INTEGRATION ON LOCALLY COMPACT NONCOMMUTATIVE SPACES

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Abstract. We present an ab initio approach to integration theory for nonunital spectral triples. This is done without reference to local units and in the full generality of semifinite noncommutative geometry. The main result is an equality between the Dixmier trace and generalised residue of the zeta function and heat kernel of suitable operators. We also examine definitions for integrable bounded elements of a spectral triple based on zeta function, heat kernel and Dixmier trace techniques. We show that zeta functions and heat kernels yield equivalent notions of integrability, which imply Dixmier traceability.

1. introduction

It is known by Connes' trace theorem, [13], that for a *compact* Riemannian manifold M of dimension p, there is an intimate connection between the residue of the zeta function of the Laplacian at its first singularity, the Dixmier trace, and (via the Wodzicki residue) integration theory of functions with respect to the standard measure. More precisely, we let Δ be a Laplace type operator acting on sections of a vector bundle and $f \in C^\infty(M)$ be a smooth function acting by multiplication on smooth sections. Then Δ extends to an essentially self-adjoint operator with compact resolvent on L^2-sections and the multiplication operator M_f extends to a bounded operator. Then if $G := (1+\Delta)^{-p/2}$, the operator $M_f G$ has singular numbers $\mu_n = O(\frac{1}{n})$, and the Dixmier trace of this operator is equal to (up to a constant depending only on the dimension p) the integral of f over M with respect to the measure in M given by the Riemannian volume form. The residue of the zeta function $\mathrm{Tr}(M_f G^s)$ at $s = 1$ and the limit of $t^{-\dim(M)/2}\,\mathrm{Tr}(M_f e^{-t\Delta})$ when $t \to 0^+$, also coincide (up to a constant) with the integral of f.

More recently in [25, 26] it has been shown that for $f \in L^1(M)$ we can define the zeta function $\mathrm{Tr}(G^{s/2} M_f G^{s/2})$ for $s > 1$ and that the (generalised or ω) residue at $s = 1$ is equal to the integral of f over M, *while the Dixmier trace of $G^{1/2} M_f G^{1/2}$ does not exist for all f in $L^1(M)$*. Related results which indicate the difficulties of noncommutative integration are that the Dixmier trace of $M_f G$ exists if and only if $f \in L^2(M)$. In summary, the generalised residue of the zeta function exists in greater generality than the Dixmier trace, and recovers integration on the manifold. When the Dixmier trace also exists, it agrees with the zeta residue (up to a constant).

When the manifold is not compact the situation is much less clear. Moreover the analogous noncommutative integration theory (for nonunital pre-C^*-algebras) has not been developed from first principles, rather, an assumption is made about the existence of a system of local units for the algebra, [36, 37]. This is a plausible assumption given that there is a local structure on a noncompact manifold M and while it allows one to prove results analogous to the unital case [20, 36, 37], such as relating the zeta function, Dixmier trace and Wodzicki residue in that setting it is clearly an unsatisfactory basis for a general framework.

We rectify this situation in this paper motivated by the fundamental question: *what is the appropriate noncommutative integral for nonunital spectral triples?* We establish here a theory of integration, integrability and spectral dimension in the nonunital case and show that in this framework we are able to relate the zeta residue to the Dixmier trace. Importantly, we will couch our results, and choose proofs, that go through for the general framework of semifinite spectral triples.

Now we summarise the main results of this paper. Let $\mathcal{N}$ be a semifinite von Neumann algebra with fixed faithful normal semifinite trace τ. In [9] we introduced a family of ideals that we denoted by $\mathcal{Z}_p := \mathcal{Z}_p(\mathcal{N}, \tau)$. These ideals are naturally defined by the asymptotics of the zeta function $s \mapsto \tau(T^s)$ as s converges from above to the infimum of values for which this trace is finite. As a consequence of our analysis we found that $\mathcal{Z}_1$ coincides with the (dual to the) Macaev ideal for which it has become standard in noncommutative geometry to use the notation $\mathcal{L}^{1,\infty}$. In this paper we will use the notation $\mathcal{Z}_p$ even in the case $p = 1$ for consistency.

Now, the problem that arises for noncommutative and nonunital integration is that we are dealing with products of operators, neither of which individually lies in $\mathcal{Z}_1$ but whose product does lie in $\mathcal{Z}_1$. Examples show that a similar phenomenon persists in the case of noncommutative algebras [20,31,32] and general semifinite traces. The difficulties posed by this situation for the analysis of Schrödinger operators have been explained in detail in [40, Chapter $f(X)g(-i\nabla)$].

Given G, a positive and injective element of $\mathcal{N}$, (in the spectral triple situation think of $G = (1+\mathcal{D}^2)^{-p/2}$ where p is the spectral dimension) define the following (partially defined) seminorm on $\mathcal{N}$

$$\|a\|_\zeta := \sup_{1\leq s\leq 2} \sqrt{s-1}\,\tau(aG^s a^*)^{1/2},$$

and the algebra $B_\zeta(G)$ of operators $a \in \mathcal{N}$ such that the norm $||a|| + \|a\|_\zeta + \|a^*\|_\zeta$ is finite. The choice of this algebra is determined by the generalisations of the results of [7, 9] that we obtain in Sections 4 and 5. Note that the definition of $B_\zeta(G)$ means we use implicitly a Hilbert algebra framework (described in Section 4), that is, our approach yields a noncommutative L^2 theory.

Our main result, the most general result we can establish relating the zeta residue and the Dixmier trace in the nonunital setting, is Theorem 4.13, proved in Section 4. It resolves a question that has been under investigation for nearly ten years. We state in this introduction a special case whose interest in the standard situation, where $\mathcal{N}$ is the algebra of bounded operators on a Hilbert space $\mathcal{H}$, was first observed by Alain Connes in [14].

Convergence Theorem. *Assume that $b \in B_\zeta(G)$ is self adjoint and non-negative with $[G, b] \in \mathcal{Z}_1^0$ (here $\mathcal{Z}_1^0$ is the closure of the trace-class operators in the norm of $\mathcal{Z}_1$). Then for any $\varepsilon > 0$, $b^{1+\varepsilon}G \in \mathcal{Z}_1$, and if $\lim_{s\to 1^+}(s-1)\tau(b^{1/2}G^s b^{1/2+\varepsilon})$ exists, then it is equal to any Dixmier trace $\tau_\omega(b^{1+\varepsilon}G)$ where we choose the generalised limit ω to satisfy the invariance conditions of [9, Theorem 4.11].*

In the unital case one may set $\varepsilon = 0$ to obtain as a corollary the main theorem in [7]. The subtlety of the nonunital case stems firstly from the fact that we need the 'symmetric' version of the limit and secondly that putting ε to zero seems impossible in general.

Other non-unital analogues of results in [7, 9] are also proved here in Sections 4 and 5. For example, for dilation invariant $\omega \in L^\infty(\mathbb{R})$ and for bounded operators $a \in \mathcal{N}$, the existence of the generalised ζ-residue $\omega - \lim_{r\to\infty} \frac{1}{r}\tau(a^*G^{1+\frac{1}{r}}a)$ implies that $a^*Ga \in \mathcal{Z}_1$ and that this limit is given by a Dixmier trace. A range of similar results shows the compatibility of the zeta residue with the ideals $\mathcal{Z}_p$ and $\mathcal{L}^{p,\infty}$. Another direction we pursue is to show that the heat kernel asymptotics imply precisely the same data and exist in the same generality as the generalised zeta residue.

There are a number of applications of our results although we will only take a few steps in our investigation of them in this short article. First we give, in Section 7, a definition of non-unital spectral triple encompassing those studied in [31, 32]. The algebras studied there, graph algebras and higher rank graph algebras, do have a quasi-local structure, namely a dense subalgebra $\mathcal{A}$ with local units. The local index formula reduces in these examples to computing a residue of a single zeta function $\zeta_b(s) = \tau(b(1+\mathcal{D}^2)^{-s/2}), \quad b \in \mathcal{A}$. The operator $\mathcal{D}$ satisfies $b(1+\mathcal{D}^2)^{-p/2} \in \mathcal{Z}_1$ for all $b \in \mathcal{A}$ and some positive integer p. The results of this paper enable us to use the Dixmier trace or heat kernel (as in Proposition 4.12) to compute this residue. Our results also generalise this index formula because we no longer need to work with $b \in \mathcal{A}$ but can allow more general $b \in B_\zeta((1+\mathcal{D}^2)^{-p/2})$.

Instances where this kind of freedom is needed include the non-unital version of the index theorem of Phillips-Raeburn [30], which will be the subject of a separate investigation. We remark that the careful analysis of the relationship of symmetric and asymmetric limits in our paper is critical for the application to [30].

The special case of the Convergence Theorem that is used for smoothly summable nonunital spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ (and in particular for [30]) is where $\varepsilon = 1$. In this instance $\mathcal{A} \subset (B_\zeta)^2$ and the convergence theorem that gives the index computation can be summarised as smoothness + summability implies that for $a \in \mathcal{A}$ non-negative, $\lim_{s\to 1^+}(s-1)\tau(a^{1/2}G^s a^{1/2}) = \tau_\omega(aG)$.

In these previous examples the index is given by the Hochschild class of the Chern character. Using the results of this paper one may also study the Hochschild class of the Chern character in the nonunital case along the lines of [8]. More importantly, this paper underlies the principle objective, the nonunital local index formula, which we establish in [6].

A further application is to a version of Connes trace theorem for noncompact manifolds along the lines of the main theorem of [27]. The idea is to use the methods of this paper in relating the residues of zeta functions to a Wodzicki residue formula via the Dixmier trace and this will be discussed elsewhere (see our last example for an indication of the class of functions to which the trace formula would apply).

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2. The ideals $\mathcal{Z}_p$

We fix a semifinite von Neumann algebra $\mathcal{N}$ acting on a separable Hilbert space $\mathcal{H}$. Fix also a faithful, semifinite, normal trace $\tau : \mathcal{N} \to \mathbb{C}$. The zeta function of a positive τ-compact operator T is given by $\zeta(s) = \tau(T^s)$ for $s \in \mathbb{C}$ with positive real part, on the assumption that there exists some $s_0 > 0$ for which the trace is finite. Note that it is then true that $|\tau(T^s)| < \infty$ for all $\Re(s) \geq s_0$. Thus, the restriction of $s \in \mathbb{C}$ to positive numbers is sufficient for the sequel.

Let us consider the space $\mathcal{Z}_1$ introduced in [9]:

$$\mathcal{Z}_1 := \mathcal{Z}_1(\mathcal{N}, \tau) := \big\{T \in \mathcal{N} :\ \|T\|_{\mathcal{Z}_1} = \limsup_{p \searrow 1} (p-1)\tau(|T|^p) < \infty\big\}.$$

Note that $||.||_{\mathcal{Z}_1}$ is only a seminorm. The next equality is easy to see (for details consult [9])

$$\|T\|_{\mathcal{Z}_1} = \limsup_{p \searrow 1} (p-1)\Big(\int_0^\infty \mu_t(|T|)^p dt\Big)^{1/p} = \limsup_{p \searrow 1} (p-1)\|T\|_p.$$

(We use the notation $\mathcal{L}^p$ for the Schatten ideals in $(\mathcal{N}, \tau)$ and $\|\cdot\|_p$ for the Schatten norms.) More generally, we define for $p \geq 1$ the spaces $\mathcal{Z}_p$, the p-convexifications of $\mathcal{Z}_1$ [24], by

$$\mathcal{Z}_p(\mathcal{N}, \tau) = \big\{T \in \mathcal{N} :\ \|T\|_{\mathcal{Z}_p} = \limsup_{q \searrow p} \big((q-p)\tau(|T|^q)\big)^{1/q} < \infty\big\}.$$

The ideal of compact operators whose partial sums of singular values are logarithmically divergent arises naturally in geometric analysis. This ideal (in the setting of general semifinite von Neumann algebras) may be described in terms of noncommutative Marcinkiewicz spaces and we refer to [9, 10, 27] for an exposition of relevant parts of this theory. Here, we set

$$M_{1,\infty}(\mathcal{N}, \tau) := \big\{T \in \mathcal{N} :\ \|T\|_{1,\infty} := \sup_{0<t<\infty} \log(1+t)^{-1} \int_0^t \mu_s(T) ds < \infty\big\}.$$

We will usually take $(\mathcal{N}, \tau)$ as fixed, and write $M_{1,\infty}$ instead of $M_{1,\infty}(\mathcal{N}, \tau)$, as this will cause no confusion. Similar comments apply to the notation for other ideals.

The Banach space $(M_{1,\infty}, \|\cdot\|_{1,\infty})$ was probably first considered by Matsaev [29]. It may be viewed as a noncommutative analogue of a Sargent (sequence) space, see [38]. In noncommutative geometry it has become customary to use the notation $\mathcal{L}^{1,\infty}$ to denote the ideal $M_{1,\infty}$. However we will avoid the $\mathcal{L}^{1,\infty}$ notation as it clashes with the well-established notation of quasi-normed weak L_1-spaces. For a fuller treatment of the history of the space $M_{1,\infty}$ and additional references, we refer the interested reader to the recent paper [33] by Pietsch.

More generally, we let $M_{p,\infty}$, $p \geq 1$, denote the p-convexification of the space $M_{1,\infty}$, defined by

$$M_{p,\infty}(\mathcal{N}, \tau) := \big\{T \in \mathcal{N} :\ \|T\|_{p,\infty}^p := \sup_{0<t<\infty} \log(1+t)^{-1} \int_0^t \mu_s(|T|^p) ds < \infty\big\}. \tag{2.1}$$

In our present context, it is important to observe that it follows from [9, Theorem 4.5] that the sets $M_{1,\infty}$ and $\mathcal{Z}_1$ coincide and that $\|T\|_0 \leq e\|G\|_{Z_1}$ and $\|T\|_{Z_1} \leq \|T\|_{1,\infty}$, where the seminorm $\|\cdot\|_0$ is the distance in the norm $\|\cdot\|_{1,\infty}$ to the subspace $M^0_{1,\infty}$ of $M_{1,\infty}$ formed by the $\|\cdot\|_{1,\infty}$-closure of the trace ideal $\mathcal{L}^1 \subset M_{1,\infty}$. To be consistent with our $\mathcal{Z}_p$ notation, we also denote the latter ideal as $\mathcal{Z}^0_1$. We also call $\mathcal{Z}^0_p$, $p > 1$, the norm closure of $\mathcal{L}^p$ in $\mathcal{Z}_p$. Of course, the spaces $\mathcal{Z}_p$ and $M_{p,\infty}$ coincide. We also stress that $\mathcal{L}^{p,\infty}$, $p > 1$, the collection of

τ-compact operators for which $\mu_t(T) = O(t^{-1/p})$, are strictly included in $\mathcal{Z}_p$, [9]. Moreover, a careful inspection of its proof, gives the following strengthening of Theorem 4.5 in [9]:

Theorem 2.1. *The norm $\|\cdot\|_{1,\infty}$ of the Marcinkiewicz space $M_{1,\infty}$ is equivalent to the ζ-norm:*

$$\sup_{p>1} (p-1)\|T\|_p\,, \quad T \in \mathcal{N}.$$

Another important feature of the ideals $\mathcal{Z}_1 = M_{1,\infty}$ is that they support singular traces. Let $M_{1,\infty}(\mathcal{H})$ denote $M_{1,\infty}$ when $(\mathcal{N}, \tau)$ is given by the algebra of all bounded linear operators equipped with standard trace. In [15], J. Dixmier constructed a non-normal semifinite trace living on the ideal $M_{1,\infty}(\mathcal{H})$ using the weight

$$\mathrm{Tr}_\omega(T) := \omega\Big(\Big\{\frac{1}{\log(1+k)}\sum_{j=1}^{k}\mu_j(T)\Big\}_{k=1}^{\infty}\Big)\ ,\ T \geq 0,$$

associated to a translation and dilation invariant state ω on $\ell^\infty(\mathbb{N})$. The seminorm $\|\cdot\|_{\mathcal{Z}_1}$ and all Dixmier traces Tr_ω vanish on $M^0_{1,\infty}(\mathcal{H})$ and this provides a first (albeit tenuous) connection between Dixmier traces and zeta functions. This connection runs much deeper however, and will be explained further in various parts of the present manuscript. To assist the reader we clarify the construction of the ideals $\mathcal{Z}_p$ in term of complex interpolation in the setting of Banach-lattices. For the definition of the two functors of complex interpolation $\bar{A}_{[\theta]}$ (the first method) and $\bar{A}^{[\theta]}$ (the second method) defined for an arbitrary Banach couple $\bar{A} = (A_0, A_1)$ we refer to [2, pp. 88-90], [28]. In general, the two spaces $\bar{A}_{[\theta]}$ and $\bar{A}^{[\theta]}$ are not equal, but always $\bar{A}_{[\theta]} \subset \bar{A}^{[\theta]}$ with a norm one injection.

Proposition 2.2. *Let $p \in (1,\infty)$. Then $\mathcal{Z}_p$ coincides with the second complex interpolation space $(\mathcal{Z}_1, \mathcal{N})^{[1-1/p]}$, associated to the Banach couple $(\mathcal{Z}_1, \mathcal{N})$.*

Proof. It follows from the results given in [16,17] that the space $(M_{1,\infty}(\mathcal{N},\tau), \mathcal{N})^{[s]}$, $s \in [0,1]$, coincides with the (fully symmetric) operator space generated by the space $(M_{1,\infty}(\mathbb{R}_+^*), L^\infty(\mathbb{R}_+^*))^{[s]}$ on the von Neumann algebra $\mathcal{N}$. From [28, Theorem 1], and the fact that $M_{1,\infty}(\mathbb{R}_+^*)$ and $L^\infty(\mathbb{R}_+^*)$ have the Fatou property, we deduce that $(M_{1,\infty}(\mathbb{R}_+^*), L^\infty(\mathbb{R}_+^*))^{[s]}$ coincides with the closure in $M_{1,\infty}(\mathbb{R}_+^*) + L^\infty(\mathbb{R}_+^*)$ of the closed unit ball of $M_{1,\infty}(\mathbb{R}_+^*)^{1-s}L^\infty(\mathbb{R}_+^*)^s = M_{1,\infty}(\mathbb{R}_+^*)^{1-s}$. Combining this result with the fact that the unit ball of $M_{1,\infty}(\mathbb{R}_+^*)^{1-s}$ is closed with respect to convergence in measure and hence also with respect to the norm convergence in $M_{1,\infty}(\mathbb{R}_+^*) + L^\infty(\mathbb{R}_+^*)$, we arrive at the equality $(M_{1,\infty}(\mathbb{R}_+^*), L^\infty(\mathbb{R}_+^*))^{[s]} = M_{1,\infty}(\mathbb{R}_+^*)^{1-s}$. Taking into account the equality above, we see that $(M_{1,\infty}(\mathcal{N},\tau), \mathcal{N})^{[s]} = M_{1,\infty}(\mathcal{N},\tau)^{1-s}$. It remains to observe that the space $M(\psi)(\mathcal{N},\tau)^{1-s}$ is precisely the space $\mathcal{Z}_{1/(1-s)}(\mathcal{N},\tau)$ described in [9]. $\square$

We introduce the ideal $\mathcal{Z}_1$ using the heat kernel as this is useful in discussing the similarities and differences between $\mathcal{Z}_1$ and the weak $\mathcal{L}^1$ ideal, denoted $\mathcal{L}_{1,w}$, defined by

$$\mathcal{L}_{1,w} := \mathcal{L}_{1,w}(\mathcal{N},\tau) := \big\{T \in \mathcal{N} : \exists C > 0 \text{ such that } \mu_t(T) \leq C/t\big\}. \tag{2.2}$$

A quasi-norm on $\mathcal{L}_{1,w}$ is given by $\|T\|_{1,w} := \inf\big\{C :\ \mu_t(T) \leq C/t\big\}$. Clearly $\mathcal{L}_{1,w} \subseteq \mathcal{Z}_1$, and in fact it was shown in [9] that the inclusion is strict.

The (multiplicative) Cesàro mean, M, on $\mathbb{R}_+^*$, is defined on $f : \mathbb{R}_+^* \to [0, \infty)$ by

$$\big(Mf\big)(\lambda) := \frac{1}{\log(\lambda)} \int_1^\lambda f(t) \frac{dt}{t}.$$

By the results of [9], $T \in \mathcal{Z}_1$ if and only if $Mf_T \in L^\infty(\mathbb{R}_+^*)$ and $T \in \mathcal{L}_{1,w}$ if and only if $f_T \in L^\infty(\mathbb{R}_+^*)$, with $f_T(\lambda) := \lambda^{-1}\tau(e^{-\lambda^{-1}|T|^{-1}})$. A natural question is whether Mf_T can be unbounded while MMf_T is bounded (and so on)? The following answers it negatively, and shows that there are no higher ideals defined by the condition $M^{\circ k} f_T \in L^\infty(\mathbb{R}_+^*)$, $k \geq 2$.

Lemma 2.3. *Let h be a positive measurable function on $(0, \infty)$ such that MMh is bounded. Then, Mh is bounded too.*

Proof. Using the definition and integrating by parts, we deduce

$$\big(MMh\big)(x) = \frac{1}{\log(x)} \int_1^x \log\left(\frac{\log(x)}{\log(\lambda)}\right) h(\lambda) \frac{d\lambda}{\lambda}.$$

Now for $x \geq 1$, we obtain

$$(MMh)(x^2) \geq \frac{1}{2\log(x)} \int_1^x \log\Big(\frac{2\log(x)}{\log(\lambda)}\Big) h(\lambda) \frac{d\lambda}{\lambda} \geq \frac{\log(2)}{2\log(x)} \int_1^x h(\lambda) \frac{d\lambda}{\lambda} = \frac{\log(2)}{2} (Mh)(x).$$

Hence $\big(MMh\big)(x^2) \geq \frac{\log(2)}{2} \big(Mh\big)(x)$ and if MMh is bounded, so too is Mh. □

Thus we have the curious situation that if h is unbounded, either one application of the Cesàro mean M will produce a bounded function, or if not, then successive applications of the Cesàro mean will never produce a bounded function. Moreover, the boundedness of f_T singles out the ideal that arises in practise, namely $\mathcal{L}_{1,w}$. The use of Cesàro invariant functionals to produce Dixmier traces has enlarged the attention of noncommutative integration theory to $\mathcal{Z}_1$, despite the fact that $\mathcal{Z}_1$ is unnatural from the point of view of most applications.

3. Banach algebras for nonunital integration

Much of the recent motivation for noncommutative integration theories comes from spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \tau)$ and their use in index theory. In Section 6, we review the definitions, but here just remind the reader that the commonest situations are when using the self-adjoint unbounded operator $\mathcal{D}$, the trace τ and suitable s, p, $t \in \mathbb{R}$, one (or more) of the maps

$$a \mapsto \tau(a(1 + \mathcal{D}^2)^{-s/2}), \quad a \mapsto \tau_\omega(a(1 + \mathcal{D}^2)^{-p/2}), \quad a \mapsto \tau(ae^{-t\mathcal{D}^2}),$$

provides a sensible functional on the algebra $\mathcal{A}$. Here, τ_ω is a semifinite Dixmier trace for $(\mathcal{N}, \tau)$. Namely, it is the linear extension of the map

$$T \in (\mathcal{Z}_1)_+ \longmapsto \omega\Big(\Big[t \longmapsto \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds\Big]\Big),$$

where $\mu_s(T)$ is the generalized singular number (function) of T (see [18]) and ω is a dilation invariant state on $L^\infty(\mathbb{R}^+)$. As we will explain, there are close links between these various situations.

Observe that for $p \geq 1$, the operator $(1+\mathcal{D}^2)^{-p/2}$ is positive, injective and has norm ≤ 1. These are the main properties we use and in the following text consider an operator G with these properties. In Section 4 we will also need some conditions on commutators $[G,a]$ between G and algebra elements a. In Section 6 we will verify these conditions for suitable spectral triples and $G=(1+\mathcal{D}^2)^{-p/2}$.

3.1. **Preliminaries.** With $(\mathcal{N},\tau)$ as above let G be a positive and injective operator in $\mathcal{N}$. Without loss of generality, we assume that $G \leq 1$. We make neither τ-compactness nor summability hypotheses on G itself. Instead we consider that a natural condition of integrability for $a \in \mathcal{N}$, relative to G, would be to ask that

$$aG^s \in \mathcal{L}^1\,, \quad \forall s>1. \tag{3.1}$$

If we consider $a \geq 0$ then the condition (3.1) is equivalent to

$$G^s a \in \mathcal{L}^1\,, \quad \forall s>1, \tag{3.2}$$

since $\mathcal{L}^1$ is a $*$-ideal. For $a \geq 0$ satisfying these equivalent conditions, it is shown by Bikchentaev in [3], that a also satisfies the equivalent conditions

$$G^{s/2}aG^{s/2} \in \mathcal{L}^1 \qquad \Longleftrightarrow \qquad a^{1/2}G^s a^{1/2} \in \mathcal{L}^1. \tag{3.3}$$

That is (3.1)⇔(3.2)⇒(3.3), so integrability of $a \geq 0$ implies square integrability of $a^{1/2}$. The implication (3.3)⇒(3.2) fails by examples in [4]. There is also a counterexample if we replace $\mathcal{L}^1$ with $\mathcal{Z}_1$ [4]. This introduces a serious difficulty because the symmetric forms in (3.3), are what we need in applications [30]. In fact functionals on $\mathcal{N}$ of the form $\mathcal{N}_+ \ni a \mapsto \tau(G^{s/2}aG^{s/2}),\ s>1$, are weights. When we determine the domain, it will not be an ideal, but rather a Hilbert algebra. Here we will avoid Hilbert algebras as we need only a Banach algebra completion. Nevertheless we make use of some Hilbert algebra ideas and this explains in part the 'square summable' flavor of some of our hypotheses and results.

3.2. **Banach Algebras from the zeta-function and heat-kernel.**

Definition 3.1. *Given G, a positive and injective element of $\mathcal{N}$, define the two families of (possibly infinite) bilinear functionals on $\mathcal{N}$*

$$\zeta(a,b;s) := \tau\big(aG^s b\big)\,, \quad g(a,b;\lambda) := \frac{1}{\lambda}\tau\big(ae^{-\lambda^{-1}G^{-1}}b\big)\,, \quad a,b \in \mathcal{N}.$$

Then, introduce the following (partially defined) seminorms on $\mathcal{N}$

$$\|a\|_\zeta := \sup_{1\leq s\leq 2} \sqrt{s-1}\,\zeta(a^*,a;s)^{1/2}\,, \quad \|a\|_{\mathrm{HK}} := \big\|Mg(a^*,a;\cdot)\big\|_\infty^{1/2},$$

where M denotes the Cesàro mean of the multiplicative group $\mathbb{R}_+^*$.

Note that since $\langle a,b\rangle_{\zeta,s} := \zeta(a^*,b,s)$ and $\langle a,b\rangle_{\mathrm{HK},\lambda} := g(a^*,b;\lambda)$ are inner products, it follows at once that $\|\cdot\|_\zeta$ and $\|\cdot\|_{\mathrm{HK}}$ are positively homogeneous and satisfy the triangle inequality. But they are also injective maps. Indeed, if for instance, $\|a\|_\zeta = 0$, then by the faithfulness of τ, we deduce that $0 = a^*G^s a = |G^{s/2}a|^2$ and thus $G^{s/2}a = 0$ too. Then, from the injectivity of G, we get that $a = 0$. A similar result holds for $\|a\|_{\mathrm{HK}}$. This shows that $\|\cdot\|_\zeta$ and $\|\cdot\|_{\mathrm{HK}}$ are true norms, not only seminorms. The finiteness of such seminorms is closely related to the zeta function and heat kernel characterizations of the ideal $\mathcal{Z}_p$ in the unital case, [9].

Definition 3.2. *Let $B_\zeta(G)$ (respectively $B_{\rm HK}(G)$) be the normed subset of $\mathcal{N}$ relative to the norm $\|a\|_\zeta+\|a^*\|_\zeta+\|a\|$ (respectively $\|a\|_{\rm HK}+\|a^*\|_{\rm HK}+\|a\|$). When no confusion can occur, we write B_ζ and $B_{\rm HK}$ instead of $B_\zeta(G)$ and $B_{\rm HK}(G)$.*

Proposition 3.3. *The sets $B_\zeta(G)$ and $B_{\rm HK}(G)$ are Banach $*$-algebras.*

Proof of Proposition 3.3. It is clear that B_ζ and $B_{\rm HK}$ are normed linear spaces with symmetric norms. Let us first show that they are sub-multiplicative. For $f:\mathbb{R}\to\mathbb{R}$ positive, we have: $\tau\big((ab)^*f(G)ab\big)\le\|b\|^2\|a^*f(G)a\|_1=\|b\|^2\tau\big(a^*f(G)a\big)$. This clearly entails that $\|ab\|_\zeta\le\|b\|\|a\|_\zeta$ and $\|(ab)^*\|_\zeta\le\|a\|\|b^*\|_\zeta$, and thus

$$\begin{aligned}\|ab\|_\zeta+\|(ab)^*\|_\zeta+\|ab\|&\le\|b\|\|a\|_\zeta+\|a\|\|b^*\|_\zeta+\|a\|\|b\|\\&\le\big(\|a\|_\zeta+\|a^*\|_\zeta+\|a\|\big)\big(\|b\|_\zeta+\|b^*\|_\zeta+\|b\|\big).\end{aligned}$$

For the completeness, let $(T_k)_{k\ge1}$ be a Cauchy sequence in B_ζ. Then $(T_k)_{k\ge1}$ converges in norm, and so there exists $T\in\mathcal{N}$ such that $T_k\to T$ in $\mathcal{N}$. By the second triangle inequality we have $\big|\,\|T_n\|_\zeta-\|T_m\|_\zeta\,\big|\le\|T_n-T_m\|_\zeta$, so we see that the numerical sequence $(\|T_k\|_\zeta)_{k\ge1}$ possesses a limit. Now since $T_k^*\,G^s\,T_k\to T^*\,G^s\,T$, for all $s>1$ in norm, it also converges in measure, and so we may apply the Fatou Lemma, [18, Theorem 3.5 (i)], to deduce that for all $s\in(1,2]$:

$$(s-1)\tau\big(T^*\,G^s\,T\big)\le\liminf_{k\to\infty}(s-1)\tau\big(T_k^*\,G^s\,T_k\big)\le\liminf_{k\to\infty}\|T_k\|_\zeta^2=\lim_{k\to\infty}\|T_k\|_\zeta^2,$$

which entails that $\|T\|_\zeta\le\lim_{k\to\infty}\|T_k\|_\zeta$. As the same conclusion holds for T^* in place of T, we have $T\in B_\zeta$. Finally, fix $\varepsilon>0$ and choose N large enough so that $\|T_n-T_m\|_\zeta\le\varepsilon$ for all $n,m>N$. Applying the Fatou Lemma to the sequence $(T_k)_{k\ge1}$, gives $\|T-T_m\|_\zeta\le\liminf_{k\to\infty}\|T_k-T_m\|_\zeta\le\varepsilon$. Hence $T_k\to T$ in the topology of B_ζ. The arguments for $B_{\rm HK}$ are similar. □

The algebras B_ζ, $B_{\rm HK}$ need not be uniformly closed, nor weakly closed, nor be ideals (even one-sided) in $\mathcal{N}$. We now prove that these two notions of 'square integrability' in fact coincide.

Lemma 3.4. *The norms $\|\cdot\|_\zeta$ and $\|\cdot\|_{\rm HK}$ are equivalent.*

Proof. Fix any $a\in\mathcal{N}$ with $\|a\|_\zeta<\infty$. We need to show that the associated function $g(a^*,a;\cdot)$ has bounded Cesàro mean. Using Fubini's Theorem to justify the inversion of the trace and the integral, we obtain

$$\big(Mg(a^*,a;\cdot)\big)(x)=\frac{1}{\log x}\tau\big(a^*Ge^{-x^{-1}G^{-1}}a\big)-\frac{1}{\log x}\tau\big(a^*Ge^{-G^{-1}}a\big)\le\frac{1}{\log x}\tau\big(a^*Ge^{-x^{-1}G^{-1}}a\big).$$

Now, making the change of variable $x=e^{1/\varepsilon}$ (i.e. $\varepsilon=s-1$) in the previous expression, we obtain, for $0<\varepsilon<1$,

$$\big|\big(Mg(a^*,a;\cdot)\big)(e^{1/\varepsilon})\big|\le\varepsilon\tau\big(a^*Ge^{-e^{-1/\varepsilon}G^{-1}}a\big)\le\varepsilon\tau\big(a^*G^{1+\varepsilon}a\big)\big\|G^{-\varepsilon}e^{-e^{-1/\varepsilon}G^{-1}}\big\|.$$

But the function $x\mapsto x^\varepsilon e^{-e^{-1/\varepsilon}x}$ has its maximum at $x=\varepsilon e^{1/\varepsilon}$, where its value is $e\varepsilon^\varepsilon e^{-\varepsilon}\le e$. Thus, $\big|\big(Mg(a^*,a;\cdot)\big)(e^{1/\varepsilon})\big|\le e\,\varepsilon\,\tau\big(a^*G^{1+\varepsilon}a\big)=e\,\varepsilon\,\zeta(a^*,a;1+\varepsilon)$, which concludes the proof of the first inclusion.

To show that $\|\cdot\|_\zeta\le C\,\|\cdot\|_{\rm HK}$, we will use the Mellin transform. Fix any $a\in\mathcal{N}$ with finite HK-norm. We have, writing again $\varepsilon=s-1$, $G^{1+\varepsilon}=\Gamma(1+\varepsilon)^{-1}\int_0^\infty t^\varepsilon\,e^{-tG^{-1}}\,dt$. Disregarding the bounded pre-factor $\Gamma(1+\varepsilon)^{-1}$ before the integral, we first decompose the integral into two pieces: $\int_0^\infty = \int_0^1 + \int_1^\infty$, to write $a^*G^{1+\varepsilon}a$ as a sum of two operators. For the second we obtain

$$\tau\Big(\int_1^\infty t^\varepsilon\, a^* e^{-tG^{-1}} a\, dt\Big) \le \int_1^\infty t^\varepsilon\, \tau\big(a^*\, e^{-G^{-1}} a\big)\big\|e^{-(t-1)G^{-1}}\big\|\, dt = g(a^*, a; 1)\int_1^\infty t^\varepsilon\, e^{-(t-1)\|G\|^{-1}}\, dt,$$

and the latter is smaller that a constant C_1, independent of $\varepsilon \in [0,1]$. For the first term, we can exchange the trace and the integral because of the finite range of the integration. This reads,

$$\tau\Big(\int_0^1 t^\varepsilon\, a^* e^{-tG^{-1}} a\, dt\Big) = \int_0^1 t^\varepsilon\, \tau\big(a^* e^{-tG^{-1}} a\big)\, dt = \int_1^\infty \lambda^{-1-\varepsilon}\, g(a^*, a; \lambda)\, d\lambda.$$

Now, we make use of the following change of variable: $0 \le y(\lambda) := \int_1^\lambda g(a^*, a; \sigma)\, \frac{d\sigma}{\sigma}$, a monotonically increasing function of λ. Observing that $y(\lambda) = \big(Mg(a^*, a; \cdot)\big)(\lambda)\log(\lambda)$, there exists a positive constant C_2, such that $y(\lambda) \le C_2 \log(\lambda)$, and thus $\lambda^{-\varepsilon} \le e^{-\varepsilon C_2^{-1} y}$. This implies that

$$\tau\Big(\int_0^1 t^\varepsilon\, a^* e^{-tG^{-1}} a\, dt\Big) \le \int_0^\infty e^{-\varepsilon C_2^{-1} y}\, dy = \varepsilon^{-1} C_2.$$

Gathering these estimates together proves that $\varepsilon\,\tau\big(a^* G^{1+\varepsilon} a\big) \le \varepsilon C_1 + C_2$, and thus the set $\big\{(s-1)\zeta\big(a^*, a; s\big) : 1 \le s \le 2\big\}$ is bounded and so $\|a\|_\zeta < \infty$. □

Applying this result to the unital case, i.e. when G alone belongs to $\mathcal{Z}_1$, and combining it with Theorem 2.1, we obtain an interesting fact.

Corollary 3.5. *The two norms*

$$T \in \mathcal{Z}_1 \longmapsto \sup_{s>1}(s-1)\tau(|T|^s) \quad \textit{and} \quad T \in \mathcal{Z}_1 \longmapsto \sup_{\lambda>0} \frac{1}{\log(\lambda)}\int_1^\lambda \mu^{-2}\tau\big(e^{-\mu^{-1}|T|^{-1}}\big)\, d\mu,$$

are equivalent to $\|\cdot\|_{1,\infty}$.

Consider the *exponentiation semigroup* P_s, $s > 0$, acting on $L^\infty((0,\infty))$ by

$$(P_s x)(t) = x(t^s), \quad t > 0.$$

A generalised limit $\omega \in L^\infty(\mathbb{R})^*_+$ is said to be exponentiation invariant if $\omega \circ P_s = \omega$ for every $s > 0$. The existence of such generalised limits follows from an invariant form of the Hahn-Banach theorem. Moreover, it was proved in [7, Theorem 1.5] that there exists a generalised limit ω which is both exponentiation invariant and $M-$invariant.

Given an exponentiation invariant generalised limit ω, we define the dilation invariant functional $\tilde\omega$ on $\mathbb{R}^*_+$, by $\tilde\omega(f) := \omega(f \circ \log)$ for $f \in L^\infty((\mathbb{R}))$.

We use the notation $\omega - \lim_{t\to\infty} f(t)$ instead of $\omega(f)$. The next result shows that the $\tilde\omega$-residue of the zeta function $\zeta(a^*, a; \cdot)$ coincides with the ω-limit of the Cesàro mean of the heat-trace function $g(a^*, a; \cdot)$. This generalises to the non-unital setting one of the main theorems of [7].

Proposition 3.6. *If $a \in B_\zeta$ and ω is an exponentiation invariant generalised limit, then*

$$\omega - \lim_{\lambda\to\infty} M\big(g(a^*, a; \cdot)\big)(\lambda) = \tilde\omega - \lim_{r\to\infty} \frac{1}{r}\,\zeta\big(a^*, a; 1 + \tfrac{1}{r}\big),$$

and if the ordinary limit of the right hand side above exists, then the ordinary limit of the left hand side exists too and they coincide. If moreover ω is M-invariant and $g(a^*,a;\cdot)$ is bounded then

$$\omega-\lim_{\lambda\to\infty} g(a^*,a;\lambda)=\tilde\omega-\lim_{r\to\infty}\frac{1}{r}\,\zeta\big(a^*,a;1+\tfrac{1}{r}\big).$$

Proof. The second assertion (i.e. when ω is M-invariant and g is bounded) is an immediate corollary of the first assertion. To prove the first assertion, observe that in the course of the proof of Lemma 3.4, we have shown that

$$\Big\|a^*G^{1+\frac{1}{r}}a-(\Gamma(1+\frac{1}{r}))^{-1}\int_0^1 t^{\frac{1}{r}}a^*e^{-tG^{-1}}a\,dt\Big\|_1\le g(a^*,a;1)\int_1^\infty t^{\frac{1}{r}}e^{-(t-1)\|G\|^{-1}}dt.$$

Since $\int_1^\infty t^{\frac{1}{r}}e^{-(t-1)\|G\|^{-1}}dt=O(1)$ as $r\to\infty$ we find

$$\lim_{r\to\infty}\frac{1}{r}\Big[\tau\big(a^*G^{1+\frac{1}{r}}a\big)-(\Gamma(1+\frac{1}{r}))^{-1}\int_0^1 t^{\frac{1}{r}}\tau\big(a^*e^{-tG^{-1}}a\big)\,dt\Big]=0, \tag{3.4}$$

and so setting $\tilde\omega(f)=\omega(f\circ\log)$ we obtain

$$\tilde\omega-\lim_{r\to\infty}\frac{1}{r}\tau\big(a^*G^{1+\frac{1}{r}}a\big)=\tilde\omega-\lim_{r\to\infty}\frac{1}{r\Gamma(1+\frac{1}{r})}\int_0^1 t^{\frac{1}{r}}\tau\big(a^*e^{-tG^{-1}}a\big)\,dt.$$

Substituting $t=e^{-\mu}$, a little computation shows that

$$\int_0^1 t^{\frac{1}{r}}\tau\big(a^*e^{-tG^{-1}}a\big)\,dt=\int_0^\infty e^{-\frac{\mu}{r}}\,d\beta(\mu)\quad\text{where}\quad\beta(\mu)=\int_0^\mu e^{-\nu}\tau\big(a^*e^{-e^{-\nu}G^{-1}}a\big)\,d\nu. \tag{3.5}$$

We now wish to use the weak-$*$ Karamata Theorem [7, Theorem 2.2], and need to check the various hypotheses. First, ω is an exponentiation invariant mean on $\mathbb{R}$, so $\tilde\omega$ is a dilation invariant mean on $\mathbb{R}_+^*$. Next, β is positive, increasing and continuous on $\mathbb{R}^+$, and satisfies $\beta(0)=0$. Finally, we need to check that $\int_0^\infty e^{-\frac{\mu}{r}}d\beta(\mu)$ is finite for any $r>0$. But this follows immediately from the first equality in Equation (3.5). Hence, the weak-$*$ Karamata Theorem gives us

$$\tilde\omega-\lim_{r\to\infty}\frac{1}{r}\int_0^\infty e^{-\frac{\mu}{r}}\,d\beta(\mu)=\tilde\omega-\lim_{\mu\to\infty}\frac{\beta(\mu)}{\mu}.$$

But

$$\frac{\beta(\mu)}{\mu}=\frac{1}{\mu}\int_1^{e^\mu}\lambda^{-2}\tau\big(a^*e^{-\lambda^{-1}G^{-1}}a\big)\,d\lambda=\big(Mg(a^*,a;\cdot)\big)(e^\mu),$$

from which the result follows, since

$$\tilde\omega-\lim_{r\to\infty}\frac{1}{r}\tau\big(aG^{1+\frac{1}{r}}a\big)=\tilde\omega-\lim_{\mu\to\infty}\big(Mg(a,a;\cdot)\big)\circ\exp(\mu)=\omega-\lim_{\mu\to\infty}\big(Mg(a,a;\mu)\big).$$

Last, if $\lim_{r\to\infty}\frac{1}{r}\tau\big(aG^{1+\frac{1}{r}}a\big)$ exists, then $\lim_{r\to\infty}\frac{1}{r}\int_0^\infty e^{-\frac{\mu}{r}}\,d\beta(\mu)$ exists too by (3.4) (and coincide with the former) and by the ordinary Karamata Theorem (see the remark right after [7, Theorem 2.2]) gives that $\lim_{\mu\to\infty}\frac{\beta(\mu)}{\mu}$ exists (and coincide with the former), which finally entails that $\lim_{\lambda\to\infty}M\big(g(a^*,a;\cdot)\big)(\lambda)$ exists (and coincide with the former). □

To conclude this discussion, we give some useful stability properties of B_ζ.

Lemma 3.7. *i) If $a \in B_\zeta$ and $f \in L^\infty(\mathbb{R})$, then $af(G) \in B_\zeta$, and when $a^* = a$, $af(a) \in B_\zeta$.*
ii) If $0 \le a, b \in \mathcal{N}$ are such that $a \in B_\zeta$ and $b^2 \le a^2$, then $b \in B_\zeta$.
iii) If $a \in B_\zeta$, then $|a|, |a^*| \in B_\zeta$.

Proof. The ideal property of the trace-norm implies $\tau\big(\bar f(G) a^* G^s a f(G)\big) \le \|f\|_\infty^2 \tau\big(a^* G^s a\big)$, and from the operator inequality $af(G) G^s \bar f(G) a^* \le \|f\|_\infty^2 a G^s a^*$, we obtain the first part of $i)$. The second part of $i)$ is even more immediate. To prove $ii)$, note that from the trace property (3.3), we have $\tau\big(bG^s b\big) = \tau\big(G^{s/2} b^2 G^{s/2}\big) \le \tau\big(G^{s/2} a^2 G^{s/2}\big) = \tau\big(aG^s a\big)$. Finally, to obtain $iii)$, let $u|a|$ and $v|a^*|$ be the polar decomposition of a and a^*. Then of course, $|a| = u^* a = a^* u$ and $|a^*| = v^* a^* = av$. Thus, $\tau\big(|a| G^s |a|\big) = \tau\big(u^* a G^s a^* u\big) = \tau\big(a G^s a^*\big)$. The proof for $|a^*|$ is entirely similar. □

3.3. Relations between $B_\zeta(G)$, $\mathcal{Z}_p$ and $\mathcal{L}^{p,\infty}$.

Proposition 3.8. *If $a, b \in B_\zeta(G)$, then $bGa \in \mathcal{Z}_1$. Moreover, when $b = a^*$, we have the following partial-trace estimate:*

$$\sigma_t\big(a^* G a\big) := \int_0^t \mu_s\big(a^* G a\big)\, ds \le \big\|a^*\, e^{-G^{-1}}\, a\big\|_1 + \|a\|^2 + \|Mg(a^*, a; \cdot)\|_\infty \log(1+t).$$

Proof. By [12, Lemma 2.3][1] we have a Cauchy-Schwarz type inequality on $\mathcal{Z}_1$. More precisely, for a, b, G as in the statement,

$$\|bGa\|_{1,\infty} \le \||bG^{1/2}|^2\|_{1,\infty}^{1/2} \||G^{1/2}a|^2\|_{1,\infty}^{1/2} = \||G^{1/2}b^*|^2\|_{1,\infty}^{1/2} \||G^{1/2}a|^2\|_{1,\infty}^{1/2} = \|bGb^*\|_{1,\infty}^{1/2} \|a^*Ga\|_{1,\infty}^{1/2},$$

so we may assume without loss of generality that $b = a^*$. Next, we recall from [7, Lemma 3.3] that if $a \in \mathcal{N}$ has norm $||a|| \le M$, then for any $1 \le s < 2$, $(a^*Ga)^s \le M^{2(s-1)} a^* G^s a$. Using this inequality we have

$$\limsup_{s\to1} (s-1)\tau((a^*Ga)^s) \le \limsup_{s\to1} \|a\|^{2(1-s)} (s-1)\tau(a^* G^s a) < \infty,$$

which shows that $a \in B_\zeta \Rightarrow a^*Ga \in \mathcal{Z}_1$.

To obtain the estimate of the partial trace, we use the Laplace transform to write

$$a^*Ga = \int_0^\infty a^*\, e^{-tG^{-1}}\, a\, dt = \Big(\int_0^{e^{-k}} + \int_{e^{-k}}^1 + \int_1^\infty\Big) a^*\, e^{-tG^{-1}}\, a\, dt =: C_k + B_k + A.$$

First, we see that A is trace-class . Indeed since $G \le 1$, we get the operator inequality: $\int_1^\infty e^{-tG^{-1}} dt = Ge^{-G^{-1}} \le e^{-G^{-1}}$, which entails that $\|A\|_1 \le \|a^*\, e^{-G^{-1}}\, a\|_1$.

and we focus on the rest. For C_k, we have the bound $\|C_k\| \le \|a\|^2\, e^{-k}$. The operator B_k can be bounded in trace-norm using

$$\|B_k\|_1 \le \int_{e^{-k}}^1 \tau\big(a^*\, e^{-tG^{-1}}\, a\big)\, dt = \ln(e^k)\, \big(Mg(a^*, a; \cdot)\big)(e^k) \le \|Mg(a^*, a; \cdot)\|_\infty\, k.$$

The K-functional associated to the Banach couple $\big(\mathcal{L}^1, \mathcal{N}\big)$ is

$$K(T, t; \mathcal{L}^1, \mathcal{N}) := \inf\big\{\|T_1\|_1 + t\|T_2\|,\, T = T_1 + T_2,\, T_1 \in \mathcal{L}^1,\, T_2 \in \mathcal{N}\big\}.$$

[1]In this reference, an index E is missing on the right-most norm of the inequality, i.e. it should be instead $\|x^*y\|_E \le \|x^*x\|_E^{1/2} \|y^*y\|_E^{1/2}$.

It is known that in this case, it can be exactly evaluated to $\sigma_t(T)$. Thus, writing $D = C_k + B_k$, we can estimate

$$\sigma_s(D) \leq \|B_k\|_1 + s\|C_k\| \leq \|Mg(a^*, a; \cdot)\|_\infty\, k + \|a\|^2 s e^{-k}.$$

Finally, given $s \in (0, \infty)$, define $k \in \mathbb{R}_+$ as $k = \ln(1+s)$, so that we have

$$\sigma_s(D) \leq \|Mg(a^*, a; \cdot)\|_\infty \ln(1+s) + \|a\|^2 s(1+s)^{-1}.$$

Gathering these estimates together, we find the bound stated in the lemma. $\square$

The next result refines our approach to obtain containments in $\mathcal{Z}_q$, $q \geq 1$.

Proposition 3.9. *Let $\delta \in (0,1]$ and $0 \leq a \in \mathcal{N}$ be such that $aGa \in \mathcal{Z}_1$. Then, for any $\varepsilon \in (0, \delta/2]$, $a^\delta G^\varepsilon \in \mathcal{Z}_{1/\varepsilon}$ with $\|a^\delta G^\varepsilon\|_{1/\varepsilon,\infty} \leq \|a\|^{\delta - 2\varepsilon}\|aGa\|^\varepsilon_{1,\infty}$.*

From Proposition 3.8, we see that the assumption $aGa \in \mathcal{Z}_1$ is satisfed for $a \in B_\zeta$.

Proof of Proposition 3.9. Note that the statement is equivalent to:

$$aGa \in \mathcal{Z}_1 \Rightarrow a^\delta G^\varepsilon a^\delta \in \mathcal{Z}_{1/\varepsilon}, \quad \forall \varepsilon \in (0, \delta].$$

Consider the holomorphic operator valued function on the open strip $S = \{z \in \mathbb{C} : \Re z \in (0,1)\}$, given by $F(z) = a^z G^z a^z$. For all $y \in \mathbb{R}$, we have $F(iy) \in \mathcal{N}$ with $\|F(iy)\| \leq 1$. Moreover, $F(1+iy) \in \mathcal{Z}_1$. Indeed, by [18, Theorem 4.2, iii)] and [12, Proposition 1.1], we obtain

$$\begin{aligned}\int_0^t \mu_s(F(1+iy))ds &= \int_0^t \mu_s(a^{1+iy}G^{1+iy}a^{1+iy})ds \leq \int_0^t \mu_s(aG^{1+iy}a)ds \\ &\leq \int_0^t \mu_s(aG^{1/2})\mu_s(G^{1/2+iy}a)ds \leq \int_0^t \mu_s(aG^{1/2})\mu_s(G^{1/2}a)ds = \int_0^t \mu_s(aGa)ds,\end{aligned}$$

and thus $\|F(1+iy)\|_{1,\infty} \leq \|aGa\|_{1,\infty}$. This shows that $a^\varepsilon G^\varepsilon a^\varepsilon$ belongs to the first complex interpolation space $(\mathcal{Z}_1, \mathcal{N})_{[\varepsilon]}$ and hence belongs to $(\mathcal{Z}_1, \mathcal{N})^{[\varepsilon]}$, the second complex interpolation space. But the latter is $\mathcal{Z}_{1/\varepsilon}$, as shown in Proposition 2.2. In summary, we have

$$\|a^\varepsilon G^\varepsilon a^\varepsilon\|_{1/\varepsilon,\infty} = \|F(\varepsilon)\|_{(\mathcal{Z}_1,\mathcal{N})^{[1-\varepsilon]}} \leq \|F(\varepsilon)\|_{(\mathcal{Z}_1,\mathcal{N})_{[1-\varepsilon]}} \leq \|F(0)\|^{1-\varepsilon}\|F(1)\|^\varepsilon_{1,\infty} = \|aGa\|^\varepsilon_{1,\infty},$$

and from the ideal property, we obtain the announced result. $\square$

According to our previous considerations, the assumption that $a, b \in B_\zeta$ is not enough to ensure that abG belongs to $\mathcal{Z}_1$. On a more positive note, the intuitive result that when $g(a^*, a; \cdot)$ is already bounded, that $a^*G^\varepsilon a$, $\varepsilon \in (0,1)$, is in the small ideal $\mathcal{L}^{1/\varepsilon,\infty}$ is true.

Proposition 3.10. *Let $\varepsilon \in (0,1)$ and let $a \in \mathcal{N}$ be such that the map $\mathbb{R}^+ \ni \lambda \mapsto g(a^*, a; \lambda)$ is bounded. Then $a^*G^\varepsilon a \in \mathcal{L}^{1/\varepsilon,\infty}$, with*

$$\mu_s\big(a^*G^\varepsilon a\big) \leq \frac{1}{\Gamma(\varepsilon)}\Big(\frac{1}{\varepsilon}\|a\|^2 + \frac{1}{1-\varepsilon}\|g(a^*, a; \cdot)\|_\infty\Big)s^{-\varepsilon}. \tag{3.6}$$

Proof. We write $a^*G^\varepsilon a = \frac{1}{\Gamma(\varepsilon)}\int_0^\infty t^{\varepsilon-1}\, a^* e^{-tG^{-1}} a\, dt$, and split $\int_0^\infty = \int_{e^{-k}}^\infty + \int_0^{e^{-k}}$, to obtain $a^*G^\varepsilon a = B_k + C_k$, $\quad k \in \mathbb{R}^*_+$. We notice that for any $S \in \mathcal{N}$,

$$\|S\|_1 = \int_0^\infty \mu_t(S)\, dt \geq \int_0^s \mu_t(S)\, dt \geq s\, \mu_s(S).$$

Then, using Fan's inequality, we obtain

$$\mu_s(B_k + C_k) \leq \mu_0(C_k) + \mu_s(B_k) \leq \|C_k\| + s^{-1}\|B_k\|_1.$$

By spectral theory, we have first that $\|C_k\| \leq (\varepsilon\Gamma(\varepsilon))^{-1}\|a\|^2 e^{-\varepsilon k}$. For the second part, we have

$$\|B_k\|_1 \leq \frac{1}{\Gamma(\varepsilon)} \int_{e^{-k}}^{\infty} g\big(a^*, a; t^{-1}\big)\, t^{\varepsilon-2}\, dt \leq \frac{1}{\Gamma(\varepsilon)} \|g(a^*, a; \cdot)\|_\infty (1-\varepsilon)^{-1} e^{k(1-\varepsilon)}.$$

Thus we have

$$\mu_s(B_k + C_k) \leq \frac{1}{\Gamma(\varepsilon)} \Big(\frac{1}{\varepsilon}\|a\|^2 + \frac{1}{1-\varepsilon}\|g(a^*, a; \cdot)\|_\infty \frac{e^k}{s}\Big) e^{-\varepsilon k}.$$

So, if for each $s \in \mathbb{R}^+$, we choose $k = \log s$, we obtain the desired estimate. $\square$

4. Zeta functions and Dixmier traces

This Section contains the main application of our previous results. We are interested in the question, first raised in [13, Chapter 4], and further studied in considerable detail in [7,9,10,25–27,39] in the unital case, concerning the relationship between singularities of the zeta function and the Dixmier trace. The extension of this result to the nonunital case without appealing to the existence of local units has interested a number of authors. The construction of our Banach algebras B_ζ was motivated by this question. The next subsection collects some general lemmas needed later.

4.1. **General facts.** We first prove a result which allows us to manipulate the commutator of fractional powers. We are indebted to Alain Connes for communicating the proof to us, which we reproduce here for completeness.

Lemma 4.1. *Let $0 \leq A, B \in \mathcal{N}$ be such that $[A, B] \in \mathfrak{S}$, where $\mathfrak{S}$ denotes any symmetrically normed (or quasi-normed) ideal of $\mathcal{N}$. Denoting by $\mathfrak{S}_p$, $p \geq 1$, the p-convexification of $\mathfrak{S}$, for all $\alpha, \beta \in (0, 1]$, we have $[A^\alpha, B^\beta] \in \mathfrak{S}_{1/\alpha\beta}$, with*

$$\|[A^\alpha, B^\beta]\|_{\mathfrak{S}_{1/\alpha\beta}} \leq \|A\|^{\alpha(1-\beta)}\|B\|^{\beta(1-\alpha)}\|[A, B]\|_{\mathfrak{S}}^{\alpha\beta}.$$

Proof. By homogeneity, we can assume that $\|A\| = \|B\| = 1$. We are going to use the Cayley transform twice to obtain a commutator estimate from a difference estimate and then use the BKS inequality [5]. To this end, let U be the unitary operator $U := (i + B)(i - B)^{-1}$. A quick computation shows that $[A, U] = 2i(i - B)^{-1}[A, B](i - B)^{-1}$, which gives

$$U^*AU - A = U^*[A, U] = 2i\, U^*(i - B)^{-1}[A, B](i - B)^{-1}.$$

Thus, we see that for all $p \geq 1$, we have $\|[A, B]\|_{\mathfrak{S}_p} \leq \|U^*AU - A\|_{\mathfrak{S}_p} \leq 2\|[A, B]\|_{\mathfrak{S}_p}$. Using finally that $(U^*AU)^\alpha = U^*A^\alpha U$, $\forall \alpha > 0$, and the BKS inequality $\|X^\alpha - Y^\alpha\|_{\mathfrak{S}_{1/\alpha}} \leq \|X - Y\|_{\mathfrak{S}}^\alpha$, we obtain

$$[A, B] \in \mathfrak{S} \;\Leftrightarrow\; U^*AU - A \in \mathfrak{S} \;\Rightarrow\; U^*A^\alpha U - A^\alpha \in \mathfrak{S}_{1/\alpha} \;\Leftrightarrow\; [A^\alpha, B] \in \mathfrak{S}_{1/\alpha}.$$

One concludes the proof using the same trick with the unitary $V := (i + A^\alpha)(i - A^\alpha)^{-1}$. $\square$

Lemma 4.2. *Let $A, B \in \mathcal{N}$, $B^* = B$, such that $[A, B] \in \mathfrak{S}$ for any symmetrically normed ideal of $\mathcal{N}$ and let $\varphi \in C_c^\infty(\mathbb{R})$. Then $[A, \varphi(B)] \in \mathfrak{S}$ with $\|[A, \varphi(B)]\|_{\mathfrak{S}} \leq \|\widehat{\varphi'}\|_1\, \|[A, B]\|_{\mathfrak{S}}$.*

Proof. Since φ is a smooth compactly supported function, it is the Fourier transform of a Schwartz function $\widehat{\varphi}$ and thus $\varphi(B) = \int_{\mathbb{R}} \widehat{\varphi}(\xi)\, e^{-2i\pi\xi B}$. The result then follows from the identity

$$[A, e^{-2i\pi\xi B}] = -2i\pi\xi \int_0^1 e^{-2i\pi\xi s B}\,[A,B]\,e^{-2i\pi\xi(1-s)B}\,ds. \qquad \square$$

Stronger estimates than that given above are available from [34, 35]. Finally we will need

Lemma 4.3. *i) If $T \in \mathcal{Z}_1^0$, then $\lim_{\varepsilon\to 0^+} \varepsilon\,\|T\|_{1+\varepsilon} = 0$.*
ii) Let $a \in B_\zeta$ and $\delta \in (0,1]$. Then the map $(0,\delta/2) \ni \varepsilon \mapsto \|a^\delta G^\varepsilon\|_{1+1/\varepsilon}$ is bounded.

Proof. The first claim follows from [9, Theorem 4.5 i)]. To prove the second part, note that for an arbitrary $T \in \mathcal{Z}_1$, by the definition of the norm in the Marcinkiewicz space $\mathcal{Z}_1$, we have $\mu_t(T) \prec\prec \|T\|_{1,\infty}/(1+t)$. Since the Schatten spaces $\mathcal{L}^p$, $1 \le p \le \infty$, are fully symmetric operator spaces we thus have $\|T\|_p \le \|T\|_{1,\infty}\|[t \mapsto (1+t)^{-1}]\|_p$, for $p > 1$ that is

$$\|T\|_p \le \|T\|_{1,\infty}(p-1)^{-1/p}. \tag{4.1}$$

Let $T := (a^\delta G^{2\varepsilon} a^\delta)^{1/2\varepsilon}$. This operator belongs to $\mathcal{Z}_1$, because $aGa \in \mathcal{Z}_1$ by Proposition 3.8 and thus $a^\delta G^{2\varepsilon} a^\delta \in \mathcal{Z}_{1/2\varepsilon}$ by Proposition 3.9. Applying the estimate (4.1), with $p = 1+\varepsilon$, to this operator yields

$$\begin{aligned}\|G^\varepsilon a^\delta\|_{1+1/\varepsilon} &= \|(a^\delta G^{2\varepsilon} a^\delta)^{1/2+1/2\varepsilon}\|_1^{\varepsilon/(1+\varepsilon)} = \|(a^\delta G^{2\varepsilon} a^\delta)^{1/2\varepsilon}\|_{1+\varepsilon}^{\varepsilon} \\ &\le \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}}\|(a^\delta G^{2\varepsilon} a^\delta)^{1/2\varepsilon}\|_{1,\infty}^{\varepsilon} = \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}}\|a^\delta G^{2\varepsilon} a^\delta\|_{1/2\varepsilon,\infty}^{1/2}.\end{aligned} \tag{4.2}$$

But Proposition 3.9 gives also the inequality

$$\|a^\delta G^{2\varepsilon} a^\delta\|_{1/2\varepsilon,\infty} \le \|a\|^{2(\delta-2\varepsilon)}\|aGa\|_{1,\infty}^{2\varepsilon}, \quad \forall \delta \in (0,1], \quad \forall \varepsilon \in (0,\delta/2). \tag{4.3}$$

Combining (4.2) with (4.3), we obtain

$$\|G^\varepsilon a^\delta\|_{1+1/\varepsilon} \le \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}}\|a^\delta G^{2\varepsilon} a^\delta\|_{1/2\varepsilon,\infty}^{1/2} \le \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}}\|a\|^{\delta-2\varepsilon}\|aGa\|_{1,\infty}^{\varepsilon}.$$

This proves the claim since $\varepsilon^{-\frac{\varepsilon}{1+\varepsilon}} \to 1$. $\square$

4.2. **Approximation schemes.** As before, without loss of generality we assume that $G^{-1} \ge 1$. In the following, we fix $0 \le a \in \mathcal{N}$ and we assume further that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$. We stress that while purely technical at the first glance, this extra δ-condition turns out to be the key assumption to get an equivalence $a \in B_\zeta \Leftrightarrow a^*Ga \in \mathcal{Z}_1$. This is explained in Section 5 where we use the factorization $a = a^\delta a^{1-\delta}$ intensively. We then construct a pair of approximation processes for a in the strong topology, the first being given by $a_n := aP_n$, where $P_n := \int_{1/n}^{\|a\|} dE_a(\lambda)$, with $\int_0^{\|a\|} \lambda\, dE_a(\lambda)$ the spectral resolution of a. Note the operator inequality $a_n^2 \ge \frac{1}{n^2}P_n$. Lemma 3.7 ii) implies then that $P_n \in B_\zeta$ as well.

Next, for each $n \in \mathbb{N}$, we pick $0 \le \varphi_n \in C_c^\infty(\mathbb{R})$ such that, restricted to the interval $[0,\|a\|]$, we have $\chi_{(1/n,\|a\|]} \le \varphi_n \le \chi_{(1/(n+1),\|a\|]}$. This immediately implies that $P_n \le \varphi_n(a)$, $\varphi_n^2(a) \le P_{n+1}$. For the second limiting process we define $a_n^\varphi := a\varphi_n(a)$. Now, since

$$a^\delta\,(1-\varphi_n(a)) \le a^\delta\,(1-\varphi_n^2(a)) \le a^\delta(1-P_n) = \int_0^{1/n} \lambda^\delta\, dE_a(\lambda),$$

we have

$$\|a^{\delta}\,(1-\varphi_n(a))\| \le \|a^{\delta}\,(1-\varphi_n^2(a))\| \le \|a^{\delta}(1-P_n)\| \le n^{-\delta}, \quad \forall\delta>0. \tag{4.4}$$

Finally, from $P_n \le \varphi_n(a) \le P_{n+1}$, we deduce that

$$\tfrac{1}{n}P_n \le a_n \le a_n^{\varphi} \le a_{n+1} \le a, \quad \varphi_n(a)P_{n+1} = \varphi_n(a)\,, \quad \varphi_n(a)P_n = P_n. \tag{4.5}$$

The reason why two approximations are required is as follows. The projection based method allows the use of several operator inequalities, most notably [7, Lemma 3.3 (ii)]. If we were then willing to assume that $[P_n, G] \in Z_1^0$, then the following proof would simplify considerably. However, this assumption is highly implausible in the examples. So we introduce the smooth approximation scheme, and a more complex proof, in order to obtain a result which is actually applicable to the examples.

The following is our main technical result from which Theorem 4.13 will follow easily.

Proposition 4.4. *Let $0 \le a \in \mathcal{N}$ be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$ and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then $\lim_{s\searrow 1}(s-1)\left\|aG^s a - (aGa)^s\right\|_1 = 0$.*

The proof of the proposition proceeds by writing

$$\begin{aligned} aG^s a - (aGa)^s &= \Big[aG^s a - a_n^{\varphi} G^s a_n^{\varphi}\Big] + \Big[a_n^{\varphi} G^s a_n^{\varphi} - a\big(\varphi_n(a)G\varphi_n(a)\big)^s a\Big] \\ &+ \Big[a\big(\varphi_n(a)G\varphi_n(a)\big)^s a - a\big(P_nGP_n\big)^s a\Big] + \Big[a\big(P_nGP_n\big)^s a - \big(a_nGa_n\big)^s\Big] + \Big[\big(a_nGa_n\big)^s - \big(aGa\big)^s\Big], \end{aligned}$$

and then controlling each successive difference in this equality in the trace norm. The following sequence of lemmas achieves this goal.

Lemma 4.5. *Let $0 \le a \in \mathcal{N}$ be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$. Then*

$$\limsup_{s\searrow 1}(s-1)\,\|aG^s a - a_n^{\varphi} G^s a_n^{\varphi}\|_1 \le \Big(n^{-2\delta} + 2\|a\|^{\delta} n^{-\delta}\Big)\,\|a^{1-\delta}\|_\zeta^2\,.$$

Proof. Since $0 \le a_n^{\varphi 2} \le a^2$ it follows from Lemma 3.7, *ii*), that $a_n^{\varphi} \in B_\zeta$ and that the function $s \mapsto (s-1)\tau\big(a_n^{\varphi} G^s a_n^{\varphi}\big)$, for $s \ge 1$, is well defined and bounded. Using the equality

$$a_n^{\varphi} G^s a_n^{\varphi} - aG^s a = (1-\varphi_n(a))aG^s a(1-\varphi_n(a)) - aG^s a(1-\varphi_n(a)) - (1-\varphi_n(a))aG^s a,$$

the result follows from $\|a^{\delta}(1-\varphi_n(a))\| \le n^{-\delta}$, by Equation (4.4). □

Lemma 4.6. *Let $0 \le a \in \mathcal{N}$ such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$. Then there exists two constants $C_1, C_2 > 0$, uniform in n, such that*

$$i) \qquad \limsup_{s\searrow 1}\,(s-1)\big\|(a_{n+1}Ga_{n+1})^s - (a_nGa_n)^s\big\|_1 \le C_1\, n^{-\delta/2},$$

$$ii) \qquad \limsup_{s\searrow 1}\,(s-1)\big\|(a_nGa_n)^s - (aGa)^s\big\|_1 \le C_2\, n^{-\delta/2}.$$

Proof. To prove i), let $A_n := a_{n+1}\,G\,a_{n+1}$ and $B_n := a_n\,G\,a_n$. Then

$$\begin{aligned}\|A_n^s - B_n^s\|_1 = \|A_n^{s/2}(A_n^{s/2} - B_n^{s/2}) + (A_n^{s/2} - B_n^{s/2})B_n^{s/2}\|_1 &\le \big(\|A_n^{s/2}\|_2 + \|B_n^{s/2}\|_2\big)\|A_n^{s/2} - B_n^{s/2}\|_2\\ &\le \big(\|A_n\|_s^{s/2} + \|B_n\|_s^{s/2}\big)\|A_n - B_n\|_s^{s/2},\end{aligned}$$

by the BKS inequality since $0 < s/2 < 1$. Then we use $\|A_n\|_s \le \|aGa\|_s$ and $\|B_n\|_s \le \|aGa\|_s$, together with

$$\|A_n - B_n\|_s = \|a_{n+1}Ga(P_{n+1} - P_n) - (P_n - P_{n+1})aGa_n\|_s \le 2\,\|a^\delta(P_{n+1} - P_n)\|\,\|a^{1-\delta}Ga\|_s,$$

to obtain

$$\|A_n^s - B_n^s\|_1 \le 2^{s/2+1}\,\|a\|^{3\delta s/2}\,\|a^\delta(P_{n+1} - P_n)\|^{s/2}\,\|a^{1-\delta}Ga^{1-\delta}\|_s^s.$$

This concludes the proof since $(s-1)\|a^{1-\delta}Ga^{1-\delta}\|_s^s$ is bounded and $\|a^\delta(P_{n+1}-P_n)\|^{s/2} \le n^{-s\delta/2}$.

To prove ii), one uses the same strategy applied to $A_n = a_nGa_n$ and $B_n = aGa$. □

The following result is inspired by [7, Lemmas 3.3-3.5]:

Lemma 4.7. *Let $P \in \mathcal{N}$ be a projector and $0 \le a \in B_\zeta$ such that $[a, P] = 0$ and $a \ge m\,P$, for some $m \in (0, 1)$. Then $\lim_{s\searrow 1}(s-1)\big\|a(PGP)^s a - (aPGPa)^s\big\|_1 = 0$.*

Proof. By [7, Lemma 3.3 i)], we have $(aPGPa)^s \le \|a\|^{2(s-1)}\,a(PGP)^s a$. The result follows if we can show that

$$(aPGPa)^s \ge m^{2(s-1)}\,a(PGP)^s a, \tag{4.6}$$

as we would then have

$$\big(m^{2(s-1)} - 1\big)a(PGP)^s a \le (aPGPa)^s - a(PGP)^s a \le \big(\|a\|^{2(s-1)} - 1\big)a(PGP)^s a, \tag{4.7}$$

and this suffices by the following reasoning. If $\|a\| \le 1$, then

$$0 \le a(PGP)^s a - (aPGPa)^s \le \big(1 - m^{2(s-1)}\big)a(PGP)^s a,$$

and the claim follows from the operator inequality $a(PGP)^s a \le aPG^sPa$, proven in [7, Lemma 3.3 i)] and from $aP \in B_\zeta$ from Lemma 3.7. So assume $\|a\| > 1$. Then,

$$-\big(\|a\|^{2(s-1)} - 1\big)a(PGP)^s a \le a(PGP)^s a - (aPGPa)^s \le \big(1 - m^{2(s-1)}\big)a(PGP)^s a.$$

Setting

$$0 \le b = \big(\|a\|^{2(s-1)} - 1\big),\ 0 \le c = \big(1 - m^{2(s-1)}\big),\ A = a(PGP)^s a,\ X = a(PGP)^s a - (aPGPa)^s,$$

we have $0 \le X + bA \le (c + b)A$, and thus $\|X\|_1 \le \|X + bA\|_1 + b\|A\|_1 \le (c + 2b)\|A\|_1$, that is

$$\big\|a(PGP)^s a - (aPGPa)^s\big\|_1 \le \Big(\big(1 - m^{2(s-1)}\big) + 2\big(\|a\|^{2(s-1)} - 1\big)\Big)\|a(PGP)^s a\|_1,$$

which gives the result since $\|a(PGP)^s a\|_1 \le \|aPG^sPa\|_1 = \|PaG^saP\|_1 \le \|aG^sa\|_1$, by [7, Lemma 3.3 i)] again. To prove (4.6), decompose $\mathcal{H}$ as $P\mathcal{H} \oplus (1 - P)\mathcal{H}$. Since $[P, a] = 0$, we know that

$$(aPGPa)^s = P(aPGPa)^s P, \quad \text{and} \quad a(PGP)^s a = Pa(PGP)^s aP,$$

and so their restrictions to $(P\mathcal{H})_n^\perp$ are zero and so (4.6) holds on $(1 - P)\mathcal{H}$. Since $a \ge m\,P$, its restriction to $P\mathcal{H}$ is an invertible element of $P\mathcal{N}P$ and [7, Lemma 3.3 ii)] gives the result. □

Next we prove some results involving both the projectors P_n and their smooth versions $\varphi_n(a)$.

Lemma 4.8. *Let $0 \le a \in \mathcal{N}$ be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$. Then there exists $C > 0$, uniform in n, such that* $\limsup_{s\searrow 1}(s-1)\big\|a(P_nGP_n)^s a - a(P_{n+1}GP_{n+1})^s a\big\|_1 \le C\, n^{-\delta/2}$.

Proof. Write

$$a(P_nGP_n)^s a - a(P_{n+1}GP_{n+1})^s a = \big(a(P_nGP_n)^s a - (a_nGa_n)^s\big) + \big((a_nGa_n)^s - (a_{n+1}Ga_{n+1})^s\big) + \big((a_{n+1}Ga_{n+1})^s - a(P_{n+1}GP_{n+1})^s a\big),$$

and apply Lemma 4.6 and Lemma 4.7. ☐

Lemma 4.9. *Let $0 \le a \in \mathcal{N}$ satisfy the hypotheses of Lemma 4.8. Then there exists $C > 0$, uniform in n, such that* $\limsup_{s\searrow 1}(s-1)\,\big\|a\big(\varphi_n(a)G\varphi_n(a)\big)^s a - a\big(P_nGP_n\big)^s a\big\|_1 \le C\, n^{-\delta/2}$.

Proof. By equation (4.5), we have $\varphi_n(a) = P_{n+1}\varphi_n(a)$, while $P_n = \varphi_n(a)P_n$. Thus,

$$\begin{aligned} a\big(\varphi_n(a)G\varphi_n(a)\big)^s a - a\big(P_nGP_n\big)^s a &= a\big(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}\big)^s a - a\big(P_n\varphi_n(a)G\varphi_n(a)P_n\big)^s a \\ &= \Big[a\big(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}\big)^s a - \big(aP_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}a\big)^s\Big] \\ &\quad + \Big[\big(aP_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}a\big)^s - \big(aP_n\varphi_n(a)G\varphi_n(a)P_na\big)^s\Big] \\ &\quad + \Big[\big(aP_n\varphi_n(a)G\varphi_n(a)P_na\big)^s - a\big(P_n\varphi_n(a)G\varphi_n(a)P_n\big)^s a\Big]. \end{aligned}$$

For the first term in parentheses, we can apply Lemma 4.7, with the modification that we replace G there by $\varphi_n(a)G\varphi_n(a)$, to obtain a vanishing contribution. Indeed, following line by line the proof of Lemma 4.7 with the indicated modification, we get the operator inequalities

$$\begin{aligned} &\big(m^{2(s-1)} - 1\big)a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a \\ &\qquad \le (aP_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}a)^s - a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a \\ &\qquad\qquad \le \big(\|a\|^{2(s-1)} - 1\big)a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a. \end{aligned}$$

Combining these operator inequalities with $a(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1})^s a \le aG^s a$, we obtain

$$\lim_{s\searrow 1}(s-1)\|a\big(P_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}\big)^s a - \big(aP_{n+1}\varphi_n(a)G\varphi_n(a)P_{n+1}a\big)^s\|_1 = 0.$$

Replacing P_{n+1} by P_n gives the same conclusion for the last term in parentheses.

For the middle term, we can apply Lemma 4.8 with the replacement $a \mapsto a_n^\varphi$, to obtain the desired trace-norm bound. Indeed, since $a_n^{\varphi\,2} = a^2\varphi_n^2(a) \le a^2$, we infer from Lemma 3.7 ii) that $a_n^\varphi \in B_\zeta$ and since $(a_n^\varphi)^{1-\delta} = a^{1-\delta}\varphi_n(a)^{1-\delta} \le a^{1-\delta}$, we see that $(a_n^\varphi)^{1-\delta}$ belongs to B_ζ too. ☐

The next lemma is the critical step in the proof of Proposition 4.4.

Lemma 4.10. *Let $0 \le a \in \mathcal{N}$ and suppose that there exists $\delta > 0$ with $a^{1-\delta} \in B_\zeta$ and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then there exists an absolute constant $C > 0$ such that*

$$\limsup_{s\searrow 1}(s-1)\,\big\|a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a\big\|_1 \le C\, n^{-\delta/2}.$$

Proof. First, from Lemma 3.7 i), $a^{1-\delta} \in B_\zeta$ implies $a \in B_\zeta$ too. Then, as $\varphi_n(a) \leq P_{n+1} \leq (n+1)a_{n+1}$, we readily see that $\varphi_n(a)G\varphi_n(a) \in \mathcal{Z}_1$ and thus $(\varphi_n(a)G\varphi_n(a))^s$ is trace-class for all $s > 1$, so that we are entitled to take the trace norm as in the statement of the lemma. Next, for $0 \leq A, B \in \mathcal{N}$, with B injective and $1 \leq s \leq 2$, we have by [19, Lemma 1]:

$$\big(ABA\big)^s = AB^{1/2}\big(B^{1/2}A^2B^{1/2}\big)^{s-1}B^{1/2}A.$$

Applying this to $A = \varphi_n(a)$ and $B = G$ (which is injective), gives

$$a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a = a_n^\varphi G^{1/2}\big(G^{s-1} - (G^{1/2}\varphi_n(a)^2G^{1/2})^{s-1}\big)G^{1/2}a_n^\varphi$$
$$= a_n^\varphi G^{1/2}\frac{\sin(\pi\varepsilon)}{\pi}\int_0^\infty \lambda^{-\varepsilon}\Big(G(1+\lambda G)^{-1} - G^{1/2}\varphi_n(a)^2G^{1/2}(1+\lambda G^{1/2}\varphi_n(a)^2G^{1/2})^{-1}\Big)d\lambda G^{1/2}a_n^\varphi$$
$$= a_n^\varphi G^{1/2}\frac{\sin(\pi\varepsilon)}{\pi}\int_0^\infty \lambda^{-\varepsilon}(1+\lambda G)^{-1}G^{1/2}\big(1-\varphi_n^2(a)\big)G^{1/2}(1+\lambda G^{1/2}\varphi_n(a)^2G^{1/2})^{-1}d\lambda G^{1/2}a_n^\varphi,$$

where we have defined $\varepsilon := s-1$ and we have used in the third equality, the identity

$$A(1+\lambda A)^{-1} - B(1+\lambda B)^{-1} = (1+\lambda A)^{-1}(A-B)(1+\lambda B)^{-1}\,, \quad 0 \leq A, B \in \mathcal{N}, \lambda > 0.$$

Hence

$$a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a = \frac{\sin(\pi\varepsilon)}{\pi}\int_0^\infty X_{\varepsilon,n}(\lambda)\,\lambda^{-\varepsilon}d\lambda,$$

with $X_{\varepsilon,n}(\lambda) = a_n^\varphi G(1+\lambda G)^{-1}\big(1-\varphi_n^2(a)\big)G^{1/2}(1+\lambda G^{1/2}\varphi_n(a)^2G^{1/2})^{-1}G^{1/2}a_n^\varphi$.

Commuting $1-\varphi_n^2(a)$ with $G(1+\lambda G)^{-1}$ on its left, we obtain $X_{\varepsilon,n}(\lambda) = Y_{\varepsilon,n}(\lambda) + Z_{\varepsilon,n}(\lambda)$, with

$$Y_{\varepsilon,n}(\lambda) := a_n^\varphi\big(1-\varphi_n^2(a)\big)G^{1/2}(G^{-1}+\lambda)^{-1}(1+\lambda G^{1/2}\varphi_n(a)^2G^{1/2})^{-1}G^{1/2}a_n^\varphi,$$
$$Z_{\varepsilon,n}(\lambda) := a_n^\varphi(1+\lambda G)^{-1}\big[G,\varphi_n^2(a)\big](G^{-1}+\lambda)^{-1}\Big(1-\lambda\varphi_n(a)^2(G^{-1}+\lambda\varphi_n(a)^2)^{-1}\Big)a_n^\varphi,$$

where we have used the two relations

$$[G(1+\lambda G)^{-1},\big(1-\varphi_n^2(a)\big)] = -\lambda^{-1}[(1+\lambda G)^{-1},\varphi_n^2(a)] = (1+\lambda G)^{-1}[G,\varphi_n^2(a)](1+\lambda G)^{-1},$$
$$G^{1/2}(1+\lambda G^{1/2}\varphi_n(a)^2G^{1/2})^{-1}G^{1/2} = (G^{-1}+\lambda\varphi_n(a)^2)^{-1} = G(1-\lambda\varphi_n(a)^2(G^{-1}+\lambda\varphi_n(a)^2)^{-1}).$$

For $\|Y_{\varepsilon,n}(\lambda)\|_1$, we use the Hölder inequality to obtain the upper bound

$$\|a^{\delta/2}\big(1-\varphi_n^2(a)\big)\|\|a^{1-\delta/2}G^{(1+\varepsilon)/2}\|_{\frac{2+\varepsilon}{1+\varepsilon}}\|G^{-\varepsilon/2}(G^{-1}+\lambda)^{-1}\|\|(1+\lambda G^{1/2}\varphi_n(a)^2G^{1/2})^{-1}\|\|G^{1/2}a\|_{2+\varepsilon}$$
$$\leq n^{-\delta/2}\,(1+\lambda)^{-1+\varepsilon/2}\|a^{1-\delta/2}G^{(1+\varepsilon)/2}\|_{\frac{2+\varepsilon}{1+\varepsilon}}\|G^{1/2}a\|_{2+\varepsilon},$$

where we have used equation (4.4) and obvious operator estimates. Next, from the operator inequality, [7, Lemma 3.3 i)], $(aGa)^{1+\varepsilon/2} \leq \|a\|^\varepsilon aG^{1+\varepsilon/2}a$, we obtain

$$\|G^{1/2}a\|_{2+\varepsilon} = \|aGa\|_{1+\varepsilon/2}^{1/2} \leq \|a\|^{\varepsilon/(2+\varepsilon)}\|aG^{1+\varepsilon/2}a\|_1^{(2+\varepsilon)^{-1}}$$
$$\leq \|a\|^{\varepsilon/(2+\varepsilon)}\|a\|_\zeta^{2/(2+\varepsilon)}\,(2/\varepsilon)^{(2+\varepsilon)^{-1}} \leq C\,\varepsilon^{-1/2}. \tag{4.8}$$

To evaluate $\|a^{1-\delta/2}G^{(1+\varepsilon)/2}\|_{(2+\varepsilon)/(1+\varepsilon)}$, we write

$$a^{1-\delta/2}G^{(1+\varepsilon)/2} = a^{\delta/2}G^{(1+\varepsilon)/2}a^{1-\delta} + a^{\delta/2}\big[a^{1-\delta},G^{(1+\varepsilon)/2}\big].$$

For the first term, we obtain

$$\|a^{\delta/2}G^{(1+\varepsilon)/2}a^{1-\delta}\|_{(2+\varepsilon)/(1+\varepsilon)} \leq \|a^{\delta/2}G^{\varepsilon/2}\|_{1+2/\varepsilon}\|G^{1/2}a^{1-\delta}\|_{2+\varepsilon} \leq C\,\varepsilon^{-1/2},$$

where we used that $\|a^{\delta/2}G^{\varepsilon/2}\|_{1+2/\varepsilon}$ remains bounded when $\varepsilon \to 0^+$, from Lemma 4.3 ii), and the estimate of equation (4.8) for the second part, since $a^{1-\delta} \in B_\zeta$ by assumption. It remains to treat the commutator term, for which Lemma 4.1 gives us

$$\big\|\big[a^{1-\delta}, G^{(1+\varepsilon)/2}\big]\big\|_{(2+\varepsilon)/(1+\varepsilon)} \le C\big\|\big[a^{1-\delta}, G\big]\big\|_{1+\varepsilon/2}^{(1+\varepsilon)/2}.$$

We conclude using Lemma 4.3 i) that $\varepsilon^{1/2}\big\|\big[a^{1-\delta}, G^{(1+\varepsilon)/2}\big]\big\|_{(2+\varepsilon)/(1+\varepsilon)} \to 0$ when $\varepsilon \to 0^+$. Hence, we have shown that

$$\|Y_{\varepsilon,n}(\lambda)\|_1 \le C\, n^{-\delta/2}\,(1+\lambda)^{-1+\varepsilon/2}\,\varepsilon^{-1},$$

and thus

$$\limsup_{\varepsilon\searrow 0}\, \varepsilon\, \frac{|\sin(\pi\varepsilon)|}{\pi}\int_0^\infty \|Y_{\varepsilon,n}(\lambda)\|_1\, \lambda^{-\varepsilon} d\lambda \le C' n^{-\delta/2}.$$

It remains to treat $Z_{\varepsilon,n}(\lambda)$ which we estimate in trace-norm as

$$\begin{aligned}\|Z_{\varepsilon,n}(\lambda)\|_1 \le \|a_n^\varphi(1+\lambda G)^{-1}[G,\varphi_n^2(a)]G(1+\lambda G)^{-1}a_n^\varphi\|_1 \\ + \|a_n^\varphi(1+\lambda G)^{-1}[G,\varphi_n^2(a)]G(1+\lambda G)^{-1}\lambda\varphi_n^2(a)(G^{-1}+\lambda\varphi_n(a)^2)^{-1}a_n^\varphi\|_1.\end{aligned}$$

We estimate the first term by

$$\begin{aligned}\|a\|\,\|[G,\varphi_n^2(a)]\|_{1+\varepsilon/2}\|G^{1-\varepsilon/2}(1+\lambda G)^{-1}\|\|G^{\varepsilon/2}a\|_{1+2/\varepsilon} \\ \le \|a\|\,\|[G,\varphi_n^2(a)]\|_{1+\varepsilon/2}\|G^{\varepsilon/2}a\|_{1+2/\varepsilon}(1+\lambda)^{-1+\varepsilon/2},\end{aligned}$$

where we have used the operator inequality

$$G^{1-\varepsilon/2}(1+\lambda G)^{-1} = G^{-\varepsilon/2}(G^{-1}+\lambda)^{-1} \le (G^{-1}+\lambda)^{-1+\varepsilon/2} \le (1+\lambda)^{-1+\varepsilon/2},$$

since $G^{-1} \ge 1$. For the second term, we obtain the bound

$$\|a\|^2\,\big\|\big[G,\varphi_n^2(a)\big]\big\|_{1+\varepsilon/2}\|G^{\varepsilon/2}\varphi_n(a)\|_{1+2/\varepsilon}\Big\|\varphi_n(a)\lambda(G^{-1}+\lambda\varphi_n(a)^2)^{-1}\varphi_n(a)\Big\|(1+\lambda)^{-1+\varepsilon/2}.$$

Since $G^{-1} \ge 1$, we have the estimate $(G^{-1}+\lambda\varphi_n(a)^2)^{-1} \le (1+\lambda\varphi_n(a)^2)^{-1}$, and thus

$$\|\varphi_n(a)\lambda(G^{-1}+\lambda\varphi_n(a)^2)^{-1}\varphi_n(a)\| \le \|\lambda\varphi_n^2(a)(1+\lambda\varphi_n(a)^2)^{-1}\| \le 1.$$

Using $\varphi_n(a) \le n\, a$, we obtain $\|G^{\varepsilon/2}\varphi_n(a)\|_{1+2/\varepsilon} \le n\|G^{\varepsilon/2}a\|_{1+2/\varepsilon}$ and so

$$\begin{aligned}\|Z_{\varepsilon,n}(\lambda)\|_1 &\le C(1+n)\big\|\big[G,\varphi_n^2(a)\big]\big\|_{1+\varepsilon/2}\|G^{\varepsilon/2}a\|_{1+2/\varepsilon}(1+\lambda)^{-1+\varepsilon/2} \\ &\le C(1+n)\|\widehat{\varphi_n^{2\prime}}\|_1\big\|\big[G,a\big]\big\|_{1+\varepsilon/2}\|G^{\varepsilon/2}a\|_{1+2/\varepsilon}(1+\lambda)^{-1+\varepsilon/2}.\end{aligned}$$

We have used Lemma 4.2, applied to the symmetrically normed ideal $\mathfrak{S} := \mathcal{L}^{1+\varepsilon}$, the operators $A := G$, $B := a \ge 0$ and the test function $\varphi := \varphi_n^2$ together with the embedding $\mathcal{Z}_1^0 \subset \mathcal{Z}_1 \subset \mathcal{L}^{1+\varepsilon}$, to obtain the last inequality. We stress that $\|\widehat{\varphi_n^{2\prime}}\|_1$ is not uniform in n since φ_n^2 pointwise-converges to a step function. However, combining Theorem 3.1 from [34] with Theorem 4 from [35], and taking into account that $\mathcal{Z}_1^0$ is an interpolation space for the couple $(\mathcal{L}^1, \mathcal{N})$, we get from $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$ that $[G,a] \in \mathcal{Z}_1^0$ as well. Thus, by Lemma 4.3 i), ii), we know that $\varepsilon\|[G,a]\|_{1+\varepsilon} \to 0$, while $\|G^\varepsilon a\|_{1+1/\varepsilon}$ remains bounded when $\varepsilon \to 0^+$. This entails that

$$\limsup_{\varepsilon\searrow 0}\, \varepsilon\, \frac{|\sin(\pi\varepsilon)|}{\pi}\int_0^\infty \|Z_{\varepsilon,n}(\lambda)\|_1\, \lambda^{-\varepsilon} d\lambda = 0.$$

Putting everything together, we obtain the announced result:

$$\limsup_{s\searrow 1}(s-1)\left\|a_n^\varphi G^s a_n^\varphi - a(\varphi_n(a)G\varphi_n(a))^s a\right\|_1$$
$$\leq \limsup_{\varepsilon\searrow 0}\varepsilon\,\frac{\sin(\pi\varepsilon)}{\pi}\int_0^\infty \big(\|Y_{\varepsilon,n}(\lambda)\|_1 + \|Z_{\varepsilon,n}(\lambda)\|_1\big)\,\lambda^{-\varepsilon}d\lambda \leq C'n^{-\delta/2}. \quad \square$$

We are now ready to complete the proof of our main technical result.

Proof of Proposition 4.4. We write:

$$aG^sa-(aGa)^s = \Big[aG^sa - a_n^\varphi G^s a_n^\varphi\Big] + \Big[a_n^\varphi G^s a_n^\varphi - a\big(\varphi_n(a)G\varphi_n(a)\big)^s a\Big]$$
$$+\Big[a\big(\varphi_n(a)G\varphi_n(a)\big)^s a - a\big(P_nGP_n\big)^s a\Big] + \Big[a\big(P_nGP_n\big)^s a - \big(a_nGa_n\big)^s\Big] + \Big[\big(a_nGa_n\big)^s - \big(aGa\big)^s\Big].$$

The lim sup, $s\to 1^+$ of the trace norm of the first bracket multiplied by $(s-1)$, is bounded by $n^{-\delta}$ by Lemma 4.5, the second is bounded by $n^{-\delta/2}$ by Lemma 4.10, the third is bounded by $n^{-\delta/2}$ by Lemma 4.9, the fourth is bounded by 0 by Lemma 4.7 and the fifth by $n^{-\delta/2}$ by Lemma 4.6 ii). This implies: $\limsup_{s\to 1+}\left\|aG^sa-(aGa)^s\right\|_1 \leq C\,n^{-\delta/2}$, $\forall n\in\mathbb{N}$. $\square$

Proposition 4.4 immediately gives us

Corollary 4.11. *Let $0\leq a\in\mathcal{N}$ satisfy the hypotheses of Lemma 4.8 and $[G,a^{1-\delta}]\in\mathcal{Z}_1^0$. Then*
(i) $\lim_{s\to 1^+}(s-1)\tau\big(aG^sa\big)$ exists iff $\lim_{s\to1^+}(s-1)\tau\big((aGa)^s\big)$ exists and then they are equal;
(ii) More generally, for any Banach limits ω, we have

$$\tilde\omega-\lim_{s\searrow 1}(s-1)\tau\big(aG^sa\big) = \tilde\omega-\lim_{s\searrow 1}(s-1)\tau\big((aGa)^s\big).$$

4.3. **Dixmier-traces computation.** We now have

Proposition 4.12. *Let $0\leq a\in\mathcal{N}$ be such that there exists $\delta>0$ with $a^{1-\delta}\in B_\zeta$ and $[G,a^{1-\delta}]\in\mathcal{Z}_1^0$. Then, if any one of the following limits exist they all do and all coincide*
(1) $\lim_{t\to\infty}\frac{1}{\log(1+t)}\int_0^t\mu_s(aGa)ds$,
(2) $\lim_{r\to\infty}\frac{1}{r}\,\tau((aGa)^{1+\frac{1}{r}})$,
(3) $\lim_{r\to\infty}\frac{1}{r}\,\zeta(a,a;1+\frac{1}{r})$,
Furthermore, the existence of any of the above limits is equivalent to
(4) every generalized limit ω which is dilation invariant yields the same value $\tau_\omega(aGa)$ and the latter value coincides with the value of the limits above.
Last, the existence of any of the limits (1), (2) or (3) implies the existence of, and the coincidence with
(5) $\lim_{\lambda\to\infty}\big(Mg(a,a;\cdot)\big)(\lambda)$.

Proof. That the limit in (5) exists and coincide with (3) when the limit in (3) exists, follows from Proposition 3.6. The simultaneous existence and equality of the limits in (2) and (3) follows from Corollary 4.11. Recall now that the assumption $a\in B_\zeta$ guarantees $aGa\in\mathcal{Z}_1$. The assertion "(2) exists if and only if (1) exists and then they are equal" is known, it follows e.g. by the same argument as in the proof of [7, Corollary 3.7] or by the argument given at the beginning of the proof of [1, Theorem 2]. If (1) exists then it equals $\tau_\omega(aGa)$ by definition. $\square$

Remark. In [11, Theorem 7], the existence and value of the various limits in Proposition 4.12 are shown to coincide with the existence and value of $\lim_{\lambda\to\infty}(M\hat{g})(\lambda)$ where

$$\hat{g}(\lambda) = \lambda^{-1}\tau(e^{-(aGa)^{-1}\lambda^{-1}}).$$

We have arrived at our main result, the nonunital analogue of [7, Theorem 3.8], [9, Theorem 4.11] and [26, Corollary 3.3] from which the convergence theorem of the introduction follows.

Theorem 4.13. *Assume that $a \in \mathcal{N}$ is self adjoint and let $a = a_+ - a_-$ be the decomposition into the difference of nonnegative operators. Assume that there exists $\delta > 0$ with $a_\pm^{1/2-\delta} \in B_\zeta$, and $[G, a_\pm^{1/2-\delta}] \in \mathcal{Z}_1^0$. Then $aG \in \mathcal{Z}_1$, and moreover,*
(i) if $\lim_{s\to1^+}(s-1)\big(\zeta(a_+^{1/2}, a_+^{1/2}; s) - \zeta(a_-^{1/2}, a_-^{1/2}; s)\big)$ exists, then it is equal to $\tau_\omega(aG)$ (defined at the beginning of Section 3) where ω is any dilation and exponentiation invariant state on $L^\infty(\mathbb{R}^+)$,
(ii) more generally, if ω is as in (i) and $\tilde{\omega} := \omega \circ \log$, then

$$\tau_\omega(aG) = \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\big(\zeta(a_+^{1/2}, a_+^{1/2}; 1+\tfrac{1}{r}) - \zeta(a_-^{1/2}, a_-^{1/2}; 1+\tfrac{1}{r})\big).$$

Proof. As observed earlier, recall that $a_\pm^{1/2-\delta} \in B_\zeta$ implies $a_\pm^{1/2} \in B_\zeta$ and $[G, a_\pm^{1/2-\delta}] \in \mathcal{Z}_1^0$ implies $[G, a_\pm^{1/2}] \in \mathcal{Z}_1^0$. Thus

$$aG = a_+G - a_-G = a_+^{1/2}Ga_+^{1/2} - a_-^{1/2}Ga_-^{1/2} + a_+^{1/2}[a_+^{1/2}, G] - a_-^{1/2}[a_-^{1/2}, G] \in \mathcal{Z}_1,$$

with $\tau_\omega(aG) = \tau_\omega(a_+^{1/2}Ga_+^{1/2}) - \tau_\omega(a_-^{1/2}Ga_-^{1/2})$, where ω (and latter $\tilde{\omega}$) has been chosen as in the proof of [9, Theorem 4.11]. We only need to prove part ii); part i) will then follow from general facts on Banach limits. By [7, Lemma 3.2(i)], [9, Theorem 4.11], Corollary 4.11 and the remark above, we have

$$\begin{aligned}\tau_\omega(aG) = \tau_\omega(a_+^{1/2}Ga_+^{1/2}) - \tau_\omega(a_-^{1/2}Ga_-^{1/2}) &= \tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\big(\tau((a_+^{1/2}Ga_+^{1/2})^{1+\frac{1}{r}}) - \tau((a_-^{1/2}Ga_-^{1/2})^{1+\frac{1}{r}})\big)\\ &= \tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\big(\tau(a_+^{1/2}G^{1+\frac{1}{r}}a_+^{1/2}) - \tau(a_-^{1/2}G^{1+\frac{1}{r}}a_-^{1/2})\big). \quad \square\end{aligned}$$

By considering independently real and imaginary parts of $a \in B_\zeta$, we get an analogous result for non-self-adjoint elements. Moreover, we could have stated a similar result using the Cesàro mean of the heat-trace function instead of the zeta function because, under the same assumptions as those of Theorem 4.13, it is true that

$$\tau_\omega(aG) = \omega - \lim_{\lambda\to\infty}\Big(\big(Mg(a_+^{1/2}, a_+^{1/2}; \cdot)\big)(\lambda) - \big(Mg(a_-^{1/2}, a_-^{1/2}; \cdot)\big)(\lambda)\Big).$$

This essentially follows from Theorem 4.13 and Proposition 3.6.

The following fact provides the tracial property of the zeta residue.

Proposition 4.14. *Let $a \in B_\zeta$ be such that $[G, a] \in \mathcal{Z}_1^0$. Then for any Dixmier trace associated with a dilation invariant state on $L^\infty(\mathbb{R}^+)$, we have*

$$0 \leq \tau_\omega(a^*aG) = \tau_\omega(aa^*G).$$

Proof. By Proposition 3.8, $a^*Ga, aGa^* \in \mathcal{Z}_1$, and since $aa^*G = aGa^* + a[a^*, G]$, $a^*aG = a^*Ga + a^*[a, G]$, aa^*G and a^*aG belong to $\mathcal{Z}_1$ as well. Now, for $A \in \mathcal{Z}_2$, we have

$$\tau_\omega(A^*A) = \omega - \lim_{t\to+\infty} \frac{\int_0^t \mu_s(A)^2 ds}{\log(1+t)} = \omega - \lim_{t\to+\infty} \frac{\int_0^t \mu_s(A^*)^2 ds}{\log(1+t)} = \tau_\omega(AA^*).$$

Now we do some rearranging

$$\begin{aligned}\tau_\omega(a^*aG) &= \tau_\omega(a^*Ga) + \tau_\omega(a^*[a,G]) = \tau_\omega(a^*Ga) = \tau_\omega(G^{1/2}aa^*G^{1/2})\\ &= \tau_\omega(aG^{1/2}a^*G^{1/2}) + \tau_\omega([G^{1/2}, a]a^*G^{1/2}) = \tau_\omega(aa^*G) + \tau_\omega(a[G^{1/2}, a^*]G^{1/2}),\end{aligned}$$

as the Dixmier trace vanishes on the ideal $\mathcal{Z}_1^0$, by Lemma 4.1 $[G^{1/2}, a] \in \mathcal{Z}_2^0$, and the fact that $\mathcal{Z}_2^0\,\mathcal{Z}_2 \subset \mathcal{Z}_1^0$. We complete the proof by observing that

$$a[G^{1/2}, a^*]G^{1/2} = a[G, a^*] - aG^{1/2}[G^{1/2}, a^*],$$

and that both terms on the right hand side of this last equation are in $\mathcal{Z}_1^0$. □

5. The converse estimate

In the previous Section we showed that $a \in B_\zeta \Rightarrow a^*Ga \in \mathcal{Z}_1$. But the converse requires more assumptions. We demonstrate this by exhibiting a counter-example. Let us introduce one more Banach $*$-sub-algebra of $\mathcal{N}$:

$$B_{\mathcal{Z}_1} = B_{\mathcal{Z}_1}(G) := \big\{a \in \mathcal{N} : \|a^*Ga\|_{1,\infty} + \|aGa^*\|_{1,\infty} + \|a\|^2 < \infty\big\}.$$

It is easy to see that in general $a \in B_{\mathcal{Z}_1} \not\Rightarrow a \in B_\zeta$ by considering the case $G = Id_{\mathcal{N}}$. Then $B_{\mathcal{Z}_1} = \mathcal{Z}_2$ and $B_\zeta = \mathcal{L}^2$. More realistic examples from spectral triples are also easy to produce. A positive result in the converse direction is the following.

Proposition 5.1. *Let $0 \le a \in \mathcal{N}$, be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_{\mathcal{Z}_1}$ and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then $a \in B_\zeta$.*

The proof of this 'converse' estimate relies essentially on the following lemma.

Lemma 5.2. *Let $0 \le a \in B_{\mathcal{Z}_1}$ and $\delta \in (0,1)$. Then, the map $\varepsilon \in (0, \delta/2] \mapsto \|a^\delta G^\varepsilon\|_{1+1/\varepsilon}$, is bounded.*

Proof. We know by assumption that $aGa \in \mathcal{Z}_1$ and thus from Proposition 3.9, we know that for $\delta \in (0,1)$, $a^\delta G^\varepsilon \in \mathcal{Z}_{1/\varepsilon}$ with $\|a^\delta G^\varepsilon\|_{1/\varepsilon,\infty} \le \|a\|^{\delta-2\varepsilon}\|aGa\|_{1,\infty}^\varepsilon$. We conclude using the same chain of estimates as in the proof of Lemma 4.3 ii):

$$\begin{aligned}\|a^\delta G^\varepsilon\|_{1+1/\varepsilon} = \||a^\delta G^\varepsilon|^{1/\varepsilon}\|_{1+\varepsilon}^\varepsilon &\le \Big(\varepsilon^{-\frac{1}{1+\varepsilon}}\||a^\delta G^\varepsilon|^{1/\varepsilon}\|_{1,\infty}\Big)^\varepsilon\\ &= \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}}\|a^\delta G^\varepsilon\|_{1/\varepsilon,\infty} \le \varepsilon^{-\frac{\varepsilon}{1+\varepsilon}}\|a\|^{\delta-2\varepsilon}\|aGa\|_{1,\infty}^\varepsilon. \quad □\end{aligned}$$

Proof of Proposition 5.1. We write $aG^{1+\varepsilon}a = a^\delta G^{1/2+\varepsilon}a^{1-\delta}G^{1/2}a + a^\delta[a^{1-\delta}, G^{1/2+\varepsilon}]G^{1/2}a$, and thus

$$\begin{aligned}\|aG^{1+\varepsilon}a\|_1 \le\ & \|a^\delta G^\varepsilon\|_{1+1/\varepsilon}\|G^{1/2}a^{1-\delta}\|_{2+2\varepsilon}\|G^{1/2}a\|_{2+2\varepsilon}\\ &+ \|a\|^\delta\|[a^{1-\delta}, G^{1/2+\varepsilon}]\|_{2(1+\varepsilon)/(1+2\varepsilon)}\|G^{1/2}a\|_{2+2\varepsilon}.\end{aligned} \tag{5.1}$$

From Lemma 4.1, we have for $\varepsilon \le 1/2$

$$\|[a^{1-\delta}, G^{1/2+\varepsilon}]\|_{2(1+\varepsilon)/(1+2\varepsilon)} \le \|a\|^{(1-\delta)(1/2-\varepsilon)} \|[a^{1-\delta}, G]\|_{1+\varepsilon}^{1/2+\varepsilon}. \tag{5.2}$$

Since $[a^{1-\delta}, G] \in \mathcal{Z}_1^0$, we know from Lemma 4.3 i) that $\|[a^{1-\delta}, G]\|_{1+\varepsilon} = o(\varepsilon^{-1})$. In particular, $\|[a^{1-\delta}, G]\|_{1+\varepsilon}^{\varepsilon} = O(1)$ and thus the inequality (5.2) gives $\|[a^{1-\delta}, G^{1/2+\varepsilon}]\|_{2(1+\varepsilon)/(1+2\varepsilon)} = o(\varepsilon^{-1/2})$. Moreover, since $a^{1-\delta}Ga^{1-\delta} \in \mathcal{Z}_1$, we know from Theorem 2.1 that $\|a^{1-\delta}Ga^{1-\delta}\|_{1+\varepsilon} = O(\varepsilon^{-1})$, which gives $\|G^{1/2}a\|_{2+2\varepsilon} \le \|a\|^{\delta}\|G^{1/2}a^{1-\delta}\|_{2+2\varepsilon} = \|a\|^{\delta}\|a^{1-\delta}Ga^{1-\delta}\|_{1+1\varepsilon}^{1/2} = O(\varepsilon^{-1/2})$. Finally, by Lemma 5.2 we know that $\|a^{\delta}G^{\varepsilon}\|_{1+1/\varepsilon} = O(1)$. Putting everything together, the inequality (5.1) gives us $\|aG^{1+\varepsilon}a\|_1 = O(\varepsilon^{-1})$, that is $\sup_{\varepsilon>0} \varepsilon\|aG^{1+\varepsilon}a\|_1 < \infty$, i.e. $a \in B_\zeta$. $\square$

Now we can state a $\mathcal{Z}_1$ version of Proposition 4.4.

Proposition 5.3. *Let $0 \le a \in \mathcal{N}$ be such that there exists $\delta > 0$ with $a^{1-\delta} \in B_{\mathcal{Z}_1}$, and $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$. Then $\lim_{s\to 1^+}(s-1)\left\|aG^s a - (aGa)^s\right\|_1 = 0$.*

Proof. Applying Proposition 5.1 to $a^{1-\delta/2}$ and $\delta/(2-\delta)$, we deduce that $a^{1-\delta/2} \in B_\zeta$. Combining Theorem 3.1 from [34] with Theorem 4 from [35], and taking into account that $\mathcal{Z}_1^0$ is an interpolation space for the couple $(\mathcal{L}^1, \mathcal{N})$, we get from $[G, a^{1-\delta}] \in \mathcal{Z}_1^0$ that $[G, a] \in \mathcal{Z}_1^0$ as well. Then the claim follows directly from Proposition 4.4. $\square$

6. Nonunital Spectral Triples

We will now use the results of the previous Sections to give an *a posteriori* definition of a finitely summable nonunital semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, relative to $(\mathcal{N}, \tau)$. A semifinite spectral triple consists of a separable Hilbert space $\mathcal{H}$ carrying a faithful representation of $\mathcal{N}$, together with an essentially self-adjoint operator $\mathcal{D}$ affiliated with $\mathcal{N}$ and a (nonunital) $*$-sub-algebra $\mathcal{A}$ of $\mathcal{N}$ such that $[\mathcal{D}, a]$ is bounded for all $a \in \mathcal{A}$ and $a(1+\mathcal{D}^2)^{-1/2} \in \mathcal{K}_\mathcal{N}$.

The main difference between the notions of finitely summable *unital* and *nonunital* spectral triple is that, in the unital case, $\mathcal{D}$ alone is enough to characterise the spectral dimension. However, the situation in the nonunital case is far more subtle since one needs a delicate interplay between $\mathcal{A}$ and $\mathcal{D}$ to obtain a good definition of spectral dimension.

Corollary 6.4 and Proposition 6.6 show that our definitions are compatible with known results from the unital case. Moreover the hypotheses are checkable in practise, and the limits are also computable in practise: we present some classical examples to illustrate this.

Despite our previous focus on 'L^1 as the square of L^2', we now define summability for spectral triples in an 'L^1' fashion. The reason for this is the local index formula [6]. However here we will quickly return, via the results of previous sections, to the L^2 type description.

Definition 6.1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a nonunital semifinite spectral triple, relative to $(\mathcal{N}, \tau)$. We then let $p := \inf\{s > 0 :$ for all $0 \le a \in \mathcal{A}$, $\tau\big(a(1+\mathcal{D}^2)^{-s/2}\big) < \infty\}$, and when it exists, we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finitely summable and call p the spectral dimension of the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.*

Remark. This definition is employed in the local index formula [6]. The use of the unsymmetrised form of the condition amounts to making the strongest possible assumption. We stress the dependence on the algebra $\mathcal{A}$ in the previous definition. Finally, note that by [3] we obtain a positive functional for each $s > 0$ since as $a(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N}, \tau)$, for $a \geq 0$,

$$\tau(a(1+\mathcal{D}^2)^{-s/2}) = \tau((1+\mathcal{D}^2)^{-s/4}a(1+\mathcal{D}^2)^{-s/4}) \geq 0.$$

If $\mathcal{A}$ is unital and the unit of $\mathcal{A}$ is also the unit of $\mathcal{N}$, then this gives the usual notion of finite summability and spectral dimension for unital spectral triples.

Definition 6.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a finitely summable nonunital semifinite spectral triple, relative to $(\mathcal{N}, \tau)$, with spectral dimension $p \geq 1$. If for all $a \in \mathcal{A}$*

$$\limsup_{s \searrow p} \left|(s-p)\tau(a(1+\mathcal{D}^2)^{-s/2})\right| < \infty,$$

we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $\mathcal{Z}_p$-summable. In this case we let $B_p := B_\zeta((1+\mathcal{D}^2)^{-p/2})$.

The special case of $\mathcal{Z}_p$-summability is the nonunital analogue of the most studied criterion in the unital case, usually called (p, ∞)-summability.

Lemma 6.3. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $\mathcal{Z}_p$-summable nonunital semifinite spectral triple, relative to $(\mathcal{N}, \tau)$ with $\mathcal{A} \subset B_p^2 = \mathrm{span}\{b_1 b_2 : b_1,\, b_2 \in B_p\}$. If $b_1[b_2, (1+\mathcal{D}^2)^{-p/2}] \in \mathcal{Z}_1^0$, $\forall\, b_1, b_2 \in B_p$, arising from elements of $\mathcal{A}$, then $a \mapsto \tau_\omega(a(1+\mathcal{D}^2)^{-p/2})$ defines a positive trace on $\mathcal{A}$.*

Proof. We make our usual abbreviation $G = (1+\mathcal{D}^2)^{-p/2}$. With the assumption above we have a finite sum $a = \sum_i b_1^i b_2^i$ and so $aG = \sum_i b_1^i b_2^i G = \sum_i b_1^i[b_2^i, G] + b_1^i G b_2^i$. Since for each i, $b_1^i[b_2^i, G] \in \mathcal{Z}_1^0$ and $b_1^i G b_2^i \in \mathcal{Z}_1$, for any Dixmier trace τ_ω, $\tau_\omega(aG) = \sum_i \tau_\omega(b_1^i G b_2^i)$ is well-defined. To see that $a \mapsto \tau_\omega(aG)$ is an (unbounded) trace we employ Proposition 4.14, and we are left to check that $[a, G] \in \mathcal{Z}_1^0$ for positive $a \in \mathcal{A}$, which is a consequence of the Leibniz rule. Positivity follows from [3] again since $\tau_\omega(aG) = \tau_\omega(G^{1/2}aG^{1/2}) \geq 0$ for $a \geq 0$. □

Corollary 6.4. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $\mathcal{Z}_p$-summable nonunital semifinite spectral triple, relative to $(\mathcal{N}, \tau)$. Suppose that $0 \leq a \in \mathcal{A}$ satisfies $a = b^2$ for $b \geq 0$ with $b^{1-\delta} \subset B_p$ for some $\delta > 0$, and with $b[b, G] \in \mathcal{Z}_1^0$. Then choosing ω, $\tilde{\omega}$ as in [9, Theorem 4.11] we have*

$$\tau_\omega(aG) = \tilde{\omega} - \lim \frac{1}{r}\tau(bG^{1+\frac{1}{r}}b).$$

Proof. This is the content of our main result Theorem 4.13. □

Our final aim is to check the validity of our assumptions in the context of spectral triples, using smoothness of the spectral triple. As usual δ is the unbounded derivation given by $\delta(T) := [|\mathcal{D}|, T]$, $T \in \mathcal{N}$ and we note that in the definition of B_p we may use $G = (1+\mathcal{D}^2)^{-p/2}$ interchangeably with $G_1 = (1+|\mathcal{D}|)^{-p}$.

Definition 6.5. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $\mathcal{Z}_p$-summable nonunital semifinite spectral triple, relative to $(\mathcal{N}, \tau)$ with $\mathcal{A} \subset B_p^2$. We say $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is smooth to order k if for all $b \in B_p$ arising from elements of $\mathcal{A}$ and for all $0 \leq j \leq k$, $\delta^j(b) \in B_p$.*

Now we check that being smooth to order $\lfloor p \rfloor + 1$ ($\lfloor \cdot \rfloor$ is the integer-part function) is enough to ensure that $b_1[b_2, (1+\mathcal{D}^2)^{-p/2}] \in \mathcal{Z}_1^0$ for b_1, $b_2 \subseteq B_p$ arising from $\mathcal{A}$.

Proposition 6.6. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a finitely summable nonunital semifinite spectral triple with spectral dimension p, and which is smooth to order $\lfloor p \rfloor + 1$. Suppose moreover that $\mathcal{A} \subset B_p^2$. Then for all b_1, $b_2 \subset B_p$ arising from $\mathcal{A}$, $b_1[b_2, (1+|\mathcal{D}|)^{-p}]$ is trace-class.*

Proof. We may write $(1+|\mathcal{D}|)^{-p} = \frac{1}{2\pi i}\int_\ell \lambda^{-p}(\lambda - 1 - |\mathcal{D}|)^{-1} d\lambda$, where $\ell = \frac{1}{2} + i\mathbb{R}$. Then we have

$$\begin{aligned}[(1+|\mathcal{D}|)^{-p}, b_2] &= \frac{1}{2\pi i}\int_\ell \lambda^{-p}(\lambda-1-|\mathcal{D}|)^{-1}[|\mathcal{D}|, b_2](\lambda-1-|\mathcal{D}|)^{-1}d\lambda \\ &= p(1+|\mathcal{D}|)^{-p-1}[|\mathcal{D}|, b_2] - \frac{1}{2\pi i}\int_\ell \lambda^{-p}(\lambda-1-|\mathcal{D}|)^{-1}[(\lambda-1-|\mathcal{D}|)^{-1}, [|\mathcal{D}|, b_2]]d\lambda.\end{aligned}$$

Repeat the previous resolvent trick to obtain

$$[(1+|\mathcal{D}|)^{-p}, b_2] = p(1+|\mathcal{D}|)^{-p-1}[|\mathcal{D}|, b_2] - \frac{1}{2\pi i}\int_\ell \lambda^{-p}(\lambda-1-|\mathcal{D}|)^{-2}[|\mathcal{D}|, [|\mathcal{D}|, b_2]](\lambda-1-|\mathcal{D}|)^{-1}d\lambda.$$

Now iterate the process and multiply on the left by b_1 to obtain

$$\begin{aligned}b_1[(1+|\mathcal{D}|)^{-p}, b_2] &= p\, b_1(1+|\mathcal{D}|)^{-p-1}[|\mathcal{D}|, b_2] - p(p+1)\, b_1(1+|\mathcal{D}|)^{-p-2}[|\mathcal{D}|, [|\mathcal{D}|, b_2]]\dots \\ &\dots - \frac{1}{2\pi i}\int_\ell \lambda^{-p}\, b_1(\lambda-1-|\mathcal{D}|)^{-\lfloor p\rfloor-1}[|\mathcal{D}|, \dots, [|\mathcal{D}|, [|\mathcal{D}|, b_2]]\cdots](\lambda-1-|\mathcal{D}|)^{-1}d\lambda,\end{aligned}$$

where there are $\lfloor p \rfloor + 1$ commutators in the integrand. Under the smoothness assumption $\delta^k(b_2) \in B_p$ for $k \le \lfloor p \rfloor + 1$ we may use Lemma 3.7 to argue that for each $\lambda \in \ell$ the product

$$b_1(\lambda-1-|\mathcal{D}|)^{-\lfloor p\rfloor-1}[|\mathcal{D}|, \dots, [|\mathcal{D}|, [|\mathcal{D}|, b_2]]\cdots],$$

is trace-class because $b_1(1+|\mathcal{D}|)^{-\lfloor p\rfloor-1}[|\mathcal{D}|, \dots, [|\mathcal{D}|, [|\mathcal{D}|, b_2]]\cdots]$ is trace-class. Simple estimates now show that the integral converges in trace norm. The result follows. □

Examples. Take $(\mathcal{N}, \tau) = (\mathcal{B}(L^2(\mathbb{R}^p, S)), \mathrm{Tr})$, where S is the trivial spinor bundle, and Tr the usual operator trace. We let $\not\partial$ be the standard Dirac operator, $G = (1+\not\partial^2)^{-p/2}$, $\mathcal{H} = L^2(\mathbb{R}^p, S)$, and $\mathcal{A}$ the bounded smooth integrable functions all of whose partial derivatives are bounded and integrable. Let $b \in L^2(\mathbb{R}^p)$ be the function $b(x) = (1+\|x\|^2)^{-p/4-\varepsilon/2}$, with $\varepsilon > 0$ fixed. Then $\mathcal{A} \ni a = b^2$ satisfies the hypotheses of Corollary 6.4 and

$$\mathrm{Tr}_\omega\left(b^2 G\right) = \tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\mathrm{Tr}\left(bG^{1+\frac{1}{r}}b\right) = \frac{2^{[p/2]}\mathrm{Vol}(\mathbb{S}^{p-1})}{p(2\pi)^p}\int_{\mathbb{R}^p}(1+\|x\|^2)^{-p/2-\varepsilon}\, d^p x,$$

the final line following from an explicit calculation similar to that in [37, Corollary 14].

Recall that an algebra $\mathcal{A}_c$ has local units when, for any finite subset of elements $\{a_1, \dots, a_k\}$ of $\mathcal{A}_c$, there exists $u \in \mathcal{A}_c$ such that $ua_i = a_i u = a_i$ for $i = 1, \dots, k$. In the example above we could take $\mathcal{A}_c$ to be the smooth compactly supported functions, and apply the theory in [37] to elements of $\mathcal{A}_c$. However, the local units based theories of [20, 37] can not acomodate the integration of the function b^2. Thus even in the classical case of manifolds our approach to integration allows a wider class of functions.

Similar examples may be constructed on any complete spin manifold M of bounded geometry, [37]. For this, consider $W^{1,\infty}$, the intersection of L^1-Sobolev spaces, endowed with the Fréchet topology associated to the countable family of norms:

$$\|f\|_{1,k} := \int_M |\Delta^{k/2} f|(x)\,\mathrm{dvol}(x),$$

where, Δ denotes the scalar Laplacian on M and dvol is the Reimannian volume form. Then, we can show with the methods developed in [6, Section 5.1.1], that the operator of point-wise multiplication by a function in $W^{1,\infty}$ acting on the L^2-sections of the spinor bundle, satisfies the assumptions of Corollary 6.4, for $\not{D}$ the Dirac operator on M. From the spinor heat kernel and its asymptotics (see [23]), we compute (with $p = \dim M$)

$$\lim_{r\to\infty} \frac{1}{r}\mathrm{Tr}(f(1+\not{D}^2)^{-(p/2)(1+1/r)}) = \frac{2}{p\Gamma(p/2)}\int_M f(x)\,a_0(x)\,\mathrm{dvol}(x).$$

Here a_0 is the first coefficient in the asymptotic expansion of the heat kernel. Kordukov shows that the coefficients are invariants depending only on the jets of the principal symbol of $\not{D}^2$. Gilkey's arguments, [22, page 334], then apply, since they are completely local. Hence $a_0 = (4\pi)^{-p/2}\mathrm{Id}_S$, where S is the spinor bundle. Thus

$$\lim_{r\to\infty} \frac{1}{r}\mathrm{Tr}(f(1+\not{D}^2)^{-(p/2)(1+1/r)}) = \frac{2^{[p/2]}\mathrm{Vol}(\mathbb{S}^{p-1})}{p(2\pi)^p}\int_M f(x)\,\mathrm{dvol}(x).$$

For example, on the upper half plane model of the hyperbolic plane, we have $\mathrm{dvol}(x+iy) = y^{-2}dx\,dy$. Then one can check that $f(x+iy) = (1+x^2)^{-1/2-\varepsilon}(1+y^2)^{-\varepsilon}y^{1+\varepsilon}$ satisfies the hypotheses of Corollary 6.4, is integrable, and is clearly not compactly supported.

Last, we mention that the results of this paper, have been intensively used in [21, Theorem 14]. Moreover, it significantly simplifies the proof of [20, Proposition 4.17] and makes it valid for a function algebra (for the Moyal product) larger than the Schwartz space, namely the one considered in [6, Lemma 6.7].

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