TWISTED CYCLIC THEORY, EQUIVARIANT KK THEORY AND KMS STATES.

ALAN L. CAREY, SERGEY NESHVEYEV, RYSZARD NEST, AND ADAM RENNIE

Dedicated to the memory of Gerard Murphy

Abstract

Recently, examples of an index theory for KMS states of circle actions were discovered, [9, 13]. We show that these examples are not isolated. Rather there is a general framework in which we use KMS states for circle actions on a C^*-algebra A to construct Kasparov modules and semifinite spectral triples. By using a residue construction analogous to that used in the semifinite local index formula we associate to these triples a twisted cyclic cocycle on a dense subalgebra of A. This cocycle pairs with the equivariant KK-theory of the mapping cone algebra for the inclusion of the fixed point algebra of the circle action in A. The pairing is expressed in terms of spectral flow between a pair of unbounded self adjoint operators that are Fredholm in the semifinite sense. A novel aspect of our work is the discovery of an eta cocycle that forms a part of our twisted residue cocycle. To illustrate our theorems we observe firstly that they incorporate the results in [9, 13] as special cases. Next we use the Araki-Woods III_λ representations of the Fermion algebra to show that there are examples which are not Cuntz-Krieger systems.

1. Introduction

1.1. **Background.** This paper presents an extension of noncommutative geometry and index theory to the purely infinite or type III case, that is, where we are looking at situations in which there are no faithful traces and we wish to replace them by KMS states. It exploits some ideas from semifinite noncommutative geometry, a recent extension of the standard type I theory of Connes [14] which was begun in [5]. Our main reference for the analytic part of these ideas is [7], however there is more background in [4, 10, 11, 12].

We build on two interesting examples. In [9, 13] it was discovered that there exist refined invariants of the Cuntz algebra and the algebra $SU_q(2)$ which arise from KMS states. In the former case there is a canonical circle action (the gauge action) and a unique associated KMS state whose GNS representation is type III while in the latter case the Haar state satisfies the KMS condition with respect to a circle action. It was shown that certain unitaries in matrix algebras over the Cuntz algebras and $SU_q(2)$ can be used to form a new type of K-group. This group then pairs with twisted cyclic cocycles to produce real valued invariants. In [9] this construction was termed 'modular index theory'. Not all unitaries in these algebras define elements of the new K-group; only those satisfying a side condition formulated in terms of the modular group of the KMS state. These index pairings have not been seen before because they use in an essential way the semifinite index theory from [7, 11, 12].

The nature of this modular theory in these examples is mysterious. The objective of this paper is to put the examples into a general framework so that, rather than being isolated phenomena, the modular index pairings in [9, 13] can be seen to arise from a more fundamental principle. The germ of the idea comes from [8] where it was observed that the examples of semifinite noncommutative geometries discovered in [28] lead to classes in the KK-theory of a mapping cone algebra. What we find in this paper is a more general framework that uses equivariant KK-theory of mapping cone algebras. The constructions we employ arise very naturally for a broad class of C^*-algebras admitting states (or weights) that are KMS for circle actions. We remark that although we do not discuss examples of the case where we start with a weight on a C^*-algebra here, we know from work in progress with M. Marcolli that such examples exist and are of considerable independent interest.

1.2. **Summary of results.** Our basic data consists of a C^*-algebra A, together with a strongly continuous action of the circle $\mathbb{T}$ by $*$-automorphisms $\sigma\colon \mathbb{T} \to \mathrm{Aut}(A)$. We let F denote the fixed point subalgebra A^σ of A, Φ the conditional expectation

$$A \ni a \mapsto \frac{1}{2\pi}\int_{\mathbb{T}} \sigma_t(a)dt \in F,$$

and set $A_k = \{a \in A \mid \sigma_t(a) = e^{ikt}a\}$. For the more algebraic part of this paper (Section 2) we make a 'spectral subspace assumption' (SSA) that the ideals $F_k = \overline{A_k A_k^*}$ are complemented in F. This generalises the notion of full spectral subspaces. When we deal with purely analytic formulae in subsequent Sections we can drop this SSA. On the analytic side we assume there is a (possibly unbounded) semifinite, norm lower semicontinuous, faithful, positive functional $\phi\colon A \to \mathbb{C}$ satisfying the KMS_β condition for the circle action σ and $\beta \neq 0$.

Associated to the pair (A, σ) there is an unbounded Kasparov module $({}_AX_F, \mathcal{D})$ constructed as follows. Define the F-valued scalar product $(a|b)_R = \Phi(a^*b)$ and ${}_AX_F$ as the corresponding C^*-module completion of A. Then $\mathcal{D}$ is the generator of the action of σ on ${}_AX_F$ induced by the action of σ on A. The pair $({}_AX_F, \mathcal{D})$ defines a class $[\mathcal{D}] := [({}_AX_F, \mathcal{D})] \in KK_1^{\mathbb{T}}(A, F)$.

Let $M = M(F, A)$ denote the mapping cone of the inclusion $F \subset A$. The mapping cone extension

$$0 \to C_0(\mathbb{R}, A) \xrightarrow{\iota} M \xrightarrow{ev} F \to 0$$

gives us an exact sequence in equivariant KK-theory, part of which is

$$KK_0^{\mathbb{T}}(M, F) \xrightarrow{\iota^*} KK_1^{\mathbb{T}}(A, F) \xrightarrow{\delta} KK_1^{\mathbb{T}}(F, F).$$

Since $\mathcal{D}$ commutes with the left action of F, we have $\delta[\mathcal{D}] = 0$, and so by exactness, there is a class $[\hat{\mathcal{D}}] \in KK_0^{\mathbb{T}}(M, F)$ with $\iota^*[\hat{\mathcal{D}}] = [\mathcal{D}]$. In the text we will give an explicit construction of such a KK-class $[\hat{\mathcal{D}}]$ using the results of [8].

The cycles $[\mathcal{D}]$ and $[\hat{\mathcal{D}}]$ define, via the Kasparov product, two index maps:

$$K_1^{\mathbb{T}}(A) \xrightarrow{\mathrm{Index}_{\mathcal{D}}} K_0^{\mathbb{T}}(F) \quad \text{and} \quad K_0^{\mathbb{T}}(M) \xrightarrow{\mathrm{Index}_{\hat{\mathcal{D}}}} K_0^{\mathbb{T}}(F).$$

compatible with $\iota_*\colon K_1^{\mathbb{T}}(A) \to K_0^{\mathbb{T}}(M)$.

We shall also introduce a closely related homomorphism into the representation ring of $\mathbb{T}$:

$$sf\colon K_0^{\mathbb{T}}(M) \to \mathbb{R}[\chi, \chi^{-1}],$$

which we call the 'equivariant spectral flow'. We will explain how the constructions of a modular index pairing for the Cuntz algebras and $SU_q(2)$ obtained previously in [9, 13] can be explained (and generalized) in terms of $\mathrm{Index}_{\hat{\mathcal{D}}}$ and sf together with the map $\tau_*\colon K_0(F) \to \mathbb{R}$ induced by the trace $\tau = \phi|_F$ and the evaluation at $\chi = e^{-\beta}$ map $\mathbb{R}[\chi, \chi^{-1}] \to \mathbb{R}$ (Theorem 4.11). Furthermore, the modular index pairing can be computed quite explicitly *for a certain subgroup of* $K_0^{\mathbb{T}}(M)$ using an analytic formula for semifinite spectral flow (Theorem 5.6).

As a new example we discuss the case of Araki-Woods factors which are obtained from KMS states for the gauge action on the Fermion algebra.

1.3. **The theorems.** For the reader's convenience we give details of the results here. Let $\mathcal{H}_\phi = L^2(A,\phi)$, let $\pi_\phi\colon A \to \mathcal{B}(\mathcal{H}_\phi)$ denote the GNS representation and $\mathcal{N}$ the commutant of $J_\phi \pi_\phi(F) J_\phi$ in $\mathcal{B}(\mathcal{H}_\phi)$. Then $\mathcal{N}$ is a semifinite von Neumann algebra and $A \simeq \pi_\phi(A) \subset \mathcal{N}$ with a positive, faithful, semifinite trace Tr_ϕ (see Lemma 3.2). Let also $\mathcal{D}$ denote the (self-adjoint) extension of the operator $\mathcal{D}$ introduced above to $\mathcal{H}_\phi$. Let Φ_k be the projection onto the k^{th} spectral subspace of $\mathcal{D}$ for each $k \in \mathbb{Z}$. Finally, set $P = \chi_{[0,\infty)}(\mathcal{D})$. Denote by $\mathcal{A}$ the algebra consisting of finite sums of σ-homogeneous elements in the domain $\mathrm{dom}(\phi)$ of ϕ. We also put $\mathcal{F} = \mathcal{A} \cap F = \mathrm{dom}(\tau)$, where $\tau = \phi|_F$.

We denote the unitization of our algebras by a superscripted $\tilde{}$. Every class in $K_0^{\mathbb{T}}(M)$ has a representative v such that $v \in (\mathcal{A}^\sim \otimes B(\mathcal{H}_U))^{\sigma\otimes \mathrm{Ad}\, U}$, vv^* and v^*v are in $\mathcal{F}^\sim \otimes B(\mathcal{H}_U)$, and $vv^* = v^*v$ modulo $\mathcal{F} \otimes B(\mathcal{H}_U)$, where $U\colon \mathbb{T} \to B(\mathcal{H}_U)$ is a finite dimensional unitary representation (Lemma 3.6). Henceforth we restrict to such v. Then we define the equivariant spectral flow as follows. Denote by χ^n the one-dimensional representation $t \mapsto e^{int}$ and write the representation ring of $\mathbb{T}$ over $\mathbb{R}$ as $\mathbb{R}[\chi,\chi^{-1}]$. Let $Q_n\colon \mathcal{H}\otimes\mathcal{H}_U \to \mathcal{H}\otimes\mathcal{H}_U$ be the projection onto the χ^n-homogeneous component and let

$$sf_n(v) = (\mathrm{Tr}_\phi \otimes \mathrm{Tr})((v^*v - vv^*)Q_n(P\otimes 1)) \in \mathbb{R}.$$

We show that $sf_n(v) = 0$ for all but a finite number of $n \in \mathbb{Z}$ and introduce the $\mathbb{T}$-equivariant spectral flow sf on $K_0^{\mathbb{T}}(M))$ with values in $\mathbb{R}[\chi,\chi^{-1}]$ defined on the class $[v]$ of v by

$$sf([v]) = \sum_{n\in\mathbb{Z}} sf_n(v)\chi^n.$$

Let $K^I(M)$ be the subgroup of $K_0^{\mathbb{T}}(M)$ generated by partial isometries whose homogeneous components are partial isometries.

Theorem 1.1. *If the spectral subspace assumption is satisfied, the equivariant spectral flow coincides with the composition*

$$K_0^{\mathbb{T}}(M) \xrightarrow{-\mathrm{Index}_{\hat{\mathcal{D}}}} K_0^{\mathbb{T}}(F) = K_0(F)[\chi,\chi^{-1}] \xrightarrow{\tau_*} \mathbb{R}[\chi,\chi^{-1}],$$

where τ_* denotes the homomorphism $K_0(F) \to \mathbb{R}$ defined by the trace τ. For elements of $K^I(M)$ this was done explicitly in [8].

It is natural to ask whether there is an analytic spectral flow formula that computes the equivariant spectral flow. There is an immediate obstacle, seen in examples: $(1+\mathcal{D}^2)^{-\frac{1}{2}}$ is almost never finitely summable with respect to Tr_ϕ. To obtain a spectral triple we use the method of [9]. We construct on $\mathcal{N}$ a faithful semifinite normal weight $\phi_{\mathcal{D}} \equiv \mathrm{Tr}_\phi(e^{-\beta\mathcal{D}/2}\cdot e^{-\beta\mathcal{D}/2})$ such that

- the modular automorphism group $\sigma^{\phi_{\mathcal{D}}}$ of $\phi_{\mathcal{D}}$ is implemented by a one parameter unitary group whose generator is $\mathcal{D}$ and, moreover, $\sigma_t^{\phi_{\mathcal{D}}}|_{\mathcal{A}} = \sigma_{-\beta t}$ for all $t \in \mathbb{R}$,
- $\phi_{\mathcal{D}}$ restricts to a faithful normal semifinite trace on the fixed point algebra $\mathcal{M}$ of $\sigma^{\phi_{\mathcal{D}}}$, $\mathcal{D}$ is affiliated to $\mathcal{M}$, and $[\mathcal{D},a]$ extends to a bounded operator (in $\mathcal{N}$) for all a in a dense subalgebra of A,
- for all $f \in F \cap \mathrm{dom}(\phi)$ and all λ in the resolvent set of $\mathcal{D}$, the operator $f(\lambda - \mathcal{D})^{-1}$ belongs to the ideal $\mathcal{K}(\mathcal{M},\phi_{\mathcal{D}})$ of compact operators in $\mathcal{M}$ relative to $\phi_{\mathcal{D}}$.

The problem now is that since A is not contained in $\mathcal{M}$, we do not have an immediate definition of a spectral flow for partial isometries in A. This has led to the definition of a new group $K_1(A,\sigma)$ [9], which is closely related to $K^I(M)$. Namely, a partial isometry $v \in A$ (or in matrices over A) is called **modular** if $[\mathcal{D},v]$ is bounded and $vQv^* \in \mathcal{M} = \mathcal{N}^\sigma$ for every spectral projection of $\mathcal{D}$. There is a semigroup $K_1(A,\sigma)$ defined as the homotopy classes of modular partial isometries in $\mathrm{Mat}_\infty(A^\sim) = \cup_n \mathrm{Mat}_n(A^\sim)$. Via the Grothendieck construction we will henceforth use the same notation for the corresponding group. The main reason to define this group is that for modular partial isometries v,

(Pvv^*, vPv^*) is a Fredholm pair in the semifinite sense [3] in $(\mathcal{M}, \phi_{\mathcal{D}})$ and hence there is a well-defined *analytic spectral flow* along the path $t \mapsto (1-t)(2Pvv^*-1) + t(2vPv^*-1)$, $t \in [0,1]$. This is equal (see [3], Section 6) to the semifinite spectral flow $sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*)$ along the linear path joining $vv^*\mathcal{D}$ to $v\mathcal{D}v^*$.

Let $v \in A$ be a modular partial isometry. Then it can be shown that the decomposition $v = \sum v_k$, $v_k \in A_k$ is finite and every v_k is a partial isometry. We can consider v_k as an operator $\mathcal{H}_\phi[k] \to \mathcal{H}_\phi$, where $\mathcal{H}_\phi[k]$ coincides with $\mathcal{H}_\phi$ as a space, but the representation of $\mathbb{T}$ is tensored by χ^k. Then v_k defines a class $\ll v_k \gg$ in $K_0^{\mathbb{T}}(M)$. There is thus a well defined homomorphism $T\colon K_1(A,\sigma) \to K^I(M)$ given by $v \mapsto \sum_k \ll v_k \gg$.

In Section 4 we obtain the following relationship between the equivariant spectral flow sf with respect to Tr_ϕ and $sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*)$.

Theorem 1.2. *The spectral flow $sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*)$ for modular partial isometries is the composition of the maps*

$$K_1(A,\sigma) \xrightarrow{T} K_0^{\mathbb{T}}(M) \xrightarrow{sf} \mathbb{R}[\chi,\chi^{-1}] \xrightarrow{\mathrm{Ev}(e^{-\beta})} \mathbb{R},$$

where $\mathrm{Ev}(e^{-\beta})$ *is the evaluation at* $\chi = e^{-\beta}$.

In Section 5 we turn to the question of providing a direct analytic formula for $sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*)$.

Theorem 1.3. *Let $v \in \mathcal{A}$ be a modular partial isometry. Then $sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*)$ is given by*

$$\mathrm{Res}_{r=1/2}\left(r \mapsto \phi_{\mathcal{D}}(v[\mathcal{D},v^*](1+\mathcal{D}^2)^{-r}) + \frac{1}{2}\int_1^\infty \phi_{\mathcal{D}}((\sigma_{-i\beta}(v^*)v - vv^*)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds\right).$$

In previous papers [9, 13] it was clear from the numerical values computed for the spectral flow for particular choices of modular partial isometries v that the mapping cone K-theory was playing a role. The preceding two theorems explain exactly these previously somewhat mysterious numerical values.

The above spectral flow formula turns out to be related to twisted cyclic cohomology as follows. Ignoring eta correction terms one can argue by analogy with the standard case of tracial weights (see [7]) to define a $\mathbb{T}$-equivariant Chern character computing the equivariant spectral flow using a Dixmier functional $\mathrm{Tr}_{\phi,\omega}$ by

$$\phi_1(g; a_0, a_1) = \frac{1}{2}\,\mathrm{Tr}_{\phi,\omega}(g a_0[\mathcal{D}, a_1]),$$

where g is an element of a group algebra of $\mathbb{T}$. In Section 5 we can make the definition of $\mathrm{Tr}_{\phi,\omega}$ precise using a residue formula and also define a twisted cocycle ϕ_1 where we choose $g = e^{-\beta}$ considered as an element of the complexification of $\mathbb{T}$. Namely, we have:

Theorem 1.4. *(i) If (A,ϕ,σ) has full spectral subspaces then for all $a_0, a_1, \in \mathcal{A}$ the residue*

$$\phi_1(a_0,a_1) := \mathrm{Res}_{r=1/2}\,\phi_{\mathcal{D}}(a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-r})$$

exists and equals $\phi(a_0[\mathcal{D},a_1])$. It defines a twisted cyclic cocycle on $\mathcal{A}$ with twisting $\sigma_{-i\beta}$, and for any modular partial isometry v

$$sf_{\phi_{\mathcal{D}}}(vv^*, v\mathcal{D}v^*) = \phi_1(v, v^*).$$

(ii) For general circle actions, the bilinear functional on $\mathcal{A}$ given by

$$\psi^r(a_0,a_1) = \phi_{\mathcal{D}}(a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-r}) + \frac{1}{2}\int_1^\infty \phi_{\mathcal{D}}((\sigma_{-i\beta}(a_1)a_0 - a_0a_1)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds$$

depends holomorphically on r for $\Re(r) > 1/2$ and modulo functions which are holomorphic for $\Re(r) > 0$ is a function valued $\sigma_{-i\beta}$-twisted (b,B)-cocycle.

We have encountered a similar situation in [27]. There the $SU_q(2)$-equivariant Chern character of the Dirac operator on the quantum sphere was evaluated at a special element $\rho \in U_q(\mathfrak{su}_2)$ and produced an explicitly computable twisted cyclic cocycle. In our current situation the evaluation at $g = e^{-\beta}$ is needed to improve summability whereas there the purpose was somewhat opposite: the evaluation at ρ prevented the dimension drop and produced a 2^+-summable spectral triple in the twisted sense instead of a 0^+-summable one. In both cases however we see that twisted cohomology appears as an analytically manageable part of an equivariant cohomology.

We make the observation that our constructions and resulting index theorems are not related to the approach of Connes and Moscovici in [15].

2. The construction of equivariant KK classes from a circle action

2.1. **A Kasparov module from a circle action.** Let A be a C^*-algebra, $\sigma\colon \mathbb{T} \to \mathrm{Aut}(A)$ a strongly continuous action of the circle. It will be convenient to consider σ as a 2π-periodic one-parameter group of automorphisms. We denote by F the fixed point algebra $\{a \in A \mid \sigma_t(a) = a \ \ \forall t \in \mathbb{R}\}$. Since $\mathbb{T}$ is a compact group, the map

$$\Phi\colon A \to F, \qquad \Phi(a) = \frac{1}{2\pi}\int_0^{2\pi} \sigma_t(a)dt$$

is a faithful conditional expectation. Next define an F-valued inner product on A by $(a|b)_R := \Phi(a^*b)$. The properties of Φ allow us to see that this is a (pre)-C^*-inner product on A, and so we may complete A in the topology determined by the norm $\|a\|_X^2 = \|(a|a)_R\|_F$ to obtain a C^*-module, for the right action of F.

Definition 2.1. *We let $X = \overline{A}$ be the C^*-module completion of A with inner product $(\cdot|\cdot)_R$.*

The circle action is defined on the dense subspace $A \subset X$ and extends to a unitary action on X. The F-module $X$is a full F-module for the right inner product. For $k \in \mathbb{Z}$, denote the eigenspaces of the action σ by

$$A_k = \{a \in A : \sigma_t(a) = e^{ikt}a \ \text{ for all } t \in \mathbb{R}\}.$$

Then $F = A_0$, which guarantees the fullness of X over F. Also A is a $\mathbb{Z}$-graded algebra in an obvious way, $A_{-k} = (A_k)^*$ and in particular, each A_k is an F-module. Note that the norm on A_k defined by the above inner product coincides with the C^*-norm. We denote by X_k the space A_k considered as a closed submodule of X. For $k \in \mathbb{Z}$ we set $F_k = \overline{A_k A_k^*}$. Some of our results using KK-theoretic constructions require the following assumption.

Definition 2.2. *The action σ on A satisfies the* **Spectral Subspace Assumption** *(SSA) if F_k is a complemented ideal in F for every $k \in \mathbb{Z}$. Equivalently, the representation $\pi_k\colon F \to \mathrm{End}_F(X_k)$ given by left multiplication satisfies $\pi_k(F) = \pi_k(F_k)$ (then $\ker \pi_k$ is the complementary ideal to F_k).*

There is a special case of this assumption which is well known, namely A is said to have **full spectral subspaces** if $F_k = F$ for all $k \in \mathbb{Z}$. The gauge action on the Cuntz algebras $\mathcal{O}_n$ provides examples where fullness holds. The quantum group $SU_q(2)$ with its Haar state and associated circle action is an example of an algebra satisfying the SSA but not having full spectral subspaces [13].

Lemma 2.3. *If $\overline{A_1A_1^*} = \overline{A_1^*A_1} = A_0$, the modules X_k and $\overline{X}_k$ are full for all $k \in \mathbb{Z}$.*

Proof. Observe that as $A_1A_1^* \subset A_0$, we have $A_1 = \overline{A_0A_1}$. So if $A_0 = \overline{A_1^*A_1}$, by induction we get $A_0 = \overline{(A_1^k)^*A_1^k}$ for $k \geq 1$. Since $A_1^k \subset A_k$, we conclude that X_k is full. Similarly, if $k \leq -1$ then $(A_1^*)^{-k} \subset A_k$, so $A_0 = \overline{A_1A_1^*}$ implies that X_k is full. $\square$

Note that as A_0A_k is dense in A_k, we always have $\overline{FA} = A$ and similarly $\overline{AF} = A$ (this also follows from the existence of a σ-invariant approximate unit in A). As there are many examples of circle actions which are not full but satisfy the SSA we will develop the theory in this generality in the present Section.

Next we remark that the general theory of C^*-modules (or Hilbert modules) is discussed in many places and we will use [26, 33]. For a right C^*-B-module Y, we let $\mathrm{End}_B(Y)$ be the C^*-algebra (for the operator norm) of adjointable endomorphisms, $\mathrm{End}^0_B(Y)$ the ideal of compact endomorphisms, which is the completion of the finite rank endomorphisms: $\mathrm{End}^{00}_B(Y)$. The latter is generated by the rank one endomorphisms $\Theta_{x,y}$, $x, y \in Y$, defined by $\Theta_{x,y}z = x(y|z)_R$, $z \in Y$.

For each $k \in \mathbb{Z}$, the projection onto the k-th spectral subspace for the circle action is defined by an operator Φ_k on X via

$$\Phi_k(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\sigma_t(x)dt, \quad x \in X.$$

The range of Φ_k is the submodule X_k. These ranges give us the natural $\mathbb{Z}$-grading of X. The operators Φ_k are adjointable endomorphisms of the F-module X such that $\Phi_k^* = \Phi_k = \Phi_k^2$ and $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$. If $K \subset \mathbb{Z}$ then the sum $\sum_{k\in K}\Phi_k$ converges strictly to a projection in the endomorphism algebra, [28]. In particular sum $\sum_{k\in\mathbb{Z}}\Phi_k$ converges strictly to the identity operator on X.

The following Lemma is the key step in obtaining a Kasparov module.

Lemma 2.4. *For a circle action on A the following conditions are equivalent:*
(i) the action satisfies the SSA;
(ii) for all $a \in A$ and $k \in \mathbb{Z}$, the endomorphism $a\Phi_k$ of the right F-module X is compact.

Proof. Assume the action satisfies the SSA. If $x, y \in A_k$ and $z \in X$, then

$$\Theta_{x,y}z = x\Phi(y^*z) = x\Phi(y^*z_k) = xy^*z_k = xy^*\Phi_k z.$$

Thus $\Theta_{x,y} = xy^*\Phi_k$. It follows that $a\Phi_k$ is compact for any $a \in A_kA_k^*$. Since $A_kA_k^*$ is dense in F_k, we see that $f\Phi_k$ is compact for any $f \in F_k$, and hence $f\Phi_k$ is compact for any $f \in F$ by the SSA. But then $af\Phi_k$ is compact for any $f \in F$ and $a \in A$. Since AF is dense in A, we can approximate $b\Phi_k$ for any $b \in A$ by (compact) endomorphisms of the form $af\Phi_k$.

Conversely, assume $f\Phi_k$ is compact for some $f \in F$, so that $f\Phi_k$ can be approximated by finite sums of operators $\Theta_{x,y}$, $x, y \in X_k$. We have seen, however, that $\Theta_{x,y} = xy^*\Phi_k$, and so $\pi_k(f)$ is in $\pi_k(F_k)$. Therefore if $f\Phi_k$ is compact for all $f \in F$ and $k \in \mathbb{Z}$, the SSA is satisfied. □

Since we have the circle action defined on X, we may use the generator of this action to define an unbounded operator $\mathcal{D}$. We will not define or study $\mathcal{D}$ from the generator point of view, instead taking a more bare-hands approach. It is easy to check that $\mathcal{D}$ as defined below is the generator of the circle action. The theory of unbounded operators on C^*-modules that we require is all contained in Lance's book, [26, Chapters 9,10]. We quote the following definitions (adapted to our situation).

Definition 2.5. [26] *Let Y be a right C^*-B-module. A densely defined unbounded operator*

$$\mathcal{D}\colon \mathrm{dom}\ \mathcal{D} \subset Y \to Y$$

is a B-linear operator defined on a dense B-submodule dom $\mathcal{D} \subset Y$. *The operator $\mathcal{D}$ is closed if the graph $G(\mathcal{D}) = \{(x, \mathcal{D}x) : x \in \mathrm{dom}\ \mathcal{D}\}$ is a closed submodule of $Y \oplus Y$.*

If $\mathcal{D}\colon \mathrm{dom}\ \mathcal{D} \subset Y \to Y$ is densely defined and unbounded, define a submodule

$$\mathrm{dom}\ \mathcal{D}^* := \{y \in Y : \exists z \in Y \text{ such that } \forall x \in \mathrm{dom}\ \mathcal{D}, (\mathcal{D}x|y)_R = (x|z)_R\}.$$

Then for $y \in \text{dom } \mathcal{D}^*$ define $\mathcal{D}^* y = z$. Given $y \in \text{dom } \mathcal{D}^*$, the element z is unique, so $\mathcal{D}^* \colon \text{dom}\mathcal{D}^* \to Y$, $\mathcal{D}^* y = z$ is well-defined, and moreover is closed.

Definition 2.6. [26] *Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D}$ is symmetric if for all $x, y \in$ dom $\mathcal{D}$ we have $(\mathcal{D}x|y)_R = (x|\mathcal{D}y)_R$. A symmetric operator $\mathcal{D}$ is self-adjoint if* dom $\mathcal{D}$ = dom $\mathcal{D}^*$ *(and so $\mathcal{D}$ is necessarily closed). A densely defined unbounded operator $\mathcal{D}$ is regular if $\mathcal{D}$ is closed, $\mathcal{D}^*$ is densely defined, and $1 + \mathcal{D}^*\mathcal{D}$ has dense range.*

The extra requirement of regularity is necessary in the C^*-module context for the continuous functional calculus, and is not automatic, [26, Chapter 9]. With these definitions in hand, we return to our C^*-module X. The following can be proved just as in [28, Proposition 4.6], or equivalently by observing that the operator $\mathcal{D}$ is presented in diagonal form.

Proposition 2.7. *Let X be the right C^*-F-module of Definition 2.1. Define $X_{\mathcal{D}} \subset X$ to be the linear space*

$$X_{\mathcal{D}} = \{x = \sum_{k \in \mathbb{Z}} x_k \in X : \|\sum_{k \in \mathbb{Z}} k^2 (x_k|x_k)_R\| < \infty\}.$$

For $x = \sum_{k \in \mathbb{Z}} x_k \in X_{\mathcal{D}}$ define $\mathcal{D}x = \sum_{k \in \mathbb{Z}} k x_k$. Then $\mathcal{D} \colon X_{\mathcal{D}} \to X$ is a self-adjoint regular operator on X.

There is a continuous functional calculus for self-adjoint regular operators, [26, Theorem 10.9], and we use this to obtain spectral projections for $\mathcal{D}$ at the C^*-module level. Let $f_k \in C_c(\mathbb{R})$ be 1 in a small neighbourhood of $k \in \mathbb{Z}$ and zero on $(-\infty, k - 1/2] \cup [k + 1/2, \infty)$. Then it is clear that $\Phi_k = f_k(\mathcal{D})$. That is the spectral projections of $\mathcal{D}$ are the same as the projections onto the spectral subspaces of the circle action.

Lemma 2.8. *If the SSA holds , then for all $a \in A$, the operator $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism of the F-module X.*

Proof. Since $a\Phi_k$ is a compact endomorphism for all $a \in A$, and $a\Phi_k$, $a\Phi_m$ have orthogonal initial spaces, the sum

$$a(1 + \mathcal{D}^2)^{-1/2} = \sum_{k \in \mathbb{Z}} (1 + k^2)^{-1/2} a\Phi_k$$

converges in norm to a compact endomorphism. □

Proposition 2.9. *If the SSA holds, the pair $(X, \mathcal{D})$ is an unbounded Kasparov module defining a class in $KK_1(A, F)$.*

Proof. We will use the approach of [22, Section 4]. Let $V = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$. We need to show that various operators belong to $\text{End}^0_F(X)$. First, $V - V^* = 0$, so $a(V - V^*)$ is compact for all $a \in A$. Also $a(1 - V^2) = a(1 + \mathcal{D}^2)^{-1}$ which is compact from Lemma 2.8 and the boundedness of $(1 + \mathcal{D}^2)^{-1/2}$. Finally, we need to show that $[V, a]$ is compact for all $a \in A$. First we suppose that $a = a_m$ is homogenous for the circle action. Then

$$\begin{aligned} [V, a] &= [\mathcal{D}, a](1 + \mathcal{D}^2)^{-1/2} - \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}[(1 + \mathcal{D}^2)^{1/2}, a](1 + \mathcal{D}^2)^{-1/2} \\ &= b_1(1 + \mathcal{D}^2)^{-1/2} + V b_2 (1 + \mathcal{D}^2)^{-1/2}, \end{aligned}$$

where $b_1 = [\mathcal{D}, a] = ma$ and $b_2 = [(1 + \mathcal{D}^2)^{1/2}, a]$. Provided that $b_2(1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism, Lemma 2.8 will show that $[V, a]$ is compact for all homogenous a. So consider the action of $[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}$ on $x=\sum_{k\in\mathbb{Z}} x_k$. We find

$$\begin{aligned}\sum_{k\in\mathbb{Z}}[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}x_k &= \sum_{k\in\mathbb{Z}}\Big((1+(m+k)^2)^{1/2}-(1+k^2)^{1/2}\Big)(1+k^2)^{-1/2}ax_k\\ &= \sum_{k\in\mathbb{Z}} f_m(k)a\Phi_k x. \qquad (1)\end{aligned}$$

The function

$$f_m(k)=\Big((1+(m+k)^2)^{1/2}-(1+k^2)^{1/2}\Big)(1+k^2)^{-1/2}$$

goes to 0 as $k\to\pm\infty$, and as the $a_m\Phi_k$ are compact with orthogonal ranges, the sum in (1) converges in the operator norm on endomorphisms and so converges to a compact endomorphism. For $a\in A$ a finite sum of homogenous terms, we apply the above reasoning to each term in the sum to find that $[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism.

Now let $a\in A$ be the norm limit of a Cauchy sequence $\{a_i\}_{i\geq 0}$ where each a_i is a finite sum of homogenous terms. Then

$$\|[V, a_i-a_j]\|_{\mathrm{End}}\leq 2\|a_i-a_j\|_{\mathrm{End}}\to 0,$$

so the sequence $[V, a_i]$ is also Cauchy in norm, and so the limit is compact. □

Corollary 2.10. *If the SSA holds, the pair $(X,\mathcal{D})$ defines a class in the equivariant KK-group $KK_1^{\mathbb{T}}(A,F)$.*

The proof of the corollary is obvious from the constructions.

2.2. **The equivariant constructions for the mapping cone algebra.** From the unbounded Kasparov A-F-module $(X,\mathcal{D})$, we shall construct a new equivariant Kasparov $M(F,A)$-F-module $(\hat{X},\hat{\mathcal{D}})$. By pairing the class of the module $(\hat{X},\hat{\mathcal{D}})$ with elements of $K_0^{\mathbb{T}}(M(F,A))$ we then get a map $K_0^{\mathbb{T}}(M(F,A))\to K_0^{\mathbb{T}}(F)$. Here $M(F,A)$ is the mapping cone C^*-algebra for the inclusion $F\hookrightarrow A$ defined by

$$M(F,A)=\{f\colon[0,\infty)\to A\,:\, f\in C_0([0,\infty),A),\ f(0)\in F\}.$$

The mapping cone algebra carries the circle action coming from the circle action on A.

In [31], Putnam showed that the K_0 group of $M(F,A)$ is given by homotopy classes of partial isometries $v\in A^\sim\otimes\mathrm{Mat}_k(\mathbb{C})$ with vv^*, $v^*v\in F^\sim\otimes\mathrm{Mat}_k(\mathbb{C})$. Before summarising the construction of $(\hat{X},\hat{\mathcal{D}})$ from [8], we adapt Putnam's description of the K-theory of the mapping cone to an equivariant setting.

Denote by $V^{\mathbb{T}}(F,A)$ the set of $\sigma\otimes\mathrm{Ad}U$ invariant partial isometries $v\in A^\sim\otimes\mathcal{B}(\mathcal{H})$, where $U\colon\mathbb{T}\to\mathcal{B}(\mathcal{H})$ is some finite dimensional unitary representation (which varies with v but which we denote generically by U), such that vv^* and v^*v belong to $F^\sim\otimes\mathcal{B}(\mathcal{H})$.

Consider the equivalence relation on $V^{\mathbb{T}}(F,A)$ generated by the following two conditions: two invariant partial isometries v_1, v_2 are equivalent if they are joined by a homotopy consisting of invariant partial isometries or if they are a pair of the form $v, v\oplus p$, where p is an invariant projection in $F^\sim\otimes\mathcal{B}(\mathcal{H})$ for some $\mathbb{T}$-module $\mathcal{H}$. For $v\in V^{\mathbb{T}}(F,A)$ we define a projection p_v in a matrix algebra over the unitization of the mapping cone algebra $M(F,A)$ by

$$p_v(t)=\begin{pmatrix} 1-vv^*+\frac{t^2vv^*}{1+t^2} & -iv\frac{t}{1+t^2}\\ iv^*\frac{t}{1+t^2} & \frac{v^*v}{1+t^2}\end{pmatrix}.$$

Then $[p_v]-\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]$ is an element of $K_0^{\mathbb{T}}(M(F,A))$ (in particular p_v is a $\sigma\otimes\mathrm{Ad}\,U$ invariant projection, for a suitable representation U of $\mathbb{T}$ coming from the representation associated with v), and the map

$$v \mapsto [p_v] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

is a bijection of $V^{\mathbb{T}}(F,A)/\sim$ onto $K_0^{\mathbb{T}}(M(F,A))$. Note that Putnam considers the non-equivariant case, but the reader can easily check that all his proofs and constructions carry over to the $\mathbb{T}$-equivariant case. A general class in $K_0^{\mathbb{T}}(M(F,A))$ is denoted by $[v]$ or equivalently by $[p_v]-[1]$.

The group $K_0^{\mathbb{T}}(M(F,A))$ is a module over the representation ring of $\mathbb{T}$, which we identify with the ring $\mathcal{R}_{\mathbb{T}} = \mathbb{R}[\chi,\chi^{-1}]$ of Laurent polynomials with real coefficients; therefore χ^n denotes the one-dimensional representation $t \mapsto e^{int}$. For a $\mathbb{T}$-module $\mathcal{H}$ we denote by $\mathcal{H}[n]$ the module with the same underlying space but with the action tensored with χ^n. Now in terms of partial isometries, the $\mathcal{R}_{\mathbb{T}}$-module structure on $K_0^{\mathbb{T}}(M(F,A))$ is described as follows: if $v \in (A^\sim \otimes \mathcal{B}(\mathcal{H}))^{\mathbb{T}}$ then $\chi[v]$ is the class of the partial isometry v considered as an element of $(A^\sim \otimes \mathcal{B}(\mathcal{H}[1]))^{\mathbb{T}}$.

The construction of $(\hat{X},\hat{\mathcal{D}})$ follows [8], where a C^*-algebra analogue of the Atiyah-Patodi-Singer (APS) theory, [2], was described. We take as our starting point the equivariant Kasparov module $(X,\mathcal{D})$ coming from the circle action σ on the C^*-algebra A.

First form the space of finite sums of elementary tensors $f = \sum_j f_j \otimes x_j$ where the f_j are compactly supported smooth functions on $[0,\infty)$ and the $x_j \in X$. Then complete this space using the C^*-module norm coming from the inner product

$$(f|g)_{L^2([0,\infty))\otimes X} = \sum_{i,j} \int_0^\infty \bar{f}_i(t) g_j(t)dt\,(x_i|y_j)_X,$$

which for convenience we write as

$$(f|g)_{L^2([0,\infty))\otimes X} := \int_0^\infty (f_t|g_t)_X dt.$$

This module is the external tensor product $L^2([0,\infty))\otimes X$. It carries an obvious left action of $M(F,A)$. We caution the reader that it is not clear that the *completion* of the space of finite sums of elementary tensors is a function space. Discussion of this matter and the proof that the next definition does in fact provide an unbounded Kasparov module can be found in [8].

Definition 2.11 ([8])**.** *Assume the SSA is satisfied. Define a graded unbounded equivariant Kasparov $M(F,A)$-F-module by*

$$(\hat{X},\hat{\mathcal{D}}) = \left(\begin{pmatrix} L^2([0,\infty))\otimes X \\ L^2([0,\infty))\otimes X \oplus \Phi_0 X \end{pmatrix}, \begin{pmatrix} 0 & -\partial_t + \mathcal{D} \\ \partial_t + \mathcal{D} & 0 \end{pmatrix} \right),$$

where we use APS boundary conditions in the sense that we take the initial domain of $\hat{\mathcal{D}}$ to be the finite linear span of elementary tensors ξ such that $\xi \in \hat{X}$ and $\hat{\mathcal{D}}\xi \in \hat{X}$ with $P\xi_1(0) = 0,\ (1-P)\xi_2(0) = 0$, where $P = \chi_{[0,+\infty)}(\mathcal{D}) = \sum_{k\geq 0}\Phi_k$ is the non-negative spectral projection of $\mathcal{D}$.

Remark. The additional copy of $\Phi_0 X$ (which has as inner product the restriction of the inner product on X) allows us to use extended L^2 functions as in [2, pp 58-60]. These are defined by considering functions f that are finite sums of elementary tensors $\sum_j f_j \otimes x_j$ where the f_j are functions on $[0,\infty)$ with a limit $f_j(\infty)$ as $t\to\infty$. Then f has a limit at infinity and we restrict our attention to those f such that $f - f(\infty)$ is in $L^2([0,\infty))\otimes X$ and $\mathcal{D}f(\infty) = 0$. The inner product in the second component is then

$$(f|f) = \int_0^\infty (f(t)-f(\infty)|f(t)-f(\infty))_X dt + (f(\infty)|f(\infty))_X.$$

The Kasparov module $(\hat{X}, \hat{\mathcal{D}})$ is equivariant using the circle action on A, which is trivial in the 'ℝ'-direction. It thus defines an element of $KK_0^{\mathbb{T}}(M(F,A),F)$. By pairing it with elements of $K_0^{\mathbb{T}}(M(F,A))$ we get a homomorphism $\mathrm{Index}_{\hat{\mathcal{D}}}\colon K_0^{\mathbb{T}}(M(F,A)) \to K_0^{\mathbb{T}}(F)$.

Theorem 2.12 ([8])**.** *Assume the SSA is satisfied. Let $U\colon \mathbb{T} \to \mathcal{B}(\mathcal{H})$ be a finite dimensional unitary representation and $v \in A^\sim \otimes \mathcal{B}(\mathcal{H})$ a $\sigma \otimes \mathrm{Ad}\, U$ invariant partial isometry with v^*v and vv^* projections in $F^\sim \otimes \mathcal{B}(\mathcal{H})$. Assume that the $\sigma \otimes \iota$ homogeneous components of v are partial isometries. Then we have*

$$\begin{aligned}&\mathrm{Index}_{\hat{\mathcal{D}}}\left([p_v] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\right]\right)\\ &= -\,\mathrm{Index}\left((P\otimes 1)v(P\otimes 1)\colon v^*v(P\otimes 1)X\otimes\mathcal{H} \to vv^*(P\otimes 1)X\otimes\mathcal{H}\right) \in K_0^{\mathbb{T}}(F).\end{aligned}$$

The proof is exactly the same as the non-equivariant result of [8], except that one must check that the kernel and cokernel projections are indeed invariant, which is immediate from the equivariance of the Kasparov module.

Remarks.

(i) In [8] we could not state the (nonequivariant version of the) above theorem for every element in $K_0(M(F,A))$, but only those with particular commutation relations with spectral projections of $\mathcal{D}$. The additional assumption on v in the above formulation is enough to get those relations satisfied. Indeed, if $v \in A^\sim \otimes \mathcal{B}(\mathcal{H})$ is homogenous then $v[\mathcal{D}\otimes 1, v^*] = kvv^*$ for some $k \in \mathbb{Z}$, and this commutes with $\mathcal{D}\otimes 1$.

(ii) One may also try to describe the class $[\hat{\mathcal{D}}]$ in the following way. First realise the class $[\mathcal{D}]$ as an extension

$$0 \to \mathcal{K}\otimes F \to E \overset{\rho}{\underset{\leftarrow}{\to}} A \to 0$$

with ρ the completely positive splitting given by $a \to PaP$, for $a \in A$, and $P = \chi_{[0,\infty)}(\mathcal{D})$. As P commutes with F, ρ is an injective homomorphism when restricted to F, and so gives us a copy of F inside E. From this we may deduce the exactness of the sequence

$$0 \to \mathcal{K}\otimes \mathcal{S}F \to M(F,E) \overset{\tilde{\rho}}{\underset{\leftarrow}{\to}} M(F,A) \to 0,$$

where $M(F,E)$, $M(F,A)$ denote the mapping cones of the respective inclusions. Corresponding to this extension is a class

$$[\tilde{\mathcal{D}}] \in KK_1^{\mathbb{T}}(M(F,A), \mathcal{S}F) = KK_1^{\mathbb{T}}(\mathcal{S}M(F,A), F) = KK_0^{\mathbb{T}}(M(F,A),F).$$

There is some evidence that the class $[\tilde{\mathcal{D}}]$ coincides with the class of $[\hat{\mathcal{D}}]$.

The theorem gives us two important tools. The first is that the pairing of $(\hat{X}, \hat{\mathcal{D}})$ is given by the Kasparov product and so enjoys all the usual functorial properties. The second is that we can compute the index pairing of the theorem by considering Toeplitz type operators $(P\otimes 1)v(P\otimes 1)$, for which the computation is much simpler. These tools use only the circle action. Next we exploit the KMS weight ϕ.

3. The equivariant spectral flow

3.1. **The induced trace.** A KMS weight provides some analytic tools that we now explain.

Definition 3.1. *A weight ϕ on a C^*-algebra A is (σ,β)-KMS weight (KMS$_\beta$ weight for short) if ϕ is a semifinite, norm lower semicontinuous, σ-invariant weight such that $\phi(aa^*) = \phi(\sigma_{i\beta/2}(a)^*\sigma_{i\beta/2}(a))$ for all $a \in \mathrm{dom}(\sigma_{i\beta/2})$.*

Here $\mathrm{dom}(\sigma_{i\beta/2})$ consists of all elements $a \in A$ such that $t \mapsto \sigma_t(a)$ extends to a continuous function from $0 \leq \Im(t) \leq \beta/2$ which is analytic in the open strip. We will assume throughout the rest of the paper that ϕ is a faithful KMS_β weight on A. Introduce the notation

$$\mathrm{dom}(\phi)_+ = \{a \in A_+ : \phi(a) < \infty\}, \quad \mathrm{dom}(\phi)^{1/2} = \{a \in A : a^*a \in \mathrm{dom}(\phi)_+\},$$

$$\mathrm{dom}(\phi) = \mathrm{span}\{\mathrm{dom}(\phi)_+\} = (\mathrm{dom}(\phi)^{1/2})^*\mathrm{dom}(\phi)^{1/2},$$

and extend ϕ to a linear functional on $\mathrm{dom}(\phi)$. Recall that we defined a conditional expectation $\Phi\colon A \to F$. We let τ be the faithful norm lower semicontinuous semifinite trace on F given by $\phi|_F$. Then $\phi = \tau \circ \Phi$, as ϕ is assumed to be σ-invariant.

The GNS construction yields a Hilbert space $\mathcal{H} := \mathcal{H}_\phi$, and a map $\Lambda\colon \mathrm{dom}(\phi)^{1/2} \to \mathcal{H}$ with dense image and $\langle \Lambda(a), \Lambda(b)\rangle = \phi(a^*b)$, where $\langle \cdot, \cdot \rangle$ is the inner product. In fact, $\Lambda(\mathrm{dom}(\phi)^{1/2} \cap (\mathrm{dom}(\phi)^{1/2})^*)$ is a left Hilbert algebra. The algebra A is represented on $\mathcal{H}$ as left multiplication operators, $a\Lambda(b) = \Lambda(ab)$, and the weight ϕ extends to a normal semifinite faithful weight on the von Neumann algebra $\pi(A)''$. We have $\sigma_t^\phi = \sigma_{-\beta t}$ on A, [23].

We now construct a semifinite von Neumann algebra from a given faithful KMS state or weight ϕ on A. We need results from [25] at this point. Namely consider the space $\mathcal{H}_\tau$ of the GNS-representation of the trace τ on F; then $\mathcal{H}$ can be identified with $X \otimes_F \mathcal{H}_\tau$ via the map $a \otimes \Lambda_\tau(f) \mapsto \Lambda_\phi(af)$. It follows that the action of A on $\mathcal{H}$ extends to a representation of $\mathrm{End}_F(X)$ on $\mathcal{H}$.

Lemma 3.2. *We let $\mathcal{N} = \mathrm{End}(X)'' \subset \mathcal{B}(\mathcal{H})$, then there is a faithful normal semifinite trace Tr_ϕ on $\mathcal{N}$ such that $\mathrm{Tr}_\phi(\Theta_{\xi,\xi}) = \tau((\xi|\xi)_R)$ for all $\xi \in X$.*

Proof. Consider first the case when $X \cong H \otimes F$ as a right Hilbert F-module, where H is a Hilbert space. Then $\mathcal{H} \cong H \otimes \mathcal{H}_\tau$, $\mathcal{N} \cong \mathcal{B}(H)\bar{\otimes}F''$ and the trace Tr_ϕ is simply $\mathrm{Tr} \otimes \tau$. We can then conclude that Tr_ϕ exists if X is only a direct summand of $H \otimes F$. This is the case when X is countably generated (in particular, when A is separable) by Kasparov's stabilization theorem, with $H = \ell^2(\mathbb{N})$. In general to construct Tr_ϕ we can argue as follows.

The commutant of $\mathcal{N}$ can be identified with the commutant of F in $\mathcal{B}(\mathcal{H}_\tau)$, that is, with the von Neumann algebra generated by elements $f \in F$ acting on the right. To put it differently,

$$\mathcal{N}' = (JFJ)'', \tag{2}$$

where J is the modular conjugation defined by ϕ and F acts on the left. Define a trace τ' on $(JFJ)''$ by $\tau'(Jf^*J) = \tau(f)$. At this moment we need to recall the notion of spatial derivative, see [34].

Assume we are given faithful normal semifinite weights ψ on $\mathcal{N}$ and ρ on $\mathcal{N}'$. A vector $\xi \in \mathcal{H}$ is called ρ-bounded if the map $\Lambda_\rho(x) \mapsto x\xi$, $x \in \mathrm{dom}(\rho)^{1/2}$, extends to a bounded map $R_\xi\colon \mathcal{H}_\rho \to \mathcal{H}$. As R_ξ is an $\mathcal{N}'$-module map, the operator $R_\xi R_\xi^*$ belongs to $\mathcal{N}$. The quadratic form

$$\{\xi \in \mathcal{H} \mid \xi \text{ is } \rho\text{-bounded}, \ \psi(R_\xi R_\xi^*) < \infty\} \ni \xi \mapsto q(\xi) := \psi(R_\xi R_\xi^*)$$

is closable and hence defines a positive self-adjoint operator $\Delta(\psi/\rho)$ such that $q(\xi) = \|\Delta(\psi/\rho)^{1/2}\xi\|^2$. The main property of spatial derivatives is that for any fixed ρ the map $\psi \mapsto \Delta(\psi/\rho)$ gives a one-to-one correspondence between faithful normal semifinite weights ψ on $\mathcal{N}$ and nonsingular positive self-adjoint operators Δ such that $\Delta^{it}x\Delta^{-it} = \sigma_{-t}^\rho(x)$ for $x \in \mathcal{N}'$.

The spatial derivative now gives us the definition of a trace Tr_ϕ on $\mathcal{N} = \mathrm{End}(X)''$ by requiring $\Delta(\mathrm{Tr}_\phi/\tau') = 1$. It is not difficult to check, see [25, Section 3], that for $\xi \in X$ we indeed have $\mathrm{Tr}_\phi(\Theta_{\xi,\xi}) = \tau((\xi|\xi)_R)$. □

The restriction of Tr_ϕ to $\mathrm{End}_F(X)$ is a strictly lower semicontinuous strictly semifinite trace, see e.g. [25, Section 3]. In addition we notice that as $\Theta_{x,x} = xx^*\Phi_0$ and $\tau(xx^*) = \tau(x^*x)$ for $x \in F$, we can conclude that

$$\mathrm{Tr}_\phi(f\Phi_0) = \tau(f) \tag{3}$$

for $f \in F_+$. Identities (2)-(3) mean that $\mathcal{N}$ is being given by the basic von Neumann algebra construction associated with the conditional expectation $\Phi\colon A'' \to F''$, while Tr_ϕ is the canonical trace on $\mathcal{N}$ defined by the trace τ on F'', [30].

Lemma 3.3. *Let A, σ, ϕ, $F = A^\sigma$ be as above. For all $f \in F$, $f \geq 0$ and $k \in \mathbb{Z}$, $k \neq 0$, we have*

$$\mathrm{Tr}_\phi(f\Phi_k) \leq e^{k\beta}\tau(f),$$

and equality holds if A has full spectral subspaces.

Proof. Consider first $f = xx^*$, $x \in A_k$. Then $f\Phi_k = \Theta_{x,x}$ and hence

$$\mathrm{Tr}_\phi(f\Phi_k) = \phi(x^*x) = e^{k\beta}\phi(xx^*) = e^{k\beta}\tau(f).$$

Therefore $\mathrm{Tr}_\phi(f\Phi_k) = e^{k\beta}\tau(f)$ if f is a finite sum of elements of the form xx^*, $x \in A_k$. Since both $\mathrm{Tr}_\phi(\cdot\,\Phi_k)$ and τ are lower semicontinuous traces on F, we conclude that, as $A_kA_k^*$ is a dense ideal in F_k, $\mathrm{Tr}_\phi(f\Phi_k) = \tau(f)$ for any $f \in F_k$, $f \geq 0$. Thus if $F_k = F$ for all $k \in \mathbb{Z}$ we get equality for all $f \geq 0$ and k.

In the more general situation consider the ideal $F_k = \overline{A_kA_k^*}$ in F. Choose an approximate unit $\{\psi_\lambda\}_\lambda$ for F_k. Since $A_kA_k^*A_k$ is dense in A_k, we have $\psi_\lambda x \to x$ for any $x \in X_k$. Hence $\psi_\lambda f\psi_\lambda$ converges strongly to the action of f on X_k for any $f \in F$. Since Tr_ϕ is strictly lower semicontinuous, for $f \geq 0$ we therefore get

$$\mathrm{Tr}_\phi(f\Phi_k) \leq \liminf_\lambda \mathrm{Tr}_\phi(\psi_\lambda f\psi_\lambda\Phi_k) = \liminf_\lambda e^{k\beta}\tau(\psi_\lambda f\psi_\lambda) = \liminf_\lambda e^{k\beta}\tau(f^{1/2}\psi_\lambda^2 f^{1/2}) \leq e^{k\beta}\tau(f).$$

□

Remark. As the proof shows we do not need compactness of the spectral projections Φ_k, only the strong convergence of $\psi_\lambda f\psi_\lambda$ to f on X_k and strict lower semicontinuity of Tr_ϕ. If the SSA holds then $\psi_\lambda f\psi_\lambda \to f$ on X_k in norm.

3.2. **The spectral flow.** Our reference for Breuer-Fredholm theory and semifinite spectral flow is [3]. We recall from Section 6 of that paper that if $\mathcal{N}$ is a semifinite von Neumann algebra with faithful normal semifinite trace τ and $\mathcal{D}_1$, $\mathcal{D}_2$ are closed self-adjoint operators affiliated with $\mathcal{N}$ which differ by a bounded operator and whose spectral projections $P_1 = \chi_{[0,+\infty)}(\mathcal{D}_1)$ and $P_2 = \chi_{[0,+\infty)}(\mathcal{D}_2)$ are such that the operator $P_1P_2 \in P_1\mathcal{N}P_2$ is Breuer-Fredholm, then the spectral flow is defined by

$$sf(\mathcal{D}_1, \mathcal{D}_2) = \mathrm{Index}_\tau(P_1P_2).$$

In the case when P_1 and P_2 are finite we clearly have $sf(\mathcal{D}_1, \mathcal{D}_2) = \tau(P_2) - \tau(P_1)$.

Now let A, $\mathcal{H}$, $\mathcal{N}$ be as in the previous Subsection. The unbounded operator $\mathcal{D}$ on X, introduced in Subsection 2.1, extends to a closed self-adjoint operator on $\mathcal{H}$, which we still denote by $\mathcal{D}$. Put $\sigma_t(x) = e^{it\mathcal{D}}xe^{-it\mathcal{D}}$ for $x \in \mathcal{N}$. The action of $\mathbb{T}$ on X extends to a unitary representation of $\mathbb{T}$ on $\mathcal{H}$, namely, $t \mapsto e^{it\mathcal{D}}$. We want to define a map from $K_0^{\mathbb{T}}(M(F, A), F)$ to the representation ring of the circle which we will call the equivariant spectral flow. Roughly speaking it will compute the spectral flow between the operators $vv^*(\mathcal{D}\otimes 1)$ and $v(\mathcal{D}\otimes 1)v^*$ on invariant subspaces for the $\mathbb{T}$-action. However, if ϕ is a weight, even the restriction of the above operators to an invariant subspace may not be enough to get a well-defined spectral flow. So we have to pay attention to domain issues.

Lemma 3.4. *Let A be a C^*-algebra and ϕ a weight on A as above. For every $n \in \mathbb{N}$ the dense subalgebra $\mathrm{Mat}_n(\mathrm{dom}(\phi)^\sim)$ of $\mathrm{Mat}_n(A^\sim)$ is closed under the holomorphic functional calculus.*

Proof. In order to prove the Lemma it suffices to show that if Γ is a closed smooth curve in $\mathbb{C}$, $a \in \mathrm{Mat}_n(\mathrm{dom}(\phi)^\sim)$ has spectrum (as an element of $\mathrm{Mat}_n(A^\sim)$) which does not intersect Γ then for any continuous function $f\colon \Gamma \to \mathbb{C}$ the integral $\int_\Gamma f(z)(z-a)^{-1}|dz|$ defines an element in $\mathrm{Mat}_n(\mathrm{dom}(\phi)^\sim)$. Let $b \in \mathrm{Mat}_n(\mathbb{C})$ be such that $c := a - b \in \mathrm{Mat}_n(\mathrm{dom}(\phi))$. Then the spectrum of b is contained in that of a, so we just have to show that

$$\int_\Gamma f(z)\big((z-a)^{-1} - \big(z-b\big)^{-1}\big)|dz| \in \mathrm{Mat}_n(\mathrm{dom}(\phi)).$$

For this observe that if $[0,1] \ni t \mapsto a_t, b_t \in \mathrm{dom}(\phi)^{1/2}$ are two continuous maps such that the functions $t \mapsto \phi(a_t^* a_t), \phi(b_t^* b_t)$ are bounded, then $\int_0^1 a_t^* b_t dt \in \mathrm{dom}(\phi)$. Indeed, by the polarization identity it is enough to consider the case $a_t = b_t$, and then the claim follows from lower semicontinuity. Observe also that for $d \in \mathrm{Mat}_n(A)$ we have $d \in \mathrm{Mat}_n(\mathrm{dom}(\phi)^{1/2})$ if and only if $(\phi \otimes \mathrm{Tr})(d^* d) < \infty$. Denote by $\mathcal{G}$ the class of continuous functions $\Gamma \to \mathrm{Mat}_n(A)$ which are finite sums of functions of the form $z \mapsto d_z^* e_z$ such that $d_z, e_z \in \mathrm{Mat}_n(A)$ depend continuously on z and the functions $z \mapsto (\phi \otimes \mathrm{Tr})(d_z^* d_z), (\phi \otimes \mathrm{Tr})(e_z^* e_z)$ are bounded. The integral of any function in $\mathcal{G}$ defines an element of $\mathrm{Mat}_n(\mathrm{dom}(\phi))$. Therefore it suffices to show that the function $z \mapsto (z-a)^{-1} - \big(z-b\big)^{-1}$ is in $\mathcal{G}$.

The class $\mathcal{G}$ contains constant $\mathrm{Mat}_n(\mathrm{dom}(\phi))$-valued functions and is stable under multiplication (from either side) by continuous $\mathrm{Mat}_n(\mathbb{C})$-valued functions. In particular, the function

$$z \mapsto c_z := c(z-b)^{-1}$$

is in $\mathcal{G}$. Furthermore, if $f_1, f_2 \in \mathcal{G}$ and $f\colon \Gamma \to \mathrm{Mat}_n(A^\sim)$ is continuous then $f_1 f f_2 \in \mathcal{G}$. The identities

$$(z-a)^{-1} - \big(z-b\big)^{-1} = (z-b)^{-1}\big((1-c_z)^{-1} - 1\big) = (z-b)^{-1}\big(c_z + c_z(1-c_z)^{-1}c_z\big)$$

show then that $(z-a)^{-1} - \big(z-b\big)^{-1}$ is indeed in $\mathcal{G}$. ☐

Observe next that if $U\colon \mathbb{T} \to \mathcal{B}(\mathcal{H}_U)$ is a finite dimensional unitary representation then any $\sigma \otimes \mathrm{Ad}\, U$ invariant element is a finite sum of homogeneous components with respect to $\sigma \otimes \iota$. So to deal with equivariant K-theory of A the following algebra is enough.

Definition 3.5. *Denote by $\mathcal{A}$ the algebra consisting of finite sums of σ-homogeneous elements in the domain $\mathrm{dom}(\phi)$ of ϕ. We also put $\mathcal{F} = \mathcal{A} \cap F = \mathrm{dom}(\tau)$.*

We next turn to equivariant K-theory of the mapping cone.

Lemma 3.6. *Every class in $K_0^{\mathbb{T}}(M(F,A))$ has a representative v such that $v \in (\mathcal{A}^\sim \otimes \mathcal{B}(\mathcal{H}_U))^{\sigma \otimes \mathrm{Ad}\, U}$, vv^* and v^*v are in $\mathcal{F}^\sim \otimes \mathcal{B}(\mathcal{H}_U)$, and $vv^* = v^*v$ modulo $\mathcal{F} \otimes \mathcal{B}(\mathcal{H}_U)$, where $U\colon \mathbb{T} \to \mathcal{B}(\mathcal{H}_U)$ is a finite dimensional unitary representation.*

Proof. By Lemma 3.4 and Putnam's description of K-theory of the mapping cone [31] we first conclude that every class has a representative v such that $v \in (\mathcal{A}^\sim \otimes \mathcal{B}(\mathcal{H}_U))^{\sigma \otimes \mathrm{Ad}\, U}$ and $vv^*, v^*v \in \mathcal{F}^\sim \otimes \mathcal{B}(\mathcal{H}_U)$. The images of the projections vv^* and v^*v in $\mathcal{B}(\mathcal{H}_U)^{\mathrm{Ad}\, U}$ under the isomorphism

$$(\mathcal{F}^\sim \otimes \mathcal{B}(\mathcal{H}_U))/(\mathcal{F} \otimes \mathcal{B}(\mathcal{H}_U)) \cong \mathcal{B}(\mathcal{H}_U)$$

are equivalent, so there exists a $\sigma \otimes \mathrm{Ad}\, U$-invariant unitary $u \in \mathcal{F}^\sim \otimes \mathcal{B}(\mathcal{H}_U)$ such that $uvv^*u = v^*v$ modulo $\mathcal{F} \otimes \mathcal{B}(\mathcal{H}_U)$. It remains to recall [31] that the classes of v and uv coincide, so that uv is the required representative. ☐

We are now ready to define the equivariant spectral flow. First consider homogeneous subspaces. Let U and v be as in the above Lemma. Let $\Psi_n\colon \mathcal{H}_U \to \mathcal{H}_U$, resp. $Q_n\colon \mathcal{H}\otimes\mathcal{H}_U \to \mathcal{H}\otimes\mathcal{H}_U$, be the projection onto the χ^n-homogeneous component, so that $Q_n = \sum_k \Phi_{n-k}\otimes\Psi_k$. We then define

$$sf_n(v) = (\mathrm{Tr}_\phi\otimes\mathrm{Tr})((v^*v - vv^*)Q_n(P\otimes 1)) \in \mathbb{R},$$

where $P = \chi_{[0,+\infty)}(\mathcal{D}) = \sum_{k\geq 0}\Phi_k$. Observe that this quantity is finite by Lemma 3.3, since $v^*v - vv^* \in \mathcal{F}\otimes\mathcal{B}(\mathcal{H}_U)$ by assumption and $\dim\mathcal{H}_U$ is finite.

Lemma 3.7. *The value $sf_n(v)$ depends only on the class of v in $K_0^{\mathbb{T}}(M(F,A))$.*

Proof. Denote by $\tilde\tau$ the normal semifinite trace $(\mathrm{Tr}_\phi\otimes\mathrm{Tr})(\cdot\, Q_n(P\otimes 1))$ on $\mathcal{N}^\sigma\otimes\mathcal{B}(\mathcal{H}_U)^{\mathrm{Ad}\,U}$. It suffices to show that if $v_t\in\mathcal{A}^\sim\otimes\mathcal{B}(\mathcal{H}_U)$, $t\in(0,1)$, is a continuous path of partial isometries satisfying the properties in the formulation of Lemma 3.3, then $\tilde\tau(v_0v_0^* - v_0^*v_0) = \tilde\tau(v_1v_1^* - v_1^*v_1)$. Since the images of the projections $v_tv_t^*$ in $\mathcal{B}(\mathcal{H}_U)$ are equivalent, we can find a continuous path of $\sigma\otimes\mathrm{Ad}\,U$-invariant unitaries $u_t\in\mathcal{F}^\sim\otimes\mathcal{B}(\mathcal{H}_U)$ such that $v_0v_0^* = u_tv_tv_t^*u_t$ modulo $\mathcal{F}\otimes\mathcal{B}(\mathcal{H}_U)$. Replacing v_t by $u_tv_tu_t^*$ we may therefore assume that the projections $v_tv_t^*$ coincide modulo $\mathcal{F}\otimes\mathcal{B}(\mathcal{H}_U)$. Then it suffices to check that if $p_t\in\mathcal{F}^\sim\otimes\mathcal{B}(\mathcal{H}_U)^{\mathrm{Ad}\,U}$, $t\in(0,1)$, is a continuous path of projections which coincide modulo $\mathcal{F}\otimes\mathcal{B}(\mathcal{H}_U)$ then $\tilde\tau(p_0-p_1)=0$. We may assume that $\|p_0-p_1\|<1$. Consider the invertible element $w = p_0p_1 + (1-p_0)(1-p_1) \in F^\sim\otimes\mathcal{B}(\mathcal{H}_U)^{\mathrm{Ad}\,U}$. Then $p_0 = wp_1w^{-1}$ and $w-1\in\mathcal{F}\otimes\mathcal{B}(\mathcal{H}_U)^{\mathrm{Ad}\,U}$. Hence

$$\tilde\tau(p_0-p_1) = \tilde\tau((w-1)p_1w^{-1}) + \tilde\tau(p_1(w^{-1}-1)) = \tilde\tau(p_1w^{-1}(w-1)) + \tilde\tau(p_1(w^{-1}-1)) = 0.$$

□

Thus we get a well-defined map $sf_n\colon K_0^{\mathbb{T}}(M(F,A))\to\mathbb{R}$.

Lemma 3.8. *For every $[v]\in K_0^{\mathbb{T}}(M(F,A))$ we have $sf_n([v])=0$ for all but a finite number of $n\in\mathbb{Z}$.*

Proof. In the notation before Lemma 3.7, we have $\Psi_k = 0$ for $|k|$ large enough. It follows that $Q_n(P\otimes 1) = 0$ for all $n\in\mathbb{Z}$ small enough and $Q_n(P\otimes 1) = Q_n$ for n sufficiently large. Therefore it suffices to check that for the normal semifinite trace $\tilde\tau = (\mathrm{Tr}_\phi\otimes\mathrm{Tr})(\cdot\, Q_n)$ on $(\mathcal{N}\otimes\mathcal{B}(\mathcal{H}_U))^{\sigma\otimes\mathrm{Ad}\,U}$ we have $\tilde\tau(vv^*-v^*v)=0$. This is true since by assumption $v-w$ belongs to the domain of $\tilde\tau$ for an element $w\in\mathbb{C}\otimes\mathcal{B}(\mathcal{H}_U)^{\mathrm{Ad}\,U}$ such that $w^*w=ww^*$. □

Definition 3.9. *The $\mathbb{T}$-equivariant spectral flow is the map $sf\colon K_0^{\mathbb{T}}(M(F,A))\to\mathbb{R}[\chi,\chi^{-1}]$ defined by*

$$sf([v]) = \sum_{n\in\mathbb{Z}} sf_n([v])\chi^n.$$

By Theorem 2.12 and definition of the induced trace we may conclude that if the SSA is satisfied then the equivariant spectral flow coincides with the composition

$$K_0^{\mathbb{T}}(M(F,A)) \xrightarrow{-\mathrm{Index}_{\hat{\mathcal{D}}}} K_0^{\mathbb{T}}(F) = K_0(F)[\chi,\chi^{-1}] \xrightarrow{\tau_*} \mathbb{R}[\chi,\chi^{-1}],$$

at least on the elements represented by $\sigma\otimes\iota$ homogeneous isometries v; here τ_* denotes the homomorphism $K_0(F)\to\mathbb{R}$ defined by the trace τ. We shall return to this in more detail in the next Section.

4. Modular index pairing

4.1. **Modular K_1.** In the previous Section we defined an equivariant spectral flow which assigns to an invariant partial isometry $v \in A^\sim \otimes \mathcal{B}(\mathcal{H}_U)$ a Laurent polynomial in χ. Being evaluated at $\chi = 1$ this polynomial gives a suitably defined spectral flow from $vv^*(\mathcal{D} \otimes 1)$ to $v(\mathcal{D} \otimes 1)v^*$ with respect to $\mathrm{Tr}_\phi \otimes \mathrm{Tr}$. We would like to obtain an analytic formula for this spectral flow. Such formulas are available under certain summability assumptions, but as Lemma 3.3 shows, even when ϕ is a state, the operator $|\mathcal{D}|^{-p}$ is not summable in general for any $p > 0$. The same Lemma suggests, however, that to improve summability it would suffice to assign the weight $e^{-n\beta}$ to every projection Φ_n. Effectively this means that we evaluate the equivariant spectral flow at $\chi = e^{-\beta}$. This was done from a different point of view in [9], where notions of a modular K_1 group and a modular pairing were introduced. Our considerations allow us to relate the results of [9] to more conventional constructions.

The following definition is essentially from [9], slightly modified and extended to adapt to our current considerations.

Definition 4.1. *A partial isometry in $A^\sim$ is modular if $v\sigma_t(v^*)$ and $v^*\sigma_t(v)$ are in $(A^\sim)^\sigma$ for all $t \in \mathbb{R}$. By a modular partial isometry over A we mean a modular partial isometry in $\mathrm{Mat}_n(A^\sim) = A^\sim \otimes \mathrm{Mat}_n(\mathbb{C})$ for some $n \in \mathbb{N}$ with respect to the action $\sigma \otimes \iota$.*

In [9] only modular unitaries were considered. Observe that every modular partial isometry v over A defines a modular unitary by

$$u_v = \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix}.$$

Define the modular K_1 group as follows.

Definition 4.2. *Let $K_1(A, \sigma)$ be the abelian group with one generator $[v]$ for each partial isometry v over A satisfying the modular condition and with the following relations:*

1) $[v] = 0$ *if v is over F,*
2) $[v] + [w] = [v \oplus w]$,
3) *if $v_t,\ t \in [0,1],$ is a continuous path of modular partial isometries in* $\mathrm{Mat}_n(A^\sim)$ *then* $[v_0] = [v_1]$.

Remarks. It is easy to show that $v \oplus w \sim w \oplus v$, see [9], however the inverse of $[v]$ is not $[v^*]$ in general. Equivalently, even though u_v is a self-adjoint unitary and hence is homotopic to the identity, such a homotopy cannot always be chosen to consist of modular unitaries.

Observe that σ-homogeneous partial isometries are modular. It turns out that they generate the whole group $K_1(A, \sigma)$. We need some preparation to prove this.

Lemma 4.3. *A unitary $u \in A^\sim$ is modular if and only if there exists a self-adjoint element $a \in F^\sim$ such that $uau^* \in F^\sim$ and $\sigma_t(u) = ue^{ita}$ for $t \in \mathbb{R}$.*

Proof. Put $u_t = u^*\sigma_t(u)$. Then

$$u_{t+s} = u^*\sigma_{t+s}(u) = u^*\sigma_t(u)\sigma_t(u^*\sigma_s(u)) = u_t u_s.$$

Thus $\{u_t\}_t$ is a norm-continuous one-parameter group of unitary operators in $F^\sim$. Hence there exists a self-adjoint element $a \in F^\sim$ such that $u_t = e^{ita}$. Therefore

$$\sigma_t(u) = ue^{ita} = e^{ituau^*}u.$$

Since u is modular, the second equality implies that $uau^* \in F^\sim$. The converse is obvious. ☐

For an element $x \in \mathrm{Mat}_n(A^\sim)$ we denote by x_k the spectral component of x with respect to $\sigma \otimes \iota$, so $(\sigma_t \otimes \iota)(x_k) = e^{ikt}x_k$.

Lemma 4.4. *A partial isometry $v \in \mathrm{Mat}_n(A^\sim)$ is modular if and only if the elements v_k are partial isometries which are zero for all but a finite number of k's and the source projections $v_k^*v_k$, $k \in \mathbb{Z}$, as well as the range projections $v_kv_k^*$, $k \in \mathbb{Z}$, are mutually orthogonal.*

Proof. Consider the modular unitary $u = u_v$. If $\sigma_t(u) = ue^{ita}$ with a as in Lemma 4.3 (but now $a \in \mathrm{Mat}_{2n}(F^\sim)$), then $u = ue^{2\pi ia}$. Hence the spectrum of a is a finite subset of $\mathbb{Z}$. Let p_k be the spectral projection of a corresponding to $k \in \mathbb{Z}$. Then $u_k = up_k$, and hence the partial isometries u_k have mutually orthogonal sources and ranges. We clearly have

$$u_0 = \begin{pmatrix} 1 - v^*v & v_0^* \\ v_0 & 1 - vv^* \end{pmatrix}, \quad u_k = \begin{pmatrix} 0 & v_{-k}^* \\ v_k & 0 \end{pmatrix} \quad \text{for} \quad k \neq 0.$$

This implies that $v_k = 0$ for all but a finite number of k, and the elements v_k, $k \neq 0$, are partial isometries with mutually orthogonal sources and ranges. Consider $w = \sum_{k\neq 0} v_k$. Then w is a partial isometry and $ww^* = \sum_{k\neq 0} v_kv_k^*$, $w^*w = \sum_{k\neq 0} v_k^*v_k$. Since

$$v^*v = v_0^*v_0 + w^*w + \sum_{k\neq 0}(v_0^*v_k + v_k^*v_0)$$

is invariant, we get $v^*v = v_0^*v_0 + w^*w$. Since v^*v and w^*w are projections, it follows that $v_0^*v_0$ is a projection orthogonal to w^*w. In other words, v_0 is a partial isometry with the source projection orthogonal to $v_k^*v_k$, $k \neq 0$. Similarly one checks that the projections $v_0v_0^*$ and $v_kv_k^*$, $k \neq 0$, are orthogonal.

The converse statement is straightforward. $\square$

Corollary 4.5. *The group $K_1(A, \sigma)$ is generated by the classes of homogeneous partial isometries.*

Proof. It suffices to observe that if v and w are modular partial isometries such that $v^*vw^*w = vv^*ww^* = 0$, then $[v + w] = [v] + [w]$. Indeed, if $R_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, then

$$v_t = \left(\begin{pmatrix} 1 - ww^* & 0 \\ 0 & 1 - ww^* \end{pmatrix} + R_tww^*\right)\begin{pmatrix} v + w & 0 \\ 0 & 0 \end{pmatrix}\left(\begin{pmatrix} 1 - w^*w & 0 \\ 0 & 1 - w^*w \end{pmatrix} + R_{-t}w^*w\right),$$

$0 \le t \le \pi/2$, is a modular homotopy from $\begin{pmatrix} v + w & 0 \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$. $\square$

We next want to relate the group $K_1(A, \sigma)$ to $K_0^{\mathbb{T}}(M(F, A))$.

Recall that if $\mathcal{K}$ is a finite dimensional Hilbert space considered with the trivial $\mathbb{T}$-module structure, we denote by $\mathcal{K}[n]$ the same space with the representation $t \mapsto e^{int}$. Assume $v \in A^\sim \otimes \mathcal{B}(\mathcal{K})$ is a partial isometry such that $v \in A_n^\sim \otimes \mathcal{B}(\mathcal{K})$, so $(\sigma_t \otimes \iota)(v) = e^{int}v$, then the partial isometry

$$w_v = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in A^\sim \otimes \mathcal{B}(\mathcal{K} \oplus \mathcal{K}[n])$$

is $\mathbb{T}$-invariant, so it defines an element of $K_0^{\mathbb{T}}(M(F, A))$. Sometimes we shall denote the class $[w_v] \in K_0^{\mathbb{T}}(M(F, A))$ by $\ll v \gg$. Note that if $n = 0$ and so v itself represents an element of $K_0^{\mathbb{T}}(M(F, A))$, there is no ambiguity in this notation as

$$\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \text{ is homotopic to } \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix},$$

and moreover, the class of v can easily be shown to be zero, see [31, Lemma 2.2(v)].

Proposition 4.6. *The map*

$$v \mapsto \sum_k \ll v_k \gg \in K_0^{\mathbb{T}}(M(F,A))$$

defined on modular partial isometries gives a homomorphism $T\colon K_1(A,\sigma) \to K_0^{\mathbb{T}}(M(F,A))$.

Proof. Since homotopic elements have homotopic spectral components, it is clear that the images of homotopic modular partial isometries coincide. It follows that we have a well-defined homomorphism $T\colon K_1(A,\sigma) \to K_0^{\mathbb{T}}(M(F,A))$; in fact, for each k the map $[u] \mapsto \ll u_k \gg$ is a homomorphism. □

This homomorphism makes it clear why $-[v] \neq [v^*]$ in $K_1(A,\sigma)$ in general. Indeed, observe first that in the group $K_0^{\mathbb{T}}(M(F,A))$ we do have $-[w] = [w^*]$, basically because u_w is an invariant self-adjoint unitary, hence there is a homotopy from u_w to 1 consisting of invariant unitaries. In particular, for homogeneous v as above we have $-[w_v] = [w_v^*]$. The class w_v^* is represented by

$$\begin{pmatrix} 0 & v^* \\ 0 & 0 \end{pmatrix} \in A^\sim \otimes \mathcal{B}(\mathcal{K}[n] \oplus \mathcal{K}), \quad \text{while} \quad w_{v^*} = \begin{pmatrix} 0 & v^* \\ 0 & 0 \end{pmatrix} \in A^\sim \otimes \mathcal{B}(\mathcal{K} \oplus \mathcal{K}[-n]).$$

Therefore $[w_v^*] = \chi^n [w_{v^*}]$. In other words, $- \ll v \gg = \chi^n \ll v^* \gg$, so that $T(-[v]) = \chi^n T([v^*])$. Equivalently, we have

$$T([u_v]) = \ll v \gg + \ll v^* \gg = (1 - \chi^{-n}) \ll v \gg .$$

4.2. **Modular index.** Recall that in Subsection 3.1 we constructed a semifinite von Neumann algebra $\mathcal{N} = \mathrm{End}(X)'' \subset \mathcal{B}(\mathcal{H})$, a faithful semifinite normal trace Tr_ϕ and an operator $\mathcal{D} = \sum_{k\in\mathbb{Z}} k\Phi_k$ on $\mathcal{H}$.

We now define a new weight on $\mathcal{N}$.

Definition 4.7. *Consider the operator $e^{-\beta\mathcal{D}} = \sum_{k\in\mathbb{Z}} e^{-k\beta}\Phi_k$. For $S \in \mathcal{N}_+$ define*

$$\phi_{\mathcal{D}}(S) = \mathrm{Tr}_\phi(e^{-\beta\mathcal{D}/2} S e^{-\beta\mathcal{D}/2}).$$

Since $e^{-\beta\mathcal{D}}$ is strictly positive and affiliated to $\mathcal{N}$, $\phi_{\mathcal{D}}$ is a faithful semifinite normal weight. Since Tr_ϕ is a trace, the modular group of $\phi_{\mathcal{D}}$ is given by $\sigma_t^{\phi_{\mathcal{D}}}(\cdot) = e^{-it\beta\mathcal{D}} \cdot e^{it\beta\mathcal{D}}$. The restriction of $\sigma_t^{\phi_{\mathcal{D}}}$ to A coincides with $\sigma_{-\beta t}$. While $\phi_{\mathcal{D}}$ is not a trace on $\mathcal{N}$, it is clearly a semifinite normal trace on the invariant subalgebra $\mathcal{M} := \mathcal{N}^\sigma$. The following Lemma captures the main reason for defining $\phi_{\mathcal{D}}$.

Lemma 4.8. *With A, σ, ϕ as above, we have $f(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \phi_{\mathcal{D}})$ if $f \in \mathcal{F} = \mathrm{dom}(\phi) \cap F$.*

Proof. This follows immediately from Lemma 3.3, since $\phi_{\mathcal{D}}(f\Phi_k) \leq \phi(f)$ for $f \geq 0$. □

We will call the data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi_{\mathcal{D}})$ the **modular spectral triple** for (A, σ, ϕ). It provides us with a way to compute the spectral flow from $vv^*\mathcal{D}$ and $v\mathcal{D}v^*$ with respect to the trace $\phi_{\mathcal{D}}$ on $\mathcal{M}$ for appropriate partial isometries in $\mathcal{A}$. The next Lemma justifies our definition of modular partial isometries.

Lemma 4.9. *Let $v \in \mathrm{Mat}_n(A^\sim)$ be a partial isometry such that $vv^*, v^*v \in \mathrm{Mat}_n(F^\sim)$. Then we have $v(Q\otimes 1)v^*, v^*(Q\otimes 1)v \in \mathrm{Mat}_n(\mathcal{M})$ for every spectral projection Q of $\mathcal{D}$ if and only if v is modular.*

Proof. Replacing v by u_v we may assume that v is unitary. Next, suppose first that v is modular. Write $\tilde\sigma$ for $\sigma_t \otimes \iota$ and $\tilde Q$ for $Q \otimes 1$. Since $\mathcal{M} = \mathcal{N}^\sigma$, we need to show that $v\tilde Q v^*$ is $\tilde\sigma$-invariant. We have

$$\tilde\sigma(vQv^*) = \tilde\sigma(v)\tilde Q\tilde\sigma(v^*) = vv^*\tilde\sigma(v)\tilde Q\tilde\sigma(v^*) = v\tilde Q v^*\tilde\sigma(v)\tilde\sigma(v^*) = v\tilde Q v^*.$$

A similar argument shows that $v^*\tilde{Q}v$ is invariant.

On the other hand, if

$$v\tilde{Q}v^* = \tilde{\sigma}(v\tilde{Q}v^*) = \tilde{\sigma}(v)\tilde{Q}\tilde{\sigma}(v^*),$$

then $v^*\tilde{\sigma}(v)$ commutes with $\tilde{Q} = Q\otimes 1$. If this is true for every spectral projection Q of the generator $\mathcal{D}$ of σ, then $v^*\tilde{\sigma}(v)$ is $(\sigma\otimes\iota)$-invariant. Similarly $v\tilde{\sigma}(v^*)$ is invariant. Hence v is modular. □

Next we show that the spectral flow is indeed well-defined for modular partial isometries.

Lemma 4.10. *For a modular partial isometry $v \in A^\sim \otimes \mathcal{B}(\mathcal{K})$ consider the projections*

$$P_1 = \chi_{[0,+\infty)}(vv^*(\mathcal{D}\otimes 1)) \quad \textit{and} \quad P_2 = \chi_{[0,+\infty)}(v(\mathcal{D}\otimes 1)v^*).$$

Then the operator $P_1P_2 \in P_1(\mathcal{M}\otimes\mathcal{B}(\mathcal{K}))P_2$ is Breuer-Fredholm and

$$\begin{aligned} &sf_{\phi_\mathcal{D}\otimes\mathrm{Tr}}(vv^*(\mathcal{D}\otimes 1), v(\mathcal{D}\otimes 1)v^*) \\ &= \sum_{k<0}\sum_{k\le n<0} e^{-\beta n}(\mathrm{Tr}_\phi\otimes\mathrm{Tr})(v_kv_k^*(\Phi_n\otimes 1)) - \sum_{k>0}\sum_{0\le n<k} e^{-\beta n}(\mathrm{Tr}_\phi\otimes\mathrm{Tr})(v_kv_k^*(\Phi_n\otimes 1)). \end{aligned}$$

Proof. By Lemma 4.4 the element v is a finite sum of its homogeneous components v_k which are partial isometries with mutually orthogonal sources and ranges. The operators $vv^*(\mathcal{D}\otimes 1)$ and $v(\mathcal{D}\otimes 1)v^*$ commute with $v_kv_k^*$ and

$$v_kv_k^*vv^*(\mathcal{D}\otimes 1) = v_kv_k^*(\mathcal{D}\otimes 1), \quad v_kv_k^*v(\mathcal{D}\otimes 1)v^* = v_k(\mathcal{D}\otimes 1)v_k^*.$$

This shows that without loss of generality we may assume that v is homogeneous, say $v = v_k$. Furthermore, for $k = 0$ the operators coincide, so we just have to consider the case $k \neq 0$.

Let $P = \chi_{[0,+\infty)}(\mathcal{D}) = \sum_{n\ge 0}\Phi_n$. Since vv^* and v^*v commute with $\mathcal{D}$, we have

$$P_1 = 1 - vv^* + vv^*(P\otimes 1) \ \text{ and } \ P_2 = 1 - vv^* + v(P\otimes 1)v^*.$$

But using homogeneity we can actually say much more and easily express these projections in terms of vv^* and Φ_n. Namely, as $v(\Phi_n\otimes 1) = (\Phi_{n+k}\otimes 1)v$, we have

$$v(P\otimes 1)v^* = \sum_{n\ge k} vv^*(\Phi_n\otimes 1). \tag{4}$$

With this information it is easy to show that P_1P_2 is Breuer-Fredholm, since this is implied by $P_1 - P_2$ being compact in $\mathcal{M}\otimes\mathcal{B}(\mathcal{K})$. However from Equation (4) we have

$$P_1 - P_2 = \sum_{n=0}^{k-1} vv^*(\Phi_n\otimes 1),\ k>0, \qquad P_1 - P_2 = -\sum_{n=k}^{-1} vv^*(\Phi_n\otimes 1),\ k<0.$$

To finish the proof it therefore remains to show that for every n the projection $vv^*(\Phi_n\otimes 1)$ has finite trace with respect to $\phi_\mathcal{D}\otimes\mathrm{Tr}$. By the same argument as in the proof of Lemma 3.3 we have

$$(\phi_\mathcal{D}\otimes\mathrm{Tr})(vv^*(\Phi_n\otimes 1)) = e^{-\beta n}(\mathrm{Tr}_\phi\otimes\mathrm{Tr})(vv^*(\Phi_n\otimes 1)) \le (\tau\otimes\mathrm{Tr})(vv^*).$$

Notice now that $v \in A\otimes\mathcal{B}(\mathcal{K})$, since $v = v_k$ is homogeneous with $k \neq 0$. Hence the projection $vv^* \in F\otimes\mathcal{B}(\mathcal{K})$ is in the domain of the semifinite trace $\tau\otimes\mathrm{Tr}$ on $F\otimes\mathcal{B}(\mathcal{K})$, since the latter domain contains the Pedersen ideal and, in particular, every projection. □

Observe that the above proof shows that if v is a modular partial isometry then $v - v_0 \in \mathcal{A}\otimes\mathcal{B}(\mathcal{K})$. Notice also that if we have a continuous path of modular partial isometries then the corresponding projections P_1 and P_2 also form norm-continuous paths. It follows that the map

$$v \mapsto sf_{\phi_\mathcal{D}\otimes\mathrm{Tr}}(vv^*(\mathcal{D}\otimes 1), v(\mathcal{D}\otimes 1)v^*)$$

defines a homomorphism $K_1(A,\sigma) \to \mathbb{R}$; this of course also follows from the explicit expression for the spectral flow. We call this homomorphism the **modular index** and denote it by $\mathrm{Index}_{\phi_{\mathcal{D}}}$. The following theorem compares $\mathrm{Index}_{\phi_{\mathcal{D}}}$ with the equivariant spectral flow.

Theorem 4.11. *The modular index map* $\mathrm{Index}_{\phi_{\mathcal{D}}} : K_1(A,\sigma) \to \mathbb{R}$ *is the composition of the maps*

$$K_1(A,\sigma) \xrightarrow{T} K_0^{\mathbb{T}}(M(F,A)) \xrightarrow{sf} \mathbb{R}[\chi,\chi^{-1}] \xrightarrow{\mathrm{Ev}(e^{-\beta})} \mathbb{R},$$

where $\mathrm{Ev}(e^{-\beta})$ *is the evaluation at* $\chi = e^{-\beta}$. *If the SSA is satisfied then, equivalently,* $\mathrm{Index}_{\phi_{\mathcal{D}}}$ *is the composition*

$$K_1(A,\sigma) \xrightarrow{[v]\mapsto\sum_k \ll v_k \gg} K_0^{\mathbb{T}}(M(F,A)) \xrightarrow{-\mathrm{Index}_{\hat{\mathcal{D}}}} K_0^{\mathbb{T}}(F) = K_0(F)[\chi,\chi^{-1}] \xrightarrow{\tau_*} \mathbb{R}[\chi,\chi^{-1}] \xrightarrow{\mathrm{Ev}(e^{-\beta})} \mathbb{R}.$$

Proof. This is a matter of bookkeeping. Let $v = v_k \in A \otimes \mathcal{B}(\mathcal{K})$ be a homogeneous partial isometry, $k \neq 0$. Recall that $\ll v \gg$ is represented by $w_v = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K}[k])$. To compute $sf([w_v])$ first observe that the projection Q_n onto the χ^n-homogeneous component of $\mathcal{H} \otimes (\mathcal{K} \oplus \mathcal{K}[k])$ is $\begin{pmatrix} \Phi_n & 0 \\ 0 & \Phi_{n-k} \end{pmatrix}$. Therefore

$$\begin{aligned} sf([w_v]) &= \sum_n (\mathrm{Tr}_\phi \otimes \mathrm{Tr})((w_v^* w_v - w_v w_v^*) Q_n (P \otimes 1))\chi^n \\ &= \sum_n (\mathrm{Tr}_\phi \otimes \mathrm{Tr})(v^*v(\Phi_{n-k} P \otimes 1) - vv^*(\Phi_n P \otimes 1))\chi^n. \end{aligned}$$

The projection $v^*v(\Phi_{n-k} \otimes 1) = v^*(\Phi_n \otimes 1)v$ is equivalent to the projection $vv^*(\Phi_n \otimes 1)$. It follows that the nth summand in the above expression is nonzero only when $n-k$ and n have different signs. More precisely, for $k < 0$ we get

$$\sum_{k \le n < 0} (\mathrm{Tr}_\phi \otimes \mathrm{Tr})(v^*v(\Phi_{n-k} \otimes 1))\chi^n = \sum_{k \le n < 0} (\mathrm{Tr}_\phi \otimes \mathrm{Tr})(vv^*(\Phi_n \otimes 1))\chi^n,$$

and for $k > 0$ we get

$$-\sum_{0 \le n < k} (\mathrm{Tr}_\phi \otimes \mathrm{Tr})(vv^*(\Phi_n \otimes 1))\chi^n.$$

For $\chi = e^{-\beta}$ these expressions coincide with those in Lemma 4.10. $\square$

5. The analytic index from spectral flow

5.1. **A spectral flow formula.** Our method of computing numerical invariants from KMS states exploits semifinite spectral flow and so we need to review the spectral flow formula of [6]. There are two versions of this formula in the unbounded setting, one for θ-summable spectral triples, and the other for finitely summable triples. It is the latter that we will want to use. First we quote [6, Corollary 8.11].

Proposition 5.1. *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D}_0)$ *be an odd unbounded* θ*-summable semifinite spectral triple relative to* $(\mathcal{M}, \tau)$, *where* τ *is a faithful semifinite normal trace on* $\mathcal{M}$. *For any* $\epsilon > 0$ *we define a one-form* α^ϵ *on the affine space* $\mathcal{M}_0 = \mathcal{D}_0 + \mathcal{M}_{sa}$ *by*

$$\alpha^\epsilon(A) = \sqrt{\frac{\epsilon}{\pi}}\,\tau(A e^{-\epsilon \mathcal{D}^2})$$

for $\mathcal{D} \in \mathcal{M}_0$ and $A \in T_{\mathcal{D}}(\mathcal{M}_0) = \mathcal{M}_{sa}$ (here $T_{\mathcal{D}}(\mathcal{M}_0)$ is the tangent space to $\mathcal{M}_0$ at $\mathcal{D}$). Then the integral of α^ϵ is independent of the piecewise C^1 path in $\mathcal{M}_0$ and if $\{\mathcal{D}_t = \mathcal{D}_a + A_t\}_{t\in[a,b]}$ is any piecewise C^1 path in $\mathcal{M}_0$ joining $\mathcal{D}_a$ and $\mathcal{D}_b$ then

$$sf(\mathcal{D}_a, \mathcal{D}_b) = \sqrt{\frac{\epsilon}{\pi}} \int_a^b \tau(\mathcal{D}_t' e^{-\epsilon\mathcal{D}_t^2})dt + \frac{1}{2}\eta_\epsilon(\mathcal{D}_b) - \frac{1}{2}\eta_\epsilon(\mathcal{D}_a) + \frac{1}{2}\tau\left([\ker(\mathcal{D}_b)] - [\ker(\mathcal{D}_a)]\right).$$

Here the truncated eta invariant is given for $\epsilon > 0$ by

$$\eta_\epsilon(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \tau(\mathcal{D}e^{-t\mathcal{D}^2})t^{-1/2}dt.$$

We want to employ this formula in a finitely summable setting, so we need to Laplace transform the various terms appearing in the formula. In fact we were able in [13] to translate the formula in [6] for the spectral flow into a residue type formula. The importance of such a formula lies in the drastic simplification of computations, since we may throw away terms that are holomorphic in a neighbourhood of the point where we take a residue.

We introduce the notation

$$C_r := \frac{\sqrt{\pi}\Gamma(r-1/2)}{\Gamma(r)} = \int_{-\infty}^\infty (1+x^2)^{-r}dx.$$

Observe that C_r is analytic for $\Re(r) > 1/2$ and has an analytic continuation to a neighbourhood of $1/2$ where it has a simple pole (cf [11]) with residue equal to 1.

Lemma 5.2. *Let $\mathcal{D}$ be a self-adjoint operator on the Hilbert space $\mathcal{H}$, affiliated to the semifinite von Neumann algebra $\mathcal{M}$. Suppose that for a fixed faithful, normal, semifinite trace τ on $\mathcal{M}$ we have*

$$(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(p,\infty)}(\mathcal{M}, \tau), \quad p \geq 1.$$

Then the Laplace transform of $\eta_\epsilon(\mathcal{D})$, the eta invariant of $\mathcal{D}$, is given by $\frac{1}{C_r}\eta_{\mathcal{D}}(r)$ where

$$\eta_{\mathcal{D}}(r) = \int_1^\infty \tau(\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds, \quad \Re(r) > 1/2 + p/2.$$

Proof. We need to Laplace transform the 'θ summable formula' for the truncated η invariant:

$$\eta_\epsilon(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \tau(\mathcal{D}e^{-t\mathcal{D}^2})t^{-1/2}dt.$$

This integral converges for all $\epsilon > 0$. First we rewrite the formula as

$$\eta_\epsilon(\mathcal{D}) = \frac{\sqrt{\epsilon}}{\sqrt{\pi}} \int_1^\infty \tau(\mathcal{D}e^{-\epsilon s\mathcal{D}^2})s^{-1/2}ds.$$

Using

$$1 = \frac{1}{\Gamma(r-1/2)} \int_0^\infty \epsilon^{r-3/2}e^{-\epsilon}d\epsilon$$

for $\Re(r) > p/2 + 1/2$, the Laplace transform of $\eta_\epsilon(\mathcal{D})$ is

$$\begin{aligned}
\frac{1}{C_r}\eta_{\mathcal{D}}(r) &= \frac{1}{\sqrt{\pi}\Gamma(r-1/2)} \int_0^\infty \epsilon^{r-1}e^{-\epsilon} \int_1^\infty \tau(\mathcal{D}e^{-\epsilon s\mathcal{D}^2})s^{-1/2}dsd\epsilon \\
&= \frac{1}{\sqrt{\pi}\Gamma(r-1/2)} \int_1^\infty s^{-1/2}\tau(\mathcal{D}\int_0^\infty \epsilon^{r-1}e^{-\epsilon(1+s\mathcal{D}^2)}d\epsilon)ds \\
&= \frac{\Gamma(r)}{\sqrt{\pi}\Gamma(r-1/2)} \int_1^\infty s^{-1/2}\tau(\mathcal{D}(1+s\mathcal{D}^2)^{-r})ds. \qquad (5)
\end{aligned}$$

□

For our final formula we restrict to $p = 1$, which is the case of interest in this paper.

Proposition 5.3. *Let $\mathcal{D}_a$ be a self-adjoint densely defined unbounded operator on the Hilbert space $\mathcal{H}$, affiliated to the semifinite von Neumann algebra $\mathcal{M}$. Suppose that for a fixed faithful, normal, semifinite trace τ on $\mathcal{M}$ we have $(1+\mathcal{D}_a^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M},\tau)$. Let $\mathcal{D}_b$ differ from $\mathcal{D}_a$ by a bounded self adjoint operator in $\mathcal{M}$. Then for any piecewise C^1 path $\{\mathcal{D}_t = \mathcal{D}_a + A_t\}$, $t \in [a,b]$ in $\mathcal{M}_0 = \mathcal{D}_a + \mathcal{M}_{sa}$ joining $\mathcal{D}_a$ and $\mathcal{D}_b$, the spectral flow $sf_\tau(\mathcal{D}_a,\mathcal{D}_b)$ is given by the formula*

$$\begin{aligned}\mathrm{Res}_{r=1/2}\, C_r sf_\tau(\mathcal{D}_a,\mathcal{D}_b) &= \mathrm{Res}_{r=1/2}\left(\int_a^b \tau(\dot{\mathcal{D}}_t(1+\mathcal{D}_t^2)^{-r})dt + \frac{1}{2}\left(\eta_{\mathcal{D}_b}(r) - \eta_{\mathcal{D}_a}(r)\right)\right)\\ &\quad + \frac{1}{2}\left(\tau(P_{\ker \mathcal{D}_b}) - \tau(P_{\ker \mathcal{D}_a})\right), \end{aligned} \tag{6}$$

where $\eta_{\mathcal{D}}(r) := \int_1^\infty \tau(\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds$, $\Re(r) > 1$. The meaning of (6) is that the function of r on the right hand side has a meromorphic continuation to a neighbourhood of $r = 1/2$ with a simple pole at $r = 1/2$ where we take the residue.

Proof. We apply the Laplace transform to the general spectral flow formula. The computation of the Laplace transform of the eta invariants is above, and the Laplace transform of the other integral is in [6]. The existence of the residue follows from the equality, for $\Re(r)$ large,

$$C_r sf_\tau(\mathcal{D}_a,\mathcal{D}_b) = \int_a^b \tau(\dot{\mathcal{D}}_t(1+\mathcal{D}_t^2)^{-r})dt + \frac{1}{2}\left(\eta_{\mathcal{D}_b}(r) - \eta_{\mathcal{D}_a}(r)\right) + C_r\frac{1}{2}\left(\tau(P_{\ker \mathcal{D}_b}) - \psi(P_{\ker \mathcal{D}_a})\right)$$

which shows that the sum of the integral and the eta terms has a meromorphic continuation as claimed. $\square$

This is the formula for spectral flow we will employ in the sequel.

5.2. **An index formula for modular partial isometries.** Having obtained a well-defined analytic index pairing, we now emulate the methods of [11] to obtain a 'local index formula' to compute this analytic pairing. Let $v \in A^\sim$ be a modular partial isometry. Recall that (as we observed after Lemma 4.10) we automatically have $v_k \in \mathcal{A}$ for $k \neq 0$. Furthermore, the same Lemma shows that v_0 does not contribute to the spectral flow. In other words, we have the following.

Lemma 5.4. *Given the modular spectral triple for (A,σ,ϕ), let $v \in A^\sim$ be a modular partial isometry so that $p = vv^* - v_0v_0^* \in \mathcal{F}$, where $v_0 \in A_0$ is the σ-invariant part of v. Then p commutes with $\mathcal{D}$ and $v\mathcal{D}v^*$, and*

$$sf_{\phi_\mathcal{D}}(vv^*\mathcal{D}, v\mathcal{D}v^*) = sf_{\phi_{\mathcal{D},p}}(p\mathcal{D}, pv\mathcal{D}v^*),$$

where $\phi_{\mathcal{D},p} = \phi_\mathcal{D}|_{p\mathcal{M}p}$ is the trace on $p\mathcal{M}p$.

In our previous examples of the Cuntz algebra and $SU_q(2)$ we showed that the operator $(1+\mathcal{D}^2)^{-1/2}$ lies in $\mathcal{L}^{(1,\infty)}(\mathcal{M},\phi_\mathcal{D})$. However in general, by Lemma 4.8, we only have $f(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M},\phi_\mathcal{D})$ for $f \in \mathcal{F} = F \cap \mathrm{dom}(\phi)$. Thanks to the Lemma above this is sufficient for our purposes, since $p(1+\mathcal{D}^2)^{-1/2}$ is in $\mathcal{L}^{(1,\infty)}(p\mathcal{M}p,\phi_{\mathcal{D},p})$, and we are justified in using our spectral flow formula.

We apply Proposition 5.3 to the path $\mathcal{D}_t = p\mathcal{D} + tpv[\mathcal{D},v^*] = p\mathcal{D} + tv[\mathcal{D},v^*]$ of operators affiliated with $p\mathcal{M}p$.

Lemma 5.5. *We have $\phi_{\mathcal{D},p}(P_{\ker \mathcal{D}_0}) - \phi_{\mathcal{D},p}(P_{\ker \mathcal{D}_1}) = 0$.*

Proof. Since we cut down by the projection p, we may assume that $v_0 = 0$ and so $v \in \mathcal{A}$ and $vv^* = p$. Then $P_{\ker \mathcal{D}_0} = vv^*\Phi_0$ and $P_{\ker \mathcal{D}_1} = v\Phi_0 v^*$. As $\phi_{\mathcal{D}}(f\Phi_0) = \phi(f)$ for $f \in \mathcal{F}$ by Equation (3), we have

$$\phi_{\mathcal{D}}(vv^*\Phi_0 - v\Phi_0 v^*) = \phi_{\mathcal{D}}((\sigma_{-i\beta}(v^*)v - vv^*)\Phi_0) = \phi(\sigma_{-i\beta}(v^*)v - vv^*) = 0.$$

□

Thus the kernel correction terms vanish for modular partial isometries. Next we obtain a residue formula for the spectral flow:

Theorem 5.6. *Given the modular spectral triple for (A, σ, ϕ) let $v \in \mathcal{A}^\sim$ be a modular partial isometry. Then $sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*)$ is given by*

$$\mathrm{Res}_{r=1/2}\left(r \mapsto \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1+\mathcal{D}^2)^{-r}) + \frac{1}{2}\int_1^\infty \phi_{\mathcal{D}}((\sigma_{-i\beta}(v^*)v - vv^*)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds\right).$$

Proof. We apply Proposition 5.3 to the path $\mathcal{D}_t = p\mathcal{D} + tv[\mathcal{D}, v^*]$. Thus by Lemma 5.4 and Lemma 5.5 we get

$$sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*) = \mathrm{Res}_{r=1/2}\left(\int_0^1 \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1+\mathcal{D}_t^2)^{-r})dt + \frac{1}{2}\left(\eta_{\mathcal{D}_1}(r) - \eta_{\mathcal{D}_0}(r)\right)\right).$$

First we observe that by [5, Proposition 10, Appendix B], the difference

$$(1 + (\mathcal{D} + tv[\mathcal{D}, v^*])^2)^{-r} - (1+\mathcal{D}^2)^{-r}$$

is (uniformly) trace class in the corner $p\mathcal{M}p$ for $r \geq 1/2$. Hence in the spectral flow formula above we may exploit analyticity in r for $\Re(r) > 1/2$ as in [11] (we are working in the semifinite algebra $p\mathcal{M}p$ with trace $\phi_{\mathcal{D}}|_{p\mathcal{M}p}$) to write

$$\int_0^1 \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1 + (\mathcal{D} + tv[\mathcal{D}, v^*])^2)^{-r})dt = \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1+\mathcal{D}^2)^{-r}) + \text{remainder}.$$

The remainder is finite at $r = 1/2$, and in fact by [11], holomorphic at $r = 1/2$.

Next consider the eta terms. We have, for $\Re(r) > 1$,

$$\begin{aligned}\eta_{\mathcal{D}_1}(r) &= \int_1^\infty \phi_{\mathcal{D}}(pv\mathcal{D}v^*(1 + s(v\mathcal{D}v^*)^2)^{-r})s^{-1/2}ds \\ &= \int_1^\infty \phi_{\mathcal{D}}(pv\mathcal{D}(1+s\mathcal{D}^2)^{-r}v^*)s^{-1/2}ds \\ &= \int_1^\infty \phi_{\mathcal{D}}(\sigma_{-i\beta}(v^*)pv\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds\end{aligned}$$

and

$$\eta_{\mathcal{D}_0}(r) = \int_1^\infty \phi_{\mathcal{D}}(p\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds.$$

Using that $\sigma_{-i\beta}(v^*)pv = \sigma_{-i\beta}(v^*)v - v_0^*v_0$ and $p = vv^* - v_0v_0^*$, we see that to finish the proof we have to check that

$$\phi_{\mathcal{D}}((v_0^*v_0 - v_0v_0^*)\mathcal{D}(1+s\mathcal{D}^2)^{-r}) = 0.$$

This is true since $\phi_{\mathcal{D}}(\cdot\,\mathcal{D}(1+s\mathcal{D}^2)^{-r})$ is a trace on $\mathcal{M}$ (note that if we considered partial isometries in a matrix algebra over $\mathcal{A}^\sim$ we would have to require in addition that $v_0^*v_0 - v_0v_0^*$ is an element over $\mathcal{F}$). □

Finally, when the circle action has full spectral subspaces, the eta corrections also vanish.

Corollary 5.7. *Assume the circle action σ has full spectral subspaces. Then for every modular partial isometry $v \in A^\sim$ we have*

$$sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*) = \mathrm{Res}_{r=1/2}\, \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1+\mathcal{D}^2)^{-r})dt.$$

Proof. Consider the modular partial isometry $w = v - v_0$. Then $w \in \mathcal{A}$, so we can apply the previous Theorem. Since the spectral flow corresponding to v and w coincide and $v[\mathcal{D}, v^*] = w[\mathcal{D}, w^*]$, all we have to do is to show that the eta term defined by w vanishes. By the assumption of full spectral subspaces we have

$$\phi_{\mathcal{D}}((\sigma_{-i\beta}(w^*)w - ww^*)\Phi_k) = \phi(\sigma_{-i\beta}(w^*)w - ww^*) = 0,$$

for all $k \in \mathbb{Z}$, and as

$$\phi_{\mathcal{D}}((\sigma_{-i\beta}(w^*)w - ww^*)\mathcal{D}(1+s\mathcal{D}^2)^{-r}) = \sum_{k\in\mathbb{Z}} \phi_{\mathcal{D}}((\sigma_{-i\beta}(w^*)w - ww^*)\Phi_k)k(1+sk^2)^{-r},$$

the eta term is indeed zero. □

5.3. **Twisted cyclic cocycles.** This subsection is motivated by the observation of [9] that when there are no eta or kernel correction terms we can define a functional on $\mathcal{A} \otimes \mathcal{A}$ by

$$(a_0, a_1) \mapsto \omega\text{-}\lim_{s\to\infty} \frac{1}{s}\phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1+\mathcal{D}^2)^{-1/s-1/2})$$

which is, at least formally, a twisted (by $\sigma_{-i\beta}$) cyclic cocycle. However we saw in [13] that in the case of $SU_q(2)$ the eta corrections created a subtle difficulty in that individually they do not have the same holomorphy properties as the term in the previous equation and that only by combining them do we obtain something we can understand in cohomological terms. Thus we set, for $a_0, a_1 \in \mathcal{A}$,

$$\eta^r_{\mathcal{D}}(a_0, a_1) = \frac{1}{2}\int_1^\infty \phi_{\mathcal{D}}((\sigma_{-i\beta}(a_1)a_0 - a_0a_1)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds.$$

This is well-defined for $\Re(r) > 1$, and as we shall see later, extends analytically to $\Re(r) > 1/2$. When we pair with a modular partial isometry we necessarily have $(r-1/2)\eta^r_{\mathcal{D}}(v, v^*)$ bounded, since the sum of the eta term and $\phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1+\mathcal{D}^2)^{-r})$ has a simple pole by Proposition 5.3.

Throughout this Section, b^σ, B^σ denote the twisted Hochschild and Connes coboundary operators in twisted cyclic theory, [24]. The twisting will always come from the regular automorphism $\sigma := \sigma_{-i\beta} = \sigma_i^{\phi_{\mathcal{D}}}$ of $\mathcal{A}$ (recall that an algebra automorphism σ is regular if $\sigma(a)^* = \sigma^{-1}(a^*)$, [24]).

In order to be able to describe the index pairing of Theorem 5.6 as the pairing of a twisted b^σ, B^σ cocycle with the modular K_1 group, we need to address the analytic difficulties we have just described. This is done in the next Lemma.

Lemma 5.8. *For a_0, $a_1 \in \mathcal{A}$, let*

$$\psi^r(a_0, a_1) = \phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1+\mathcal{D}^2)^{-r}) + \eta^r_{\mathcal{D}}(a_0, a_1).$$

Then for $a_0, a_1, a_2 \in \mathcal{A}$ the functions $r \mapsto \phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1+\mathcal{D}^2)^{-r})$ and $r \mapsto \eta^r_{\mathcal{D}}(a_0, a_1)$ are analytic for $\Re(r) > 1/2$, while $r \mapsto (b^\sigma\psi^r)(a_0, a_1, a_2)$ is analytic for $\Re(r) > 0$.

Proof. Recall that the algebra $\mathcal{A}$ consists of finite sums of homogeneous elements in the domain of ϕ. Therefore we may assume that a_0, a_1, a_2 are homogeneous. Consider the conditional expectation $\Psi\colon \mathcal{N} \to \mathcal{N}^\sigma$, $\Psi(x) = \sum_n \Phi_n x \Phi_n$. Then $\phi_{\mathcal{D}} = \phi_{\mathcal{D}} \circ \Psi$. It follows that if $a_0 \in A_k$ and $a_1 \in A_m$ then $\phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1+\mathcal{D}^2)^{-r}) = 0$ unless $k = -m$, and in the latter case we have

$$\phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1+\mathcal{D}^2)^{-r}) = \sum_{n\in\mathbb{Z}} \frac{s_n}{(1+n^2)^r},$$

where $s_n = m\,\phi_{\mathcal{D}}(a_0a_1\Phi_n)$. By Lemma 4.8 the sequence $\{s_n\}_n$ is bounded. Hence the function $\phi_{\mathcal{D}}(a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-r})$ is analytic for $\Re(r)>1/2$.

Consider now $\eta^r(a_0,a_1)$. If $a_0\in A_k$ and $a_1\in A_m$ then $\eta^r(a_0,a_1)=0$ unless $k=-m$. In the latter case put $s_n=\phi_{\mathcal{D}}(a_0a_1\Phi_n)$. Notice that

$$\phi_{\mathcal{D}}(\sigma(a_1)a_0\Phi_n)=\phi_{\mathcal{D}}(a_0\Phi_na_1)=\phi_{\mathcal{D}}(a_0a_1\Phi_{n-m})=s_{n-m}.$$

The sequence $\{s_n\}_n$ is bounded. Assume $m\geq 0$. Then for $\Re(r)>1$ we have

$$\int_1^\infty \phi_{\mathcal{D}}((\sigma(a_1)a_0-a_0a_1)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds=\sum_{n\in\mathbb{Z}}\int_1^\infty\frac{(s_{n-m}-s_n)n}{(1+sn^2)^r}s^{-1/2}ds$$

which we may write as

$$\begin{aligned}
&2\sum_{n>0}(s_{n-m}-s_n)\int_n^\infty\frac{dt}{(1+t^2)^r}-2\sum_{n<0}(s_{n-m}-s_n)\int_{-n}^\infty\frac{dt}{(1+t^2)^r}\\
&=2\sum_{n=-m+1}^{0}s_n\int_{n+m}^\infty\frac{dt}{(1+t^2)^r}-2\sum_{n>0}s_n\int_n^{n+m}\frac{dt}{(1+t^2)^r}\\
&+2\sum_{n=-m}^{-1}s_n\int_{-n}^\infty\frac{dt}{(1+t^2)^r}-2\sum_{n<-m}s_n\int_{-n-m}^{-n}\frac{dt}{(1+t^2)^r}.
\end{aligned}$$

The above series of functions analytic on $\Re(r)>1/2$ converge uniformly on $\Re(r)>1/2+\epsilon$ for every $\epsilon>0$. A similar argument works for $m\leq 0$. Hence the function $r\mapsto\eta^r(a_0,a_1)$ extends analytically to $\Re(r)>1/2$.

Turning to $b^\sigma\psi^r$, first notice that $b^\sigma\eta^r_{\mathcal{D}}=0$, since $r\mapsto b^\sigma\eta^r_{\mathcal{D}}$ is analytic for $\Re(r)>1/2$ and $\eta^r_{\mathcal{D}}=b^\sigma\theta^r_{\mathcal{D}}$ for $\Re(r)>1$, where

$$\theta^r_{\mathcal{D}}(a_0)=-\frac{1}{2}\int_1^\infty\phi_{\mathcal{D}}(a_0\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds.$$

It follows that $(b^\sigma\psi^r)(a_0,a_1,a_2)$ is given by

$$\begin{aligned}
&\phi_{\mathcal{D}}(a_0a_1[\mathcal{D},a_2](1+\mathcal{D}^2)^{-r}-\phi_{\mathcal{D}}(a_0[\mathcal{D},a_1a_2](1+\mathcal{D}^2)^{-r})+\phi_{\mathcal{D}}(\sigma(a_2)a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-r})\\
&=-\phi_{\mathcal{D}}(a_0[\mathcal{D},a_1]a_2(1+\mathcal{D}^2)^{-r})+\phi_{\mathcal{D}}(\sigma(a_2)a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-r}).
\end{aligned}$$

If $a_0\in A_k$, $a_1\in A_l$ and $a_2\in A_m$, then the above expression is zero unless $k+l+m=0$. In the latter case put $s_n=l\,\phi_{\mathcal{D}}(a_0a_1a_2\Phi_n)$. Then a computation similar to that for η^r yields, for $\Re(r)>1/2$,

$$\begin{aligned}
(b^\sigma\psi^r)(a_0,a_1,a_2)&=\sum_{n\in\mathbb{Z}}s_n((1+(n+m)^2)^{-r}-(1+n^2)^{-r})\\
&=\sum_{n\in\mathbb{Z}}s_n(1+n^2)^{-r}\left(\left(1+\frac{2mn+m^2}{1+n^2}\right)^{-r}-1\right)
\end{aligned}$$

Using that if Ω is a compact subset of $\Re(r)>0$ then $|(1+x)^{-r}-1|\leq C\,|x|$ for some $C>0$, sufficiently small x and all $r\in\Omega$, we see that the above series converges uniformly on Ω. Hence $(b^\sigma\psi^r)(a_0,a_1,a_2)$ extends analytically to $\Re(r)>0$. □

The following result links our analytic constructions to twisted cyclic theory.

Proposition 5.9. *Given the modular spectral triple for (A,σ,ϕ) define a bilinear functional on $\mathcal{A}$ with values in the functions holomorphic for $\Re(r)>1$ by*

$$a_0,a_1\mapsto\left(r\mapsto\left(\phi_{\mathcal{D}}(a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-r})+\frac{1}{2}\int_1^\infty\phi_{\mathcal{D}}((\sigma(a_1)a_0-a_0a_1)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds\right)\right)$$

This functional continues analytically to $\Re(r)>1/2$ and is a twisted b,B-cocycle modulo functions holomorphic for $\Re(r)>0$. The twisting is given by the regular automorphism $\sigma:=\sigma_{-i\beta}=\sigma_i^{\phi_{\mathcal{D}}}$.

Remark. If ϕ is a state then the domain of the cocycle of the Proposition is much larger, but to prove this requires more work.

Proof. As before, for $\Re(r)>1$ we define the functional ψ^r by the formula

$$\psi^r(a_0,a_1)=\phi_{\mathcal{D}}(a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-r})+\frac{1}{2}\int_1^\infty\phi_{\mathcal{D}}((\sigma(a_1)a_0-a_0a_1)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds,$$

and then extend ψ^r analytically to $\Re(r)>1/2$, which is possible by Lemma 5.8. Then $(B^\sigma\psi^r)(a_0)=\psi^r(1,a_0)$ and for $\Re(r)>1$ is given by

$$(B^\sigma\psi^r)(a_0)=\phi_{\mathcal{D}}([\mathcal{D},a_0](1+\mathcal{D}^2)^{-r})+\frac{1}{2}\int_1^\infty\phi_{\mathcal{D}}((\sigma(a_0)-a_0)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds.$$

The first term vanishes since $\Psi([\mathcal{D},a_0])=0$ for any $a_0\in\mathcal{A}$, while the second terms vanishes by σ_t-invariance of $\phi_{\mathcal{D}}$. That $b^\sigma\psi^r$ is analytic for $\Re(r)>0$ was proved in the last Lemma. □

Corollary 5.10. *If the circle action has full spectral subspaces then for all $a_0,a_1\in\mathcal{A}$ the residue*

$$\phi_1(a_0,a_1):=\mathrm{Res}_{r=1/2}\,\phi_{\mathcal{D}}(a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-r})$$

exists and equals $\phi(a_0[\mathcal{D},a_1])$. It defines a twisted cyclic cocycle on $\mathcal{A}$, and for any modular partial isometry $v\in\mathcal{A}$

$$sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D},v\mathcal{D}v^*)=\phi_1(v,v^*)=\phi(v[\mathcal{D},v^*]).$$

Proof. Under the full spectral subspaces assumption we have $\phi_{\mathcal{D}}(f\Phi_n)=\phi(f)$ for $f\in\mathcal{F}$, whence

$$\phi_{\mathcal{D}}(a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-r})=\phi_{\mathcal{D}}(\Psi(a_0[\mathcal{D},a_1])(1+\mathcal{D}^2)^{-r})=\phi(a_0[\mathcal{D},a_1])\sum_{n\in\mathbb{Z}}\frac{1}{(1+n^2)^r}.$$

This shows that the residue exists and equals $\phi(a_0[\mathcal{D},a_1])$. That it defines a twisted cyclic cocycle follows from the proof of Proposition 5.9. That $\phi_1(v,v^*)$ computes the spectral flow follows from Corollary 5.7. □

Remark. It is of course easy to see directly that $\phi(a_0[\mathcal{D},a_1])$ is a twisted cyclic cocycle, while the fact that it computes the spectral flow agrees with Lemma 4.10.

Finally, we have the following reformulation of Lemma 4.8 as an analogue of the result of A. Connes on continuity (in ω) of the Dixmier trace on pseudodifferential operators on a compact manifold in the KMS-weight context.

Proposition 5.11. *Given the modular spectral triple for (A,σ,ϕ) and a Dixmier functional (see Section 3 of [7]), $\omega\in(L^\infty(\mathbf{R}))^*$, define $\phi_{\mathcal{D},\omega}:A_+\to[0,\infty]$ by*

$$\phi_{\mathcal{D},\omega}(a)=\omega\text{-}\lim_{r\to\infty}\frac{1}{r}\phi_{\mathcal{D}}(\Psi(a)(1+\mathcal{D}^2)^{-1/2-1/2r}),$$

where $\Psi\colon\mathcal{N}\to\mathcal{M}$ is the $\phi_{\mathcal{D}}$-preserving conditional expectation. Then $\phi_{\mathcal{D},\omega}(a)\le2\phi(a)$ for any $a\in A_+\cap\mathrm{dom}(\phi)$, with equality if A has full spectral subspaces. In particular, when the circle action has full spectral subspaces, $\phi_{\mathcal{D},\omega}(a)$ is independent of ω.

Proof. Recall that $\Psi(x) = \sum_{n\in\mathbb{Z}} \Phi_n x \Phi_n$. Observe also that $\Phi_n a \Phi_n = \Phi(a)\Phi_n$. Hence for $a \in A_+$,

$$\phi_{\mathcal{D}}(\Phi_n a \Phi_n) = \phi_{\mathcal{D}}(\Phi(a)\Phi_n) \leq \phi(\Phi(a)) = \phi(a)$$

by Lemma 4.8, with equality if A has full spectral subspaces. This shows that

$$\phi_{\mathcal{D},\omega}(a) = \omega\text{-}\lim_{r\to\infty} \frac{1}{r} \sum_{n\in\mathbb{Z}} \phi_{\mathcal{D}}(\Phi_n a \Phi_n)(1+n^2)^{-1/2-1/2r}$$

exists and when A has full spectral subspaces

$$\phi_{\mathcal{D},\omega}(a) = \lim_{r\to\infty} \frac{\phi(a)}{r} \sum_{n\in\mathbb{Z}} (1+n^2)^{-1/2-1/2r} = 2\phi(a).$$

□

6. Examples

Our first two examples are covered in detail in [9, 13] so we will only present a summary here.

Example 1. For the algebra $\mathcal{O}_n$ (with generators $S_1, \ldots, S_n$) we write S_α for the product $S_{\mu_1} \ldots S_{\mu_k}$ and $k = |\alpha|$. We take the usual gauge action σ, and the unique KMS state ϕ for this circle action.

In the Cuntz algebra case we have full spectral subspaces. Due to the absence of eta terms, the analytic formula is the easiest to apply, so we can compute the pairing with $S_\alpha S_\beta^*$ using the residue cocycle, Corollary 5.10, and get

$$sf(S_\alpha S_\alpha^* \mathcal{D}, S_\alpha S_\beta^* \mathcal{D} S_\beta S_\alpha^*) = (|\beta| - |\alpha|)\frac{1}{n^{|\alpha|}}.$$

Example 2. For $SU_q(2)$ we used the graph algebra description of Hong and Szymanski, [18], and we use the notation and computations from [13]. There we introduced a new set of generators T_k, $\tilde{T}_k$, U_n for this algebra. The generators T_k and $\tilde{T}_k$ are non-trivial homogenous partial isometries for the modular group of the Haar state, h, which is a $\mathrm{KMS}_{-\log q^2}$ state.

For $SU_q(2)$ there are eta correction terms. Given the explicit computations in [13] and the description of the fixed point algebra as the unitization of an infinite direct sum of copies of $C(S^1)$ (that is the C^*-algebra of the one point compactification of an infinite union of circles of radius q^{2k}, $k \geq 0$) it is not hard to see that our SSA is satisfied for $SU_q(2)$.

The presence of the eta corrections makes the analytic computation of spectral flow from the twisted cocycle harder (it can still be done explicitly as in [13]). Instead we employ the factorisation through the KK-pairing. Taking the value of the trace $\mathrm{Tr}_\phi(T_k^* T_k \Phi_j) = q^{2(|j|+1)}$ from [13] we have

$$sf_{\phi_{\mathcal{D}}}(T_k^* T_k \mathcal{D}, T_k^* \mathcal{D} T_k) = \mathrm{Ev}(e^{\log q^2}) \circ \tau_* \left(\sum_{j=-k}^{-1} [T_k^* T_k \Phi_j] \chi^j \right) = \sum_{j=-k}^{-1} \mathrm{Tr}_\phi(T_k^* T_k \Phi_j) q^{2j} = kq^2.$$

The point of this example is that there are naturally occurring examples satisfying the SSA but without full spectral subspaces.

Example 3: the Araki-Woods factors. We will follow the treatment of the Araki-Woods factors in Pedersen [29], Subsection 8.12 and the subsequent discussion. We let A be the Fermion algebra, that is the C^*-inductive limit of the matrix algebras $\mathrm{Mat}_{2^n}(\mathbb{C})$ which is the n-fold tensor product of the matrix algebra of 2×2 matrices $\mathrm{Mat}_2(\mathbb{C})$.

For $0 < \lambda < 1/2$ let

$$h_n = \otimes_{k=1}^n \begin{pmatrix} 2(1-\lambda) & 0 \\ 0 & 2\lambda \end{pmatrix}$$

Let ϕ be the tracial state on A (given by the tensor product of the normalised traces on $\mathrm{Mat}_2(\mathbb{C})$) and define

$$\phi_\lambda(x) = \phi(h_n x),\ x \in \mathrm{Mat}_{2^m}(\mathbb{C}),\ m \leq n.$$

Then ϕ_λ is a state on $\mathrm{Mat}_{2^m}(\mathbb{C})$ and is independent of n. By continuity it extends to a state on A. Consider the automorphism group defined by $\mathrm{Ad}\, h_n^{-it}$. It is not hard to see that ϕ_λ satisfies the KMS condition with respect to $\mathrm{Ad}\, h_n^{-it}$ at 1 for this group or equivalently at $\beta = \ln \frac{1-\lambda}{\lambda}$ for the gauge action $\sigma_t = \mathrm{Ad}\, h_n^{-it/\beta}$. Everything extends by continuity to A. Then the GNS representation corresponding to ϕ_λ generates a type $\mathrm{III}_{\lambda'}$ factor, where $\lambda' = \lambda/(1-\lambda)$ (for a proof see [29] 8.15.13).

The simplest way to see the we have full spectral subspaces for the circle action σ is to replace this version of the Fermion algebra by the isomorphic copy given by annihilation and creation operators, see e.g. [17].

To describe the isomorphism, we let $\sigma_j,\ j = 1, 2, 3$ be the Pauli matrices in their usual representation:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the isomorphism is given by defining $a_j = \sigma_3 \otimes \ldots \sigma_3 \otimes (\sigma_1 + i\sigma_2)/2$, where the last term is in the j-th tensorial factor. Then the $a_j,\ j \in \mathbb{N}$, and their adjoints a_j^* satisfy the usual relations of the C^*-algebra of the canonical anticommutation relations (i.e. the Fermion algebra):

$$a_j a_k^* + a_k^* a_j = \delta_{jk}, \quad a_j a_k + a_k a_j = 0.$$

The gauge invariant algebra is generated by monomials in the a_j, a_k^* which have equal numbers of creation and annihilation operators. Clearly A_1 is generated by monomials with one more creation operator than annihilation operator. From the anticommutation relations above it is now clear that $A_1^* A_1$ and $A_1 A_1^*$ are dense in the gauge invariant subalgebra. So we have full spectral subspaces. Thus the main results of the paper apply to this example.

Modular partial isometries are easy to find, since each a_j is an homogenous partial isometry in A_1. For a single a_j we can employ the twisted cyclic cocycle to get the index

$$sf_{\phi_{\mathcal{D}}}(a_j a_j^* \mathcal{D}, a_j \mathcal{D} a_j^*) = -\phi(a_j a_j^*) = -\lambda = -(1 + e^\beta)^{-1}.$$

Similarly if we have the partial isometry v formed by taking the product of n distinct a_j's we obtain

$$sf_{\phi_{\mathcal{D}}}(vv^* \mathcal{D}, v\mathcal{D}v^*) = -n(1 + e^\beta)^{-n}.$$

In [9] we made the observation that for modular unitaries u_v, $sf_{\phi_{\mathcal{D}}}(\mathcal{D}, u_v \mathcal{D} u_v^*)$ is just Araki's relative entropy [1] of the two KMS weights $\phi_{\mathcal{D}}$ and $\phi_{\mathcal{D}} \circ \mathrm{Ad}\, u_v$. In this example of the Fermion algebra we see that the relative entropy depends on two physical parameters, the inverse temperature β and the modulus of the charge n carried by the product of Fermion annihilation or creation operators appearing in v.

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Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia

E-mail address: acarey@maths.anu.edu.au

Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, 0316 Oslo, Norway

E-mail address: sergeyn@math.uio.no

Department of Mathematics, Copenhagen University, Universitetsparken 5, 2100 Copenhagen, Denmark

E-mail address: rnest@math.ku.dk

Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia

E-mail address: rennie@maths.anu.edu.au