KK-Theory and Spectral Flow in von Neumann Algebras

by

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Abstract

We present a definition of spectral flow for any norm closed ideal J in any von Neumann algebra N. Given a path of selfadjoint operators in N which are invertible in N/J, the spectral flow produces a class in $K_0(J)$.

Given a semifinite spectral triple $(\mathcal{A}, H, \mathcal{D})$ relative to (N, τ) with $\mathcal{A}$ separable, we construct a class $[\mathcal{D}] \in KK^1(A, \mathcal{K}(N))$. For a unitary $u \in \mathcal{A}$, the von Neumann spectral flow between $\mathcal{D}$ and $u^*\mathcal{D}u$ is equal to the Kasparov product $[u] \hat{\otimes}_A [\mathcal{D}]$, and is simply related to the numerical spectral flow, and a refined C^*-spectral flow.

1. Introduction

The theory of analytic spectral flow has received a great deal of attention in recent years, with significant progress being made by many authors, [2, 4, 8, 9, 10, 22, 23, 24, 28]. The article [28] contains a much more detailed review of other aspects of spectral flow.

Here we take a slightly different tack, replacing numerical measures of spectral flow by K-theory valued measures, as in [18, 28]. The advantages of this approach are the great generality in which it can be defined, and its compatibility with the various numerical notions.

This compatability yields constraints on the possible values of spectral flow, which, for example, in the semifinite setting of [22, 23], is a priori any real number. Our description of spectral flow allows one to factor through a K-theory group, and so constrain the possible values of the spectral flow. The more refined we can be about the target K-theory group, the more refined our information.

We define von Neumann spectral flow for any norm closed ideal J in any von Neumann algebra N. Given a path of selfadjoint operators in N which are invertible in N/J, we obtain a class in $K_0(J)$. In order to be able to work in such a general context, we need to develop a $K_0(J)$-valued index theory for any such pair N, J. Such an index theory is developed in Section 2, and then in Section 3 we define

and study the von Neumann spectral flow. We then follow the approach of [22, 23], defining spectral flow in terms of relative indices of projections.

A closely related idea which we introduce is a von Neumann spectral triple, modelled on the definition of semifinite spectral triples, but valid for any von Neumann algebra N and ideal J. We show that such a triple defines a Kasparov class, and relate the spectral flow to Kasparov products.

In particular, every semifinite spectral triple represents a KK-class, just as ordinary spectral triples represent K-homology classes. This extends the observed relation in [18, 19].

In Section 5 we discuss the consequences of refining our target K-theory group to $K_0(\mathcal{B})$, where $\mathcal{B} \subset J$ is a σ-unital subalgebra. We show that this can always be done for a von Neumann spectral triple, and so we can define a C^*-spectral flow. We relate this spectral flow to our previously defined von Neumann spectral flow.

Section 6 relates both von Neumann and C^* spectral flow for a semifinite spectral triple to the numerical spectral flow obtained from a trace.

The Appendix summarises some results from KK-theory that we require, and proves an explicit formula for certain odd pairings in KK-theory, which plays a key role throughout the paper.

2. K-theory-valued von Neumann Index Theory

Throughout this section, we let N be a von-Neumann algebra acting on a Hilbert space H and let J be a norm closed ideal in N. Let $\pi : N \to N/J$ be the quotient map.

In all the following, we will distinguish between the kernel of an operator $S : H \to H$ called $\ker(S)$ and the projection onto the kernel called $N(S) \in \mathcal{L}(H)$. Likewise we have the image of S, $\mathrm{Im}(S)$ and the projection onto the norm closure of $\mathrm{Im}(S)$, denoted $R(S) \in \mathcal{L}(H)$. If S is in N then $N(S)$ and $R(S)$ are in N also.

For any two projections $p, q \in \mathcal{L}(H)$ we denote the projection onto $\mathrm{Im}(p) \cap \mathrm{Im}(q)$ by $p \cap q \in \mathcal{L}(H)$. If $p, q \in N$ and $S \in N'$, then $Sp = pS$ and $Sq = qS$ thus $Sp \cap q = p \cap qS$. It follows easily that $p \cap q \in N'' = N$.

We recall some facts about the polar decomposition of an operator. Let $S \in N$. The partial isometry $u \in N$ from the polar decomposition of S is called the *phase*

of S. The phase of S has the following properties

$$
\begin{aligned}
u|S| &= S & S^* &= u^*|S^*| \\
uu^* = R(u) &= R(S) & u^*u = R(u^*) &= R(S^*) \\
1 - uu^* = N(u^*) &= N(S^*) & 1 - u^*u = N(u) &= N(S).
\end{aligned}
$$

See [12, Appendice III] for more details. Since K-theory is well-defined for non-separable C^*-algebras, we can ask what the generalised index map in K-theory gives us for an invertible in the 'Calkin algebra' N/J.

Lemma 2.1 *Let $[\pi(S)] \in K_1(N/J)$ be a class in $K_1(N/J)$ represented by the unitary $\pi(S)$, where $S \in M_n(N)$ for some $n \in \mathbb{N}$. Then*

$$
\partial[\pi(S)] = [N(S)] - [N(S^*)] \in K_0(J),
$$

where $\partial : K_1(N/J) \to K_0(J)$ is the boundary map in K-theory. See for instance [3, Definition 8.3.1].

Proof: The algebra $M_n(N) = M_n(\mathbb{C}) \otimes N$ is a von-Neumann algebra acting on the Hilbert space $\oplus_{i=1}^n H$ so S can be polar decomposed in $M_n(N)$. Let $u \in M_n(N)$ be the phase of S. Now, u is a lift of $\pi(S)$ since

$$
\pi(S) = \pi(u|S|) = \pi(u)\pi(S^*S)^{1/2} = \pi(u).
$$

And we conclude from the definition of the boundary map, [3, 14, 26], that

$$
\partial[\pi(S)] = [1 - u^*u] - [1 - uu^*] = [N(S)] - [N(S^*)]
$$

as claimed. $\qquad\square$

The generic situation where the index of an operator S is relevant for applications is when $S : H_1 \to H_2$. Even to define 'odd' index pairings one requires such operators. Thus one must consider operators not in a von Neumann algebra N, but in a skew-corner qNp where $p,q \in N$ are projections. This situation was first considered in [11] for semifinite von Neumann algebras. The following definition generalises the semifinite notion of Fredholm.

Definition 2.2 *Let p,q be projections in N. Then $S \in qNp$ is a $(q$-$p)$-Fredholm operator if there exists a $T, R \in pNq$ such that*

$$
\pi(TS) = \pi(p) \quad \text{and} \quad \pi(SR) = \pi(q).
$$

Since $\pi(T) = \pi(TSR) = \pi(R)$, we can choose $R = T$. The operator T is called a *parametrix* for S.

Remark 2.3 Suppose we have an operator $S \in qNp$. Then

$$N(S) \cap p = N(S)p \quad \text{and} \quad N(S^*) \cap q = N(S^*)q.$$

This follows immediately since

$$(1 - p)N(S) = (1 - p) = N(S)(1 - p) \Rightarrow pN(S) = N(S)p$$

so $N(S) \cap p = N(S)p$. Similar comments apply to the projections $N(S^*)$ and q.

Lemma 2.4 *Let $S \in qNp$ and let $u \in N$ be the phase of S. Then $u \in qNp$ and we have the identities*

$$p - u^*u = N(S) - (1 - p) = N(S) \cap p$$
$$q - uu^* = N(S^*) - (1 - q) = N(S^*) \cap q.$$

Furthermore if $S \in qNp$ is a $(q\text{-}p)$-Fredholm operator then $\pi(u^*u) = \pi(p)$ and $\pi(uu^*) = \pi(q)$. In particular u is $(q\text{-}p)$-Fredholm and $N(S) \cap p, N(S^*) \cap q \in J$.

Proof: First, u is in qNp since $(1-p)H \subseteq \text{Ker}(S) = \text{Ker}(u)$ and $\text{Im}(u) = \overline{\text{Im}(S)} \subseteq qH$. Next, $(1 - p)N(S) = (1 - p)$ so $N(S) - (1 - p) = N(S) - N(S)(1 - p) = N(S)p = N(S) \cap p$ by Remark 2.3. The statement concerning $N(S^*)$ and q is proved in the same way.

Now, suppose that $S \in qNp$ is a $(q\text{-}p)$-Fredholm operator with parametrix $T \in pNq$. Then $S^*S \in pNp$ is a $(p\text{-}p)$-Fredholm operator with parametrix $TT^* \in pNp$. This means that $\pi(S^*S)$ is invertible in the C^*-algebra $\pi(p)(N/J)\pi(p)$. Similarly $\pi(SS^*)$ is invertible in the C^*-algebra $\pi(q)(N/J)\pi(q)$ Clearly, then the phase $u \in qNp$ of $S \in qNp$ is a lift of $\pi(S)\pi(S^*S)^{-1/2} \in \pi(q)(N/J)\pi(p)$. This allows us to deduce the identities

$$\pi(u^*u) = \pi(S^*S)^{-1/2}\pi(S^*S)\pi(S^*S)^{-1/2} = \pi(p) \quad \text{and}$$
$$\pi(uu^*) = \pi(S)\pi(S^*S)^{-1}\pi(S^*) = \pi(q)$$

as desired.   □

The result allows us to make the following definition.

Definition 2.5 Let $S \in qNp$ be a $(q\text{-}p)$-Fredholm operator. We define the $(q\text{-}p)$-*index* of S as the class

$$Ind_{(q\text{-}p)}(S) = [N(S) \cap p] - [N(S^*) \cap q]$$

in $K_0(J)$.

Let $S \in qNp$ be a $(q\text{-}p)$-Fredholm operator and let $u \in qNp$ be the phase of S. The triple (p,q,u) is thus a relative K-cycle for $(N,N/J)$ and thus defines the class $[S] := [p,q,u] \in K_0(N,N/J)$ in relative K-theory. Recall that '(p,q,u) is a relative K-cycle for $(N,N/J)$' means that, [14, Definition 4.3.1], p,q are projections in N, and $\pi(p) = \pi(u^*u)$, $\pi(q) = \pi(uu^*)$, where $\pi : N \to N/J$ is the quotient map. The relative K-cycle (p,q,u) is degenerate if u is a Murray-von Neumann equivalence between p and q, in which case the K-theory class defined by (p,q,u) is zero.

The relative K-theory $K_0(N,N/J)$ is related to the K-theory of the ideal $K_0(J)$ through the excision map

$$\mathrm{Exc} : K_0(J) \to K_0(N,N/J)$$

as defined in [14, Definition 4.3.7]. The excision map is an isomorphism, [14, Theorem 4.3.8]. In the next theorem we will see that the $(q\text{-}p)$-index of S is simply the inverse of the excision map applied to the class $[S] \in K_0(N,N/J)$. Many properties of the $(q\text{-}p)$-index will thus follow immediately, and we will state the ones we need as corollaries.

Theorem 2.6 *Let $S \in qNp$ be a $(q\text{-}p)$-Fredholm operator and let $u \in qNp$ be the phase of S. Then the identity*

$$\mathrm{Exc}^{-1}[S] = \mathrm{Ind}_{q\text{-}p}(S)$$

is valid in $K_0(J)$.

Proof: We can express the class $[S] \in K_0(N,N/J)$ as a sum of classes

$$[S] = [p,q,u] = [p - u^*u, q - uu^*, 0] + [u^*u, uu^*, u].$$

The relative K-cycle (u^*u, uu^*, u) is degenerate so actually

$$[S] = [p - u^*u, q - uu^*, 0].$$

The projections $p - u^*u = N(S) \cap p$ and $q - uu^* = N(S^*) \cap q$ are in J by Lemma 2.4, so

$$\mathrm{Exc}^{-1}[S] = [p - u^*u] - [q - uu^*] = \mathrm{Ind}_{q\text{-}p}(S)$$

as desired. $\qquad\qquad\square$

Corollary 2.7 *Let $S_0 \in qNp$ and $S_1 \in qNp$ be $(q\text{-}p)$-Fredholm operators. Suppose that there is a norm-continuous path of $(q\text{-}p)$-Fredholm operators connecting S_0 and S_1. Then*

$$\mathrm{Ind}_{(q\text{-}p)}(S_0) = \mathrm{Ind}_{(q\text{-}p)}(S_1).$$

Proof: Let $t \mapsto S_t \in qNp$ be the norm-continuous path of q-p Fredholm operators. Then $t \to \pi(S_t)\pi(S_t^* S_t)^{-1/2}$ is continuous where the inverse is taken in $\pi(p)(N/J)\pi(p)$. Then $\pi(u_t) = \pi(S_t)\pi(S_t^* S_t)^{-1/2}$, where u_t is the polar decomposition of S_t. Next, $\pi(u_0^*)\pi(u_t)$ is actually unitary in $\pi(p)(N/J)\pi(p)$ for all t since $\pi(u_t u_t^*) = \pi(q)$ and $\pi(u_t^* u_t) = \pi(p)$ by Lemma 2.4. The continuous path of unitaries $\pi(u_0^*)\pi(u_t) + \pi(1-p)$ thus lies in the connected component of the identity, since $\pi(u_0^*)\pi(u_0) + \pi(1-p) = \pi(1)$, and so can be lifted to a continuous path (of unitaries) $\{w_t\}$ in N. Thus we obtain a lifting $\{v_t\} := \{u_0 w_t\}$ of $\{\pi(u_t)\}$.

Now (p, q, v_t) is a relative K-cycle for each $t \in [0,1]$, [14, Lemma 4.3.13]. If $u_0 \in qNp$ and $u_1 \in qNp$ are the phases of S_0 and S_1 respectively, then $\pi(u_0) = \pi(v_0)$ and $\pi(u_1) = \pi(v_1)$ so we have the identity

$$[S_0] = [p, q, u_0] = [p, q, v_0] = [p, q, v_1] = [p, q, u_1] = [S_1]$$

in $K_0(N, N/J)$. It thus follows immediately by Theorem 2.6 that

$$\mathrm{Ind}_{(q\text{-}p)}(S_0) = \mathrm{Exc}^{-1}[S_0] = \mathrm{Exc}^{-1}[S_1] = \mathrm{Ind}_{(q\text{-}p)}(S_1)$$

as desired. $\square$

Corollary 2.8 *Let $S \in qNp$ be a $(q\text{-}p)$-Fredholm operator and let $T \in rNq$ be an $(r\text{-}q)$-Fredholm operator. Then TS is an $(r\text{-}p)$-Fredholm operator and*

$$\mathrm{Ind}_{(r\text{-}q)}(T) + \mathrm{Ind}_{(q\text{-}p)}(S) = \mathrm{Ind}_{(r\text{-}p)}(TS).$$

Proof: Set $U_\theta = \begin{pmatrix} q\cos\theta & q\sin\theta \\ -q\sin\theta & q\cos\theta \end{pmatrix}$. Then, by homotopy invariance,

$$\mathrm{Ind}_{(r\text{-}p)}(TS) = \mathrm{Ind}_{(r\oplus q\text{-}p\oplus q)}((T\oplus q)U_0(S\oplus q)) = \mathrm{Ind}_{(r\oplus q\text{-}p\oplus q)}((T\oplus q)U_{\frac{\pi}{2}}(S\oplus q)).$$

But $(T \oplus q)U_{\frac{\pi}{2}}(S \oplus q)^*(T \oplus q)U_{\frac{\pi}{2}}(S \oplus q) = S^*S \oplus T^*T$, so if we set $v =$ phase of T, and $u =$ phase of S, we find that

$$\mathrm{Ind}_{(r\oplus q\text{-}p\oplus q)}((T \oplus q)U_{\frac{\pi}{2}}(S \oplus q)) = \mathrm{Ind}_{(r\oplus q\text{-}p\oplus q)}((v \oplus q)U_{\frac{\pi}{2}}(u \oplus q)).$$

Again by homotopy invariance

$$\mathrm{Ind}_{(r\text{-}p)}(TS) = \mathrm{Ind}_{(r\text{-}p)}(vu).$$

The claimed result follows since, in $K_0(N, N/J)$, $[p, r, vu] = [q, r, v] + [p, q, u]$. $\square$

Let N be a semifinite von Neumann algebra equipped with a fixed normal, semifinite, faithful trace τ. Let $\mathcal{K}_N$ be the τ-compact operators as defined in Definition 6.1. All projections in $\mathcal{K}_N$ have finite trace by Theorem 6.5. Applying the homomorphism $\tau_* : K_0(\mathcal{K}_N) \to \mathbb{R}$ from Theorem 6.4 to the main theorems of this section we obtain some of the important results from Breuer-Fredholm theory, [2, 5, 6, 8, 9, 10, 11, 22, 23, 24].

3. Von Neumann Spectral Flow

3.1. Basic Definitions and Properties

Having set up the appropriate index theory for Fredholm operators in skew-corners pNq, [11], we now analyse spectral flow. This is associated with odd index pairings, and so self-adjoint operators. Specialising our definition of p-q-Fredholm to the case $p = q = 1$ we have the following.

Definition 3.1 An operator $T \in N$ is said to be J-*Fredholm* if $\pi(T)$ is invertible in N/J. The space of J-Fredholm operators is denoted by $\mathcal{F}$. The space of selfadjoint J-Fredholm operators is denoted by $\mathcal{F}_{sa}$.

In order to develop our notion of spectral flow, we follow the work of Phillips, [22, 23]. Indeed most of the results, and proofs, of this section are present, at least implicitly, in Phillips' work. Due to small differences in detail, we repeat the statements and basic arguments for the reader's benefit.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be the indicator function for the interval $[0, \infty)$ defined by

$$\chi(t) = \begin{cases} 1 & t \in [0, \infty) \\ 0 & t \in (-\infty, 0). \end{cases}$$

Lemma 3.2 *Let* $T \in \mathcal{F}_{sa}$ *then* $\pi\big(\chi(T)\big) = \chi\big(\pi(T)\big)$.

Proof: Note that $\chi\big(\pi(T)\big)$ makes sense since $0 \notin \mathrm{Sp}\big(\pi(T)\big)$ thus we can find an $\varepsilon > 0$ such that the interval $[-\varepsilon, \varepsilon]$ is included in the resolvent set of $\pi(T)$. Now define the continuous functions $f_1 : \mathbb{R} \to \mathbb{R}$

$$f_1(t) = \begin{cases} 0 & t \in (-\infty, -\varepsilon] \\ \varepsilon^{-1}t + 1 & t \in [-\varepsilon, 0] \\ 1 & t \in [0, \infty) \end{cases}$$

and $f_2 : \mathbb{R} \to \mathbb{R}$ by

$$f_2(t) = \begin{cases} 0 & t \in (-\infty, 0] \\ \varepsilon^{-1}t & t \in [0, \varepsilon] \\ 1 & t \in [\varepsilon, \infty). \end{cases}$$

So $f_1 = \chi = f_2$ on $\mathrm{Sp}(\pi(T))$ while $f_1 \geq \chi \geq f_2$ on $\mathrm{Sp}(T)$. Thus

$$\chi\big(\pi(T)\big) = f_1\big(\pi(T)\big) = \pi\big(f_1(T)\big) \geq \pi\big(\chi(T)\big) \geq \pi\big(f_2(T)\big) = f_2\big(\pi(T)\big) = \chi\big(\pi(T)\big)$$

yielding $\chi\big(\pi(T)\big) = \pi\big(\chi(T)\big)$ as desired. $\qquad\square$

Lemma 3.3 *Let* $t \mapsto B_t$ *be a norm continuous path in* $\mathcal{F}_{sa}$. *Then* $t \mapsto \chi\big(\pi(B_t)\big)$ *is a norm continuous path in the* C^*-*algebra* N/J.

To prove this lemma we need a general result from the theory of C^*-algebras. The result is probably well-known to the experts, but as we could not find a reference, we include a proof.

Lemma 3.4 *Let A be a C^*-algebra and let U be an open subset of $\mathbb{R}$. Denote by A_{sa} the real subspace of selfadjoint elements with the induced topology from A. Then the set*

$$\{a \in A_{sa} \,|\, \mathrm{Sp}(a) \subseteq U\}$$

is open in A_{sa}.

Proof: Let $a \in A_{sa}$ with $\mathrm{Sp}(a) \subseteq U$. The function $\mathrm{dist}(\cdot, U^c) : \mathbb{C} \to [0, \infty[$ defined by

$$\mathrm{dist}(\lambda, U^c) = \inf\{|\lambda - \mu| \,|\, \mu \in U^c\}$$

for all $\lambda \in \mathbb{C}$ is continuous. It attains thus its minimum on the compact set $\mathrm{Sp}(a)$. Furthermore for $\lambda \in \mathrm{Sp}(a)$ we have $\mathrm{dist}(\lambda, U^c) > 0$ because $\lambda \notin \overline{U^c} = U^c$, so the minimum is strictly positive. Set

$$\varepsilon = \frac{\mathrm{dist}\big(\mathrm{Sp}(a), U^c\big)}{2} = \frac{\inf\{|\lambda - \mu| \,|\, \lambda \in \mathrm{Sp}(a), \mu \in U^c\}}{2} > 0.$$

Now take $b \in A_{sa}$ with $\|b - a\| < \frac{\varepsilon}{2}$ and suppose for contradiction that there exists a $\lambda \in \mathrm{Sp}(b)$ with

$$B_\varepsilon(\lambda) \cap \mathrm{Sp}(a) = \emptyset.$$

Here $B_\varepsilon(\lambda)$ denotes the ball of radius $\varepsilon > 0$ and center λ. Let $\mu \in B_{\varepsilon/4}(\lambda)$. Then $\mu \notin \mathrm{Sp}(a)$ and

$$\|(\mu - a)^{-1}\|^{-1} = \sup\{|\mu - \alpha|^{-1} \,|\, \alpha \in \mathrm{Sp}(a)\}^{-1} = \mathrm{dist}\big(\mu, \mathrm{Sp}(a)\big) \geq \frac{3\varepsilon}{4}.$$

Furthermore

$$\|(\lambda - b) - (\mu - a)\| \leq |\lambda - \mu| + \|a - b\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \leq \|(\mu - a)^{-1}\|^{-1}.$$

So $\lambda - b$ is actually invertible which is a contradiction, see [17, Proposition 17.3]. Hence for $\lambda \in \mathrm{Sp}(b)$ we cannot have

$$B_\varepsilon(\lambda) \cap \mathrm{Sp}(a) = \emptyset.$$

Because of the way the ε was chosen we conclude that $\mathrm{Sp}(b) \subseteq U$. Thus $\mathrm{Sp}(b) \subseteq U$ for any $b \in A_{sa}$ with $\|b - a\| < \varepsilon/2$. $\square$

Proof: *of Lemma 3.3.* Let $t_0 \in [0,1]$. Choose an $\varepsilon > 0$ such that the interval $[-\varepsilon, \varepsilon]$ is included in the resolvent set of $\pi(B_{t_0})$. Now

$$\mathrm{Sp}\big(\pi(B_{t_0})\big) \subseteq (-\infty, -\varepsilon) \cup (\varepsilon, \infty).$$

By Lemma 3.4 and the continuity of $t \mapsto \pi(B_t)$ there is a $\delta > 0$ such that

$$\mathrm{Sp}\big(\pi(B_t)\big) \subseteq (-\infty, -\varepsilon) \cup (\varepsilon, \infty)$$

for all $t \in (t_0 - \delta, t_0 + \delta) \cap [0,1]$. So for all $t \in (t_0 - \delta, t_0 + \delta) \cap [0,1]$ we have the identity

$$\chi\big(\pi(B_t)\big) = f\big(\pi(B_t)\big)$$

where f is some fixed continuous function (for instance the function f_1 from the proof of Lemma 3.2). But the function

$$t \mapsto f\big(\pi(B_t)\big)$$

is clearly continuous and the lemma is thereby proved. □

With these tools at hand we can now define spectral flow as a class in $K_0(J)$.

Definition 3.5 (Spectral flow) Let $t \mapsto B_t$ be a norm continuous path in $\mathcal{F}_{sa}$. By Lemma 3.3 the path

$$t \mapsto \pi\big(\chi(B_t)\big) = \chi\big(\pi(B_t)\big)$$

is norm continuous. Find a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that

$$\big\| \pi\big(\chi(B_t)\big) - \pi\big(\chi(B_s)\big) \big\| < 1/2 \quad \text{for all} \quad t, s \in [t_{i-1}, t_i].$$

Set $p_i = \chi(B_{t_i})$. We now define the *spectral flow* of the path $\{B_t\}$ to be

$$sf\{B_t\} = \sum_{i=1}^{n} \big[(1 - p_i) \cap p_{i-1} \big] - \big[(1 - p_{i-1}) \cap p_i \big] \in K_0(J).$$

This definition raises several questions which we will answer in the following lemmas.

1. Are the elements $p_i\, p_{i-1} \in p_i\, N p_{i-1}$ (p_i-p_{i-1})-Fredholm operators for all $i \in \{1, \ldots, n\}$?

2. Is the spectral flow independent of the partition chosen ?

3. Is the spectral flow invariant under homotopies of the path $\{B_t\}$?

4. Is the spectral flow of $\{B_t\}$ equal to the spectral flow of $\{C_t\}$ if $B_t - C_t \in J$ for all $t \in [0,1]$?

Lemma 3.6 *Suppose that $p, q \in N$ are two projections such that $\|\pi(p) - \pi(q)\| < 1$. Then $qp \in qNp$ is a (q-p)-Fredholm operator. Thus, by Lemma 2.4, we have $(1 - q) \cap p \in J$ and $(1 - p) \cap q \in J$.*

Proof: The inequality

$$\|\pi(pqp) - \pi(p)\| \leq \|\pi(p) - \pi(q)\| < 1$$

shows that $\pi(pqp)$ is invertible in $\pi(pNp)$, so there is an operator $T \in pNp$ such that $\pi(Tpqp) = \pi(p)$. Likewise the inequality

$$\|\pi(qpq) - \pi(q)\| \leq \|\pi(q) - \pi(q)\| < 1$$

shows that $\pi(qpq)$ is invertible in $\pi(qNq)$, so there is an operator $R \in qNq$ such that $\pi(qpqR) = \pi(q)$. It follows that qp is a $(q\text{-}p)$-Fredholm operator. $\square$

Corollary 3.7 *For a path $\{B_t\}$ in $\mathcal{F}_{sa}$ and a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that*

$$\|\pi\big(\chi(B_t)\big) - \pi\big(\chi(B_s)\big)\| < 1/2 \qquad \text{for all } t,s \in [t_{i-1}, t_i]$$

for all $i \in \{1,\ldots,n\}$ we can express the spectral flow of the path as the sum of $(p_i\text{-}p_{i-1})$-indices

$$\mathrm{sf}\{B_t\} = \sum_{i=1}^{n} \mathrm{Ind}_{(p_i\text{-}p_{i-1})}(p_i\, p_{i-1})$$

where $p_i = \chi(B_{t_i})$ for all $i \in \{0,\ldots,n\}$. Thus by Theorem 2.8 we actually have

$$\mathrm{sf}\{B_t\} = \mathrm{Ind}_{(p_n\text{-}p_0)}(p_n \ldots p_0) = [N(p_n \ldots p_0) \cap p_0] - [N(p_0 \ldots p_n) \cap p_n].$$

Lemma 3.8 *Suppose that p,q,r are three projections in N with*

$$\|\pi(p) - \pi(q)\| < 1/2 \quad , \quad \|\pi(q) - \pi(r)\| < 1/2 \quad \text{and} \quad \|\pi(r) - \pi(p)\| < 1/2$$

then

$$\mathrm{Ind}_{(r\text{-}q)}(rq) + \mathrm{Ind}_{(q\text{-}p)}(qp) = \mathrm{Ind}_{(r\text{-}p)}(rp).$$

Thus the spectral flow is independent of the partition chosen -it doesn't change if a finer one is chosen.

Proof: We want to prove that

$$\mathrm{Ind}_{(r\text{-}q)}(rq) + \mathrm{Ind}_{(q\text{-}p)}(qp) - \mathrm{Ind}_{(r\text{-}p)}(rp) = 0.$$

By Theorem 2.8 this amounts to show that

$$\mathrm{Ind}_{(r\text{-}r)}(rqpr) = 0.$$

Verify the inequality

$$\|\pi(rqpr) - \pi(r)\| \leq \|\pi(qp) - \pi(r)\|$$
$$\leq \|\pi(qp) - \pi(q)\| + \|\pi(q) - \pi(r)\|$$
$$\leq \|\pi(p) - \pi(q)\| + \|\pi(q) - \pi(r)\|$$
$$< 1.$$

Let $t \in [0,1]$, then

$$\|\pi\big((1-t)rqpr + tr\big) - \pi(r)\| = (1-t)\|\pi(rqpr) - \pi(r)\| < (1-t)$$

thus $\pi\big((1-t)rqpr + tr\big)$ is invertible in $\pi(rNr)$ for all $t \in [0,1]$. This means that the path $t \mapsto (1-t)rqpr + tr$ consists entirely of $(r\text{-}r)$-Fredholm operators and it connects $rqpr$ with r. To finish the proof we simply refer to Theorem 2.7 which gives

$$0 = \mathrm{Ind}_{(r\text{-}r)}(r) = \mathrm{Ind}_{(r\text{-}r)}(rqpr)$$

as desired. □

Lemma 3.9 *[2, 23] Let $\{B_t\}$ and $\{C_t\}$ be two paths of selfadjoint J-Fredholm operators. Let $H : [0,1] \times [0,1] \to \mathcal{F}_{sa}$ be a homotopy connecting $\{B_t\}$ and $\{C_t\}$ leaving the endpoints fixed. That is H is norm-continuous with $H(t,0) = B_t$, $H(t,1) = C_t$ for all $t \in [0,1]$ and $H(0,s) = B_0$, $H(1,s) = B_1$ for all $s \in [0,1]$. In particular $B_0 = C_0$ and $B_1 = C_1$. Then $\mathrm{sf}\{B_t\} = \mathrm{sf}\{C_t\}$.*

Proof: The map $\zeta : [0,1] \times [0,1] \to N/J$ defined by

$$\zeta(t,s) = \pi\Big(\chi\big(H(t,s)\big)\Big)$$

is continuous and thus uniformly continuous, so we can choose a grid

$$0 = t_0 < t_1 \ldots < t_n = 1 \quad , \quad 0 = s_0 < s_1 \ldots < s_n = 1$$

of $[0,1] \times [0,1]$ such that for any $(t,s),(u,v) \in [t_{i-1},t_i] \times [s_{j-1},s_j]$ we have $\|\zeta(t,s) - \zeta(u,v)\| < \frac{1}{2}$ where $i, j \in \{1,\ldots,n\}$ are fixed.

Now look at the spectral flow along the borders of the squares. That is, for $i, j \in \{1,\ldots,n\}$ there are eight paths of selfadjoint J-Fredholm operators. For instance we have

$$u \mapsto H\big((1-u)t_{i-1} + ut_i, s_{j-1}\big)$$

as one of them. The spectral flow of this path will be denoted by

$$\mathrm{sf}_H\big((t_{i-1},s_{j-1}),(t_i,s_{j-1})\big).$$

Likewise for the spectral flow of the other paths. Applying Lemma 3.8 and the definition of spectral flow gives

$$
\mathrm{sf}_H\big((t_{i-1},s_{j-1}),(t_i,s_{j-1})\big) + \mathrm{sf}_H\big((t_i,s_{j-1}),(t_i,s_j)\big)
$$
$$
+ \,\mathrm{sf}_H\big((t_i,s_j),(t_{i-1},s_j)\big) + \mathrm{sf}_H\big((t_{i-1},s_j),(t_{i-1},s_{j-1})\big) = 0.
$$

Furthermore

$$
sf_H\big((t_{i-1},s_{j-1}),(t_i,s_{j-1})\big) = -sf_H\big((t_i,s_{j-1}),(t_{i-1},s_{j-1})\big).
$$

And an easy combinatorial argument yields the result.   □

Remark 3.10 Suppose that $p,q \in N$ are two projections with $\|p - q\| < 1$, then

$$
Ker(p) \cap Im(q) = 0 = Ker(q) \cap Im(p)
$$

so the J-index of the projections

$$
Ind_{(p\text{-}q)}(pq) = [(1-p)\cap q] - [(1-q)\cap p] = 0.
$$

To see this we start by deducing that $1 - p + pqp$ is invertible in N from the inequality

$$
\|p - pqp\| \le \|p - q\| < 1.
$$

If now x is in $Ker(q) \cap Im(p)$ we immediately have

$$
(1 - p + pqp)x = 0
$$

but $1 - p + pqp$ was invertible so $x = 0$. Therefore $Ker(q) \cap Im(p) = 0$. To prove that $Ker(p) \cap Im(q) = 0$ simply interchange p and q.

Lemma 3.11 *Let $\{B_t\}$ and $\{C_t\}$ be two paths of self adjoint J-Fredholm operators with $B_t - C_t \in J$ for all $t \in [0,1]$ and*

$$
Ind_{(p_0\text{-}q_0)}(p_0 q_0) = Ind_{(q_1\text{-}p_1)}(q_1 p_1) = 0
$$

where $p_0 = \chi(B_0)$, $p_1 = \chi(B_1)$, $q_0 = \chi(C_0)$ and $q_1 = \chi(C_1)$. Then $\mathrm{sf}\{B_t\} = \mathrm{sf}\{C_t\}$. The condition (3.11) is true if for instance

$$
\|\chi(B_0) - \chi(C_0)\| < 1 \quad and \quad \|\chi(C_1) - \chi(B_1)\| < 1
$$

by Remark 3.10.

Proof: Choose a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that

$$\|\pi(\chi(B_t)) - \pi(\chi(B_s))\| < \frac{1}{4}$$

and

$$\|\pi(\chi(C_t)) - \pi(\chi(C_s))\| < \frac{1}{4}$$

for all $t,s \in [t_{i-1},t_i]$, $i \in \{1,\ldots,n\}$.

Now join the elements B_{t_i} and C_{t_i} by a straight line for each $i \in \{0,\ldots,n\}$ denoted by $(BC)_i$. The straight line from C_{t_i} to B_{t_i} is denoted by $(CB)_i$.

Notice that the lines are paths of selfadjoint J-Fredholm operators because

$$\pi\big((1-t)B_{t_i} + tC_{t_i}\big) = \pi(B_{t_i})$$

for all $t \in [0,1]$ and $i \in \{1,\ldots,n\}$.

Now, almost by definition, the spectral flow along the square

$$
\begin{array}{ccc}
C_{t_{i-1}} & \longleftarrow & C_{t_i} \\
(CB)_{i-1} \downarrow & & (BC)_i \uparrow \\
B_{t_{i-1}} & \longrightarrow & B_{t_i}
\end{array}
$$

is zero. Since too the spectral flow along the lines $(BC)_0$ and $(BC)_1$ is zero by assumption we can use the same combinatorial argument as in the proof of Lemma 3.9 to reach the desired conclusion, namely $sf\{B_t\} = sf\{C_t\}$. $\qquad\square$

3.2. *Von Neumann Spectral Triples and Spectral Flow*

Definition 3.12 A *von Neumann spectral triple* $(\mathcal{A}, H, \mathcal{D})$ relative to (N, J) consists of a representation of the $*$-algebra $\mathcal{A}$ in the von Neumann algebra N acting on the Hilbert space H, together with a norm closed ideal J and a self-adjoint operator $\mathcal{D}$ affiliated to N such that

1. $[\mathcal{D},a]$ is defined on $\mathrm{Dom}(\mathcal{D})$ and extends to a bounded operator on H for all $a \in \mathcal{A}$.

2. $a(\lambda - \mathcal{D})^{-1} \in J$ for all $\lambda \notin \mathbb{R}$ and $a \in \mathcal{A}$.

The J-spectral triple is said to be **unital** if the unit of N is in $\mathcal{A}$.

While type III factors have no nontrivial norm closed ideals, such ideals exist in the non-factor case. We have no particular examples in the type III case, but the

definition of J spectral triple emphasises that the type of the von Neumann algebra is not the issue once we have a nontrivial ideal J.

If $(\mathcal{A}, H, \mathcal{D})$ is a **unital** J-spectral triple, we use the spectral theorem to define the bounded operator in N

$$F_{\mathcal{D}} := \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}.$$

Let $t \mapsto A_t$ be a path of selfadjoint operators in N. We claim the path

$$t \mapsto \mathcal{D}_t := \mathcal{D} + A_t$$

is a continuous path of unbounded selfadjoint J-Fredholm operators in the sense that the path

$$t \mapsto F_{\mathcal{D}_t} = \mathcal{D}_t(1 + \mathcal{D}_t^2)^{-\frac{1}{2}}$$

is a norm continuous path of self-adjoint J-Fredholm operators. The self-adjointness and boundedness follows for all $t \in [0,1]$ from the spectral theorem applied to the function

$$x \mapsto x(1 + x^2)^{-1/2}.$$

So we need to prove the claims of continuity and J-Fredholmness.

For continuity, let $t, s \in [0,1]$, and apply [8, Appendix A, Theorem 8] to find

$$\|F_{\mathcal{D}_t} - F_{\mathcal{D}_s}\| = \|\mathcal{D}_t(1 + \mathcal{D}_t^2)^{-\frac{1}{2}} - \mathcal{D}_s(1 + \mathcal{D}_s^2)^{-\frac{1}{2}}\| \leq \|A_t - A_s\|$$

proving continuity.

To prove the J-Fredholmness, let $t \in [0,1]$. Then [8, Lemma 2.7] says that for $0 < \varepsilon < 1/4$ we have

$$F_{\mathcal{D}_t} - F_{\mathcal{D}_0} = B_\varepsilon (1 + \mathcal{D}_0^2)^{-(1/2-\varepsilon)}$$

where $B_\varepsilon \in N$ and $\|B_\varepsilon\| \leq C(\epsilon)\|A_t - A_0\|$. For $\varepsilon = 1/4$ we get

$$F_{\mathcal{D}_t} - F_{\mathcal{D}_0} = B_{1/4}(1 + \mathcal{D}_0^2)^{-1/4}.$$

By [8, Appendix, Lemma 6], and defining $f(x) = 1 + \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2 + 4}$, we have

$$(1 + \mathcal{D}_0^2)^{-1} \leq f(\|A_0\|)(1 + \mathcal{D}^2)^{-1} \in J,$$

and as $f(\|A_0\|)$ is scalar,

$$(1 + \mathcal{D}_0^2)^{-1/4} \leq f(\|A_0\|)^{1/4}(1 + \mathcal{D}^2)^{-1/4} \in J.$$

Since $B_{1/4}$ is in N we conclude

$$\pi(F_{\mathcal{D}_0}) = \pi(F_{\mathcal{D}_t}).$$

At last for each $t \in [0,1]$,

$$\pi(F_{\mathcal{D}_t})\pi(F_{\mathcal{D}_t}) = \pi\big(\mathcal{D}_t^2(1+\mathcal{D}_t^2)^{-1}\big) = \pi\big((1+\mathcal{D}_t^2)(1+\mathcal{D}_t^2)^{-1}\big) = \pi(1)$$

so $\pi(F_{\mathcal{D}_t})$ is invertible for all $t \in [0,1]$.

These considerations allow us to define spectral flow for such paths of unbounded Fredholm operators.

Definition 3.13 (Unbounded Spectral Flow) Let $\{A_t\}_{t\in[0,1]}$ be a norm continuous path of self-adjoint operators in N, and $(\mathcal{A},H,\mathcal{D})$ a von Neumann spectral triple relative to (N,J). The spectral flow of the "continuous" path of unbounded selfadjoint J-Fredholm operators $t \mapsto \mathcal{D} + A_t$ is defined to be

$$sf\{\mathcal{D}_t\} := sf\{F_{\mathcal{D}_t}\}.$$

The relationship of this definition to the Riesz topology on unbounded self-adjoint operators and spectral flow can be found in [4].

Theorem 3.14 *Let $\{A_t\}_{t\in[0,1]}$ be a norm continuous path of self-adjoint operators in N, and $(\mathcal{A},H,\mathcal{D})$ a von Neumann spectral triple relative to (N,J). Let*

$$p_1 = \chi(F_{\mathcal{D}+A_1}) \quad and \quad p_0 = \chi(F_{\mathcal{D}+A_0}).$$

The spectral flow of the path $t \mapsto \mathcal{D} + A_t$ only depends on the end points $\mathcal{D} + A_0$ and $\mathcal{D} + A_1$ and is the class

$$sf\{\mathcal{D}_t\} = sf\{F_{\mathcal{D}_t}\} = [(1-p_1) \cap p_0] - [(1-p_0) \cap p_1]$$
$$= \mathrm{Ind}_{(p_1-p_0)}(p_1 p_0) \in K_0(J).$$

Proof: Notice that

$$\big\|\pi\big(\chi(F_{\mathcal{D}_t})\big) - \pi\big(\chi(F_{\mathcal{D}_s})\big)\big\| = \big\|\chi\big(\pi(F_{\mathcal{D}_t})\big) - \chi\big(\pi(F_{\mathcal{D}_s})\big)\big\| = 0$$

for all $s,t \in [0,1]$ so by definition

$$sf\{\mathcal{D}_t\} = sf\{F_{\mathcal{D}_t}\} = \big[(1-p_1) \cap p_0\big] - \big[(1-p_0) \cap p_1\big].$$

From this formula it is obvious that the spectral flow only depends on the end points.

$\square$

4. Kasparov Modules from Spectral Triples

In this section we show that from any von Neumann spectral triple $(\mathcal{A}, H, \mathcal{D})$ relative to (N, J), with J σ-unital, we can construct a Kasparov module $(\mathcal{M}_A, F_{\mathcal{D}}) \in \mathbb{E}(A, J)$, where $\mathcal{M}_A : A \to \mathcal{L}(J)$ is left multiplication by elements in A. Defining $p_F = \frac{F_{\mathcal{D}}+1}{2}$ we get the class $[\mathcal{M}_A, p_F]^1 \in KK^1(A, J)$. We will then show that for any unitary $u \in \mathcal{A}$, the unbounded spectral flow from $\mathcal{D}$ to $u^*\mathcal{D}u$ is given by the Kasparov product $[u] \hat{\otimes}_A [\mathcal{M}_A, p_F]^1$. For an explanation of the terminology we refer to the appendix.

In general the ideal J of a von Neumann spectral triple will not be σ-unital. (For example, if N is a factor, J is σ-unital only if it is $\{0\}$ or $N = B(H)$ for some Hilbert space H and $J = K(H)$). We give one method of avoiding the lack of σ-unitality in the next Section.

4.1. Construction of a Kasparov Module

Let $(\mathcal{A}, H, \mathcal{D})$ be a von Neumann spectral triple relative to (N, J) where the ideal J is arbitrary.

The ideal J is a right Hilbert J-module when equipped with the inner product $\langle x, y \rangle = x^*y$ and the action of J from the right given by multiplication. Since A, the norm closure of $\mathcal{A}$, is represented in N, and $F_{\mathcal{D}} \in N$, we see that $F_{\mathcal{D}} \in \mathcal{L}(J)$ and that there is a $*$-homomorphism $\mathcal{M}_A : A \to \mathcal{L}(J)$ given by left multiplication. We remark that the Hilbert module J is countably generated if and only if J is σ-unital.

Theorem 4.1 *For all $a \in A$ the operators $[F_{\mathcal{D}}, a]$, $a(1 - F_{\mathcal{D}}^2)$ and $a(F_{\mathcal{D}} - F_{\mathcal{D}}^*)$ are in J. Thus, the pair $(\mathcal{M}_A, F_{\mathcal{D}})$ is a Kasparov A-J-module when J is σ-unital.*

Proof: We have already noticed that $F_{\mathcal{D}} = F_{\mathcal{D}}^*$. Let $a \in A$. Calculating modulo J

$$aF_{\mathcal{D}}^2 = a\big(\mathcal{D}^2(1 + \mathcal{D}^2)^{-1}\big) \sim a\big(\mathcal{D}^2(1 + \mathcal{D}^2)^{-1} + (1 + \mathcal{D}^2)^{-1}\big) = a.$$

So $a(F_{\mathcal{D}}^2 - 1) \in J$ for all $a \in A$.

Let $a, b \in \mathcal{A}$. We have

$$[F_{\mathcal{D}}, a]b = \mathcal{D}\big[(1 + \mathcal{D}^2)^{-1/2}, a\big]b + [\mathcal{D}, a](1 + \mathcal{D}^2)^{-1/2}b.$$

As $[\mathcal{D}, a] \in N$, see [10, p.456], we have

$$[\mathcal{D}, a](1 + \mathcal{D}^2)^{-1/2}b \in J.$$

Thus we only need to show that

$$\mathcal{D}\big[(1 + \mathcal{D}^2)^{-1/2}, a\big]b \in J.$$

Now, we employ integral formula, [20, p.8],

$$(1 + \mathcal{D}^2)^{-1/2} = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (1 + \mathcal{D}^2 + \lambda)^{-1} d\lambda.$$

Denote the resolvent $(1 + \mathcal{D}^2 + \lambda)^{-1}$ by $R(\lambda)$. Then provided

$$\frac{1}{\pi} \int_0^\infty \lambda^{-1/2} \mathcal{D}[R(\lambda), a] b \, d\lambda$$

is convergent in operator norm, it is equal to

$$\mathcal{D}[(1 + \mathcal{D}^2)^{-1/2}, a] b.$$

Applying some basic commutator identities yields

$$\begin{aligned}
\mathcal{D}[R(\lambda), a] b &= \mathcal{D}R(\lambda)[a, \mathcal{D}^2] R(\lambda) b \\
&= \mathcal{D}R(\lambda)[a, \mathcal{D}]\mathcal{D}R(\lambda) b + \mathcal{D}R(\lambda)\mathcal{D}[a, \mathcal{D}] R(\lambda) b.
\end{aligned}$$

To establish the required norm estimates we require some inequalities. The following inequalities can be proved using the spectral theorem for unbounded operators, see [8, Appendix A],

1. $\|R(\lambda)\| = \|(1 + \mathcal{D}^2 + \lambda)^{-1}\| \le \frac{1}{1+\lambda}$

2. $\|\mathcal{D}R(\lambda)\| = \|\mathcal{D}(1 + \mathcal{D}^2 + \lambda)^{-1}\| \le \frac{1}{2\sqrt{1+\lambda}}$

3. $\|\mathcal{D}^2 R(\lambda)\| = \|\mathcal{D}^2 (1 + \mathcal{D}^2 + \lambda)^{-1}\| \le 1$

for all $\lambda \in [0, \infty)$. Thus

$$\begin{aligned}
\frac{1}{\pi} \int_0^\infty &\lambda^{-1/2} \|\mathcal{D}[R(\lambda), a] b\| \, d\lambda \\
&\le \frac{1}{\pi} \|b\| \|[a, \mathcal{D}]\| \int_0^\infty \lambda^{-1/2} \left(\frac{1}{4(1 + \lambda)} + \frac{1}{1 + \lambda} \right) d\lambda < \infty.
\end{aligned}$$

That is

$$\mathcal{D}[(1 + \mathcal{D}^2)^{-1/2}, a] b = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} \mathcal{D}[R(\lambda), a] b \, d\lambda$$

where the integral is convergent in operator norm. At last

$$\begin{aligned}
\mathcal{D}[R(\lambda), a] b &= \mathcal{D}R(\lambda) ab - [\mathcal{D}, a] R(\lambda) b - a\mathcal{D}R(\lambda) b \\
&= \mathcal{D}R(\lambda)^{1/2} R(\lambda)^{1/2} ab - [\mathcal{D}, a] R(\lambda) b - a\mathcal{D}R(\lambda)^{1/2} R(\lambda)^{1/2} b \in J
\end{aligned}$$

for all $\lambda \in [0,\infty)$ since all the terms are in J. Thus we conclude that

$$\mathcal{D}\big[(1+\mathcal{D}^2)^{-1/2},a\big]b \in J$$

and thus that $[F_\mathcal{D},a]b \in J$ for all $a,b \in \mathcal{A}$. By taking norm limits $[F_\mathcal{D},a]b \in J$ for all $a,b \in A = \overline{\mathcal{A}}$. $\qquad\square$

The argument used in the preceding proof is almost identical with the argument of S. Baaj and P. Julg used in [1] to build a bounded Kasparov module out of an unbounded one.

4.2. The Pairing with $K_1(A)$ and Spectral Flow

There is a certain case of unbounded spectral flow which is particularly interesting. Suppose that $(\mathcal{A},H,\mathcal{D})$ is a unital von Neumann spectral triple relative to (N,J). Let $u \in \mathcal{A}$ be unitary and consider the path

$$t \mapsto \mathcal{D}_t := (1-t)\mathcal{D} + tu^*\mathcal{D}u = \mathcal{D} + t[u^*,\mathcal{D}]u.$$

The function $t \mapsto t[u^*,\mathcal{D}]u$ is a continuous path of selfadjoint elements in N, so we can calculate the spectral flow of the path $\mathcal{D}_t$ via the transformation $\mathcal{D}_t \mapsto F_{\mathcal{D}_t}$.

Lemma 4.2 Let $(\mathcal{A},H,\mathcal{D})$ be a unital von Neumann spectral triple relative to (N,J). Setting $p = \chi(F_\mathcal{D})$ and letting $u \in \mathcal{A}$ be unitary, we have $up - pu \in J$.

Proof: Polar decomposition of $F_\mathcal{D}$ gives

$$F_\mathcal{D} = (2p-1)|F_\mathcal{D}|.$$

So the image of $F_\mathcal{D}$ in the Calkin algebra is

$$\pi(F_\mathcal{D}) = \pi(2p-1)\pi(|F_\mathcal{D}|) = \pi(2p-1)\pi(F_\mathcal{D}^2)^{1/2} = \pi(2p-1).$$

It follows that

$$2[u,p] - [u,F_\mathcal{D}] = [u,(2p-1)-F_\mathcal{D}] \in J.$$

By Theorem 4.1 we have $[u,F_\mathcal{D}] \in J$ so $[u,p] \in J$ as claimed. $\qquad\square$

Theorem 4.3 Suppose that $(\mathcal{A},H,\mathcal{D})$ is a unital von Neumann spectral triple relative to (N,J), and $u \in \mathcal{A}$ is a unitary. For the path $t \mapsto \mathcal{D}_t$ from above we have

$$\mathrm{sf}\{\mathcal{D}_t\} = \mathrm{sf}\{F_{\mathcal{D}_t}\} = \partial\big[\pi(pup) + \pi(1-p)\big]$$
$$= \mathrm{Ind}_{(p\text{-}p)}(pup)$$

where $p = \chi(F_\mathcal{D})$. From now on the spectral flow from $\mathcal{D}$ to $u^*\mathcal{D}u$ will be denoted by $\mathrm{sf}(\mathcal{D},u^*\mathcal{D}u)$.

Proof: From Theorem 3.14

$$\text{sf}\{F_{\mathcal{D}_t}\} = \big[(1-u^*pu)\cap p\big] - \big[(1-p)\cap u^*pu\big]$$

since

$$\chi(F_{u^*\mathcal{D}u}) = \chi(u^*F_{\mathcal{D}}u) = u^*\chi(F_{\mathcal{D}})u = u^*pu.$$

Now

$$x \in \text{Ker}(u^*pu) \cap \text{Im}(p) \Leftrightarrow (px = x \;\; \text{and} \;\; pux = 0) \Leftrightarrow x \in \text{Ker}(pup) \cap \text{Im}(p)$$

and

$$\begin{aligned}
x \in \text{Ker}(p) \cap \text{Im}(u^*pu) &\Leftrightarrow (px = 0 \;\; \text{and} \;\; ux = pux) \\
&\Leftrightarrow (pu^*pux = 0 \;\; \text{and} \;\; ux = pux) \\
&\Leftrightarrow ux \in \text{Ker}(pu^*p) \cap \text{Im}(p).
\end{aligned}$$

Thus

$$(1 - u^*pu) \cap p = N(pup) \cap p$$

and

$$u\big((1-p) \cap u^*pu\big)u^* = N(pu^*p) \cap p.$$

Since $N(pup + 1 - p) = N(pup) \cap p$ and $N(pu^*p + 1 - p) = N(pu^*p) \cap p$ we conclude by Lemma 2.1 that

$$\text{sf}\{F_{\mathcal{D}_t}\} = \big[N(pup) \cap p\big] - \big[N(pu^*p) \cap p\big] = \partial\big[\pi(pup) + \pi(1 - p)\big].$$

Remark that $\pi(pup) + \pi(1 - p)$ is unitary in N/J since $pu - up \in J$. $\square$

Corollary 4.4 *Setting* $p_F = \frac{F_{\mathcal{D}}+1}{2}$ *we actually have*

$$\text{sf}(\mathcal{D}, u^*\mathcal{D}u) = \partial[\pi(p_F u p_F + 1 - p_F)].$$

Proof: In the proof of Lemma 4.2 we saw that $\pi(2p - 1) = \pi(F_{\mathcal{D}})$ so $\pi(p) = \pi(p_F)$ and the corollary follows easily. $\square$

The last theorem of this section, which is the main theorem of the paper, expresses spectral flow from $\mathcal{D}$ to $u^*\mathcal{D}u$ in terms of a Kasparov product. We will need to use three different boundary maps namely

$$\begin{aligned}
&\partial : K_1(N/J) \to K_0(J) \\
&\partial_{J\otimes\mathcal{K}} : K_1\big(\mathcal{C}(J \otimes \mathcal{K})\big) \to K_0(J \otimes \mathcal{K}) \quad \text{and} \\
&\partial_J : K_1\big(\mathcal{C}(J)\big) \to K_0(J).
\end{aligned}$$

Note that for any C^*-algebra B, $\mathcal{C}(B)$ denotes the Calkin algebra $\mathcal{L}(B)/B$. Likewise we have the quotient maps

$$\pi : N \to N/J \quad \pi_{J\otimes\mathcal{K}} : \mathcal{L}(J \otimes \mathcal{K}) \to \mathcal{C}(J \otimes \mathcal{K}) \quad \text{and} \quad \pi_J : \mathcal{L}(J) \to \mathcal{C}(J).$$

Theorem 4.5 *Suppose that the norm closed ideal J is σ-unital and that the C^*-algebra $A = \overline{A}$ is separable. Denote by $[\mathcal{D}] = [\mathcal{M}_A, p_F]^1$ the class in $KK^1(A, J)$ of the Kasparov module $(\mathcal{M}_A, F_{\mathcal{D}}) \in \mathbb{E}(A, J)$ constructed in Theorem 4.1. Recall that $p_F = \frac{F_{\mathcal{D}}+1}{2}$.*

Let $u \in \tilde{A}$ be unitary and denote by $[u]$ its class in $K_1(A)$. Then we have the identity

$$\mathrm{sf}(\mathcal{D}, u^* \mathcal{D} u) = \partial[\pi(p_F u p_F + 1 - p_F)] = [u] \hat{\otimes}_A [\mathcal{D}].$$

Proof: We start by stabilizing using the isomorphisms $K_1(A) \cong K_1(A \otimes \mathcal{K})$ and $KK^1(A, J) \cong KK^1(A \otimes \mathcal{K}, J \otimes \mathcal{K})$. In this way we obtain the classes

$$[u \otimes e_{11} + e] \quad \text{and} \quad [\mathcal{M}_{A \otimes \mathcal{K}}, p_F \otimes 1]^1$$

in $K_1(A \otimes \mathcal{K})$ and $KK^1(A \otimes \mathcal{K}, J \otimes \mathcal{K})$ respectively, where e_{11} is a minimal projection in $\mathcal{K}$ and $e = 1 - 1 \otimes e_{11}$. See [21, Corollary 7.1.9] and [3, Corollary 17.8.8]. Thus, by Theorem 7.8 the product is given by

$$
\begin{aligned}
[u] \hat{\otimes}_A [\mathcal{D}] &= \partial_{J \otimes \mathcal{K}} \big[\pi_{J \otimes \mathcal{K}} \big(p_F \otimes 1 (u \otimes e_{11} + e) p_F \otimes 1 + 1 - p_F \otimes 1 \big) \big] \\
&= \partial_{J \otimes \mathcal{K}} \big[\pi_{J \otimes \mathcal{K}} \big((p_F u p_F) \otimes e_{11} + p_F \otimes 1 - p_F \otimes e_{11} + 1 - p_F \otimes 1 \big) \big] \\
&= \partial_{J \otimes \mathcal{K}} \big[\pi_{J \otimes \mathcal{K}} \big((p_F u p_F + 1 - p_F) \otimes e_{11} + e \big) \big]
\end{aligned}
$$

in $K_0(J \otimes \mathcal{K})$. Recall that $\pi_J(p_F^2 - p_F) = 0$ since $[\mathcal{M}_A, p_F]^1 \in KK^1(A, J)$ and A is unital.

But this is precisely the element $\partial_J[\pi_J(p_F u p_F + 1 - p_F)] \in K_0(J)$ under the isomorphism of $K_0(J)$ with $K_0(J \otimes \mathcal{K})$ [14, Lemma 4.2.4]. Therefore the proof is finished if we can prove the identity

$$\partial_J[\pi_J(p_F u p_F + 1 - p_F)] = \partial[\pi(p_F u p_F + 1 - p_F)].$$

To do so, let $x \in N$ be a norm-one lift of $\pi(p_F u p_F + 1 - p_F) \in N/J$. Then, as N acts on J by multiplication we have $N \subseteq \mathcal{L}(J)$, so $x \in \mathcal{L}(J)$ is too a norm-one lift of $\pi_J(p_F u p_F + 1 - p_F) \in \mathcal{C}(J)$. Recalling the description of the boundary map using norm-one lifts given in [14, Proposition 4.8.10], the desired identity follows. $\qquad\square$

5. C^*-Spectral Flow

A problem with the construction of the Kasparov module in the last section is that it only works for σ-unital ideals J. For an arbitrary ideal in a von Neumann this may very well not be the case. When we can replace J by a σ-unital C^*-algebra B, we not only ensure the existence of the KK-class, but can obtain stronger constraints on the values of the spectral flow.

5.1. Basic Definitions

Let $(\mathcal{A}, H, \mathcal{D})$ be a unital von Neumann spectral triple relative to (N, J), and let $A = \overline{\mathcal{A}}$ be the norm closure of $\mathcal{A}$. *We assume throughout this section that A is a separable C^*-algebra.*

Suppose that $B \subseteq J$ is a σ-unital C^*-algebra such that $(\mathcal{M}_A, F_\mathcal{D}) \in \mathbb{E}(A, B)$, where $\mathcal{M}_A : A \rightarrow \mathcal{L}(B)$, thus in particular A is supposed to act on B by left-multiplication.

Note that B is a countably generated right Hilbert B-module when equipped with the inner product $\langle x, y \rangle = x^* y$ for all $x, y \in B$ and the action of B from the right given by multiplication. The class $[\mathcal{M}_A, p_F]^1 \in KK^1(A, B)$ is denoted by $[\mathcal{D}_B]$ where $p_F = \frac{F_\mathcal{D} + 1}{2}$.

Let $\partial_B : K_1\big(\mathcal{C}(B)\big) \rightarrow K_0(B)$ be the boundary map, where $\mathcal{C}(B)$ is the Calkin algebra $\mathcal{L}(B)/B$. Let $\pi_B : \mathcal{L}(B) \rightarrow \mathcal{C}(B)$ denote the quotient map.

Definition 5.1 Let $(\mathcal{A}, H, \mathcal{D})$ and $B \subseteq J \subseteq N$ be as above. We define the C^*-spectral flow as the quantity

$$sf_B(\mathcal{D}, u^* \mathcal{D} u) = \partial_B[\pi_B(p_F u p_F + 1 - p_F)] \in K_0(B).$$

The reason for supposing the existence of the Kasparov module class $[\mathcal{D}_B]$ is that we want to describe the C^*-spectral flow using a Kasparov product. In fact we have

Theorem 5.2 *Let $(\mathcal{A}, H, \mathcal{D})$ be a unital von Neumann spectral triple relative to (N, J). Suppose that $B \subseteq J$ is a σ-unital C^*-algebra such that $(\mathcal{M}_A, F_\mathcal{D}) \in \mathbb{E}(A, B)$ where $\mathcal{M}_A : A \rightarrow \mathcal{L}(B)$. Let $u \in \mathcal{A}$ be unitary. The C^*-spectral flow from $\mathcal{D}$ to $u^* \mathcal{D} u$ is equal to the product of $[\mathcal{D}_B] \in KK^1(A, B)$ and the class of the unitary $[u] \in K_1(A)$. That is*

$$[u] \hat{\otimes}_A [\mathcal{D}_B] = \partial_B[\pi_B(p_F u p_F + 1 - p_F)] = sf_B(\mathcal{D}, u^* \mathcal{D} u).$$

Proof: The proof is similar to the one given in Theorem 4.5. □

To justify the definition of C^*-spectral flow, we must show that there exists a σ-unital C^*-algebra B such that $(\mathcal{M}_A, F_\mathcal{D})$ is a Kasparov A-B-module.

Theorem 5.3 *Let $(\mathcal{A}, H, \mathcal{D})$ be a unital von Neumann spectral triple relative to (N, J) with A separable. Let $\mathcal{B}$ be the smallest C^*-algebra in $\mathcal{L}(H)$ containing the elements*

$$F_\mathcal{D}[F_\mathcal{D}, a] \quad b[F_\mathcal{D}, a]$$
$$F_\mathcal{D} b[F_\mathcal{D}, a] \quad a\varphi(\mathcal{D})$$

for all $a, b \in \mathcal{A}$ and $\varphi \in C_0(\mathbb{R})$. Then $\mathcal{B}$ is separable, contained in J and the pair $(\mathcal{M}_A, F_\mathcal{D})$ is a Kasparov A-$\mathcal{B}$-module. In particular $\mathcal{B}$ is σ-unital.

Proof: Recall that $\mathcal{A}$ is supposed to be unital. The C^*-algebra $C_0(\mathbb{R})$ is generated by the resolvent function $x \mapsto (i+x)^{-1}$ and the operator $(i+\mathcal{D})^{-1}$ is in J so $a\varphi(\mathcal{D})$ is in J for all $\varphi \in C_0(\mathbb{R})$. By Theorem 4.1, $[F_{\mathcal{D}},a] \in J$ so all of the generators of $\mathcal{B}$ are in J and thus $\mathcal{B} \subseteq J$. Observe that $\mathcal{B}$ is separable, and so σ-unital.

Now, clearly A acts on $\mathcal{B}$ by multiplication from the left. Furthermore $F_{\mathcal{D}}$ is in $\mathcal{L}(\mathcal{B})$ since

$$1 - F_{\mathcal{D}}^2 = (1 + \mathcal{D}^2)^{-1} \in \mathcal{B} \quad \text{and} \quad F_{\mathcal{D}}\varphi(\mathcal{D}) \in \mathcal{B}$$

for any $\varphi \in C_0(\mathbb{R})$.

Proving that $(\mathcal{M}_A, F_{\mathcal{D}})$ is a Kasparov A-$\mathcal{B}$-module is now straightforward. $\square$

5.2. *The Relationship Between C^*-Spectral Flow and von Neumann Spectral Flow*

In this section we want to compare the C^*-spectral flow with the von Neumann spectral flow. Let $(\mathcal{A}, H, \mathcal{D})$ be a unital von Neumann spectral triple relative to (N, J) and let $u \in \mathcal{A}$ be unitary. In Corollary 4.4 we found the expression

$$\mathrm{sf}(\mathcal{D}, u^*\mathcal{D}u) = \partial[\pi(p_F u p_F + 1 - p_F)] \in K_0(J)$$

for the von Neumann spectral flow.

Let B be a σ-unital C^*-algebra contained in J such that $(\mathcal{M}_A, F_{\mathcal{D}})$ is a Kasparov A-B-module. By definition we have the following expression

$$\mathrm{sf}_B(\mathcal{D}, u^*\mathcal{D}u) = \partial_B[\pi_B(p_F u p_F + 1 - p_F)] \in K_0(B)$$

for the C^*-spectral flow.

These two notions should coincide in $K_0(J)$ when we apply the map

$$i_* : K_0(B) \to K_0(J)$$

induced by the inclusion $i : B \to J$.

Lemma 5.4 *Let $(\mathcal{A}, H, \mathcal{D})$ be a unital von Neumann spectral triple relative to (N, J), and let B be a σ-unital C^*-algebra contained in J such that $(\mathcal{M}_A, F_{\mathcal{D}}) \in \mathbb{E}(A, B)$ where $\mathcal{M}_A : A \to \mathcal{L}(B)$ is left-multiplication. The inclusion of B in $\mathcal{L}(H)$ can be extended to an injective $*$-homomorphism*

$$i : \mathcal{L}(B) \to \mathcal{L}(H)$$

such that $i(T)(bx) = (Tb)x$ for all $T \in \mathcal{L}(B)$, $b \in B$ and $x \in H$. The image of the extension i is contained in the double commutant of $B \subseteq \mathcal{L}(H)$. In particular $\mathcal{L}(B)$ can be realized inside N.

Proof: Since $(\mathcal{M}_A, F_\mathcal{D})$ is a Kasparov A-B-module and A is unital, we must have $1 - F_\mathcal{D}^2 = (1 + \mathcal{D}^2)^{-1} \in B$. The image of $(1 + \mathcal{D}^2)^{-1} \in \mathcal{L}(H)$ is the domain of $\mathcal{D}^2$ which is dense in H. The representation of B on H by i is thus seen to be non-degenerate. Therefore, by [16, Proposition 2.1], the inclusion extends to $\mathcal{L}(B)$ giving an injective $*$-homomorphism

$$i : \mathcal{L}(B) \to \mathcal{L}(H)$$

such that $i(T)(bx) = (Tb)x$ for all $T \in \mathcal{L}(B)$, $b \in B$ and $x \in H$.

Let $S \in B'$ and let $T \in \mathcal{L}(B)$. Suppose that $x \in H$ is of the form $x = by$ for some $b \in B$ and $y \in H$. Now

$$i(T)Sby = i(T)bSy = (Tb)Sy = S(Tb)y = Si(T)by$$

so $i(T)S = Si(T)$ on a dense subspace of H and we conclude that $i(T) \in B'' \subseteq N'' = N$. $\qquad\square$

Theorem 5.5 *Let $(\mathcal{A}, H, \mathcal{D})$ be a unital von Neumann spectral triple relative to (N, J), and let B be a σ-unital C^*-algebra contained in J such that $(\mathcal{M}_A, F_\mathcal{D})$ is a Kasparov A-B-module. The von Neumann spectral flow coincides with the C^*-spectral flow under the homomorphism $i_* : K_0(B) \to K_0(J)$. More precisely for $u \in \mathcal{A}$ unitary*

$$\mathrm{sf}(\mathcal{D}, u^*\mathcal{D}u) = i_*\big(\mathrm{sf}_B(\mathcal{D}, u^*\mathcal{D}u)\big).$$

Proof: By Lemma 5.4 there are isometric maps

$$i : B \to J \quad \text{and} \quad i : \mathcal{L}(B) \to N$$

which allow us to also define the map (not necessarily injective)

$$i : \mathcal{C}(B) \to N/J.$$

Now, let $x \in \mathcal{L}(B)$ be a norm-one lift of the unitary $\pi_B(p_F u p_F + 1 - p_F) \in \mathcal{C}(B)$, then $i(x) \in N$ is a norm-one lift of the unitary $\pi(p_F u p_F + 1 - p_F) \in N/J$. By [14, Proposition 4.8.10] we have

$$i_*\left[\partial_B\big(\pi_B(p_F u p_F + 1 - p_F)\big)\right]$$

$$= i_*\left(\begin{bmatrix} xx^* & x(1-x^*x)^{1/2} \\ x^*(1-xx^*)^{1/2} & 1-x^*x \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} i(x)i(x)^* & i(x)\big(1-i(x)^*i(x)\big)^{1/2} \\ i(x)^*\big(1-i(x)i(x)^*\big)^{1/2} & 1-i(x)^*i(x) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \partial[\pi(p_F u p_F + 1 - p_F)].$$

Which is the desired identity. $\qquad\square$

Corollary 5.6 *Let $(\mathcal{A}, H, \mathcal{D})$ be a unital von Neumann spectral triple relative to (N, J), and let B be a σ-unital C^*-algebra contained in J such that $(\mathcal{M}_A, F_{\mathcal{D}})$ is a Kasparov A-B-module. For $u \in \mathcal{A}$ unitary, the von Neumann spectral flow from $\mathcal{D}$ to $u^*\mathcal{D}u$ can be expressed in terms of the Kasparov product of $[\mathcal{D}_B] \in KK^1(A, B)$ and the class $[u] \in K_1(A)$ of the unitary $u \in \mathcal{A}$. More precisely*

$$\mathrm{sf}(\mathcal{D}, u^*\mathcal{D}u) = i_*([u] \hat{\otimes}_A [\mathcal{D}_B]).$$

Proof: This follows immediately by Theorem 5.5 and Theorem 5.2.   □

6. Numerical Spectral Flow

Our aim in this section is to relate the von Neumann spectral flow to the numerical spectral flow in semifinite von Neumann algebras studied in [10, 22, 23].

In this section we let N denote a semifinite von Neumann algebra equipped with a fixed semifinite, faithful, normal trace τ. Furthermore, for any $*$-algebra $\mathcal{F} \subseteq N$ we let $\mathcal{F}^+$ denote the $*$-algebra generated in N by $\mathcal{F}$ and the unit in N. When $\mathcal{F}$ is non-unital, we write the elements of $\mathcal{F}^+$ as pairs $x + \lambda Id$, where $x \in \mathcal{F}$ and $\lambda \in \mathbb{C}$.

Definition 6.1 Let $\mathcal{F}_N$ be the $*$-algebra in N generated by the projections p with finite trace, $\tau(p) < \infty$. By [13, Section 1.8], $\mathcal{F}_N$ is an ideal in N. The τ-*compact operators*, $\mathcal{K}_N$, is the norm-closure of $\mathcal{F}_N$.

Let $(\mathcal{A}, H, \mathcal{D})$ be a semifinite spectral triple relative to (N, τ) as defined in [10, Definition 2.1]. Notice that $(\mathcal{A}, H, \mathcal{D})$ is a von Neumann spectral triple relative to $(N, \mathcal{K}_N)$ in an obvious way. For semifinite spectral triples, spectral flow is defined as a real number, whereas our methods produce a class in $K_0(\mathcal{K}_N)$. The problem is solved by establishing a homomorphism $\tau_* : K_0(\mathcal{K}_N) \to \mathbb{R}$ by means of the trace τ. The existence and nature of such a homomorphism is of course well known, but as the link to the semifinite case is very important we will carry out the details.

Lemma 6.2 *Let $n \in \mathbb{N}$. For each finite set of elements $\{x_1, \ldots, x_m\} \subseteq M_n(\mathcal{F}_N)$ there is a projection $p \in M_n(\mathcal{F}_N)$ with $px_i = x_i$ for all $i \in \{1, \ldots, m\}$. The projection p is called a* local unit *for $\{x_1, \ldots, x_m\}$.*

Proof: For any finite set of projections $\{p_1, \ldots, p_m\}$ in $\mathcal{F}_N$, the inequality $\sup\{p_1, \ldots, p_m\} \leq p_1 + \ldots + p_m$ holds so $\sup\{p_1, \ldots, p_m\} \in \mathcal{F}_N$. Furthermore, for each $i \in \{1, \ldots, m\}$ we have $p_i \leq \sup\{p_1, \ldots, p_m\}$ yielding $\sup\{p_1, \ldots, p_m\}p_i = p_i$, so $\sup\{p_1, \ldots, p_m\}$ is a local unit for $\{p_1, \ldots, p_m\}$. To obtain the desired property for $\mathcal{F}_N$, note that each element in $\mathcal{F}_N$ is a complex polynomial of finite degree, where the variables are projections with finite trace.

Now, let $n \in \mathbb{N}$ and fix a finite set of matrices $\{x_1, \ldots, x_m\} \subseteq M_n(\mathcal{F}_N)$. Choose a projection $p \in \mathcal{F}_N$, such that $px_i^{kl} = x_i^{kl}$ for all $i \in \{1, \ldots, m\}$ and $k, l \in \{1, \ldots n\}$,

where x_i^{kl} is the matrix entry corresponding to row k and column l. Then obviously $\mathrm{diag}(p,\dots p)x_i = x_i$ for all $i \in \{1,\dots,m\}$ as desired. $\qquad\square$

Lemma 6.3 *For each $n \in \mathbb{N}$ the $*$-algebra $\mathcal{F}_N$ is stable under the holomorphic functional calculus. That is, for $x \in M_n(\mathcal{F}_N)$ and for f a holomorphic function in a neighborhood of the spectrum of x in $M_n(\mathcal{K}_N)$ with $f(0) = 0$ we have $f(x) \in M_n(\mathcal{F}_N)$. In other words, $\mathcal{F}_N$ equipped with the C^*-norm from $\mathcal{K}_N$ becomes a pre-C^*-algebra. In particular it has a well-defined K-theory and, by [7, Proposition 3], the inclusion $i : \mathcal{F}_N \to \mathcal{K}_N$ induces an isomorphism $i_* : K_0(\mathcal{F}_N) \to K_0(\mathcal{K}_N)$.*

Proof: We employ the technique of [25, Proposition 4]. First, notice that $f(x) \in M_n(\mathcal{K}_N)$ because $M_n(\mathcal{K}_N)$ is a C^*-algebra. Now, for a closed curve γ winding once around the spectrum of x in $M_n(\mathcal{K}_N)$ not touching 0, the identity

$$f(x) = \frac{1}{2\pi i} \int_\gamma f(\lambda)(\lambda - x)^{-1} d\lambda$$

is valid. Let p be a local unit for x, let λ be in the resolvent of x and check that

$$(1-p) = (1-p)(x-\lambda)(x-\lambda)^{-1} = -\lambda(1-p)(x-\lambda)^{-1}.$$

Thus for $\lambda \neq 0$ we have

$$(1-p)(x-\lambda)^{-1} = -\frac{1}{\lambda}(1-p).$$

This enables us to calculate

$$\begin{aligned}
(1-p)f(x) &= \frac{1}{2\pi i} \int_\gamma f(\lambda)(1-p)(\lambda - x)^{-1} d\lambda \\
&= \frac{1}{2\pi i} \int_\gamma \frac{-f(\lambda)}{\lambda}(1-p) d\lambda \\
&= (1-p)f(0) = 0.
\end{aligned}$$

It follows that $pf(x) = f(x)$. As $M_n(\mathcal{F}_N)$ is an ideal in $M_n(\mathcal{K}_N)$ and $p \in M_n(\mathcal{F}_N)$ we conclude that $f(x) \in M_n(\mathcal{F}_N)$ as desired. $\qquad\square$

Theorem 6.4 *There is a homomorphism $\tau_* : K_0(\mathcal{K}_N) \to \mathbb{R}$ given by*

$$\tau_*\big([x + \lambda Id] - [y + \mu Id]\big) = \tau_n(x) - \tau_n(y)$$

for each pair of projections $x + \lambda Id$, $y + \mu Id \in M_n(\mathcal{F}_N^+)$ with $[\lambda] = [\mu]$ in $K_0(\mathbb{C})$. Remark that $\tau_n = \tau \otimes \mathrm{Tr}$ on the algebraic tensor product $\mathcal{F}_N \otimes M_n(\mathbb{C}) = M_n(\mathcal{F}_N)$ where Tr is the canonical trace on $M_n(\mathbb{C})$.

Proof: Define $\hat{\tau} : \mathcal{F}_N^+ \to \mathbb{R}$ by $\hat{\tau}(x + \lambda Id) = \tau(x)$ then $\hat{\tau}$ satisfies the relation $\hat{\tau}(u^*xu) = \hat{\tau}(x)$ for all unitaries $u \in \mathcal{F}_N^+$. Indeed, write $u = v + \alpha Id$, where $v \in \mathcal{F}_N$ and $\alpha \in \mathbb{C}$, then

$$\overline{\alpha}\alpha = 1 \quad \text{and} \quad v^*v + v^*\alpha + v\overline{\alpha} = 0 = vv^* + v^*\alpha + v\overline{\alpha}.$$

Now, we simply calculate

$$(v^* + \overline{\alpha}Id)(x + \lambda Id)(v + \alpha Id)$$
$$= (v^*xv + v^*x\alpha + v^*\lambda v + v^*\lambda\alpha + \overline{\alpha}xv + x + \overline{\alpha}\lambda v + \lambda Id)$$
$$= (v^*xv + v^*x\alpha + \overline{\alpha}xv + x + \lambda Id)$$

thus applying our extended $\hat{\tau}$ yields

$$\hat{\tau}\big((v^* + \overline{\alpha}Id)(x + \lambda Id)(v + \alpha Id)\big) = \tau(v^*xv + v^*x\alpha + \overline{\alpha}xv + x) = \tau(x).$$

Now, clearly, there is a well-defined homomorphism $\tau_* : K_0(\mathcal{F}_N^+) \to \mathbb{R}$ given by $\hat{\tau}([x + \lambda Id] - [y + \mu Id]) = \tau_n(x) - \tau_n(y)$ for each pair of projections $(x + \lambda Id), (y + \mu Id) \in M_n(\mathcal{F}_N^+)$. Since $K_0(\mathcal{F}_N)$ is the kernel of the homomorphism $\pi_* : K_0(\mathcal{F}_N^+) \to K_0(\mathbb{C})$ induced by the projection $\pi : \mathcal{F}_N^+ \to \mathbb{C}$ we get the desired map by restriction and a reference to Lemma 6.3. $\square$

Theorem 6.5 *Let p be a projection in $M_n(\mathcal{K}_N)$, then actually $p \in M_n(\mathcal{F}_N)$.*

Proof: Since $M_n(\mathcal{F}_N)$ is dense in $M_n(\mathcal{K}_N)$, there is a positive element $e \in M_n(\mathcal{F}_N)$ such that $\|e - p\| < \frac{1}{24}$. In particular $\|e\| < 2$. The estimate

$$\|e^2 - e\| \leq \|e(e - p)\| + \|(e - p)p\| + \|p - e\| < \frac{1}{4}$$

shows that e is almost a projection. It follows that $1/2 \notin \mathrm{Sp}(e)$, creating a gap in the spectrum of e. There is thus an $\varepsilon > 0$ such that

$$\mathrm{Sp}(e) \subseteq [0, 1/2 - \varepsilon] \cup [1/2 + \varepsilon, 5/4]$$

and the function $f : \mathbb{R}/\{\frac{1}{2}\} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} 0 & t < \frac{1}{2} \\ 1 & t > \frac{1}{2} \end{cases}$$

is holomorphic on a neighborhood of $\mathrm{Sp}(e)$ with $f(0) = 0$. By Lemma 6.3 the projection $f(e)$ is in $M_n(\mathcal{F}_N)$. Moreover,

$$\sup\{|f(t) - t| \,|\, t \in \mathrm{Sp}(e)\} \leq \sup\{1/2 - \varepsilon, 1/4\}$$

so $\|f(e) - e\| < \frac{1}{2}$. This gives us the inequality

$$\|p - f(e)\| \leq \|p - e\| + \|e - f(e)\| < 1$$

but then p and $f(e)$ must be equivalent, i.e. there exist a unitary u in $M_n(\mathcal{K}_N^+)$ such that $u^* f(e)u = p$ by [14, Proposition 4.1.7]. The proof is finished by recalling that $M_n(\mathcal{F}_N)$ is an ideal in $M_n(N)$. $\qquad\square$

Definition 6.6 [22, 23] Let $\{B_t\}$ be a path of selfadjoint operators in N such that $\pi(B_t) \in N/\mathcal{K}_N$ is invertible for all t. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of $[0,1]$ such that for each $i \in \{1 \ldots, n\}$ we have

$$\|\pi(\chi(B_t)) - \pi(\chi(B_s))\| < 1/2$$

for all $t, s \in [t_{i-1}, t_i]$. Recalling that all projections in $\mathcal{K}_N$ have finite trace by Theorem 6.5, we define the semifinite spectral flow of the path $\{B_t\}$ as the real number

$$sf\{B_t\} = \sum_{i=1}^{n} \left(\tau\big(N(p_i) \cap p_{i-1}\big) - \tau\big(N(p_{i-1}) \cap p_i\big) \right)$$

where $p_i = \chi(B_{t_i})$.

Theorem 6.7 *The semifinite spectral flow of the path $\{B_t\}$ from above can be expressed as*

$$\mathrm{sf}\{B_t\} = \tau\big(N(p_n \ldots p_0) \cap p_0\big) - \tau\big(N(p_0 \ldots p_n) \cap p_n\big).$$

Moreover it is independent of the partition chosen and is invariant under homotopies of the path $\{B_t\}$ keeping the endpoints fixed.

Proof: This follows immediately by applying our homomorphism $\tau_* : K_0(\mathcal{K}_N) \to \mathbb{R}$ from Theorem 6.4 to the results in Corollary 3.7, Lemma 3.8 and Lemma 3.9 recalling that each projection $p \in \mathcal{K}_N$ has finite trace by Theorem 6.5. $\qquad\square$

Definition 6.8 [8, 9, 22, 23] Let $(\mathcal{A}, H, \mathcal{D})$ be a unital semifinite spectral triple relative to (N, τ). Suppose that the norm closure $A = \overline{\mathcal{A}}$ of $\mathcal{A}$, is a separable C^*-algebra. For each path of selfadjoint operators $\{A_t\}$ in N, we define the semifinite spectral flow of the path $t \mapsto \mathcal{D} + A_t := \mathcal{D}_t$ as the real number $sf\{\mathcal{D}_t\} := sf\{F_{\mathcal{D}_t}\}$.

We can now state our main theorem relating the three different notions of spectral flow we have discussed.

Theorem 6.9 *Let $(\mathcal{A}, H, \mathcal{D})$ be a unital semifinite spectral triple relative to (N, τ). Suppose that the norm closure $A = \overline{\mathcal{A}}$ of $\mathcal{A}$, is a separable C^*-algebra. Let $u \in \mathcal{A}$*

be unitary. Set $\mathcal{D}_t = (1-t)\mathcal{D} + tu^*\mathcal{D}u = \mathcal{D} + tu^*[\mathcal{D},u]$, then the unbounded semifinite spectral flow of the path $t \mapsto \mathcal{D}_t$ is given by

$$\mathrm{sf}\{\mathcal{D}_t\} = \tau\big(\partial[\pi(pup + 1 - p)]\big) = \tau\big(N(pup + 1 - p)\big) - \tau\big(N(pu^*p + 1 - p)\big)$$

where $\tau_* : K_0(\mathcal{K}_N) \to \mathbb{R}$ is the homomorphism from Theorem 6.4 and $p = \chi(F_\mathcal{D})$. In addition there exists a separable C^*-algebra $\mathcal{B} \subseteq \mathcal{K}_N$ and a class $[\mathcal{D}_\mathcal{B}] \in KK^1(A,\mathcal{B})$ such that

$$\mathrm{sf}\{\mathcal{D}_t\} = \tau_*\big(i_*([u]\hat{\otimes}_A[\mathcal{D}_\mathcal{B}])\big)$$

where $i : \mathcal{B} \to \mathcal{K}_N$ is the inclusion and $[u] \in K_1(A)$ is the class of the unitary.

Proof: This follows immediately by applying our $\tau_* : K_0(\mathcal{K}_N) \to \mathbb{R}$ from Theorem 6.4 to both sides of the equalities in Theorem 4.3 and in Corollary 5.6, keeping in mind that each projection in $\mathcal{K}_N$ has finite trace by Theorem 6.5.   □

Theorem 6.9 shows that semifinite spectral triples represent KK-classes in a precise sense. While this is really proved here only for odd spectral triples, the discussion in [19] and some simple adaptations of these proofs show that such a representation theorem is also true in the even case.

7. Appendix on Kasparov Products

In this appendix we give explicit forms for odd pairings in KK-theory. In order to do this, we need to recall some basic definitions and results.

Definition 7.1 Let A and B be $\mathbb{Z}_2$-graded C^*-algebras. A Kasparov A-B-module is a pair (ψ,V) consisting of a graded $*$-homomorphism $\psi : A \to \mathcal{L}(E)$, with E a countably generated, graded right Hilbert B-module, together with an odd operator $V \in \mathcal{L}(E)$ such that

1. $\psi(a)(V^2 - 1) \in \mathcal{K}(E)$

2. $\psi(a)(V - V^*) \in \mathcal{K}(E)$

3. $[V,\psi(a)] \in \mathcal{K}(E)$

for all $a \in A$. The set of Kasparov A-B-module is denoted by $\mathbb{E}(A,B)$. An element $(\psi,V) \in \mathbb{E}(A,B)$ is called degenerate when $a(V^2 - 1) = a(V - V^*) = [V,a] = 0$.

The set $\mathbb{E}(A,B)$ becomes the even KK-theory group $KK(A,B)$ when equipped with direct sum and the equivalence relation $\sim_{oh}$ generated by operator homotopy, unitary equivalence and addition of degenerate elements. Unitary equivalence is denoted by $\sim_u$. The class represented by the pair $(\psi,V) \in \mathbb{E}(A,B)$ is denoted by $[\psi,V] \in KK(A,B)$. [3, Proposition 17.3.3].

To define the odd KK-theory group we introduce the Clifford algebra $\mathbb{C}_1$, that is the C^*-algebra $\mathbb{C} \oplus \mathbb{C}$ equipped with the standard odd grading, and we set $KK^1(A, B) = KK(A, B \hat{\otimes} \mathbb{C}_1)$ where $\hat{\otimes}$ denotes the graded tensor product as defined in [3, Chapter 14.4].

For ungraded C^*-algebras A and B there is a description of the odd KK-theory using extensions of C^*-algebras. More precisely

Theorem 7.2 *[3, Proposition 17.6.5] There is an isomorphism*

$$\mathrm{Ext}^{-1}(A, B \otimes \mathcal{K}) \cong KK^1(A, B \otimes \mathcal{K}).$$

An invertible extension given by the $*$-homomorphism $\psi : A \to \mathcal{L}(B \otimes \mathcal{K})$ and the element $p \in \mathcal{L}(B \otimes \mathcal{K})$ that is with Busby-invariant $\tau : a \mapsto \pi(p\psi(a)p) \in \mathcal{C}(B \otimes \mathcal{K})$ is mapped to the Kasparov A-$(B \otimes \mathcal{K})$-module $(\psi \hat{\otimes} 1, (2p-1)\hat{\otimes}\varepsilon) \in \mathbb{E}(A, (B \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1)$ with $\psi \hat{\otimes} 1 : A \to \mathcal{L}((B \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1)$ and $\varepsilon = (1, -1) \in \mathbb{C}_1$.

With this in mind we will employ the notation $[\psi, p]^1$ for the class $[\psi \hat{\otimes} 1, (2p-1)\hat{\otimes}\varepsilon] \in KK^1(A, B \otimes \mathcal{K}) = KK(A, (B \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1)$ where $\psi : A \to \mathcal{L}(B \otimes \mathcal{K})$ and $p \in \mathcal{L}(B \otimes \mathcal{K})$ have the properties

1. $\psi(a)(p^2 - p) \in B \otimes \mathcal{K}$

2. $\psi(a)(p - p^*) \in B \otimes \mathcal{K}$

3. $[p, \psi(a)] \in B \otimes \mathcal{K}$

for all $a \in A$.

Let A, B and D be graded C^*-algebras. Suppose that A and D are separable and that B is σ-unital. A fundamental property of KK-theory is the existence of a bilinear associative product

$$\hat{\otimes}_A : KK^i(D, A) \times KK^j(A, B) \to KK^{i+j}(D, B).$$

The aim of this appendix is to give a concrete description of a certain instance of this product namely the one between $K_1(A) = KK^1(\mathbb{C}, A)$ and $KK^1(A, B)$, [3, Chapter 18].

We will need to quote a couple of results. First of all, since the aim is to form products with K-theory we will use the isomorphism of K-theory with KK-theory.

Lemma 7.3 *Let A be an ungraded C^*-algebra. The groups $KK^1(\mathbb{C}, A \otimes \mathcal{K})$ and $K_1(A \otimes \mathcal{K})$ are isomorphic. The isomorphism is given by*

$$[\mathcal{M}_{\mathbb{C}}, p]^1 \mapsto \partial[\pi(p)]$$

where $\mathcal{M}_{\mathbb{C}} : \mathbb{C} \to \mathcal{L}(A \otimes \mathcal{K})$ is left multiplication by the complex numbers, $\pi :$ $\mathcal{L}(A \otimes \mathcal{K}) \to C(A \otimes \mathcal{K})$ is the quotient map and $\partial : K_0\big(C(A \otimes \mathcal{K})\big) \to K_1(A \otimes \mathcal{K})$ is the boundary map, [3, Proposition 17.5.7].

From the definition of ∂ it follows that, for each class $[u] \in K_1(A \otimes \mathcal{K})$, there exists a selfadjoint $q \in \mathcal{L}(A \otimes \mathcal{K})$ with $\|q\| \leq 1$ such that $[u] = [\exp(2\pi i q)]$, [26, Proposition 12.2.2].

Likewise the groups $KK(\mathbb{C}, A \otimes \mathcal{K})$ and $K_0(A \otimes \mathcal{K})$ are isomorphic. The isomorphism is given by

$$[\mathcal{M}_{\mathbb{C}}, V] \mapsto \partial[\pi(T)]$$

where $\mathcal{M}_{\mathbb{C}} : \mathbb{C} \to \mathcal{L}\big((A \otimes \mathcal{K}) \oplus (A \otimes \mathcal{K})\big)$, the element $V \in \mathcal{L}\big((A \otimes \mathcal{K}) \oplus (A \otimes \mathcal{K})\big)$ is the matrix

$$V = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$$

and $\partial : K_1\big(C(A \otimes \mathcal{K})\big) \to K_0(A \otimes \mathcal{K})$ is the boundary map. Note that the grading on $(A \otimes \mathcal{K}) \oplus (A \otimes \mathcal{K})$ is given by $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, [3, Proposition 17.5.5].

Let A, B and D be graded C^*-algebras, with A and D separable and B σ-unital. The Kasparov product can be constructed using the notion of connections. Let $(\psi_1, V_1) \in \mathbb{E}(D, A)$ with $\psi_1 : D \to \mathcal{L}(E_1)$ and let $(\psi_2, V_2) \in \mathbb{E}(A, B)$ with $\psi_2 : A \to \mathcal{L}(E_2)$. We can form the graded interior tensor product $E = E_1 \hat{\otimes}_{\psi_2} E_2$ in the sense of [3, Chapter 14.4]. For each $x \in E_1$ there is a map $T_x \in \mathcal{L}(E_2, E)$ such that $T_x(y) = x \hat{\otimes} y$ for all $y \in E_2$, [16, Proposition 4.6].

Definition 7.4 An odd operator $F \in \mathcal{L}(E)$ is called a V_2-connection for E_1 if, for any homogeneous $x \in E_1$, we have

$$T_x V_2 - (-1)^{\partial x} F T_x \in \mathcal{K}(E_2, E)$$

where ∂x denotes the degree of x in E_1.

Now we are ready to state the most important background result. It gives a concrete description of the product under an assumption on commutators. Later on the C^*-algebra D is going to be the complex numbers so the assumption will be trivially satisfied.

Theorem 7.5 [3, Proposition 18.10.1] Let $x = (\psi_1, V_1) \in \mathbb{E}(D, A)$ with $V_1 = V_1^*$ and $\|V_1\| \leq 1$. Let $y = (\psi_2, V_2) \in \mathbb{E}(A, B)$. Let F be a V_2-connection for E_1. Set $E = E_1 \hat{\otimes}_{\psi_2} E_2$, $\psi = \psi_1 \hat{\otimes} 1 : A \to \mathcal{L}(E)$ and

$$V = V_1 \hat{\otimes} 1 + \big((1 - V_1^2)^{1/2} \hat{\otimes} 1\big) F.$$

If $[V_1 \hat{\otimes} 1, \psi(a)] \in \mathcal{K}(E)$ for all $a \in A$, then $z = (\psi, V) \in \mathbb{E}(D, B)$ is operator homotopic to the Kasparov product of x and y, i.e. $[x] \hat{\otimes}_A [y] = [z]$ in $KK(D, B)$.

To form the product in KK^1 we need to be able to move the Clifford algebra from the second coordinate to the first. This is accomplished by the following lemma.

Lemma 7.6 *Let A and B be ungraded C^*-algebras. There is a group isomorphism*

$$\varphi : KK^1(A, B \otimes \mathcal{K}) = KK\big(A, (B \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1\big) \to KK(A\hat{\otimes}\mathbb{C}_1, B \otimes \mathcal{K})$$

such that

$$\varphi[\psi, p]^1 = \left[\sigma, \begin{pmatrix} 0 & 2p-1 \\ 2p-1 & 0 \end{pmatrix}\right]$$

where the graded $*$-homomorphism $\sigma : A\hat{\otimes}\mathbb{C}_1 \to \mathcal{L}\big((B \otimes \mathcal{K}) \oplus (B \otimes \mathcal{K})\big)$ is given by

$$\sigma(a, -a) = \begin{pmatrix} 0 & -i\psi(a) \\ i\psi(a) & 0 \end{pmatrix}$$

and

$$\sigma(a, a) = \begin{pmatrix} \psi(a) & 0 \\ 0 & \psi(a) \end{pmatrix}.$$

The grading on $(B \otimes \mathcal{K}) \oplus (B \otimes \mathcal{K})$ is given by the grading operator $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, [15].

7.1. Product between K_1 and KK^1

The starting point is a translation of Theorem 7.5 suited to handle the odd case.

Theorem 7.7 *Let A, B and D be ungraded C^*-algebras, with A and D separable and B σ-unital. Let $[x]$ be a class in $KK^1(D, A \otimes \mathcal{K})$. By Theorem 7.2 we can assume that $[x]$ is represented by the Kasparov module*

$$x = (\psi_1\hat{\otimes}1, (2q-1)\hat{\otimes}\varepsilon) \in \mathbb{E}\big(D, (A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1\big)$$

where $\psi_1\hat{\otimes}1 : D \to \mathcal{L}(E_1)$, with $E_1 = (A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1$. By [3, Proposition 17.4.3] we may assume that $q = q^*$ and that $\|q\| \leq 1$.

Let $[y]$ be a class in $KK\big((A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1, B\big) \cong KK^1(A \otimes \mathcal{K}, B)$. See Lemma 7.6. Suppose that $[y]$ is represented by the module

$$y = (\psi_2, V_2) \in \mathbb{E}\big((A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1, B\big)$$

with $\psi_2 : (A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1 \to \mathcal{L}(E_2)$.

Set $E = E_1 \hat{\otimes}_{\psi_2} E_2$ and $\psi = (\psi_1 \hat{\otimes} 1) \hat{\otimes} 1$. Let $F \in \mathcal{L}(E)$ be a V_2-connection for E_1. Define

$$V = -\big(\cos(\pi q) \hat{\otimes} \varepsilon\big) \hat{\otimes} 1 + \Big(\big(\sin(\pi q) \hat{\otimes} 1\big) \hat{\otimes} 1\Big) F \in \mathcal{L}(E).$$

Suppose that

$$\Big[\big(\cos(\pi q) \hat{\otimes} \varepsilon\big) \hat{\otimes} 1, \psi(d)\Big] \in \mathcal{K}(E)$$

for all $d \in D$. Then (ψ, V) is a Kasparov D-B-module which is operator homotopic to the Kasparov product of x and y. That is $[\psi, V] = [x] \hat{\otimes}_{A \otimes \mathcal{K}} [y]$.

Proof: Let $d \in D$. Remark that $\psi_1(d)(q^2 - q) \in A \otimes \mathcal{K}$, thus modulo $A \otimes \mathcal{K}$ we have

$$\psi_1(d) \cos(\pi q) = \psi_1(d) \sum_{k=1}^{\infty} (-1)^k \frac{(\pi q)^{2k}}{(2k)!} + \psi_1(d)$$

$$\sim \psi_1(d) \sum_{k=1}^{\infty} (-1)^k \frac{\pi^{2k}}{(2k)!} q + \psi_1(d)$$

$$= \psi_1(d)\big(\cos(\pi)q - q + 1\big)$$

$$= -\psi_1(d)(2q - 1)$$

for all $d \in D$. It follows that x is a compact pertubation of

$$(\psi_1 \hat{\otimes} 1, -\cos(\pi q) \hat{\otimes} \varepsilon) \in \mathbb{E}\big(D, (A \otimes \mathcal{K}) \hat{\otimes} \mathbb{C}_1\big)$$

so they determine the same class in $KK\big(D, (A \otimes \mathcal{K}) \hat{\otimes} \mathbb{C}_1\big)$.

By assumption the last module fulfils the conditions of Theorem 7.5 so the theorem is proved if

$$\Big(1 - \big(\cos(\pi q) \hat{\otimes} \varepsilon\big)^2\Big)^{1/2} = \sin(\pi q) \hat{\otimes} 1$$

but this is clear since we supposed that $\|q\| \leq 1$ and $q = q^*$ so $\mathrm{Sp}(q) \subseteq [-1, 1]$, a fact which yields the positivity of $\sin(\pi q) \hat{\otimes} 1$. $\square$

Theorem 7.8 *Suppose that A and B are ungraded C^*-algebras, with A separable and B σ-unital. Let $[u] \in K_1(A \otimes \mathcal{K})$. The isomorphism from Lemma 7.3 sends $[u]$ to a class $[\mathcal{M}_{\mathbb{C}}, q]^1 \in KK^1(\mathbb{C}, A \otimes \mathcal{K})$ where $q = q^*$ and $\|q\| \leq 1$. In particular $[u] = [\exp(2\pi i q)]$ so without loss of generality we can assume that $u = \exp(2\pi i q)$. Let $y = [\psi, p]^1 \in KK^1(A \otimes \mathcal{K}, B \otimes \mathcal{K})$ and assume that $\psi : A \otimes \mathcal{K} \to \mathcal{L}(B \otimes \mathcal{K})$ is non-degenerate. Then the product $[u] \hat{\otimes}_{A \otimes \mathcal{K}} y$ is equal to*

$$\partial\Big[\pi\big(p\psi(u)p + (1 - p)\big)\Big] \in K_0(B \otimes \mathcal{K})$$

where ψ is extended to $(A \otimes \mathcal{K})^+$ and $\partial : K_1(\mathcal{C}(B \otimes \mathcal{K})) \to K_0(B \otimes \mathcal{K})$ is the boundary map.

Proof: Applying the isomorphism

$$\varphi : KK^1(A \otimes \mathcal{K}, B \otimes \mathcal{K}) \to KK((A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1, B \otimes \mathcal{K})$$

from Lemma 7.6 to y we get

$$\varphi y = \varphi[\psi, p]^1 = [\sigma, V_2]$$

with $V_2 = \begin{pmatrix} 0 & 2p-1 \\ 2p-1 & 0 \end{pmatrix}$ and $\sigma : (A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1 \to \mathcal{L}((B \otimes \mathcal{K}) \oplus (B \otimes \mathcal{K}))$ given by

$$\sigma(a, -a) = \begin{pmatrix} 0 & -i\psi(a) \\ i\psi(a) & 0 \end{pmatrix} \quad \text{and} \quad \sigma(a, a) = \begin{pmatrix} \psi(a) & 0 \\ 0 & \psi(a) \end{pmatrix}$$

which thus canonically represents y in $KK((A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1, B \otimes \mathcal{K})$.

Recall that $[\mathcal{M}_{\mathbb{C}}, q]^1 \in KK^1(\mathbb{C}, A \otimes \mathcal{K})$ is notation for the class $[x] \in KK(\mathbb{C}, (A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1)$ represented by the module

$$x = (\mathcal{M}_{\mathbb{C}}, (2q-1)\hat{\otimes}\varepsilon) \in \mathbb{E}(\mathbb{C}, (A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1).$$

We are now in position to form the product $z = [u]\hat{\otimes}_{A \otimes \mathcal{K}} y = [x]\hat{\otimes}_{A \otimes \mathcal{K}} \varphi y$. Set $E_1 = (A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1$, $E_2 = (B \otimes \mathcal{K}) \oplus (B \otimes \mathcal{K})$ and $E = E_1\hat{\otimes}_\sigma E_2$. Recall that the grading on $(B \otimes \mathcal{K}) \oplus (B \otimes \mathcal{K}) = E_2$ is given by the grading operator $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since ψ is assumed to be non-degenerate, σ is also non-degenerate and there is an even unitary isomorphism

$$w : E_1\hat{\otimes}_\sigma E_2 = ((A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1)\hat{\otimes}_\sigma((B \otimes \mathcal{K}) \oplus (B \otimes \mathcal{K}))$$
$$\to (B \otimes \mathcal{K}) \oplus (B \otimes \mathcal{K}) = E_2$$

given by

$$w(x_1\hat{\otimes}_\sigma x_2) = \sigma(x_1)x_2 \qquad x_1 \in E_1, x_2 \in E_2.$$

Let $x \in E_1$ be homogeneous. Clearly $wT_x = \sigma(x)$ so

$$wT_x V_2 - (-1)^{\partial x} V_2 wT_x = [\sigma(x), V_2] \in \mathcal{K}(E_2).$$

Thus $w^* V_2 w \in E$ is a V_2-connection for E_1. By Theorem 7.7 we can represent the product z by the module $(\mathcal{M}_{\mathbb{C}}, V)$ where

$$V = -(\cos(\pi q)\hat{\otimes}\varepsilon)\hat{\otimes}1 + ((\sin(\pi q)\hat{\otimes}1)\hat{\otimes}1)w^* V_2 w \in \mathcal{L}(E).$$

But $(\mathcal{M}_{\mathbb{C}}, V) \sim_u (\mathcal{M}_{\mathbb{C}}, wVw^*)$ so actually

$$z = [\mathcal{M}_{\mathbb{C}}, wVw^*] \in KK(\mathbb{C}, B \otimes \mathcal{K})$$

where

$$
\begin{aligned}
wVw^* &= -\sigma\big(\cos(\pi q)\hat{\otimes}\varepsilon\big) + \sigma\big(\sin(\pi q)\hat{\otimes}1\big)V_2 \\
&= -\begin{pmatrix} 0 & -i\psi\big(\cos(\pi q)\big) \\ i\psi\big(\cos(\pi q)\big) & 0 \end{pmatrix} + \begin{pmatrix} \psi\big(\sin(\pi q)\big) & 0 \\ 0 & \psi\big(\sin(\pi q)\big) \end{pmatrix}V_2 \\
&= \begin{pmatrix} 0 & i\psi\big(\cos(\pi q)\big) + \psi\big(\sin(\pi q)\big)(2p-1) \\ -i\psi\big(\cos(\pi q)\big) + \psi\big(\sin(\pi q)\big)(2p-1) & 0 \end{pmatrix} \\
&\in \mathcal{L}\big((B \otimes \mathcal{K}) \oplus (B \otimes \mathcal{K})\big).
\end{aligned}
$$

Here $\sigma : \mathcal{L}\big((A \otimes \mathcal{K})\hat{\otimes}\mathbb{C}_1\big) \to \mathcal{L}(B \otimes \mathcal{K})$ and $\psi : \mathcal{L}(A \otimes \mathcal{K}) \to \mathcal{L}(B \otimes \mathcal{K})$ denotes the extensions as in [16, Proposition 2.1].

Applying the isomorphism $KK(\mathbb{C}, B \otimes \mathcal{K}) \cong K_0(B \otimes \mathcal{K})$ from Lemma 7.3 we get that the product is nothing but the element

$$\partial\Big[\pi\big(-i\psi\big(\cos(\pi q)\big) + \psi\big(\sin(\pi q)\big)(2p-1)\big)\Big] \in K_0(B \otimes \mathcal{K}).$$

Set $v = i\exp(i\pi q) = i\cos(\pi q) - \sin(\pi q)$. The element v is a unitary in $\mathcal{L}(A \otimes \mathcal{K})$ and thus homotopic to 1 so

$$
\begin{aligned}
\partial\Big[\pi\big(-i\psi\big(\cos(\pi q)\big) + \psi\big(\sin(\pi q)\big)(2p-1)\big)\Big] = \\
\partial\Big[\pi\big(-i\psi\big(v\cos(\pi q)\big) + \psi\big(v\sin(\pi q)\big)(2p-1)\big)\Big].
\end{aligned}
$$

Furthermore, with the same argument as in the proof of Theorem 7.7, we have

$$\pi\big(\cos(\pi q)\cos(\pi q)\big) = \pi\big((1-2q)(1-2q)\big) = \pi(1)$$

so $\cos^2(\pi q) - 1 \in A \otimes \mathcal{K}$. Moreover $\sin(\pi q) \geq 0$ since $q = q^*$ and $\|q\| \leq 1$, so $\sin(\pi q) = \big(1 - \cos^2(\pi q)\big)^{1/2} \in A \otimes \mathcal{K}$. We thus have $v\cos(\pi q) \in (A \otimes \mathcal{K})^+$. By assumption $\psi(a)p - p\psi(a) \in B \otimes \mathcal{K}$ for all $a \in (A \otimes \mathcal{K})^+$ so

$$
\begin{aligned}
\pi\Big(-i\psi\big(v\cos(\pi q)\big) &+ \psi\big(v\sin(\pi q)\big)(2p-1)\Big) \\
&= \pi\Big(-ip\psi\big(v\cos(\pi q)\big)p - i(1-p)\psi\big(v\cos(\pi q)\big)(1-p) \\
&\quad + p\psi\big(v\sin(\pi q)\big)p - (1-p)\psi\big(v\sin(\pi q)\big)(1-p)\Big) \\
&= \pi\Big(p\psi(-v^2)p + (1-p)\psi\big(v[-i\cos(\pi q) - \sin(\pi q)]\big)(1-p)\Big) \\
&= \pi\big(p\psi(u)p + (1-p)\big).
\end{aligned}
$$

That is

$$z = \partial\Big[\pi\big(-i\,\psi\big(\cos(\pi q)\big) + \psi\big(\sin(\pi q)\big)(2p-1)\big)\Big] = \partial\Big[\pi\big(p\psi(u)p + (1-p)\big)\Big]$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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