $p \geq 1$ is given by

$$\operatorname{OP}_0^r(\mathcal{D}) := (1 + \mathcal{D}^2)^{r/2} \mathcal{B}_1^\infty(\mathcal{D}, p), \quad r \in \mathbb{R}, \quad \operatorname{OP}_0^*(\mathcal{D}) := \bigcup_{r \in \mathbb{R}} \operatorname{OP}_0^r(\mathcal{D}).$$

We topologise $\text{OP}_0^r(\mathcal{D})$ with the family of norms

$$\mathcal{P}_{n,l}^r(T) := \mathcal{P}_{n,l}((1 + \mathcal{D}^2)^{-r/2}T), \quad n, l \in \mathbb{N},$$

where the norms $\mathcal{P}_{n,l}$ are as in Definition 2.9.

Remark 2.11. To lighten the notation, we do not make explicit the important dependence on the real number $p \geq 1$ in the definition of the tame pseudodifferential operators. We also observe that the definition of all the spaces $\mathcal{B}_2(\mathcal{D}, p)$, $\mathcal{B}_1(\mathcal{D}, p)$ and $\text{OP}^r(\mathcal{D})$ depends only on $(1 + \mathcal{D}^2)^{1/2}$.

Since $\mathcal{B}_1^\infty(\mathcal{D}, p)$ is *a priori* a nonunital algebra, functions of \mathcal{D} alone do not belong to OP_0^* . In particular, not all ‘differential operators’, such as powers of \mathcal{D} , are tame pseudodifferential operators.

Definition 2.12. Let \mathcal{D} be a densely defined self-adjoint operator on the Hilbert space \mathcal{H} and $p \geq 1$. The set of **regular order- r pseudodifferential operators** is

$$\text{OP}^r(\mathcal{D}) := (1 + \mathcal{D}^2)^{r/2} \left(\bigcap_{n \in \mathbb{N}} \text{dom } \delta^n \right), \quad r \in \mathbb{R}, \quad \text{OP}^*(\mathcal{D}) := \bigcup_{r \in \mathbb{R}} \text{OP}^r(\mathcal{D}).$$

The natural topology of $\text{OP}^r(\mathcal{D})$ is associated with the family of norms

$$\sum_{k=0}^l \|\delta^k((1 + \mathcal{D}^2)^{-r/2}T)\|, \quad l \in \mathbb{N}.$$

With this definition, $\text{OP}_0^r(\mathcal{D})$ is a Fréchet space and both $\text{OP}^0(\mathcal{D})$ and $\text{OP}_0^0(\mathcal{D})$ are Fréchet $*$ -algebras. Moreover it is proved in [CGRS2] that for $r > p$, we have $\text{OP}_0^{-r}(\mathcal{D}) \subset \mathcal{L}^1(\mathcal{H})$ and that for all $r, t \in \mathbb{R}$ we have $\text{OP}_0^r(\mathcal{D}) \text{OP}^t(\mathcal{D})$, $\text{OP}^t(\mathcal{D}) \text{OP}_0^r(\mathcal{D}) \subset \text{OP}_0^{r+t}(\mathcal{D})$. Thus the tame pseudodifferential operators form an ideal within the regular pseudodifferential operators.

The other main ingredient of the pseudodifferential calculus is the one (complex) parameter group

$$(5) \quad \sigma^z(T) := (1 + \mathcal{D}^2)^{z/2} T (1 + \mathcal{D}^2)^{-z/2}, \quad z \in \mathbb{C}, \quad T \in \text{OP}^*(\mathcal{D}).$$

The group σ restricts to a strongly continuous one parameter group on each $\text{OP}^r(\mathcal{D})$ and $\text{OP}_0^r(\mathcal{D})$, and has a Taylor expansion, as first proved in [CoM]. The following version of the Taylor expansion is adapted to tame pseudodifferential operators, and can be found in [CGRS2, Proposition 2.35].

Proposition 2.13. Let \mathcal{D} be a densely defined self-adjoint operator on the Hilbert space \mathcal{H} and $p \geq 1$. Let $T \in \text{OP}_0^r(\mathcal{D})$ and $z = n + 1 - \alpha$ with $n \in \mathbb{N}$ and $\Re(\alpha) \in (0, 1)$. Then we have

$$\sigma^{2z}(T) - \sum_{k=0}^n C_k(z) (\sigma^2 - \text{Id})^k(T) \in \text{OP}_0^{r-n-1}(\mathcal{D}) \quad \text{with} \quad C_k(z) := \frac{z(z-1) \cdots (z-k+1)}{k!}.$$

We also prove here a lemma, which will be useful to us later on.

Lemma 2.14. Let \mathcal{D} be a densely defined self-adjoint operator on the Hilbert space \mathcal{H} and $p \geq 1$. Let $T \in \text{OP}^0(\mathcal{D})$ have bounded inverse. Then $T^{-1} \in \text{OP}^0(\mathcal{D})$.

Proof. We need to show that $\delta^n(T^{-1})$ is bounded for all $n \geq 1$. We first check that

$$\delta(T^{-1}) = -T^{-1}\delta(T)T^{-1}$$

is given by a product of bounded operators. Iterating this formula shows that there are combinatorial constants $C_{l,n,k}$ such that

$$\delta^n(T^{-1}) = \sum_{1 \leq l \leq n} \sum_{1 \leq k_1, \dots, k_l \leq n, |k|=n} C_{l,n,k} T^{-1} \delta^{k_1}(T) T^{-1} \delta^{k_2}(T) T^{-1} \dots T^{-1} \delta^{k_l}(T) T^{-1}.$$

Since $\delta^k(T) \in \text{OP}^0(\mathcal{D})$ is bounded for all k , we see that $\delta^n(T^{-1})$ is bounded for all n . Hence T^{-1} is indeed an element of $\text{OP}^0(\mathcal{D})$. \square

In [CGRS2] it was shown that Dirac type operators on complete Riemannian manifolds of bounded geometry give rise to smoothly summable spectral triples, with spectral dimension given by the dimension of the underlying manifold. The bounded geometry hypothesis arises from the need to employ heat kernel techniques to prove finite summability.

The next section explores the kind of analytic data that Dirac type operators on suitable pseudo-Riemannian manifolds give rise to. We use this section as motivation for the discussion of pseudo-Riemannian spectral triples to come.

3. DIRAC OPERATORS ON PSEUDO-RIEMANNIAN MANIFOLDS

Let (M, g) be an n -dimensional time- and space-oriented pseudo-Riemannian manifold of signature (t, s) . We will assume that we are given an orthogonal direct sum decomposition of the tangent bundle $TM = E_t \oplus E_s$ (with $\dim E_t = t$ and $\dim E_s = s$) such that the metric g is negative definite on E_t and positive definite on E_s . We shall consider elements of E_t to be ‘purely timelike’ and elements of E_s to be ‘purely spacelike’. We shall denote by T the projection $TM \rightarrow E_t$ onto the ‘purely timelike’ subbundle. This projection is orthogonal with respect to g , which means that $g(v - Tv, Tw) = 0$ for all $v, w \in TM$. We then also have a *spacelike reflection* $r := 1 - 2T$ which acts as $(-\mathbb{I}) \oplus \mathbb{I}$ on the decomposition $E_t \oplus E_s$.

Definition 3.1. *Given a direct sum decomposition $TM = E_t \oplus E_s$ (or, equivalently, given a spacelike reflection r , or a timelike projection T), we define a new metric g_E on TM by*

$$g_E(v, w) := g(rv, w).$$

Since $r(1 - T) = 1 - T$, we readily check that T is also an orthogonal projection with respect to the new metric g_E . Furthermore, g_E is positive definite, and hence (M, g_E) is a Riemannian manifold. Alternatively, we can also write

$$g(v, w) = g_E(rv, w) = g_E((1 - T)v, (1 - T)w) - g_E(Tv, Tw).$$

3.1. The Clifford algebras. Let $\text{Cliff}(TM, g_E)$ denote the real Clifford algebra with respect to g_E , and $\text{Cliff}(TM, g)$ the real Clifford algebra with respect to g . Their complexifications are equal (since a complex Clifford algebra is independent of the signature of the chosen metric) and will be denoted $\mathbb{C}\text{Cliff}(TM)$. Let the inclusion $TM \hookrightarrow \text{Cliff}(TM, g_E) \subset \mathbb{C}\text{Cliff}(TM)$ be denoted by γ_E . Our conventions are such that $\gamma_E(v)^* = -\gamma_E(v)$ and $\gamma_E(v)\gamma_E(w) + \gamma_E(w)\gamma_E(v) = -2g_E(v, w)$. We can now define a linear map $\gamma: TM \rightarrow \mathbb{C}\text{Cliff}(TM)$ by

$$(6) \quad \gamma(v) := \gamma_E(v - Tv) - i\gamma_E(Tv).$$

Since T is a projection, we have $g_E(v - Tv, Tv) = 0$. We then verify that $\gamma(v)^2 = -g(v, v)$, since

$$\begin{aligned}\gamma(v)^2 &= \gamma_E(v - Tv)^2 - \gamma_E(Tv)^2 - i\gamma_E(v - Tv)\gamma_E(Tv) - i\gamma_E(Tv)\gamma_E(v - Tv) \\ &= -g_E(v - Tv, v - Tv) + g_E(Tv, Tv) + 2ig_E(v - Tv, Tv) = -g(v, v).\end{aligned}$$

Note that $(1 - T - iT)^2 = r$. An alternative calculation for the complex-linear continuation $\gamma: TM \otimes \mathbb{C} \rightarrow \text{Cliff}(TM)$ and the complexified metric then yields the same result:

$$\gamma(v)^2 = \gamma_E((1 - T - iT)v)^2 = -g_E((1 - T - iT)v, (1 - T - iT)v) = -g_E(rv, v) = -g(v, v).$$

By the universal properties of Clifford algebras, this ensures that γ extends to an algebra homomorphism $\gamma: \text{Cliff}(TM, g) \rightarrow \text{Cliff}(TM)$. Furthermore, by calculating

$$\begin{aligned}\gamma(v)\gamma(w)^* &= -\gamma_E(v - Tv)\gamma_E(w - Tw) - \gamma_E(Tv)\gamma_E(Tw) \\ &\quad - i\gamma_E(v - Tv)\gamma_E(Tw) + i\gamma_E(Tv)\gamma_E(w - Tw),\end{aligned}$$

we find that γ also satisfies

$$(7) \quad \gamma(v)\gamma(w)^* + \gamma(w)^*\gamma(v) = 2g_E(v, w)$$

with respect to the Riemannian metric.

3.2. Dirac operators. Take a local pseudo-orthonormal basis $\{e_i\}_{i=1}^n$ of TM such that $\{e_i\}_{i=1}^t$ is a basis for E_t , and $\{e_i\}_{i=t+1}^n$ is a basis for E_s , i.e.

$$g(e_i, e_j) = \delta_{ij}\kappa(j), \quad \kappa(j) = \begin{cases} -1 & j = 1, \dots, t; \\ 1 & j = t + 1, \dots, n. \end{cases}$$

The basis $\{e_i\}_{i=1}^n$ is then automatically also an orthonormal basis with respect to g_E . We shall denote by h the map $T^*M \rightarrow TM$ which maps $\alpha \in T^*M$ to its dual in TM with respect to the metric g . That is:

$$h(\alpha) = v, \quad \iff \quad \alpha(w) = g(v, w) \quad \text{for all } w \in TM.$$

Let $\{\theta^i\}_{i=1}^n$ be the basis of T^*M dual to $\{e_i\}_{i=1}^n$, so that $\theta^i(e_j) = \delta_j^i$. We then see that $h(\theta^i) = \kappa(i)e_i$.

Fix a bundle $S \rightarrow M$, such that $\text{Cliff}(TM) \simeq \text{End}(S)$. We shall now discuss the construction of the standard Dirac operator on $\Gamma(S)$. Let us denote by c the pseudo-Riemannian Clifford multiplication $\Gamma(T^*M \otimes S) \rightarrow \Gamma(S)$ given by

$$c(\alpha \otimes \psi) := \gamma(h(\alpha))\psi.$$

Let ∇ be the Levi-Civita connection for the pseudo-Riemannian metric g , and let ∇^S be its lift to the spinor bundle. We shall consider the Dirac operator on $\Gamma(S)$ defined as the composition

$$\not{D}: \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \xrightarrow{c} \Gamma(S), \quad \not{D} := c \circ \nabla^S = \sum_{j=1}^n \kappa(j)\gamma(e_j)\nabla_{e_j}^S.$$

The pseudo-Riemannian Dirac operator \not{D} is closeable, but it will not be symmetric, and not even normal! However, in the next section we will employ the notion of a Krein space to show that this operator will (under suitable assumptions) be Krein-self-adjoint with respect to an indefinite scalar product.

3.3. Krein spaces. Let us recall the definition and some basic properties of Krein spaces from [Bo, page 100].

Definition 3.2 ([Bo, §V.1]). *A Krein space is a vector space V with an indefinite inner product (\cdot, \cdot) that admits a fundamental decomposition of the form $V = V^+ \oplus V^-$ into a positive-definite subspace V^+ and a negative-definite subspace V^- , where V^+ and V^- are intrinsically complete, i.e. complete with respect to the norms $\|v\|_{V^\pm} := |(v, v)|^{1/2}$.*

Let P^\pm denote the projections on V^\pm . Then the operator $\mathcal{J} := P^+ - P^-$ is self-adjoint and unitary, and defines a positive-definite inner product $(\cdot, \cdot)_{\mathcal{J}} := (\cdot, \mathcal{J}\cdot)$. Such an operator \mathcal{J} (which depends on the choice of fundamental decomposition) is called a *fundamental symmetry* of the Krein space V .

A decomposable, non-degenerate inner product space V is a Krein space if and only if, for every fundamental symmetry \mathcal{J} , the inner product $(\cdot, \cdot)_{\mathcal{J}}$ turns V into a Hilbert space.

Let T be a linear operator with dense domain in the Krein space V . Then the Krein-adjoint T^+ of T is defined on the domain

$$\text{dom } T^+ := \{v \in V \mid \exists w \in V : \forall u \in \text{dom } T : (Tu, v) = (u, w)\},$$

and we define $T^+v := w$. Let \mathcal{J} be a fundamental symmetry of V , and let T^* be the adjoint of T with respect to the Hilbert space inner product $(\cdot, \cdot)_{\mathcal{J}}$. We then have the relation $T^+ = \mathcal{J}T^*\mathcal{J}$. An operator T is Krein-self-adjoint if and only if $\mathcal{J}T$ and $T\mathcal{J}$ are self-adjoint with respect to $(\cdot, \cdot)_{\mathcal{J}}$.

3.3.1. The Krein space of spinors. We shall now briefly describe the Krein space of spinors on a pseudo-Riemannian spin manifold (M, g) . For more details we refer to [Bau, §3.3].

Recall the spacelike reflection r which acts as $(-\mathbb{I}) \oplus \mathbb{I}$ on the decomposition $E_t \oplus E_s$ of the tangent bundle TM . Let $\Gamma_c(S)$ be the space of compactly supported smooth sections on the spinor bundle $S \rightarrow M$. There exists a positive-definite hermitian product (depending on the decomposition $E_t \oplus E_s$)

$$(\cdot, \cdot) : \Gamma_c(S) \times \Gamma_c(S) \rightarrow C_c^\infty(M),$$

which gives rise to the inner product

$$\langle \psi_1, \psi_2 \rangle := \int_M (\psi_1, \psi_2) \nu_g, \quad \text{for } \psi_1, \psi_2 \in \Gamma_c(S),$$

where ν_g is the canonical volume form of (M, g) . The completion of $\Gamma_c(S)$ with respect to this inner product is denoted $L^2(M, S)$.

As before, we have a (pseudo)-orthonormal basis $\{e_i\}_{i=1}^n$ such that $\{e_i\}_{i=1}^t$ is a basis for E_t and $\{e_i\}_{i=t+1}^n$ is a basis for E_s . We shall define an operator \mathcal{J}_M on $L^2(M, S)$ by

$$\mathcal{J}_M := i^{t(t-1)/2} \gamma(e_1) \cdots \gamma(e_t).$$

This operator \mathcal{J}_M is a self-adjoint unitary operator, i.e. $\mathcal{J}_M^* = \mathcal{J}_M$ and $\mathcal{J}_M^2 = 1$. Furthermore, for the spacelike reflection r as above and for a vector $v \in TM$, we have the relation

$$(8) \quad \mathcal{J}_M \gamma(v) \mathcal{J}_M = (-1)^t \gamma(rv).$$

We can now define an indefinite hermitian product

$$(\cdot, \cdot)_{\mathcal{J}_M} := (\mathcal{J}_M \cdot, \cdot).$$

Then $L^2(M, S)$ with indefinite inner product $\langle \cdot, \cdot \rangle_{\mathcal{J}_M} := \int_M (\cdot, \cdot)_{\mathcal{J}_M} \nu_g$ is a *Krein space* with *fundamental symmetry* \mathcal{J}_M .

Let us quote a few facts from [Bau, §3.3]. Using the spacelike reflection r we obtain a new connection $\nabla^r := r \circ \nabla \circ r$, and its spin lift is given by $\nabla^{r,S} = \mathcal{J}_M \nabla^S \mathcal{J}_M$. For the Dirac operator \mathcal{D} we denote its (formal) Hilbert space adjoint (with respect to $\langle \cdot, \cdot \rangle$) as \mathcal{D}^* and its (formal) Krein space adjoint (with respect to $\langle \cdot, \cdot \rangle_{\mathcal{J}_M}$) as \mathcal{D}^\dagger . We shall define its real and imaginary parts as $\Re \mathcal{D} := \frac{1}{2}(\mathcal{D} + \mathcal{D}^*)$ and $\Im \mathcal{D} := -\frac{i}{2}(\mathcal{D} - \mathcal{D}^*)$.

Proposition 3.3 ([Bau, Satz 3.17]). *The formal adjoints \mathcal{D}^* and \mathcal{D}^\dagger are locally of the form*

$$\mathcal{D}^\dagger = (-1)^t \mathcal{D}, \quad \mathcal{D}^* = \sum_{j=1}^n \gamma(e_j) \nabla_{e_j}^{r,S},$$

where (e_1, \dots, e_n) is a local (pseudo)-orthonormal frame.

Proposition 3.4 ([Bau, Satz 3.19]). *Let (M, g) be an n -dimensional time- and space-oriented pseudo-Riemannian spin manifold of signature (t, s) . Let r be a spacelike reflection, such that the associated Riemannian metric g_E is complete. Consider the Dirac operator $\mathcal{D} := c \circ \nabla^S$. Then*

- (1) *the operators $\Re \mathcal{D}$ and $\Im \mathcal{D}$ are essentially self-adjoint with respect to $\langle \cdot, \cdot \rangle$; and*
- (2) *the operator $i^t \mathcal{D}$ is essentially Krein-self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathcal{J}_M}$.*

In fact the sum

$$\mathcal{D}_E := \Re \mathcal{D} + \Im \mathcal{D}$$

is also essentially self-adjoint (see Corollary 4.8), so to a Dirac operator \mathcal{D} we can associate a self-adjoint operator \mathcal{D}_E . Its square is given by

$$\mathcal{D}_E^2 = (\Re \mathcal{D})^2 + (\Im \mathcal{D})^2 + (\Re \mathcal{D})(\Im \mathcal{D}) + (\Im \mathcal{D})(\Re \mathcal{D}) = \frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}) - \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2}).$$

We shall write this as $\mathcal{D}_E^2 = \langle \mathcal{D} \rangle^2 - \mathcal{R}_{\mathcal{D}}$, where we define

$$\langle \mathcal{D} \rangle^2 := \frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}), \quad \mathcal{R}_{\mathcal{D}} := \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2}).$$

Remark 3.5. We will refer to the map $\mathcal{D} \mapsto \mathcal{D}_E$ as *Wick rotation* of the Dirac operator. Let us justify this terminology by considering the basic example of a product space-time $M = \mathbb{R} \times N$, where \mathbb{R} represents time and N is an $n-1$ -dimensional Riemannian manifold with metric g_N . Consider the pseudo-Riemannian metric on M given by $g_M(t, \vec{x}) := \text{diag}(-f(t), g_N(\vec{x}))$ for a strictly positive function $f: \mathbb{R} \rightarrow \mathbb{R}^+$. Let ∇ be the Levi-Civita connection associated to g_M . Then the product-form of g_M implies that $\nabla^r = \nabla$, and hence that the Dirac operator $\mathcal{D} = \sum_{j=1}^n \kappa(j) \gamma(e_j) \nabla_{e_j}^S$ yields the Wick rotation $\mathcal{D}_E = \sum_{j=1}^n \gamma_E(e_j) \nabla_{e_j}^S$. In this case, the map $\mathcal{D} \mapsto \mathcal{D}_E$ is therefore simply achieved through replacing $\kappa(j) \gamma(e_j)$ by $\gamma_E(e_j)$, which indeed resembles a Wick rotation of the time-coordinate.

Lemma 3.6. *Let \mathcal{D} be as in Proposition 3.4. Then the operator $\langle \mathcal{D} \rangle^2$ defined on the domain $\text{Dom} \langle \mathcal{D} \rangle^2 := \text{Dom} \mathcal{D}\mathcal{D}^* \cap \text{Dom} \mathcal{D}^*\mathcal{D}$ is essentially self-adjoint, elliptic and commutes with the fundamental symmetry \mathcal{J}_M . Hence it is also essentially Krein-self-adjoint.*

Proof. Consider

$$T := (\not{D}, \not{D}^*) : \text{dom } \not{D} \cap \text{dom } \not{D}^* \subset L^2(M, S) \rightarrow L^2(M, S) \oplus L^2(M, S).$$

Then the operator

$$T^*T = \overline{\not{D}\not{D}^* + \not{D}^*\not{D}} = 2\overline{\langle \not{D} \rangle^2}$$

has principal symbol $2g_E(\xi, \xi)$ (which easily follows using (7)) and is therefore elliptic. Hence T must also be elliptic, and then T^*T is densely defined and essentially self-adjoint by [BrMSh, Corollary 2.10] (noting that the metric g^{TM} defined therein is equal to $2g_E$, which is complete by assumption). By Krein-selfadjointness of $i^t \not{D}$ (Proposition 3.4) we know that $\not{D}^* = (-1)^t \mathcal{J}_M \not{D} \mathcal{J}_M$, from which it easily follows that $[\langle \not{D} \rangle^2, \mathcal{J}_M] = 0$. \square

3.4. The triple of a pseudo-Riemannian manifold. To a Riemannian spin manifold (M, g_E) one associates the spectral triple $(C_c^\infty(M), L^2(M, S), \not{D})$,¹ where \not{D} is the standard Dirac operator corresponding to the Riemannian metric g_E . In a similar manner, we associate to the pseudo-Riemannian manifold (M, g) the triple $(C_c^\infty(M), L^2(M, S), \not{D})$, where now \not{D} is the standard Dirac operator defined with respect to the pseudo-Riemannian metric g , as considered above. The following few results derive some basic properties of this triple which will motivate our abstract definition of a pseudo-Riemannian spectral triple.

To show that we obtain a spectral triple using the operator \mathcal{D}_E , we need to assume that the Riemannian metric g_E is well-behaved. Recall that the injectivity radius $r_{\text{inj}} \in [0, \infty)$ of the Riemannian manifold (M, g_E) is defined as

$$r_{\text{inj}} := \inf_{x \in M} \sup\{r_x > 0\},$$

where r_x is such that the exponential map \exp_x (defined w.r.t. the Riemannian metric g_E) is a diffeomorphism from the ball $B(0, r_x) \subset T_x M$ to an open neighborhood U_x of $x \in M$. We observe that there is a related notion of injectivity radius for Lorentzian manifolds which is adapted to the pair of metrics (g, g_E) ; see [CLeF].

Definition 3.7. *Let (M, g) be an n -dimensional time- and space-oriented pseudo-Riemannian spin manifold of signature (t, s) . Let r be a spacelike reflection such that (M, g_E) is complete. We say that (M, g, r) has bounded geometry if (M, g_E) has strictly positive injectivity radius, and all the covariant derivatives of the (pseudo-Riemannian) curvature tensor of (M, g) are bounded (w.r.t. g_E) on M . A Dirac bundle on M is said to have bounded geometry if in addition all the covariant derivatives of Ω^S , the curvature tensor of the connection ∇^S , are bounded (w.r.t. g_E) on M . For brevity, we simply say that (M, g, r, S) has bounded geometry.*

A differential operator is said to have *uniform C^∞ -bounded coefficients*, if for any atlas consisting of charts of normal coordinates, the derivatives of all order of the coefficients are bounded on the chart domain and the bounds are uniform on the atlas. It is shown in [R, Propositions 5.4 & 5.5] that the assumption of bounded geometry is equivalent to the existence of a *good coordinate ball*, that is a ball B with center 0 in \mathbb{R}^n which is the domain of a normal coordinate system at every point of M , such that the Christoffel symbols of ∇ and ∇^S lie in a bounded subset of the

¹Note that in fact, one should consider here the closure $\bar{\not{D}}$ of \not{D} , but to simplify notation we will just write \not{D} for its closure as well.

Fréchet space $C^\infty(B)$. Thus bounded geometry implies that the Dirac operator \mathcal{D} has uniform C^∞ -bounded coefficients.

Proposition 3.8. *Let (M, g) be an n -dimensional time- and space-oriented pseudo-Riemannian spin manifold of signature (t, s) . Let r be a spacelike reflection such that (M, g_E) is complete and (M, g, r, S) has bounded geometry. Then the triple $(C_c^\infty(M), L^2(M, S), \mathcal{D})$ satisfies*

- 1) $\text{dom}\langle \mathcal{D} \rangle^2 := \text{dom } \mathcal{D}\mathcal{D}^* \cap \text{dom } \mathcal{D}^*\mathcal{D}$ is dense in $L^2(M, S)$ and $\langle \mathcal{D} \rangle^2 := \frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D})$ is essentially self-adjoint on this domain;
- 2a) $(\mathcal{D}^2 - \mathcal{D}^{*2}) \in \text{OP}^1(\langle \mathcal{D} \rangle)$ and $[\langle \mathcal{D} \rangle^2, \mathcal{D}^2 - \mathcal{D}^{*2}] \in \text{OP}^2(\langle \mathcal{D} \rangle)$;
- 2b) $a(\mathcal{D}^2 - \mathcal{D}^{*2})(1 + \langle \mathcal{D} \rangle^2)^{-1}$ is compact for all $a \in C_c^\infty(M)$;
- 3) $[\mathcal{D}, a]$ and $[\mathcal{D}^*, a]$ extend to bounded operators on \mathcal{H} for all $a \in C_c^\infty(M)$ (in particular, all $a \in C_c^\infty(M)$ preserve $\text{dom } \mathcal{D}$, $\text{dom } \mathcal{D}^*$);
- 4) $a(1 + \langle \mathcal{D} \rangle^2)^{-1/2} \in \mathcal{K}(L^2(M, S))$ for all $a \in C_c^\infty(M)$.

Proof. From Lemma 3.6 we know that $\langle \mathcal{D} \rangle^2$ is essentially self-adjoint, which proves 1), and elliptic, which proves 4). Since $\mathcal{D}, \mathcal{D}^*$ are first order differential operators, 3) is immediate. Consider the operator $\mathcal{R}_{\mathcal{D}} := \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2})$, initially defined on the dense subset of compactly supported smooth sections $\Gamma_c(M, S)$. Since $\mathcal{D}^{*2} = \mathcal{J}_M \mathcal{D}^2 \mathcal{J}_M$ by Proposition 3.4, we see that $\mathcal{D}^2 - \mathcal{D}^{*2} = [\mathcal{D}^2, \mathcal{J}_M] \mathcal{J}_M$. As \mathcal{D}^2 is a second-order differential operator whose principal symbol commutes with \mathcal{J}_M , we see that $\mathcal{R}_{\mathcal{D}}$ is a first-order differential operator. Hence $\mathcal{R}_{\mathcal{D}}(1 + \langle \mathcal{D} \rangle^2)^{-1/2}$ is a bounded operator. Ellipticity of $\langle \mathcal{D} \rangle^2$ then implies 2b). Under the assumption of bounded geometry, $\mathcal{R}_{\mathcal{D}}(1 + \langle \mathcal{D} \rangle^2)^{-1/2}$ is in fact a smooth uniformly bounded operator with uniformly bounded covariant derivatives, which implies 2a). \square

Remark 3.9. So far we have only used the boundedness of the coefficients of \mathcal{D} and the ellipticity of $\langle \mathcal{D} \rangle$. Later we will use information from the bounded geometry hypothesis to deduce summability properties for $\langle \mathcal{D} \rangle$.

4. PSEUDO-RIEMANNIAN SPECTRAL TRIPLES

4.1. Preliminaries on unbounded operators. Our definition of pseudo-Riemannian spectral triple relies on some results about unbounded operators. Here we recall some standard facts, and use these to prove some perturbation results that we will require.

We observe that questions about non-self-adjoint unbounded operators often revolve around the issue of ‘symmetric versus self-adjoint’, and have a flavour of boundary value problems.

Here we will aim to control all ‘symmetric versus self-adjoint’ issues, focusing instead on the ‘algebraic’ lack of symmetry that we saw was typical of Dirac operators on pseudo-Riemannian manifolds. Naturally we must make fairly strong assumptions to ensure that ‘symmetric versus self-adjoint’ issues do not interfere with the purely algebraic aspects.

First recall that if $\mathcal{D} : \text{dom } \mathcal{D} \rightarrow \mathcal{H}$ is a densely defined closed operator, then both $\mathcal{D}\mathcal{D}^*$ and $\mathcal{D}^*\mathcal{D}$ are densely defined (with obvious domains), self-adjoint and positive. The operator $\langle \mathcal{D} \rangle^2 := \frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D})$ is defined on $\text{dom } \mathcal{D}^*\mathcal{D} \cap \text{dom } \mathcal{D}\mathcal{D}^*$, and when this is dense, $\langle \mathcal{D} \rangle^2$ is a densely defined symmetric operator. The symmetric operator $\langle \mathcal{D} \rangle^2$ is positive, since it is the operator associated with the quadratic form

$$q(\phi, \psi) = \langle \mathcal{D}\phi, \mathcal{D}\psi \rangle + \langle \mathcal{D}^*\phi, \mathcal{D}^*\psi \rangle, \quad \phi, \psi \in \text{dom } \mathcal{D} \cap \text{dom } \mathcal{D}^*.$$

Then, by [RS, Theorem X.23], q is a closeable form, and its closure is associated with a unique positive operator which we denote by $\langle \mathcal{D} \rangle_F^2$, the Friedrich's extension.

By considering instead $\frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}) + 1$, and applying [RS, Theorem X.26], we see that $\frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D})$ is essentially self-adjoint if and only if the Friedrich's extension is the only self-adjoint extension, which occurs if and only if the range of $\frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}) + 1$ is dense. The key assumption we shall make to avoid 'symmetric versus self-adjoint' issues is that $\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}$ is essentially self-adjoint.

Definition 4.1. *Given $\mathcal{D} : \text{dom } \mathcal{D} \rightarrow \mathcal{H}$ a densely defined closed unbounded operator with $\text{dom } \mathcal{D} \cap \text{dom } \mathcal{D}^*$ dense in \mathcal{H} , we define the 'Wick rotated' operator*

$$\mathcal{D}_E = \frac{e^{i\pi/4}}{\sqrt{2}}(\mathcal{D} - i\mathcal{D}^*) = \frac{1}{2}(\mathcal{D} + \mathcal{D}^*) + \frac{i}{2}(\mathcal{D} - \mathcal{D}^*)$$

with initial domain $\text{dom } \mathcal{D} \cap \text{dom } \mathcal{D}^*$. Then \mathcal{D}_E is symmetric and so closeable. On $\text{dom } \mathcal{D}_E^2$ we have

$$\mathcal{D}_E^2 = \frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D}) + \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2}) =: \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}},$$

where we write $\mathcal{R}_{\mathcal{D}} = \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2})$.

The next lemma appears as [RS, exercise 28, Chapter X] (see also the proof of [B, Lemma 3]).

Lemma 4.2. *Suppose that T is a symmetric operator on the Hilbert space \mathcal{H} with $\text{dom } T^2$ dense in \mathcal{H} . If T^2 is essentially self-adjoint on $\text{dom } T^2$ then T is essentially self-adjoint.*

Our main technical tool for passing from pseudo-Riemannian spectral triples to spectral triples is the commutator theorem, [RS, Theorem X.36]. We restate a slightly weaker version of this result using the language of pseudodifferential operators from Section 2.

Theorem 4.3. *Let $N \geq 1$ be a positive self-adjoint operator on the Hilbert space \mathcal{H} . If $A \in \text{OP}^2(N^{1/2})$ is closed and symmetric and furthermore $[N, A] \in \text{OP}^2(N^{1/2})$ then*

1) $\text{dom } N \subset \text{dom } N + A$ and there exists a constant $C > 0$ such that for all $\xi \in \text{dom } N$

$$\|(N + A)\xi\| \leq C \|N\xi\|;$$

2) the operator $N + A$ is essentially self-adjoint on any core for N .

4.2. The definition and immediate consequences. The following definitions are intended to give an analytic framework for dealing with the non-normal unbounded operators arising in pseudo-Riemannian geometry. We have endeavoured to make this framework as general as possible, and it is easy to see from Proposition 3.8 that the triples we constructed from (suitable) pseudo-Riemannian manifolds satisfy this definition. After presenting definitions of smoothness and summability by analogy with spectral triples, we give a broad range of examples.

Recall that given a closed densely defined operator $\mathcal{D} : \text{dom } \mathcal{D} \subset \mathcal{H} \rightarrow \mathcal{H}$, we define $\langle \mathcal{D} \rangle^2 = \frac{1}{2}(\mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D})$ and $\mathcal{R}_{\mathcal{D}} = \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2})$ when defined. We also set $\mathcal{H}_{\infty} = \bigcap_{k \geq 0} \text{dom } \langle \mathcal{D} \rangle^k$.

Definition 4.4. *A pseudo-Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of a $*$ -algebra \mathcal{A} represented on the Hilbert space \mathcal{H} as bounded operators, along with a densely defined closed operator $\mathcal{D} : \text{dom } \mathcal{D} \subset \mathcal{H} \rightarrow \mathcal{H}$ such that*

1) $\text{dom } \mathcal{D}^*\mathcal{D} \cap \text{dom } \mathcal{D}\mathcal{D}^*$ is dense in \mathcal{H} and $\langle \mathcal{D} \rangle^2$ is essentially self-adjoint on this domain;

- 2a) $\mathcal{R}_{\mathcal{D}} : \mathcal{H}_{\infty} \rightarrow \mathcal{H}_{\infty}$, and $[\langle \mathcal{D} \rangle^2, \mathcal{R}_{\mathcal{D}}] \in \text{OP}^2(\langle \mathcal{D} \rangle)$;
 2b) $a\mathcal{R}_{\mathcal{D}}(1 + \langle \mathcal{D} \rangle^2)^{-1}$ is compact for all $a \in \mathcal{A}$;
 3) $[\mathcal{D}, a]$ and $[\mathcal{D}^*, a]$ extend to bounded operators on \mathcal{H} for all $a \in \mathcal{A}$ (in particular, all $a \in \mathcal{A}$ preserve $\text{dom } \mathcal{D}$, $\text{dom } \mathcal{D}^*$);
 4) $a(1 + \langle \mathcal{D} \rangle^2)^{-1/2} \in \mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{A}$;
 The pseudo-Riemannian triple is said to be even if there exists $\Gamma \in \mathcal{B}(\mathcal{H})$ such that $\Gamma = \Gamma^*$, $\Gamma^2 = 1$, $\Gamma a = a\Gamma$ for all $a \in \mathcal{A}$ and $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$. Otherwise the pseudo-Riemannian triple is said to be odd.

The definition implies that the operator $\mathcal{R}_{\mathcal{D}}$ is in fact in $\text{OP}^2(\langle \mathcal{D} \rangle)$.

Lemma 4.5. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple. Then*

$$(1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{R}_{\mathcal{D}} (1 + \langle \mathcal{D} \rangle^2)^{-1/2} \in \text{OP}^0(\langle \mathcal{D} \rangle).$$

Hence $\mathcal{R}_{\mathcal{D}} \in \text{OP}^2(\langle \mathcal{D} \rangle)$.

Proof. The operators $\mathcal{D}\mathcal{D}^*$ and $\mathcal{D}^*\mathcal{D}$ are positive and bounded by $1 + \langle \mathcal{D} \rangle^2$, from which it follows that

$$\|\mathcal{D}(1 + \langle \mathcal{D} \rangle^2)^{-1/2}\|^2 = \|(1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{D}^* \mathcal{D} (1 + \langle \mathcal{D} \rangle^2)^{-1/2}\| \leq 2.$$

Hence \mathcal{D} (and similarly \mathcal{D}^*) is bounded by $(1 + \langle \mathcal{D} \rangle^2)^{1/2}$. Thus for $\mathcal{R}_{\mathcal{D}} = \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2})$ we obtain

$$\begin{aligned} & \| (1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{R}_{\mathcal{D}} (1 + \langle \mathcal{D} \rangle^2)^{-1/2} \| \\ & \leq \frac{1}{2} \| (1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{D}^2 (1 + \langle \mathcal{D} \rangle^2)^{-1/2} \| + \frac{1}{2} \| (1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{D}^{*2} (1 + \langle \mathcal{D} \rangle^2)^{-1/2} \| \leq 2. \end{aligned}$$

Applying $L_{\langle \mathcal{D} \rangle} = (1 + \langle \mathcal{D} \rangle^2)^{-1/2} [\langle \mathcal{D} \rangle^2, \cdot]$ repeatedly, using Equation (4), and recalling that $\mathcal{R}_{\mathcal{D}}^{(n)} = [\langle \mathcal{D} \rangle^2, \mathcal{R}_{\mathcal{D}}^{(n-1)}] \in \text{OP}^{n+1}(\langle \mathcal{D} \rangle)$ for all $n \geq 1$ from part 2a) of Definition 4.4, we see that

$$L_{\langle \mathcal{D} \rangle}^n ((1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{R}_{\mathcal{D}} (1 + \langle \mathcal{D} \rangle^2)^{-1/2}) = (1 + \langle \mathcal{D} \rangle^2)^{-1/2} L_{\langle \mathcal{D} \rangle}^n (\mathcal{R}_{\mathcal{D}}) (1 + \langle \mathcal{D} \rangle^2)^{-1/2}$$

is bounded for all $n \geq 1$. □

Remarks 4.6. In Definition 4.4, both parts of condition 2) are intended to force $\mathcal{D}^2 - \mathcal{D}^{*2}$ to be a ‘first order operator’, regarding $\langle \mathcal{D} \rangle^2$ as second order. There are two things to control here: the order of the ‘differential operators’ appearing in $\mathcal{D}^2 - \mathcal{D}^{*2}$, and the growth of the ‘coefficients’. All of these quotation marks can be understood quite literally in the classical case described in the last section.

If we were to restrict attention, in the classical case, to differential operators with bounded coefficients, we would expect the easiest assumptions to force $\mathcal{D}^2 - \mathcal{D}^{*2}$ to be first order would be

$$(9) \quad \mathcal{D}^2 - \mathcal{D}^{*2} \in \text{OP}^1(\langle \mathcal{D} \rangle)$$

and

$$(10) \quad a(\mathcal{D}^2 - \mathcal{D}^{*2}) \in \text{OP}_0^1(\langle \mathcal{D} \rangle) \text{ for all } a \in \mathcal{A}.$$

In fact, Equation (9) actually implies 2a), since $\text{OP}^1(\langle \mathcal{D} \rangle) \subset \text{OP}^2(\langle \mathcal{D} \rangle)$, while Equation (10) together with 4) implies 2b).

The reason for weakening the assumptions so that *a priori* $\mathcal{R}_{\mathcal{D}} = \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2}) \in \text{OP}^2(\langle \mathcal{D} \rangle)$ only, is to allow for unbounded coefficients and also to allow for non-smooth elements in our algebra. (For instance the condition (10) forces $a \in \mathcal{A}$ to be smooth). The harmonic oscillator, treated in detail later, is an example of where unbounded coefficients occur.

Ultimately we will restrict attention to smoothly summable pseudo-Riemannian spectral triples, but we would like to have a definition with minimal smoothness requirements. We have found numerous not-quite-equivalent ways of specifying that $\mathcal{D}^2 - \mathcal{D}^{*2}$ and/or $a(\mathcal{D}^2 - \mathcal{D}^{*2})$ are order one operators, each with slightly different (smoothness) hypotheses. All these different approaches allow us to prove that we can obtain a spectral triple from a pseudo-Riemannian spectral triple, though the methods differ in each case. For instance, some of the other possible assumptions allow the use of the Kato-Rellich theorem in place of the commutator theorem we employ here.

Corollary 4.7. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple. Then the operator $\langle \mathcal{D} \rangle^2 + \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2})$ is essentially self-adjoint on any core for $\langle \mathcal{D} \rangle^2$, and $\text{dom } \langle \mathcal{D} \rangle^2 \subset \text{dom } \langle \mathcal{D} \rangle^2 + \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2})$. Moreover there is $C > 0$ such that for all $\xi \in \text{dom } \langle \mathcal{D} \rangle^2$, we have*

$$\|(1 + \langle \mathcal{D} \rangle^2 + \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2}))\xi\| \leq C\|(1 + \langle \mathcal{D} \rangle^2)\xi\|.$$

Hence we also have

$$\|(1 + \langle \mathcal{D} \rangle^2 + \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2}))(1 + \langle \mathcal{D} \rangle^2)^{-1}\| \leq C.$$

Proof. All of these statements follow from [RS, Theorem X.36], quoted here as Theorem 4.3, whose hypotheses follow from the condition 2a). \square

Corollary 4.8. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple. Then the operator $\mathcal{D}_E = \frac{e^{i\pi/4}}{\sqrt{2}}(\mathcal{D} - i\mathcal{D}^*)$ is essentially self-adjoint. Hence $\mathcal{D}_E^2 = \langle \mathcal{D} \rangle^2 + \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2})$ is a positive operator.*

Proof. The first statement follows from Lemma 4.2, while the second is a consequence of the reality of the spectrum of \mathcal{D}_E . \square

Ultimately we will be interested in obtaining spectral triples from pseudo-Riemannian spectral triples which satisfy the hypotheses of the local index formula. For this we introduce the following notion of spectral dimension and smooth summability for pseudo-Riemannian spectral triples.

Definition 4.9. *A pseudo-Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, is said to be finitely summable if there exists $s > 0$ such that for all $a \in \mathcal{A}$, $a(1 + \langle \mathcal{D} \rangle^2)^{-s/2} \in \mathcal{L}^1(\mathcal{H})$. In such a case, we let*

$$p := \inf \{s > 0 : \forall a \in \mathcal{A}, \text{Trace}(|a|(1 + \langle \mathcal{D} \rangle^2)^{-s/2}) < \infty\},$$

and call p the spectral dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Definition 4.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple. Define the set $S^0 := \mathcal{A} \cup [\mathcal{D}, \mathcal{A}] \cup [\mathcal{D}^*, \mathcal{A}]$, and then recursively define the sets $S^n := [\langle \mathcal{D} \rangle^2, S^{n-1}] \cup [\mathcal{R}_{\mathcal{D}}, S^{n-1}]$ for $n \geq 1$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k summable if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finitely summable with spectral dimension p and*

$$S^n \subset \mathcal{B}_1^{k-n}(\langle \mathcal{D} \rangle, p)(1 + \langle \mathcal{D} \rangle^2)^{n/2} \quad \forall 0 \leq n \leq k.$$

We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is smoothly summable if it is QC^k summable for all $k \in \mathbb{N}$ or, equivalently, if

$$S^n \subset \mathcal{B}_1^\infty(\langle \mathcal{D} \rangle, p)(1 + \langle \mathcal{D} \rangle^2)^{n/2} = \text{OP}_0^n(\langle \mathcal{D} \rangle) \quad \forall n \geq 0.$$

Remark 4.11. If $\mathcal{R}_{\mathcal{D}} = \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2}) = 0$, so that $\langle \mathcal{D} \rangle^2 = \mathcal{D}_E^2$, then the definition of smooth summability here would reduce to $S^0 \subset \text{OP}_0^0(\langle \mathcal{D} \rangle) = \text{OP}_0^0(|\mathcal{D}_E|)$.

4.3. Examples.

4.3.1. *Pseudo-Riemannian manifolds.* Proposition 3.8 shows that those pseudo-Riemannian manifolds arising by reflection from complete Riemannian manifolds of bounded geometry have a Dirac operator satisfying our definition of pseudo-Riemannian spectral triples.

For finite summability, we observe that the bounded geometry hypothesis ensures that $\text{Trace}(a(1 + \langle \mathcal{D} \rangle^2)^{-s/2})$ is finite for $s > n = \dim M$, where $a \in C_c^\infty(M)$ is a compactly supported smooth function. Hence the spectral dimension p is equal to the metric dimension n ; see [CGRS2].

For smooth summability, we also need $\text{Trace}(\gamma(e_j)a(1 + \langle \mathcal{D} \rangle^2)^{-s/2})$ to be finite, which holds since a is compactly supported, and so $\gamma(e_j)a$ is also compactly supported and bounded. Furthermore, we observe that $\langle \mathcal{D} \rangle^2$ is a uniformly elliptic second order differential operator with scalar principal symbol (the metric). Hence $\langle \mathcal{D} \rangle^2$ determines the usual order of compactly supported pseudodifferential operators.

The correction $\mathcal{R}_{\mathcal{D}} = \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2})$ is a first order operator, and the bounded geometry hypothesis implies that $\mathcal{R}_{\mathcal{D}} \in \text{OP}^1(\langle \mathcal{D} \rangle)$. Hence $[\mathcal{R}_{\mathcal{D}}, \text{OP}_0^r(\langle \mathcal{D} \rangle)] \subset \text{OP}_0^{r+1}(\langle \mathcal{D} \rangle)$. It thus follows that the triple $(C_c^\infty(M), L^2(M, S), \mathcal{D})$ is a smoothly summable pseudo-Riemannian spectral triple.

4.3.2. *Previous work.* Paschke and Sitarz, [PS], used equivariance with respect to a group or quantum group of isometries to produce examples of Lorentzian spectral triples, which are a special case of the definition presented above. In particular they constructed examples on noncommutative tori and noncommutative 3-spheres (both isospectral deformations of S^3 and $SU_q(2)$).

Strohmaier and Van Suijlekom, [S, vS], used a Krein space formulation for their definition of semi-Riemannian spectral triples, but this is largely equivalent to the approach we have adopted, as can be seen from Sections 3.3 and 5.4. Strohmaier also examined noncommutative tori, while Van Suijlekom examined deformations of generalised cylinders. All of these examples can be put in our framework.

4.3.3. *Finite Geometries.* Just as there are virtually no constraints to the existence of a spectral triple for a finite dimensional algebra, pseudo-Riemannian spectral triples are easily constructed in this case. So let \mathcal{A} be a finite dimensional complex algebra, i.e. a direct sum of simple matrix algebras. Choose two representations of \mathcal{A} on finite dimensional Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 . Let

$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with respect to this decomposition. All we need now is

a ‘Dirac’ operator. Choose any linear map $B : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, and set $\mathcal{D} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$. Then the definition of an even pseudo-Riemannian spectral triple is trivially fulfilled. Likewise it is trivially smoothly summable.

4.3.4. *First order differential operators.* We consider a constant coefficient first order differential operator of the form

$$\mathcal{D} = \sum_{j=1}^n M_j \frac{\partial}{\partial x^j} + K$$

where $K, M_j \in M_d(\mathbb{C})$. The operator \mathcal{D} , acting on the smooth compactly supported sections in $L^2(\mathbb{R}^n, \mathbb{C}^d)$, extends to a closed and densely defined operator. One may check that $(C_c^\infty(\mathbb{R}^n), L^2(\mathbb{R}^n, \mathbb{C}^d), \mathcal{D})$ yields a pseudo-Riemannian spectral triple provided that for all $j, k = 1, \dots, n$ and for all $0 \neq \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ the following three conditions hold:

$$\begin{aligned} \sum_{j,k=1}^n (M_j^* M_k + M_j M_k^*) \xi_j \xi_k &\in GL_d(\mathbb{C}), \\ \{M_j, M_k\} &= \{M_j^*, M_k^*\}, \\ \left[\sum_{j,k=1}^n (M_j^* M_k + M_j M_k^*) \xi_j \xi_k, \sum_{j=1}^n (\{M_j, K\} + \{M_j^*, K^*\}) \xi_j \right] &= 0. \end{aligned}$$

Provided that in addition

$$[M_j, M_k M_l^* + M_k^* M_l] = 0, \quad \text{and} \quad [K, M_k M_l^* + M_k^* M_l] = 0,$$

the pseudo-Riemannian spectral triple is in fact smoothly summable.

4.3.5. *The harmonic oscillator.* We consider (the creation operator associated to) the harmonic oscillator

$$\mathcal{D} = \frac{d}{dx} + x,$$

acting on $L^2(\mathbb{R})$. To show that we obtain a pseudo-Riemannian spectral triple, we observe that the operator

$$\langle \mathcal{D} \rangle^2 = -\frac{d^2}{dx^2} + x^2$$

is well-known to be essentially self-adjoint and to have compact resolvent. For all $a \in C_1^\infty(\mathbb{R})$, the smooth functions with integrable derivatives, the commutators $[\mathcal{D}, a]$ and $[\mathcal{D}^*, a]$ are bounded. The other items to check involve the operator

$$\mathcal{R}_{\mathcal{D}} := \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2}) = i(1 + 2x \frac{d}{dx}),$$

which is first order as a differential operator, but has an unbounded coefficient. With $\mathcal{D}_E = \frac{1}{2}(\mathcal{D} + \mathcal{D}^*) + \frac{i}{2}(\mathcal{D} - \mathcal{D}^*)$ we have $\mathcal{D}_E = i \frac{d}{dx} + x$, but we could just as easily take $\widetilde{\mathcal{D}}_E = \frac{1}{2}(\mathcal{D} + \mathcal{D}^*) - \frac{i}{2}(\mathcal{D} - \mathcal{D}^*)$ to get $\widetilde{\mathcal{D}}_E = -i \frac{d}{dx} + x$. Our results apply to both possibilities.

Lemma 4.12. *For all $n \geq 0$ the operator $\mathcal{R}_{\mathcal{D}}^{(n)}$ is an element of $\text{OP}^2(\langle \mathcal{D} \rangle)$, and for all $a \in C_1^\infty(\mathbb{R})$ the operator $a \mathcal{R}_{\mathcal{D}}(1 + \langle \mathcal{D} \rangle^2)^{-1}$ is compact.*

Proof. Straightforward calculations show that

$$\mathcal{R}_{\mathcal{D}}^{(1)} = [\langle \mathcal{D} \rangle^2, \mathcal{R}_{\mathcal{D}}] = 4i \langle \mathcal{D} \rangle^2 - 8ix^2, \quad \mathcal{R}_{\mathcal{D}}^{(2)} = 16\mathcal{R}_{\mathcal{D}},$$

and so it suffices to check the first claim for $n = 0, 1$. We begin by observing that $0 \leq x^2 \leq x^2 - \frac{d^2}{dx^2} + 1$ implies

$$(11) \quad \left\| x \left(1 + x^2 - \frac{d^2}{dx^2} \right)^{-1/2} \right\|^2 = \left\| \left(1 + x^2 - \frac{d^2}{dx^2} \right)^{-1/2} x \right\|^2 \leq 1.$$

Similarly $\left\| \frac{d}{dx} \left(1 + x^2 - \frac{d^2}{dx^2} \right)^{-1/2} \right\| \leq 1$. Hence

$$\left(1 + \langle \mathcal{D} \rangle^2 \right)^{-1/2} \mathcal{R}_{\mathcal{D}} \left(1 + \langle \mathcal{D} \rangle^2 \right)^{-1/2} \text{ and } \left(1 + \langle \mathcal{D} \rangle^2 \right)^{-1/2} \mathcal{R}_{\mathcal{D}}^{(1)} \left(1 + \langle \mathcal{D} \rangle^2 \right)^{-1/2}$$

are bounded operators. Hence $\mathcal{R}_{\mathcal{D}} \in \text{OP}^2(\langle \mathcal{D} \rangle)$ as is $\mathcal{R}_{\mathcal{D}}^{(1)}$. If a is a bounded integrable function, the product ax is bounded, and so by the compactness of $\left(1 + \langle \mathcal{D} \rangle^2 \right)^{-1/2}$ we see that $a\mathcal{R}_{\mathcal{D}} \left(1 + \langle \mathcal{D} \rangle^2 \right)^{-1}$ is compact. \square

Thus the harmonic oscillator gives rise to a pseudo-Riemannian spectral triple.

Proposition 4.13. *The pseudo-Riemannian spectral triple $(C_1^\infty(\mathbb{R}), L^2(\mathbb{R}), \frac{d}{dx} + x)$ is smoothly summable with spectral dimension 1.*

Proof. Let $a : \mathbb{R} \rightarrow [0, \infty)$ be a smooth integrable function, and let $(x, y) \mapsto k_t(x, y)$ be the integral kernel of $e^{-t\langle \mathcal{D} \rangle^2}$ for $\mathcal{D} = \frac{d}{dx} + x$. Mehler's formula gives

$$k_t(x, y) = \frac{1}{\sqrt{2\pi \sinh(2t)}} e^{-\frac{1}{2} \coth(2t)(x^2 + y^2) + \text{cosech}(2t)xy}.$$

Then for $s > 2$

$$\begin{aligned} \text{Trace}(a(1 + \langle \mathcal{D} \rangle^2)^{-s/2}) &= \frac{1}{\Gamma(s/2)} \int_{\mathbb{R}} a(x) \int_0^\infty t^{s/2-1} e^{-t} k_t(x, x) dt dx \\ &\leq \frac{1}{\Gamma(s/2)} \int_{\mathbb{R}} a(x) dx \int_0^\infty t^{s/2-1} e^{-t} \frac{1}{\sqrt{2\pi \sinh(2t)}} dt \end{aligned}$$

and this remains finite for $s > 1$. Thus the spectral dimension is ≤ 1 . To see that the spectral dimension is ≥ 1 , and so is precisely 1, one computes this trace for the function $a : x \mapsto e^{-x^2}$.

For the algebra $\mathcal{A} = C_1^\infty(\mathbb{R})$, we find that the commutators $[\mathcal{D}, a]$ and $[\mathcal{D}^*, a]$ are again elements of this algebra. With the notation of Definition 4.10, we thus find that $S^0 := \mathcal{A} \cup [\mathcal{D}, \mathcal{A}] \cup [\mathcal{D}^*, \mathcal{A}] = \mathcal{A}$. The above computations now allow us to see that S^0 lies in $B_1(\langle \mathcal{D} \rangle, 1)$.

Finally, we need to check that $S^n \subset \text{OP}_0^n(\langle \mathcal{D} \rangle)$ for all n . First, by Equation (11), and the relations

$$\left[\langle \mathcal{D} \rangle^2, \frac{d}{dx} \right] = 2x, \quad [\langle \mathcal{D} \rangle^2, x] = -2\frac{d}{dx} \quad \left[\mathcal{R}_{\mathcal{D}}, x^n \frac{d^m}{dx^m} \right] = 2i(n - m)x^n \frac{d^m}{dx^m},$$

we can see that both multiplication by x and $\frac{d}{dx}$ lie in $\text{OP}^1(\langle \mathcal{D} \rangle)$. For $a \in C_1^\infty(\mathbb{R})$, we use the computation

$$[\langle \mathcal{D} \rangle^2, a] = -a'' - 2a' \frac{d}{dx}$$

and a simple induction to see that $a \in \text{OP}^0(\langle \mathcal{D} \rangle)$. Hence $C_1^\infty(\mathbb{R}) \subset \text{OP}_0^0(\langle \mathcal{D} \rangle)$. It is then straightforward to check that any element $T \in S^n$ can be written in the form

$$T = \sum_{k+l \leq n} a_{k,l} x^k \frac{d^l}{dx^l}$$

for functions $a_{k,l} \in C_1^\infty(\mathbb{R})$. This is obviously true for $n = 0$. Assuming it holds for all $T \in S^n$ for some n , one shows it also holds for $n + 1$ by explicitly calculating the commutators $[\langle \mathcal{D} \rangle^2, T]$ and $[\mathcal{R}_{\mathcal{D}}, T]$. So it follows by induction that indeed $T = \sum_{k+l \leq n} a_{k,l} x^k \frac{d^l}{dx^l}$ for all $T \in S^n$, for all n . Since $a_{k,l} \in \text{OP}_0^0(\langle \mathcal{D} \rangle)$, $x^k \in \text{OP}^k(\langle \mathcal{D} \rangle)$, and $\frac{d^l}{dx^l} \in \text{OP}^l(\langle \mathcal{D} \rangle)$, it follows that $S^n \subset \text{OP}_0^n(\langle \mathcal{D} \rangle)$ for all n . Therefore we conclude that the pseudo-Riemannian spectral triple $(C_1^\infty(\mathbb{R}), L^2(\mathbb{R}), \frac{d}{dx} + x)$ is smoothly summable. \square

4.3.6. Semifinite examples. There is a notion of semifinite spectral triple, [BeF, CPRS1, CPRS2, CPRS3], where $(\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H}), \text{Trace})$ are replaced by $(\mathcal{N}, \mathcal{K}(\mathcal{N}, \tau), \tau)$ where \mathcal{N} is an arbitrary semifinite von Neumann algebra, $\mathcal{K}(\mathcal{N}, \tau)$ is the ideal of τ -compact operators in \mathcal{N} , and τ is a faithful, semifinite, normal trace. Thus we require \mathcal{D} affiliated to \mathcal{N} , and the compact resolvent condition is relative to $\mathcal{K}(\mathcal{N}, \tau)$. Examples of semifinite spectral triples arising from graph and k -graph C^* -algebras were described in [PR, PRS]. These were constructed using the natural action of the torus T^k on a k -graph algebra, by ‘pushing forward’ the Dirac operator on the torus. More sophisticated examples coming from covering spaces of manifolds of bounded geometry were considered in [CGRS2] also.

For the k -graph algebras, $k \geq 2$, we may of course take a Lorentzian Dirac operator (or more generally pseudo-Riemannian Dirac operator) and push this forward instead. This gives rise to a ‘semifinite pseudo-Riemannian spectral triple’, but as the details would take us too far afield, we leave this to the reader to explore.

5. CONSTRUCTING A SPECTRAL TRIPLE FROM A PSEUDO-RIEMANNIAN SPECTRAL TRIPLE

Our main theorem associates to a pseudo-Riemannian spectral triple a bona fide spectral triple. As before, we write $\mathcal{R}_{\mathcal{D}} := \frac{i}{2}(\mathcal{D}^2 - \mathcal{D}^{*2})$ for brevity.

Theorem 5.1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple. Consider*

$$\mathcal{D}_E := \frac{1}{2}(\mathcal{D} + \mathcal{D}^*) + \frac{i}{2}(\mathcal{D} - \mathcal{D}^*) = \frac{e^{i\pi/4}}{\sqrt{2}}(\mathcal{D} - i\mathcal{D}^*)$$

as in Definition 4.1. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$ is a spectral triple.

Proof. We already know from Corollary 4.8 that \mathcal{D}_E is essentially selfadjoint on $\text{dom } \mathcal{D} \cap \text{dom } \mathcal{D}^*$. From the definition of pseudo-Riemannian spectral triple, the commutator

$$[\mathcal{D}_E, a] = \frac{1+i}{2}[\mathcal{D}, a] + \frac{1-i}{2}[\mathcal{D}^*, a]$$

is bounded for all $a \in \mathcal{A}$. Thus, to show that we have a spectral triple, we need to consider the resolvent condition. We first need to show, for $a \in \mathcal{A}$, the compactness of

$$a(1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^{-1} = a(1 + \langle \mathcal{D} \rangle^2)^{-1} - a(1 + \langle \mathcal{D} \rangle^2)^{-1} \mathcal{R}_{\mathcal{D}} (1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^{-1}.$$

The first term is compact by condition 4) in the definition of pseudo-Riemannian spectral triple. For the second term we write

$$a(1 + \langle \mathcal{D} \rangle^2)^{-1} \mathcal{R}_{\mathcal{D}} = -a(1 + \langle \mathcal{D} \rangle^2)^{-1} [\langle \mathcal{D} \rangle^2, \mathcal{R}_{\mathcal{D}}] (1 + \langle \mathcal{D} \rangle^2)^{-1} + a \mathcal{R}_{\mathcal{D}} (1 + \langle \mathcal{D} \rangle^2)^{-1}.$$

Both terms here are compact, the first by 2a) and 4), the second by 2b). Thus $a(1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^{-1}$ is compact. Finally we employ the integral formula for fractional powers to complete the proof

that we have a spectral triple. The same reasoning as above gives the compactness of the integrand in

$$a(1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^{-1/2} = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} a(1 + \lambda + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^{-1} d\lambda,$$

and as the integral converges in norm, the left hand side is a compact operator. Thus $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$ is a spectral triple. \square

5.1. Smooth summability. We now consider the smooth summability of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$. While this can be checked directly for each example, as we do for the harmonic oscillator, we first present a sufficient condition guaranteeing the smooth summability of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$ given the smooth summability of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, with the same spectral dimension. Our sufficient condition requires an additional assumption on the boundedness of $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$. The harmonic oscillator (see §5.3) suggests that this condition is not necessary. We proceed by proving a few lemmas about the structure of pseudodifferential operators associated to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Lemma 5.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple. Then $(1 + \mathcal{D}_E^2)$ is an element of $\text{OP}^2(\langle \mathcal{D} \rangle)$. Furthermore, if $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$ is bounded, then $(1 + \mathcal{D}_E^2)^{-1}$ is an element of $\text{OP}^{-2}(\langle \mathcal{D} \rangle)$.*

Proof. Since $\mathcal{R}_{\mathcal{D}} \in \text{OP}^2(\langle \mathcal{D} \rangle)$ by hypothesis, $1 + \mathcal{D}_E^2 = 1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}} \in \text{OP}^2(\langle \mathcal{D} \rangle)$. If furthermore $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$ is bounded, it follows from Lemma 2.14 that $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1} \in \text{OP}^0(\langle \mathcal{D} \rangle)$, and hence that $(1 + \mathcal{D}_E^2)^{-1} \in \text{OP}^{-2}(\langle \mathcal{D} \rangle)$. \square

In the following discussion of smooth summability, the boundedness of $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$ plays a crucial role. We pause to give a sufficient condition for this boundedness to hold.

Lemma 5.3. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple with*

$$\|(1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{R}_{\mathcal{D}} (1 + \langle \mathcal{D} \rangle^2)^{-1/2}\| < 1.$$

Then $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$ lies in $\text{OP}^0(\langle \mathcal{D} \rangle)$, and hence in particular is bounded.

Proof. We know that $(1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{R}_{\mathcal{D}} (1 + \langle \mathcal{D} \rangle^2)^{-1/2}$ is bounded since $\mathcal{R}_{\mathcal{D}} \in \text{OP}^2(\langle \mathcal{D} \rangle)$, and assuming that $\|(1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{R}_{\mathcal{D}} (1 + \langle \mathcal{D} \rangle^2)^{-1/2}\| < 1$ ensures that -1 is not in the spectrum. Hence the operator

$$(1 + (1 + \langle \mathcal{D} \rangle^2)^{-1/2} \mathcal{R}_{\mathcal{D}} (1 + \langle \mathcal{D} \rangle^2)^{-1/2})^{-1} = (1 + \langle \mathcal{D} \rangle^2)^{1/2} (1 + \mathcal{D}_E^2)^{-1} (1 + \langle \mathcal{D} \rangle^2)^{1/2}$$

is bounded. Observe that this also implies the boundedness of $(1 + \mathcal{D}_E^2)^{-1/2} (1 + \langle \mathcal{D} \rangle^2)^{1/2}$ and $(1 + \langle \mathcal{D} \rangle^2)^{1/2} (1 + \mathcal{D}_E^2)^{-1/2}$.

From Lemma 5.2 we know that $(1 + \langle \mathcal{D} \rangle^2)^{-1/2} (1 + \mathcal{D}_E^2) (1 + \langle \mathcal{D} \rangle^2)^{-1/2}$ lies in $\text{OP}^0(\langle \mathcal{D} \rangle)$, and since this operator has bounded inverse, it follows from Lemma 2.14 that $(1 + \langle \mathcal{D} \rangle^2)^{1/2} (1 + \mathcal{D}_E^2)^{-1} (1 + \langle \mathcal{D} \rangle^2)^{1/2}$ lies in $\text{OP}^0(\langle \mathcal{D} \rangle)$. This also implies that $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$ lies in $\text{OP}^0(\langle \mathcal{D} \rangle)$. \square

Lemma 5.4. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple such that $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$ is bounded. Then the ratios*

$$\begin{aligned} & (1 + \langle \mathcal{D} \rangle^2)^{-s} (1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^s, \quad (1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^s (1 + \langle \mathcal{D} \rangle^2)^{-s}, \\ & (1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^{-s} (1 + \langle \mathcal{D} \rangle^2)^s, \quad (1 + \langle \mathcal{D} \rangle^2)^s (1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^{-s} \end{aligned}$$

are bounded for all $0 \leq s \in \mathbb{R}$.

Proof. Let $\sigma^r(T) := (1 + \langle \mathcal{D} \rangle^2)^{r/2} T (1 + \langle \mathcal{D} \rangle^2)^{-r/2}$ be the one parameter group associated to $\langle \mathcal{D} \rangle$. Consider, for instance, the ratio $(1 + \langle \mathcal{D} \rangle^2)^{-s} (1 + \mathcal{D}_E^2)^s$. Then for $s \in \mathbb{R}$ we have

$$\begin{aligned} (1 + \langle \mathcal{D} \rangle^2)^{-s} (1 + \mathcal{D}_E^2)^s &= (1 + \langle \mathcal{D} \rangle^2)^{-s+1} (1 + \langle \mathcal{D} \rangle^2)^{-1} (1 + \mathcal{D}_E^2) (1 + \mathcal{D}_E^2)^{s-1} \\ &= \sigma^{-2s+2}((1 + \langle \mathcal{D} \rangle^2)^{-1} (1 + \mathcal{D}_E^2)) (1 + \langle \mathcal{D} \rangle^2)^{-s+1} (1 + \mathcal{D}_E^2)^{s-1} \\ &=: K_s (1 + \langle \mathcal{D} \rangle^2)^{-s+1} (1 + \mathcal{D}_E^2)^{s-1}, \end{aligned}$$

where $K_s \in \text{OP}^0(\langle \mathcal{D} \rangle)$ because $(1 + \langle \mathcal{D} \rangle^2)^{-1} (1 + \mathcal{D}_E^2) \in \text{OP}^0(\langle \mathcal{D} \rangle)$ by Lemma 5.2. Repeating this process shows that we can assume that $0 < s < 1$. Similar arguments hold for the other ratios. For $0 < s < 1$, the function $t \mapsto t^s$ is operator monotone, and elementary techniques then yield the result. \square

Lemma 5.5. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a pseudo-Riemannian spectral triple such that $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$ is bounded. Then for $s \geq 0$ the ratio*

$$(1 + \langle \mathcal{D} \rangle^2)^s (1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^{-s}$$

is an element of $\text{OP}^0(\langle \mathcal{D} \rangle)$.

Proof. As in the proof of Lemma 5.4 we can reduce the problem to the case $0 < s < 1$. We already know from Lemma 5.4 that $(1 + \langle \mathcal{D} \rangle^2)^s (1 + \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}})^{-s}$ is bounded. Let us write $T := (1 + \mathcal{D}_E^2)$ for brevity. As in the proof of Lemma 2.14, there are (combinatorial) constants $C_{l,n,k}$ such that

$$\begin{aligned} \delta_{\langle \mathcal{D} \rangle}^n((\lambda + T)^{-1}) &= \sum_{1 \leq l \leq n} \sum_{1 \leq k_1, \dots, k_l \leq n, |k| = n} C_{l,n,k} \\ &\quad (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_1}(T) (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_2}(T) (\lambda + T)^{-1} \cdots (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_l}(T) (\lambda + T)^{-1}. \end{aligned}$$

Using the integral formula for fractional powers, we can then write

$$\begin{aligned} (1 + \langle \mathcal{D} \rangle^2)^s \delta_{\langle \mathcal{D} \rangle}^n(T^{-s}) &= \frac{1}{\pi} \int_0^\infty \lambda^{-s} \sum_{1 \leq l \leq n} \sum_{1 \leq k_1, \dots, k_l \leq n, |k| = n} C_{l,n,k} \\ &\quad (1 + \langle \mathcal{D} \rangle^2)^s (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_1}(T) (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_2}(T) (\lambda + T)^{-1} \cdots (\lambda + T)^{-1} \delta_{\langle \mathcal{D} \rangle}^{k_l}(T) (\lambda + T)^{-1} d\lambda. \end{aligned}$$

Since $\delta_{\langle \mathcal{D} \rangle}^k(T) \in \text{OP}^2(\langle \mathcal{D} \rangle)$ so that $\delta_{\langle \mathcal{D} \rangle}^k(T) (\lambda + T)^{-1} = \delta_{\langle \mathcal{D} \rangle}^k(T) (1 + \langle \mathcal{D} \rangle^2)^{-1} (1 + \langle \mathcal{D} \rangle^2) T^{-1} T (\lambda + T)^{-1}$ is uniformly bounded in λ for each k , we see that the integral converges to a bounded operator for all n , so $(1 + \langle \mathcal{D} \rangle^2)^s (1 + \mathcal{D}_E^2)^{-s}$ is in $\text{OP}^0(\langle \mathcal{D} \rangle)$. \square

Theorem 5.6. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a smoothly summable pseudo-Riemannian spectral triple such that $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$ is bounded. Then the corresponding spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$ is smoothly summable, with the same spectral dimension.*

Proof. Since the operators $(1 + \mathcal{D}_E^2)^{s/2} (1 + \langle \mathcal{D} \rangle^2)^{-s/2}$ and $(1 + \langle \mathcal{D} \rangle^2)^{s/2} (1 + \mathcal{D}_E^2)^{-s/2}$ are bounded (by Lemma 5.4), we see that $a(1 + \langle \mathcal{D} \rangle^2)^{-s/2} \in \mathcal{L}^1(\mathcal{H})$ implies $a(1 + \mathcal{D}_E^2)^{-s/2} \in \mathcal{L}^1(\mathcal{H})$ and conversely. Hence $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$ is finitely summable, with the same spectral dimension.

Similarly, it is straightforward to show that $\mathcal{B}_2(\mathcal{D}_E, p)$ coincides with $\mathcal{B}_2(\langle \mathcal{D} \rangle, p)$, and so also that $\mathcal{B}_1(\mathcal{D}_E, p) = \mathcal{B}_1(\langle \mathcal{D} \rangle, p)$. Now suppose that $T \in S^0$, which by assumption is contained in $\text{OP}_0^0(\langle \mathcal{D} \rangle)$. Let us write $T^{(0)} := T$ and $T^{(n)} := [\mathcal{D}_E^2, T^{(n-1)}] = [\langle \mathcal{D} \rangle^2, T^{(n-1)}] + [\mathcal{R}_{\mathcal{D}}, T^{(n-1)}]$. It then follows that $T^{(n)}$ is a finite linear combination of elements of S^n , which by assumption is contained in $\text{OP}_0^n(\langle \mathcal{D} \rangle)$. Because we can write

$$R_{\mathcal{D}_E}^k(T) = T^{(k)}(1 + \mathcal{D}_E^2)^{-k/2} = T^{(k)}(1 + \langle \mathcal{D} \rangle^2)^{-k/2}(1 + \langle \mathcal{D} \rangle^2)^{k/2}(1 + \mathcal{D}_E^2)^{-k/2},$$

and because $(1 + \langle \mathcal{D} \rangle^2)^{k/2}(1 + \mathcal{D}_E^2)^{-k/2}$ lies in $\text{OP}_0^0(\langle \mathcal{D} \rangle)$ by Lemma 5.5, we see that $R_{\mathcal{D}_E}^k(T)$ is an element of $\text{OP}_0^0(\langle \mathcal{D} \rangle) = \mathcal{B}_1^\infty(\langle \mathcal{D} \rangle, p)$, so in particular $R_{\mathcal{D}_E}^k(T) \in \mathcal{B}_1(\langle \mathcal{D} \rangle, p) = \mathcal{B}_1(\mathcal{D}_E, p)$ for all k . Therefore T is an element of $\mathcal{B}_1^\infty(\mathcal{D}_E, p)$, and hence $S^0 \subset \mathcal{B}_1^\infty(\mathcal{D}_E, p)$. Since $\mathcal{A} \cup [\mathcal{D}_E, \mathcal{A}]$ consists of linear combinations of elements of S^0 , we conclude that $\mathcal{A} \cup [\mathcal{D}_E, \mathcal{A}]$ is also contained in $\mathcal{B}_1^\infty(\mathcal{D}_E, p)$, and hence that $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$ is smoothly summable. \square

Given a pseudo-Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ we have seen that there is a more or less canonical construction of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$, and this construction preserves the property of smooth summability needed for the local index formula.

This means that to each smoothly summable pseudo-Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ we can associate a K -homology class $[(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)] \in K^*(A)$, where A is the norm completion of \mathcal{A} , and compute the pairing of this class with $K_*(A)$ using the (nonunital) local index formula, [CGRS2].

Remark 5.7. Theorem 5.1 shows that the operator $\mathcal{D}_E = \frac{e^{i\pi/4}}{\sqrt{2}}(\mathcal{D} - i\mathcal{D}^*)$ can be used to obtain a spectral triple, where $\mathcal{D}_E^2 = \langle \mathcal{D} \rangle^2 + \mathcal{R}_{\mathcal{D}}$. However the proof is exactly the same for the operator $\widetilde{\mathcal{D}}_E = \frac{e^{-i\pi/4}}{\sqrt{2}}(\mathcal{D} + i\mathcal{D}^*)$, with $\widetilde{\mathcal{D}}_E^2 = \langle \mathcal{D} \rangle^2 - \mathcal{R}_{\mathcal{D}}$.

This means that we potentially have two K -homology classes associated to a pseudo-Riemannian spectral triple. In Section 5.4 we will see a refinement of the definition, from [PS], which guarantees that these two classes are in fact negatives of each other.

5.2. Examples. We present a few examples showing what happens when a pseudo-Riemannian spectral triple is converted into a spectral triple using Theorem 5.1.

5.2.1. Pseudo-Riemannian manifolds. We have already shown in Section 3 that for pseudo-Riemannian manifolds which arise from complete Riemannian manifolds of bounded geometry, the triple $(C_c^\infty(M), L^2(M, S), \not{D})$ is a pseudo-Riemannian spectral triple. Theorem 5.1 then returns the self-adjoint operator \not{D}_E . (Actually the signs we have used mean that we obtain $\not{D}_E = \Re \not{D} - \Im \not{D}$, but by the remark above we still get a smoothly summable spectral triple. Using $\frac{e^{-i\pi/4}}{\sqrt{2}}(\not{D} + i\not{D}^*)$ would return the operator $\widetilde{\not{D}}_E = \Re \not{D} + \Im \not{D}$.)

In these examples the operator $\mathcal{R}_{\not{D}}$ is a first order differential operator, and so $\not{D}_E^2 = \langle \not{D} \rangle^2 - \mathcal{R}_{\not{D}}$ is a second order positive elliptic differential operator. This implies that $(1 + \langle \not{D} \rangle^2)(1 + \not{D}_E^2)^{-1}$ is bounded, since $(1 + \not{D}_E^2)^{-1}$ is an order -2 pseudodifferential operator. Then Theorem 5.6 tells us that the associated spectral triple is smoothly summable.

5.2.2. *Finite geometries.* With the same notation as in §4.3.3, we find the operator

$$\mathcal{D}_E = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 0 & -iB^* \\ B & 0 \end{pmatrix}.$$

5.2.3. *First order differential operators.* With the same notation as in §4.3.4, we find that

$$\mathcal{D}_E = \sum_j \tilde{M}_j \partial_j + \frac{1}{2}(K + K^*) + \frac{i}{2}(K - K^*)$$

where $\tilde{M}_j = M_j$ if $M_j^* = -M_j$ and $\tilde{M}_j = iM_j$ if $M_j = M_j^*$. Since $\mathcal{R}_{\mathcal{D}}$ is a first order differential operator in these examples, and \mathcal{D}_E^2 is second order and uniformly elliptic, one can show that $\mathcal{R}_{\mathcal{D}}(1 + \mathcal{D}_E^2)^{-1}$ is bounded, and Theorem 5.6 gives us a smoothly summable spectral triple.

5.3. **The K -homology class of the harmonic oscillator.** Here we consider the K -homology class of the spectral triple obtained from the harmonic oscillator. Theorem 5.1 gives

$$\mathcal{D}_E = i \frac{d}{dx} + x, \quad \mathcal{D}_E^2 = -\frac{d^2}{dx^2} + x^2 + i(1 + 2x \frac{d}{dx}).$$

We let $C_1^\infty(\mathbb{R})$ be the smooth functions all of whose derivatives are integrable on \mathbb{R} . We have seen in Proposition 4.13 that the pseudo-Riemannian spectral triple $(C_1^\infty(\mathbb{R}), L^2(\mathbb{R}), \frac{d}{dx} + x)$ is smoothly summable with spectral dimension $p = 1$.

In order to conclude from Theorem 5.6 that the spectral triple $(C_1^\infty(\mathbb{R}), L^2(\mathbb{R}), \mathcal{D}_E)$ is also smoothly summable, we would need to check that $(1 + \langle \mathcal{D} \rangle^2)(1 + \mathcal{D}_E^2)^{-1}$ is a bounded operator. We have been unable to prove this, and at present have no reason to believe it is true.

On the other hand, we can simply check that $C_1^\infty(\mathbb{R}) \subset B_1^\infty(\mathcal{D}_E, 1)$. However, this follows in the same way as in the proof of Proposition 4.13. The integral kernels of $e^{-t\mathcal{D}_E^2}$ and $(1 + \mathcal{D}_E^2)^{-s/2}$ are given by

$$k_t(x, y) = \frac{1}{2\sqrt{\pi t}} e^{-(x-y)^2/4t + i(x^2 - y^2)/2}, \quad k_s(x, y) = \frac{1}{\Gamma(s/2)} \int_0^\infty t^{s/2-1} e^{-t} k_t(x, y) dt.$$

Then for any $g \in C_1^\infty(\mathbb{R})$ one finds that

$$\text{Trace}(g(1 + \mathcal{D}_E^2)^{-s/2}) = \frac{\Gamma(s/2 - 1/2)}{2\sqrt{\pi}\Gamma(s/2)} \int_{\mathbb{R}} g(x) dx.$$

Thus the spectral dimension is ≥ 1 , and taking $g(x) = e^{-x^2}$ shows that the spectral dimension is precisely 1. The smoothness is an easy check, using the same computations in the proof of Proposition 4.13. To show that the spectral triple is smoothly summable, we observe that for all $f \in C_1^\infty(\mathbb{R})$ we have $[\mathcal{D}_E^2, f] = -f'' + 2if' \mathcal{D}_E$. Then a straightforward induction shows that $C_1^\infty(\mathbb{R}) \subset B_1^\infty(\mathcal{D}_E, 1)$. Hence the spectral triple $(C_1^\infty(\mathbb{R}), L^2(\mathbb{R}), \mathcal{D}_E)$ is also smoothly summable with spectral dimension 1. We can then apply the local index formula of [CGRS2, Theorem 4.33].

For a unitary u in the unitization of $C_1^\infty(\mathbb{R})$, the local index formula computes the pairing of the class of the spectral triple with the K -theory class of u as

$$\langle [u], [(C_1^\infty(\mathbb{R}), L^2(\mathbb{R}), \mathcal{D}_E)] \rangle = - \lim_{s \rightarrow 1/2} (s - 1/2) \text{Trace}(u^* [\mathcal{D}_E, u] (1 + \mathcal{D}_E^2)^{-s}).$$

The odd K -theory of the real line is $K_1(C_0(\mathbb{R})) = \mathbb{Z}$. For $m \in \mathbb{Z}$ we choose representatives of these classes to be $u = e^{2im \tan^{-1}(x)}$, and this gives $u^*[\mathcal{D}_E, u] = \frac{-2m}{1+x^2} =: g_m(x)$. Using the trace calculations above we have

$$\begin{aligned} \lim_{s \rightarrow 1} \frac{s-1}{2} \text{Trace}(u^*[\mathcal{D}_E, u](1 + \mathcal{D}_E^2)^{-s/2}) &= \lim_{s \rightarrow 1} \frac{1}{2\sqrt{\pi}\Gamma(s/2)} \int_{\mathbb{R}} g(x) \int_0^\infty \frac{s-1}{2} t^{s/2-1/2-1} e^{-t} dt dx \\ &= \lim_{s \rightarrow 1} \frac{1}{2\sqrt{\pi}\Gamma(s/2)} \int_{\mathbb{R}} g(x) \int_0^\infty \frac{d}{dt} (t^{s/2-1/2}) e^{-t} dt dx = \frac{-m}{\pi} \int_{\mathbb{R}} \frac{1}{1+x^2} dx = -m. \end{aligned}$$

Hence the spectral triple $(C_1^\infty(\mathbb{R}), L^2(\mathbb{R}), i\frac{d}{dx} + x)$ has a nontrivial K -homology class, and it coincides with the class of $(C_1^\infty(\mathbb{R}), L^2(\mathbb{R}), i\frac{d}{dx})$, see [HR, page 298]. From the perspective of principal symbols this is not surprising, however the unboundedness of the perturbation means that the result is not immediate.

5.4. Application to compact manifolds. We give an index theoretic application in a special case closely related to the Lorentzian spectral triples of [PS].

Definition 5.8. A Lorentz-type spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \beta)$ is given by a $*$ -algebra \mathcal{A} represented on the Hilbert space \mathcal{H} as bounded operators, along with a bounded operator β on \mathcal{H} and a densely defined closed operator $\mathcal{D} : \text{dom } \mathcal{D} \subset \mathcal{H} \rightarrow \mathcal{H}$ such that

- 0) $\text{dom } \mathcal{D}^* \mathcal{D} \cap \text{dom } \mathcal{D} \mathcal{D}^*$ is dense in \mathcal{H} and $\langle \mathcal{D} \rangle^2$ is essentially self-adjoint on this domain;
- 1) $\beta = -\beta^*$, $\beta^2 = -1$, $\beta a = a\beta$ for all $a \in \mathcal{A}$ and $\beta \mathcal{D}$ is essentially self-adjoint on $\text{dom } \mathcal{D}^* \cap \text{dom } \mathcal{D}$;
- 2a) $\beta[\mathcal{D}^2, \beta] : \mathcal{H}_\infty \rightarrow \mathcal{H}_\infty$ and $[\langle \mathcal{D} \rangle^2, \beta[\mathcal{D}^2, \beta]] \in \text{OP}^2(\langle \mathcal{D} \rangle)$;
- 2b) $a\beta[\mathcal{D}^2, \beta](1 + \langle \mathcal{D} \rangle^2)^{-1} \in \mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{A}$;
- 3) $[\mathcal{D}, a]$ and $[\mathcal{D}^*, a]$ extend to bounded operators on \mathcal{H} for all $a \in \mathcal{A}$;
- 4) $a(1 + \langle \mathcal{D} \rangle^2)^{-1/2} \in \mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{A}$.

The triple is said to be even if there exists $\Gamma \in \mathcal{B}(\mathcal{H})$ such that $\Gamma = \Gamma^*$, $\Gamma^2 = 1$, $\Gamma a = a\Gamma$ for all $a \in \mathcal{A}$, $\beta\Gamma + \Gamma\beta = 0$ and $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$. Otherwise the triple is said to be odd.

Remark 5.9. As noted in [PS, page 5], this is not really capturing Lorentzian signature, but rather that the number of timelike directions is odd. This can be refined using a real structure.

The essential self-adjointness of $\beta\mathcal{D}$ implies that $\mathcal{D}^* = \beta\mathcal{D}\beta$ on $\text{dom } \mathcal{D}^* \cap \text{dom } \mathcal{D}$, and that β preserves this domain. In particular, \mathcal{D} is Krein self-adjoint as in Section 3.3 for the fundamental symmetry $\mathcal{J} = i\beta$. The condition on $\beta[\mathcal{D}^2, \beta]$ can be understood in terms of the definition of pseudo-Riemannian spectral triples via

$$\beta[\mathcal{D}^2, \beta] = \beta\mathcal{D}^2\beta + \mathcal{D}^2 = -\mathcal{D}^{*2} + \mathcal{D}^2.$$

In particular a Lorentz-type spectral triple is a pseudo-Riemannian spectral triple.

Lemma 5.10. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \beta)$ be a Lorentz-type spectral triple. Then the K -homology classes arising from the operators

$$\mathcal{D}_E = \frac{e^{i\pi/4}}{\sqrt{2}}(\mathcal{D} - i\mathcal{D}^*) \quad \text{and} \quad \widetilde{\mathcal{D}}_E = \frac{e^{-i\pi/4}}{\sqrt{2}}(\mathcal{D} + i\mathcal{D}^*)$$

are negatives of one another.

Proof. One simply computes that $\beta\mathcal{D}_E\beta^* = -\widetilde{\mathcal{D}}_E$, which shows that $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$ is unitarily equivalent to $(\mathcal{A}, \mathcal{H}, -\widetilde{\mathcal{D}}_E)$. \square

Proposition 5.11. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \beta, \Gamma)$ be an even Lorentz-type spectral triple with \mathcal{A} unital. Assume that there exists at least one even function f with $f(0) \neq 0$ and $f(\mathcal{D}_E)$ trace class. If $\mathcal{D}^2 - \mathcal{D}^{*2} = \beta[\mathcal{D}^2, \beta] = 0$, then*

$$\text{Index} \left(\frac{1 - \Gamma}{2} \mathcal{D}_E \frac{1 + \Gamma}{2} \right) = 0.$$

That is, the pairing of $[(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)] \in K^0(\mathcal{A})$ with the class $[1] \in K_0(\mathcal{A})$ is zero.

Remark 5.12. By analogy with the classical case we call β Lorentz harmonic, since it commutes with the Laplacian of the indefinite metric. The hypothesis on the summability of \mathcal{D}_E is implied by θ -summability or finite summability of course.

Proof. Since $\mathcal{D}^2 - \mathcal{D}^{*2}$ vanishes, the fundamental symmetry β commutes with $\mathcal{D}_E^2 = \langle \mathcal{D} \rangle^2$, and so also with any function of \mathcal{D}_E^2 . The index can be computed using the McKean-Singer formula, and we refer to [CPRS3] for this version. For any even function f not vanishing at 0 and such that $f(\mathcal{D}_E)$ is trace class we have

$$\text{Index} \left(\frac{1 - \Gamma}{2} \mathcal{D}_E \frac{1 + \Gamma}{2} \right) = \frac{1}{f(0)} \text{Trace}(\Gamma f(\mathcal{D}_E)).$$

Then using $\beta^2 = -1$, $\beta\Gamma + \Gamma\beta = 0$, $\beta\mathcal{D}_E^2 = \mathcal{D}_E^2\beta$, and cyclicity of the trace, it is straightforward to show that $\text{Trace}(\Gamma f(\mathcal{D}_E))$ must vanish identically. Hence the index vanishes, and the proof is complete. \square

We thank Christian Bär for the following example.

Corollary 5.13. *There exists a compact spin manifold which admits a time-oriented Lorentzian metric (and so has zero Euler characteristic) but does not possess any Lorentz harmonic fundamental symmetries. In particular, it has no nowhere vanishing Lorentz harmonic one-forms.*

Proof. Suppose we have a compact Riemannian spin manifold (M, g_E) with vanishing Euler characteristic. Let z be a normalised non-vanishing vector field, and $z^\#$ the dual one-form. Then we get a Lorentzian metric $g(v, w) := g_E(v, w) - 2z^\#(v)z^\#(w)$. With the same notation as in Section 3, we set $\beta = \gamma_E(z)$, and so we obtain the Lorentz-type spectral triple $(C^\infty(M), L^2(M, S), \mathcal{D}, \beta)$, using Proposition 3.4.

Now suppose that we can choose some fundamental symmetry $\tilde{\beta}$ to be Lorentz harmonic. Then $\frac{1}{4}\text{Index}((1 - \Gamma)\mathcal{D}_E(1 + \Gamma)) = 0$, and we have shown previously that \mathcal{D}_E is the Dirac operator of the Riemannian metric \tilde{g}_E canonically associated to our Lorentzian metric and fundamental symmetry $\tilde{\beta}$. Hence if we can choose $\tilde{\beta}$ Lorentz harmonic, we must have $\hat{A}(M) = 0$. Equivalently, if $\hat{A}(M) \neq 0$ we see that we can not choose $\tilde{\beta}$ harmonic, and in particular $\beta = \gamma_E(z)$ is not Lorentz harmonic.

Let $M_1 = S^2 \times F_g$ where F_g is a compact orientable surface of genus $g \geq 2$. Then $\chi(M_1) = 2(2 - g)$ and $\hat{A}(M_1) = 0$. Let M_2 be a K3 space, so $\chi(M_2) = 24$ and $\hat{A}(M_2) = 2$. Finally let $M = M_1 \# M_2$ be the connected sum. Then $\chi(M) = \chi(M_1) + \chi(M_2) - 2 = 26 - 2g$ and $\hat{A}(M) = \hat{A}(M_1) + \hat{A}(M_2) = 2$. So for $g = 13$ we may construct a Lorentzian metric on M , and

since the A-roof genus does not vanish, there do not exist any Lorentz harmonic fundamental symmetries. \square

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