TWISTED CYCLIC THEORY AND AN INDEX THEORY FOR THE GAUGE INVARIANT KMS STATE ON CUNTZ ALGEBRAS

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Abstract

This paper presents, by example, an index theory appropriate to algebras without trace. Whilst we work exclusively with the Cuntz algebras the exposition is designed to indicate how to develop a general theory. Our main result is an index theorem (formulated in terms of spectral flow) using a twisted cyclic cocycle where the twisting comes from the modular automorphism group for the canonical gauge action on the Cuntz algebra. We introduce a modified K_1-group of the Cuntz algebra so as to pair with this twisted cocycle. As a corollary we obtain a noncommutative geometry interpretation for Araki's notion of relative entropy in this example. We also note the connection of this example to the theory of noncommutative manifolds.

Contents

1. Introduction

In this paper we initiate an extension of index theory to algebras without trace. We take the Cuntz algebras O_n [Cu] as basic examples. In the absence of a non-trivial trace on the Cuntz algebras, our approach is to use a KMS state, [BR2], to define an index pairing using spectral flow. The state we use is the unique KMS state for the canonical $\mathbb{T}^1$ gauge action on O_n. As O_n is a graph algebra, we can import many of the techniques of [PR] where the semifinite version of the local index formula was used to calculate spectral flow invariants of a class of Cuntz-Krieger algebras. The Cuntz algebras give us an excellent testing ground for the ideas required to deal with index theory in a type III setting.

The approach is motivated by [CPRS2] where a semifinite local index formula in noncommutative geometry is proved. This semifinite theory is reviewed in Section 2 together with notation for the Cuntz algebras. In [PR] the semifinite theory was applied to certain graph C^*-algebras. The new idea explained there was the construction of a Kasparov A, F-module for the graph algebra A of a locally finite graph with no sources where $F = A^{\mathbb{T}^1}$ is the fixed point algebra for the natural $\mathbb{T}^1$ gauge action. This construction applies to the Cuntz algebra because it is a graph algebra of this type. A K-theoretic refinement for the index theorem in [PR] was developed in [CPR] where the odd Kasparov module of [PR] is 'doubled up' on a half infinite cylinder to an even Kasparov $M(F, A), F$-module, where $M(F, A)$ is the mapping cone algebra for the inclusion of the fixed point algebra. Our idea is to modify this tracial case so as to extend, as far as is possible, these results to the Cuntz algebra.

We easily observe that there is a Kasparov module for the Cuntz algebra and hence that we have a $K_0(F)$-valued pairing with $M(F, A)$. However, in the absence of a trace we need a new idea. **The primary result of this paper introduces a modified spectral triple (referred to as a 'modular spectral triple') with which we can compute an index pairing.** Our method, of employing a KMS functional instead of a trace, leads to various subtleties. Restricting the KMS state to the fixed point algebra F gives a trace on F, and so a homomorphism on $K_0(F)$. However, when we pass to the Morita equivalent algebra of compact endomorphisms on our Kasparov module, we find that the functional we are forced to employ on this new algebra does not respect all Murray-von Neumann equivalences. It is this fact that leads to the consideration of finer invariants than those obtained from ordinary K-theory in the KMS or 'twisted setting'.

We show that modular spectral triples lead to 'twisted residue cocycles' using a variation on the semifinite residue cocycle of [CPRS2]. It is well known that such twisted cocycles cannot pair with ordinary K_1, rather we introduce, in Section 4, a substitute which we term 'modular K_1'. It is a semigroup and, as is explained in our main theorem (Theorem 5.5), there is a general spectral flow formula which defines the pairing of modular K_1 with our 'twisted residue cocycle'. There is an analogy with the local index formula of noncommutative geometry in the $\mathcal{L}^{1,\infty}$-summable case, however, there are important differences: the usual residue cocycle is replaced by a twisted residue cocycle and the Dixmier trace arising in the standard situation is replaced by a KMS-Dixmier functional. The common ground with [CPRS2] stems from the use of the general spectral flow formula of [CP2] to derive the twisted residue cocycle and this has the corollary that we have a homotopy invariant.

For the Cuntz algebras the main result is Theorem 5.6 and its Corollary where we compute, for particular modular unitaries in matrix algebras over the Cuntz algebras, the precise numerical values arising from the general formalism. We use [CPR] to see that these numerical values provide strong evidence that the mapping cone KK-theory of Section 2 is playing a (yet to be fully understood) role. In the final Section we note that there is a physical interpretation of the spectral flow invariant we are calculating in terms of Araki's notion of relative entropy of two KMS states. We also show that our modular spectral triples for the Cuntz algebras satisfy twisted versions of Connes' axioms for noncommutative manifolds.

We plan to return to this matter and to the appropriate cohomological setting for our index theorem elsewhere. Already, in work in progress [CRT], we have uncovered further examples which indicate there is a complex and interesting theory to be understood.

The organisation is summarised in the Contents list. Section 2 is review material which places this article in context. The Cuntz algebra example begins on Section 3 and the main new material is in Sections 4 and 5.

2. Some background

2.1. **Semifinite noncommutative geometry.** We begin with some semifinite versions of standard definitions and results following [CPRS2]. Let ϕ be a fixed faithful, normal, semifinite trace on a von Neumann algebra $\mathcal{N}$. Let $\mathcal{K}_{\mathcal{N}}$ be the ϕ-compact operators in $\mathcal{N}$ (that is the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\phi(E) < \infty$).

Definition 2.1. *A* **semifinite spectral triple** $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$ and $(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_{\mathcal{N}}$ for all $\lambda \notin \mathbf{R}$. The triple is said to be* **even** *if there is $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$ and $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Otherwise it is* **odd**.

Note that if $T \in \mathcal{N}$ and $[\mathcal{D}, T]$ is bounded, then $[\mathcal{D}, T] \in \mathcal{N}$.

We recall from [FK] that if $S \in \mathcal{N}$, the **t-th generalized singular value** of S for each real $t > 0$ is given by
$$\mu_t(S) = \inf\{\|SE\| \; : \; E \text{ is a projection in } \mathcal{N} \text{ with } \phi(1 - E) \le t\}.$$
The ideal $\mathcal{L}^1(\mathcal{N}, \phi)$ consists of those operators $T \in \mathcal{N}$ such that $\| T \|_1 := \phi(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will denote the norm on $\mathcal{L}^1(\mathcal{N}, \phi)$ by $\| \cdot \|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1(\mathcal{N}, \phi)$ if $\|T\|_1 := \int_0^\infty \mu_t(T)dt < \infty$. When $\mathcal{N} \ne \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1(\mathcal{N}, \phi)$ need not be complete in this norm but it is complete in the norm $\|.\|_1 + \|.\|_\infty$. (where $\|.\|_\infty$ is the uniform norm). We use the notation
$$\mathcal{L}^{(1,\infty)}(\mathcal{N}, \phi) = \left\{ T \in \mathcal{N} \; : \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds < \infty \right\}.$$

The reader should note that $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \phi)$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of ϕ-measurable operators affiliated to $\mathcal{N}$. Our notation is however consistent with that of [C] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of ϕ-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that $\mu_\infty(T) := \lim_{t\to\infty} \mu_t(T) = 0$.

Definition 2.2. *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ relative to $(\mathcal{N}, \phi)$ with $\mathcal{A}$ unital is $(1, \infty)$-summable if $(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(1,\infty)}(\mathcal{N}, \phi)$ for all $\lambda \in \mathbf{C} \setminus \mathbf{R}$.*

It follows that if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is $(1, \infty)$-summable then it is n-summable (with respect to the trace ϕ) for all $n > 1$. We next need to briefly discuss Dixmier traces. For more information on semifinite Dixmier traces, see [CPS2]. For $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N}, \phi)$, $T \ge 0$, the function
$$F_T : t \to \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds$$
is bounded. There are certain $\omega \in L^\infty(\mathbf{R}_*^+)^*$, [CPS2, C], which define (Dixmier) traces on $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \phi)$ by setting
$$\phi_\omega(T) = \omega(F_T), \quad T \ge 0$$

and extending to all of $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \phi)$ by linearity. For each such ω we write ϕ_ω for the associated Dixmier trace. Each Dixmier trace ϕ_ω vanishes on the ideal of trace class operators. Whenever the function F_T has a limit at infinity, all Dixmier traces return that limit as their value. This leads to the notion of a measurable operator [C, LSS], that is, one on which all Dixmier traces take the same value.

We now introduce (a special case of) the analytic spectral flow formula of [CP1, CP2]. This formula starts with a semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and computes the ϕ spectral flow from $\mathcal{D}$ to $u\mathcal{D}u^*$, where $u \in \mathcal{A}$ is unitary with $[\mathcal{D}, u]$ bounded, in the case where $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is n-summable for $n > 1$ (Theorem 9.3 of [CP2]):

$$(1) \qquad sf_\phi(\mathcal{D}, u\mathcal{D}u^*) = \frac{1}{C_{n/2}} \int_0^1 \phi(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-n/2}) dt,$$

with $C_{n/2} = \int_{-\infty}^{\infty} (1 + x^2)^{-n/2} dx$. This real number $sf_\phi(\mathcal{D}, u\mathcal{D}u^*)$ is a pairing of the K-homology class $[\mathcal{D}]$ of $\mathcal{A}$ with the $K_1(\mathcal{A})$ class $[u]$ [CPRS2]. There is a geometric way to view this formula. It is shown in [CP2] that the functional $X \mapsto \phi(X(1 + (\mathcal{D} + Y)^2)^{-n/2})$ determines an exact one-form for X in the tangent space, $\mathcal{N}_{sa}$, of an affine space $\mathcal{D} + \mathcal{N}_{sa}$ modelled on $\mathcal{N}_{sa}$. Thus (1) represents the integral of this one-form along the path $\{\mathcal{D}_t = (1 - t)\mathcal{D} + tu\mathcal{D}u^*\}$ provided one appreciates that $\dot{\mathcal{D}}_t = u[\mathcal{D}, u^*]$ is a tangent vector to this path. In [CPRS2], the local index formula in noncommutative geometry of [CM] was extended to semifinite spectral triples. In the simplest terms, the local index formula is a pairing of a finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with the K-theory of the C^*-algebra $\overline{\mathcal{A}}$. Our approach in this paper is inspired by the following theorem (see also [CPRS2, CM, H]).

Theorem 2.3 ([CPS2]). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd $(1, \infty)$-summable semifinite spectral triple, relative to $(\mathcal{N}, \phi)$. Then for $u \in \mathcal{A}$ unitary the pairing of $[u] \in K_1(\overline{\mathcal{A}})$ with $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by*

$$\langle [u], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = sf_\phi(\mathcal{D}, u\mathcal{D}u^*) = \lim_{s \to 0^+} s \, \phi(u[\mathcal{D}, u^*](1 + \mathcal{D}^2)^{-1/2-s}).$$

In particular, the limit on the right exists.

2.2. **The Cuntz algebras and the canonical Kasparov module.** For $n \geq 2$, the Cuntz algebra [Cu] on n generators, O_n, is the (universal) C^*-algebra generated by n isometries S_i, $i = 1, ..., n$, subject only to the relation $\sum_{i=1}^n S_i S_i^* = 1$. The projections $S_i S_i^*$ will be denoted by P_i and more generally we will write $P_\mu = S_\mu S_\mu^*$. For $\mu \in \{1, 2, ..., n\}^k = \mathbf{n}^k$ we write $S_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_k}$ and $S_\mu^* = S_{\mu_k}^* S_{\mu_{k-1}}^* \cdots S_{\mu_1}^*$. Using the fact that $S_i^* S_j = \delta_{ij}$, one can show that every word in the S_i, S_j^* can be written in the form $S_\mu S_\nu^*$, where $\mu \in \mathbf{n}^k$ and $\nu \in \mathbf{n}^l$ are multi-indices. We will write $|\mu| = k$ and $|\nu| = l$ for the length of such multi-indices. As the family of monomials $\{S_\mu S_\nu^*\}$ is closed under multiplication and involution, we have

$$(2) \qquad O_n = \overline{\text{span}}\{S_\mu S_\nu^* : \mu \in \mathbf{n}^k, \ \nu \in \mathbf{n}^m, \ k, m \geq 0\}.$$

If $z \in \mathbb{T}^1$, then the family $\{z S_j\}$ is another Cuntz-Krieger family which generates O_n, and the universal property of O_n gives a homomorphism $\sigma_z : O_n \to O_n$ such that $\sigma_z(S_e) = z S_e$. The homomorphism $\sigma_{\overline{z}}$ is an inverse for σ_z, so $\sigma_z \in \text{Aut} \, O_n$, and a routine argument shows that σ is a strongly continuous action of $\mathbb{T}^1$ on O_n. It is called the *gauge action*. Averaging over σ with respect to normalised Haar measure gives a positive, faithful expectation Φ of O_n onto the fixed-point algebra $F := O_n^\sigma$:

$$\Phi(a) := \frac{1}{2\pi} \int_{\mathbb{T}^1} \sigma_z(a) \, d\theta \ \text{ for } \ a \in O_n, \ \ z = e^{i\theta}.$$

To simplify notation, we let $A = O_n$ be the Cuntz algebra and $F = A^\sigma$, the fixed point algebra for the $\mathbb{T}^1$ gauge action. The algebras A_c, F_c are defined as the finite linear span of the generators. Right

multiplication makes A into a right F-module, and similarly A_c is a right module over F_c. We define an F-valued inner product $(\cdot|\cdot)_R$ on both these modules by $(a|b)_R := \Phi(a^*b)$.

Definition 2.4. *Let X be the right F C^*-module obtained by completing A (or A_c) in the norm*

$$\|x\|_X^2 := \|(x|x)_R\|_F = \|\Phi(x^*x)\|_F.$$

The algebra A acting by left multiplication on X provides a representation of A as adjointable operators on X. Let X_c be the copy of $A_c \subset X$. The $\mathbb{T}^1$ action on X_c is unitary and extends to X, [PR]. For all $k \in \mathbf{Z}$, the projection onto the k-th spectral subspace of the $\mathbb{T}^1$ action is the operator Φ_k on X:

$$\Phi_k(x) = \frac{1}{2\pi} \int_{\mathbb{T}^1} z^{-k}\sigma_z(x)d\theta, \quad z = e^{i\theta}, \quad x \in X.$$

Observe that Φ_0 restricts to Φ on A and on generators of O_n we have

$$(3) \qquad \Phi_k(S_\alpha S_\beta^*) = \begin{cases} S_\alpha S_\beta^* & |\alpha| - |\beta| = k \\ 0 & |\alpha| - |\beta| \neq k \end{cases}.$$

We quote the following result from [PR].

Lemma 2.5. *The operators Φ_k are adjointable endomorphisms of the F-module X such that $\Phi_k^* = \Phi_k = \Phi_k^2$ and $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$. If $K \subset \mathbf{Z}$ then the sum $\sum_{k \in K} \Phi_k$ converges strictly to a projection in the endomorphism algebra. The sum $\sum_{k \in \mathbf{Z}} \Phi_k$ converges to the identity operator on X. For all $x \in X$, the sum $x = \sum_{k \in \mathbf{Z}} \Phi_k x = \sum_{k \in \mathbf{Z}} x_k$ converges in X.*

The unbounded operator of the next proposition is of course the generator of the $\mathbb{T}^1$ action on X. We refer to Lance's book, [L, Chapters 9,10], for information on unbounded operators on C^*-modules.

Proposition 2.6. [PR] *Let X be the right C^*-F-module of Definition 2.4. Define $\mathcal{D} : X_\mathcal{D} \subset X$ to be the linear space*

$$X_\mathcal{D} = \{x = \sum_{k \in \mathbf{Z}} x_k \in X : \|\sum_{k \in \mathbf{Z}} k^2(x_k|x_k)_R\| < \infty\}.$$

For $x \in X_\mathcal{D}$ define $\mathcal{D}(x) = \sum_{k \in \mathbf{Z}} kx_k$. Then $\mathcal{D} : X_\mathcal{D} \to X$ is a is self-adjoint, regular operator on X.

Remark. On generators in O_n (regarded as elements of $X_c \subset X$) we have $\mathcal{D}(S_\alpha S_\beta^*) = (|\alpha| - |\beta|)S_\alpha S_\beta^*$.

We will need the following technical result from [PR] later:

Lemma 2.7. *For all $a \in A$ and $k \in \mathbf{Z}$, $a\Phi_k \in End_F^0(X)$, the compact F linear endomorphisms of the right F module X. If $a \in A_c$ then $a\Phi_k$ is finite rank.*

Introduce the rank one operator $\Theta_{x,y}^R$ by $\Theta_{x,y}^R z = x(y|z)_R$. Then by [PR, Lemma 4.7], for $k \geq 0$, $\Phi_k = \sum_{|\mu|=k} \Theta_{S_\mu,S_\mu}^R$ where for the Cuntz algebras the sum is finite. For the negative subspaces the formula in [PR] gives, in the Cuntz algebras $\Phi_{-k} = \frac{1}{n^k} \sum_{|\mu|=k} \Theta_{S_\mu^*,S_\mu^*}^R$.

Theorem 2.8. [PR] *Let X be the right F module of Definition 2.4. Let $V = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$. Then (X,V) is an odd Kasparov module for A-F and so defines an element of $KK^1(A,F)$.*

Given the hypotheses of the Theorem, we may write $\mathcal{D}$ as $\mathcal{D} = \sum_{k \in \mathbf{Z}} k\Phi_k$.

Remarks. The constructions in [PR] imply immediately that we obtain a class in $KK^1(O_n, F)$. Theorem 2.8 is part of an index theorem proved in [PR]. The pairing of (X,V) with unitaries u in $K_1(A)$ gives a $K_0(F)$ valued index, and writing $P = \mathcal{X}_{[0,\infty)}(\mathcal{D})$, it is given by

$$(4) \qquad \langle [u], [(X,V)] \rangle = [\ker(PuP)] - [\operatorname{coker}(PuP)]$$

where the square brackets denote the K_0 class of the relevant kernel projections. However, the main result of [PR] (which fails for the Cuntz algebras) requires a faithful semifinite gauge invariant lower semi-continuous trace ϕ on A.

2.3. **The mapping cone algebra and APS boundary conditions.** In [CPR] we refined [PR] by showing that $K_0(F)$-valued indices could also be obtained from an even index pairing in KK-theory using APS boundary conditions, similar to [APS3]. We briefly review this result, as it provides an interpretation of the modular index pairings in Section 5. We use the notation $M_k(B)$ to denote the algebra of $k \times k$ matrices over an algebra B. If $F \subset A$ is a sub-C^*-algebra of the C^*-algebra A, then the mapping cone algebra for the inclusion is

$$M(F, A) = \{f : \mathbf{R}_+ = [0, \infty) \to A : f \text{ is continuous and vanishes at infinity, } f(0) \in F\}.$$

When F is an ideal in A it is known that $K_0(M(F, A)) \cong K_0(A/F)$, [Pu]. In general, $K_0(M(F, A))$ is the set of homotopy classes of partial isometries $v \in M_k(A)$ with range and source projections vv^*, v^*v in $M_k(F)$, with operation the direct sum and inverse $-[v] = [v^*]$. All this is proved in [Pu].

Following [CPR] we now explain our noncommutative analogue of the Atiyah-Patodi-Singer index theorem [APS1]. Note that when we are working with matrix algebras over A or $M(F, A)$ we inflate $\mathcal{D}$ to $\mathcal{D} \otimes I_k$ and so on.

Definition 2.9. *Let $(X, \mathcal{D})$ be an unbounded Kasparov module and form the algebraic tensor product of $\mathcal{L}^2(\mathbf{R}_+)$ and X. We complete the linear span of the elementary tensors in the algebraic tensor product (these are functions from $\mathbf{R}_+$ to X) in the norm arising from the inner product*

$$\langle \xi, \eta \rangle := \int_0^\infty (\xi(t)|\eta(t))_X dt$$

*and write the completion as $\mathcal{L}^2(\mathbf{R}_+) \otimes X$ and denote this space by $\mathcal{E}$. An **extended $\mathcal{L}^2$-function** $f : \mathbf{R}_+ \to X$ is a function of the form $f = g + x_0$ such that g is in $\mathcal{L}^2(\mathbf{R}_+) \otimes X$ and x_0 is a constant function with $\mathcal{D}(x_0) = 0$, that is $x_0 \in \ker(\mathcal{D}) = \Phi_0(X)$. We denote the space of extended $\mathcal{L}^2$-functions by $\hat{\mathcal{E}}$ and define the F-valued inner product on $\hat{\mathcal{E}}$ by $\langle g + x | h + y \rangle_{\hat{\mathcal{E}}} := \langle g | h \rangle_{\mathcal{E}} + \langle x | y \rangle_X$.*

Now, certain Kasparov A, F-modules extend to Kasparov $M(F, A), F$-modules:

Proposition 2.10 ([CPR])**.** *Let $(X, \mathcal{D})$ be an ungraded unbounded Kasparov module for C^*-algebras A, F with $F \subset A$ a subalgebra such that $\overline{AF} = A$. Suppose that $\mathcal{D}$ also commutes with the left action of $F \subset A$, and that $\mathcal{D}$ has discrete spectrum. Then the pair*

$$(\hat{X}, \hat{\mathcal{D}}) = \left(\begin{pmatrix} \mathcal{E} \\ \hat{\mathcal{E}} \end{pmatrix}, \begin{pmatrix} 0 & -\partial_t + \mathcal{D} \\ \partial_t + \mathcal{D} & 0 \end{pmatrix} \right)$$

with APS boundary conditions is a graded unbounded Kasparov module for the mapping cone algebra $M(F, A)$.

By APS boundary conditions we mean let $P = \mathcal{X}_{\mathbf{R}_+}(\mathcal{D})$ and take the domain of $\hat{\mathcal{D}}$ to initially be

$$\text{dom}\hat{\mathcal{D}} = \{\xi \in \text{span of elementary tensors in } \hat{X} : P\xi_1(0) = 0, \ (1 - P)\xi_2(0) = 0, \ \hat{\mathcal{D}}\xi \in \hat{X}\}.$$

In [CPR] we show that APS boundary conditions make sense for the self adjoint closure of $\hat{\mathcal{D}}$ and no technical obstructions exist to working with this closure on its natural domain. Strictly speaking we should also mention that the unbounded Kasparov module is defined for a certain smooth algebra $\mathcal{A} \subset A$, and we will suppose that this is the case, and that $F \subset \mathcal{A}$. To explain the appearance in the second component of $\hat{X}$ of the right F, C^*-module $\hat{\mathcal{E}}$, we have to recall that the different treatment of $\ker(\mathcal{D})$ (which has the restriction of the inner product on X) is to account for 'extended $\mathcal{L}^2$ solutions' corresponding to the zero eigenvalue of $\mathcal{D}$, just as in [APS1, pp 58-60].

If v is a partial isometry in $M_k(\mathcal{A})^\sim$ (the minimal unitization) setting

$$e_v(t) = \begin{pmatrix} 1 - \frac{vv^*}{1+t^2} & -iv\frac{t}{1+t^2} \\ iv^*\frac{t}{1+t^2} & \frac{v^*v}{1+t^2} \end{pmatrix},$$

defines e_v as a projection in $M_k(F, A)$ (the $k \times k$ matrices over the mapping cone). When v is a unitary, denoted u say, then u is trivially a partial isometry with range and source in (the unitization of) $M_k(F)$, so we obtain a class in $K_0(M(F, A))$ which we denote by $[e_u] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$. In the statement of the next result (which is a special case of the main theorem of [CPR]) we suppress the subscript k.

Proposition 2.11 ([CPR]). *Let $(X, \mathcal{D})$ be an ungraded unbounded Kasparov module for (pre-)C^*-algebras $\mathcal{A}, F$ with $F \subset \mathcal{A}$ a subalgebra such that $\overline{AF} = F$. Suppose that $\mathcal{D}$ also commutes with the left action of $F \subset \mathcal{A}$, and that $\mathcal{D}$ has discrete spectrum. Let $(\hat{X}, \hat{\mathcal{D}})$ be the unbounded Kasparov $M(F, \mathcal{A}), F$ module of Proposition 2.10. Then for any unitary $u \in \mathcal{A}$ such that $u^*[\mathcal{D}, u]$ **is bounded and commutes with** $\mathcal{D}$ we have the following equality of index pairings with values in $K_0(F)$:*

$$\langle [u^*], [(X, \mathcal{D})] \rangle \quad := \quad \mathrm{Index}(Pu^*P) = \langle [e_u] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right], [(\hat{X}, \hat{\mathcal{D}})] \rangle \in K_0(F).$$

Moreover, if $v \in \mathcal{A}$ is a partial isometry, with $vv^*, v^*v \in F$ and $v^*[\mathcal{D}, v]$ bounded and commuting with $\mathcal{D}$ we have

$$\langle [e_v] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right], [(\hat{X}, \hat{\mathcal{D}})] \rangle = -\mathrm{Index}(PvP : v^*vP(X) \to vv^*P(X)) \in K_0(F)$$

$$(5) \hspace{3cm} = \mathrm{Index}(Pv^*P : vv^*P(X) \to v^*vP(X)) \in K_0(F).$$

We remark that the hypothesis that $\mathcal{D}$ and $v^*[\mathcal{D}, v]$ commute can be considerably relaxed (with considerable effort). We will see later how this theorem assists us when $\bar{\mathcal{A}} = O_n$ and O_n is equipped with its natural KMS state. Before turning to this we compute $K_0(M(F, A))$ for the examples we have in mind. Let $A = O_n$ and let $\mathcal{A}$ be any dense smooth subalgebra such that the fixed point algebra for the modular automorphism, F, is contained in $\mathcal{A}$. Using $K_1(\mathcal{A}) = K_1(F) = 0$, the six term sequence in K-theory becomes

$$0 \to K_0(\mathcal{M}(F, \mathcal{A})) \to K_0(F) \to K_0(\mathcal{A}) \to K_1(\mathcal{M}(F, \mathcal{A})) \to 0.$$

Now $K_0(F) = \mathbf{Z}[1/n]$ while $K_0(A) = \mathbf{Z}_{n-1}$, [Dav]. A careful analysis of the map $K_0(F) \to K_0(\mathcal{A})$ shows that it is induced by inclusion, [CPR]. Since $K_0(\mathcal{A}) = \{0, Id, 2Id, ..., (n-2)Id\}$ for the Cuntz algebra, this map is onto. Hence $K_1(\mathcal{M}(F, \mathcal{A})) = 0$ and $K_0(\mathcal{M}(F, \mathcal{A})) = (n-1)\mathbf{Z}[1/n]$.

3. The modular spectral triple of the Cuntz algebras

The Cuntz algebras do not possess a faithful gauge invariant trace. There is however a unique state which is KMS for the gauge action, namely $\psi := \tau \circ \Phi : O_n \to \mathbf{C}$, where $\Phi : O_n \to F$ is the expectation and $\tau : F \to \mathbf{C}$ the unique faithful normalised trace. As the Cuntz algebras satisfy the hypotheses of [PR] (they are graph algebras of a locally finite graph with no sources), the generator of the gauge action $\mathcal{D}$ acting on the right C^*-F-module X gives us a Kasparov module $(X, \mathcal{D})$. As with tracial graph algebras, we take this class as our starting point. However we immediately encounter a difficulty that there are no unitaries to pair with, since $K_1(O_n) = 0$. Nevertheless, there are many partial isometries with range and source in the fixed point algebra (O_n is generated by such elements), so the APS pairing of the previous section is available. We would like to compute a numerical pairing using a spectral triple and we use the Kasparov module for this purpose.

Let $\mathcal{H} = \mathcal{L}^2(O_n)$ be the GNS Hilbert space given by the faithful state $\psi = \tau \circ \Phi$. That is, the inner product on O_n is defined by $\langle a, b \rangle = \psi(a^*b) = (\tau \circ \Phi)(a^*b)$. Then $\mathcal{D}$ extends to a self-adjoint unbounded operator on $\mathcal{H}$, [PR], and we denote this closure by $\mathcal{D}$ from now on. The representation π of O_n on $\mathcal{H}$ by left multiplication is bounded and nondegenerate, and the dense subalgebra $\text{span}\{\pi(S_\mu S_\nu^*)\}$ is in the smooth domain of the derivation $\delta = \text{ad}(|\mathcal{D}|)$. We denote the left action of an element $a \in O_n$ by $\pi(a)$ so that $\pi(a)b = ab$ for all $b \in O_n$. This distinction between elements of O_n as vectors in $\mathcal{L}^2(O_n)$ and operators on $\mathcal{L}^2(O_n)$ is sometimes crucial. Thus we see that the central algebraic structures of the gauge spectral triple on a tracial graph algebra are mirrored in this construction.

What differs from the tracial situation is the analytic information. We begin by obtaining some information about the trace on F, the corresponding state on O_n and the associated modular theory.

Lemma 3.1. *The trace $\tau : F \to \mathbf{C}$ satisfies $\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu} \frac{1}{n^{|\mu|}}$.*

Proof. First of all, we must have $|\mu| = |\nu|$ in order that $S_\mu S_\nu^* \in F$, and then

$$
\begin{aligned}
\tau(S_\mu S_\nu^*) &= \tau(S_\mu S_\mu^* S_\mu S_\nu^*) = & \tau(S_\mu S_\nu^* S_\nu S_\nu^*) \\
&= \tau(S_\mu S_\nu^* S_\mu S_\mu^*) = & \tau(S_\nu S_\nu^* S_\mu S_\nu^*) \\
&= \delta_{\mu,\nu} \tau(S_\mu S_\mu^*) = & \delta_{\mu,\nu} \tau(S_\nu S_\nu^*).
\end{aligned}
$$

Thus whenever $|\mu| = |\nu|$ we have $\tau(S_\mu S_\mu^*) = \tau(S_\nu S_\nu^*)$. Since there are exactly n^k distinct S_μ all with orthogonal ranges so that $\sum_{|\mu|=k} S_\mu S_\mu^* = 1$, the result follows. $\square$

Let $\mathcal{S}$ first denote the operator $a \mapsto a^*$ defined on O_{nc} as a subspace of $\mathcal{L}^2(O_n)$. The conjugate-linear adjoint of $\mathcal{S}$ exists, is denoted $\mathcal{F}$ and will be explicitly calculated on the subspace O_{nc} in the next lemma. It satisfies

$$\mathcal{F}(S_\mu S_\nu^*) = n^{(|\mu|-|\nu|)} S_\nu S_\mu^*.$$

In particular, $\mathcal{F}$ is densely defined so that $\mathcal{S}$ is closable. So we use the same symbol $\mathcal{S}$ to denote the closure and also $\mathcal{F}$ will denote the closure of $\mathcal{F}$ restricted to O_{nc}. Then $\mathcal{S}$ has a polar decomposition as

$$\mathcal{S} = J\Delta^{1/2} = \Delta^{-1/2}J, \quad \mathcal{F} = J\Delta^{-1/2} = \Delta^{1/2}J, \quad \text{where} \quad \Delta = \mathcal{F}\mathcal{S},$$

where J is an antilinear map, $J^2 = 1$. The Tomita-Takesaki modular theory, [KR], shows that

$$\Delta^{-it}\pi(O_n)''\Delta^{it} = \pi(O_n)'', \quad J\pi(O_n)''J^* = (\pi(O_n)'')',$$

where $\pi(O_n)''$ is the weak closure of the left action of O_n on $\mathcal{L}^2(O_n)$. However, all of these operators can be explicitly calculated on the subspace O_{nc} which is in fact a Tomita algebra.

Lemma 3.2. *The algebra $O_{nc} = \text{span}\{S_\mu S_\nu^*\}$ with the inner product, $\langle a|b \rangle = \psi(a^*b) = \tau \circ \Phi(a^*b)$ arising from the state $\psi = \tau \circ \Phi$ is a Tomita algebra (except for the trivial difference that our inner product is linear in the second coordinate).*

Proof. Since the inner product on O_{nc} comes from the GNS construction given by the faithful state $\psi = \tau \circ \Phi$ the left action of O_{nc} on itself is involutive, faithful (and hence isometric), and nondegenerate. This takes care of Takesaki's Axioms (I), (II), (III) for a Tomita algebra [Ta]. Next, as mentioned above, the operator $\mathcal{S}$ on O_{nc} is just the mapping on O_n, $a \mapsto \mathcal{S}(a) = a^*$, which, on generators, is $\mathcal{S}(S_\mu S_\nu^*) = S_\nu S_\mu^*$. We define the conjugate linear map $\mathcal{F}$ on generators (and extend by linearity) via:

$$\mathcal{F}(S_\mu S_\nu^*) = n^{(|\mu|-|\nu|)} S_\nu S_\mu^*.$$

To see that $\mathcal{F}$ is the adjoint of $\mathcal{S}$ it suffices to check the defining equation: $\langle \mathcal{S}(a)|b \rangle = \langle \mathcal{F}(b)|a \rangle$ on generators $a = S_\alpha S_\beta^*$ and $b = S_\mu S_\nu^*$. Then,

$$\langle \mathcal{S}(a)|b \rangle = \tau \circ \Phi(S_\alpha S_\beta^* S_\mu S_\nu^*) \quad \text{while} \quad \langle \mathcal{F}(b)|a \rangle = \tau \circ \Phi(S_\mu S_\nu^* S_\alpha S_\beta^*)n^{(|\mu|-|\nu|)}.$$

Now, if $|\mu| + |\alpha| - |\nu| - |\beta| \neq 0$ then both terms are 0 hence equal. While if $|\mu| + |\alpha| - |\nu| - |\beta| = 0$, then

$$\langle \mathcal{S}(a)|b\rangle = \tau(S_\alpha S_\beta^* S_\mu S_\nu^*) \quad \text{while} \quad \langle \mathcal{F}(b)|a\rangle = \tau(S_\mu S_\nu^* S_\alpha S_\beta^*) n^{(|\mu|-|\nu|)}.$$

In the second case where $|\beta| - |\mu| = |\alpha| - |\nu|$, we assume that $|\beta| - |\mu| = |\alpha| - |\nu| \geq 0$ as the case ≤ 0 is very similar.

Now, $S_\alpha S_\beta^* S_\mu S_\nu^* = 0$ unless $\beta = \mu\lambda$, whence $S_\alpha S_\beta^* S_\mu S_\nu^* = S_\alpha S_\lambda^* S_\nu^*$ and $|\lambda| = |\beta| - |\mu|$. Then since $S_\alpha S_\lambda^* S_\nu^* \neq 0$ we must have $\alpha = \nu\lambda$, and hence we have

$$S_\alpha S_\beta^* S_\mu S_\nu^* = S_\alpha S_\alpha^* \quad \text{where} \quad \beta = \mu\lambda \quad \text{and} \quad \alpha = \nu\lambda \quad \text{and} \quad |\lambda| = |\beta| - |\mu|.$$

Similary, $S_\mu S_\nu^* S_\alpha S_\beta^* = 0$ unless $S_\mu S_\nu^* S_\alpha S_\beta^* = S_\beta S_\beta^*$ where $\alpha = \nu\gamma$, $\beta = \mu\gamma$ and $|\gamma| = |\alpha| - |\nu|$. We note that since $|\lambda| = |\beta| - |\mu| = |\alpha| - |\nu| = |\gamma|$, we have that the two expressions $S_\alpha S_\beta^* S_\mu S_\nu^*$ and $S_\mu S_\nu^* S_\alpha S_\beta^*$ are both nonzero at the same time with the same condition: $\beta = \mu\lambda$ and $\alpha = \nu\lambda$. Finally, $\tau(S_\alpha S_\beta^* S_\mu S_\nu^*) = \tau(S_\alpha S_\alpha^*) = n^{-|\alpha|}$ while

$$\tau(S_\mu S_\nu^* S_\alpha S_\beta^*) n^{(|\mu|-|\nu|)} = \tau(S_\beta S_\beta^*) n^{(|\mu|-|\nu|)} = n^{-|\beta|} n^{(|\mu|-|\nu|)} = \cdots = n^{-|\alpha|}.$$

Thus, $\mathcal{F}$ is the adjoint of $\mathcal{S}$ and so both are closable. This takes care of Takesaki's Axiom (IX).

We immediately deduce that $\Delta(S_\mu S_\nu^*) = \mathcal{F}\mathcal{S}(S_\mu S_\nu^*) = n^{(|\nu|-|\mu|)} S_\mu S_\nu^*$ so that

$$\Delta^{1/2}(S_\mu S_\nu^*) = n^{(1/2)(|\nu|-|\mu|)} S_\mu S_\nu^*$$

and $J(S_\mu S_\nu^*) = n^{(1/2)(|\nu|-|\mu|)} S_\nu S_\mu^*$ so that $\mathcal{S} = J\Delta^{1/2}$, $\mathcal{F} = \Delta^{1/2}J$, as required. Moreover, for all $z \in \mathbf{C}$ we have:

$$\Delta^z(S_\mu S_\nu^*) = n^{z(|\nu|-|\mu|)} S_\mu S_\nu^*$$

where for $w \in \mathbf{C}$ we take $n^w := e^{w \log(n)}$. We remark that each $S_\mu S_\nu^*$ is an eigenvector of Δ for the nonzero eigenvalue $n^{(|\nu|-|\mu|)}$, and so each eigenvalue has infinite multiplicity.

We quickly review Takesaki's remaining axioms for a Tomita algebra. First, there is the un-numbered axiom that each $\Delta^z : O_{nc} \to O_{nc}$ is an algebra homomorphism. Clearly, each Δ^z is a linear isomorphism, and it suffices to check multiplicativity on the generators. This is a calculation based on the following fact: $(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = 0$ unless **either** $|\nu| \geq |\alpha|$ and $\nu = \alpha\lambda$ where $(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = S_\mu S_{\beta\lambda}^*$ **or** $|\nu| \leq |\alpha|$ and $\alpha = \nu\gamma$ where $(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = S_{\mu\gamma} S_\beta^*$. We remind the reader that this axiom says that as operators on O_n:

$$\pi(\Delta^z(a)) = \Delta^z \pi(a)\Delta^{-z} \quad \text{in particular,} \quad \pi(\Delta^{it}(a)) = \Delta^{it}\pi(a)\Delta^{-it}.$$

Axiom (IV): $\mathcal{S}(\Delta^z(a)) = \Delta^{-\bar{z}}(\mathcal{S}(a))$ for all $a \in O_{nc}$ and all $z \in \mathbf{C}$. This is a straightforward calculation.

Axiom (V): $\langle \Delta^z(a)|b\rangle = \langle a|\Delta^{\bar{z}}(b)\rangle$ for all $a, b \in O_{nc}$, and $z \in \mathbf{C}$. Another easy calculation.

Axiom (VI): $\langle \Delta(\mathcal{S}(a))|\mathcal{S}(b)\rangle = \langle b|a\rangle$ for all $a, b \in O_{nc}$. This is equivalent to $\langle \mathcal{F}(a)|\mathcal{S}(b)\rangle = \langle b|a\rangle$.

Axiom (VII): The function $z \mapsto \langle a|\Delta^z(b)\rangle$ is analytic on $\mathbf{C}$ for each $a, b \in O_{nc}$. Again an easy calculation since our inner products are **linear in the second** variable. Finally we have:

Axiom (VIII): For each $t \in \mathbf{R}$ the subspace $(1+\Delta^t)(O_{nc})$ is dense in O_{nc}. In fact, each generator $S_\mu S_\nu^*$ is an eigenvector of $(1+\Delta^t)$ with positive eigenvalue: $1+n^{t(|\nu|-|\mu|)}$, and hence $(1+\Delta^t)(O_{nc}) = O_{nc}$. $\square$

Lemma 3.3. *The group of modular automorphisms of the von Neumann algebra O_n'' generated by the left action of O_n on $\mathcal{L}^2(O_n)$ (which is the same as the von Neumann algebra generated by the left action of O_{nc} on $\mathcal{L}^2(O_{nc}) = \mathcal{L}^2(O_n)$) is given on the generators by*

$$(6) \qquad \sigma_t(\pi(S_\mu S_\nu^*)) := \Delta^{it}\pi(S_\mu S_\nu^*)\Delta^{-it} = \pi(\Delta^{it}(S_\mu S_\nu^*)) = n^{it(|\nu|-|\mu|)}\pi(S_\mu S_\nu^*).$$

Proof. This is a straightforward calculation obtained by evaluating these operators on a generator $S_\alpha S_\beta^*$ and using the un-numbered Axiom that Δ^{it} is an algebra homomorphism on O_{nc}. $\qquad\square$

Remarks. A special case of the KMS condition on the modular automorphism group of the state ψ, [Ta], (for $t = i$) is the following: $\psi(xy) = \psi(\sigma_i(y)x)$ for $x, y \in \pi(O_n)$. The proof is elementary:

$$\tau \circ \Phi(xy) = \langle x^*|y \rangle = \langle \mathcal{S}(x)|y \rangle = \langle \mathcal{F}(y)|x \rangle = \tau \circ \Phi(\mathcal{SF}(y)x) = \tau \circ \Phi(\Delta^{-1}(y)x) = \tau \circ \Phi(\sigma(y)x).$$

From now on we refer to this as the KMS condition for the state ψ.

Corollary 3.4. *With O_n acting on $\mathcal{H} := \mathcal{L}^2(O_n)$ we let $\mathcal{D}$ be the generator of the natural unitary implementation of the gauge action of $\mathbb{T}^1$ on O_n. Then we have*

$$\Delta = n^{-\mathcal{D}} \text{ or } e^{it\mathcal{D}} = \Delta^{-it/\log n}.$$

To continue, we recall the underlying right C^*-F-module, X, which is the completion of O_n for the norm $\|x\|_X^2 = \|\Phi(x^*x)\|_F$.

Lemma 3.5. *Any F-linear endomorphism T of the module X which preserves the copy of O_n inside X, extends uniquely to a bounded operator on the Hilbert space $\mathcal{H} = \mathcal{L}^2(O_n)$.*

Proof. For any $x \in X$ we have, by [L, Proposition 1.2], $(Tx|Tx)_R \leq \|T\|_{End}^2 (x|x)_R$ in F^+. Letting $\|T\|_\infty$ denote the operator norm on $\mathcal{H}$ we estimate using $x \in O_n$:

$$\|T\|_\infty^2 = \sup_{\|x\|_{\mathcal{H}} \leq 1} \langle Tx|Tx \rangle_{\mathcal{H}} = \sup_{\|x\|_{\mathcal{H}} \leq 1} \tau((Tx|Tx)_R) \leq \sup_{\|x\|_{\mathcal{H}} \leq 1} \|T\|_{End}^2 \tau((x|x)_R) = \|T\|_{End}^2.$$

$\qquad\square$

In particular, the finite rank endomorphisms of the pre-C^* module O_{nc} (acting on the left) satisfy this condition, and we denote the algebra of all these endomorphisms by $End_F^{00}(O_{nc})$.

Proposition 3.6. *Let $\mathcal{N}$ be the von Neumann algebra $\mathcal{N} = (End_F^{00}(O_{nc}))''$, where we take the commutant inside $\mathcal{B}(\mathcal{H})$. Then $\mathcal{N}$ is semifinite, and there exists a faithful, semifinite, normal trace $\tilde{\tau} : \mathcal{N} \to \mathbf{C}$ such that for all rank one endomorphisms $\Theta_{x,y}^R$ of O_{nc},*

$$\tilde{\tau}(\Theta_{x,y}^R) = (\tau \circ \Phi)(y^*x), \quad x, y \in O_{nc}.$$

In addition, $\mathcal{D}$ is affiliated to $\mathcal{N}$ and O_n, acting on the left on X, is a subalgebra of $\mathcal{N}$.

Proof. We define $\tilde{\tau}$ as a supremum of an increasing sequence of vector states, as in [PR], which ensures that $\tilde{\tau}$ is normal. First for $|\mu| \neq 0$ we define for $T \in \mathcal{N}$

$$\omega_\mu(T) := \langle S_\mu, T S_\mu \rangle + \frac{1}{n^{|\mu|}} \langle S_\mu^*, T S_\mu^* \rangle.$$

Together with $\omega_1(T) := \langle 1, T1 \rangle$, this gives a collection of positive vector states on $\mathcal{N}$. We define

$$\tilde{\tau}(T) = \omega_1(T) + \lim_{L \nearrow} \sum_{\mu \in L} \omega_\mu(T),$$

where L ranges over the finite subsets of the finite path space E^* of the graph underlying O_n. With this definition, the proof in [PR, Lemma 5.11] can be applied almost verbatim to this case. The only real change in the proof occurs on page 121 of [PR]: the line before the phrase "the last inequality following" should be replaced by:

$$= \|T\| \sum_{s(\mu)=v, |\mu|=k} \tau(p_{r(\mu)}) = \|T\| n^k \tau(p_v) < \infty.$$

Rather than repeat the proof here, we simply observe for the reader's benefit that to check the trace property (on endomorphisms) only requires that τ is a trace on F, not all of O_n. Here is the formal calculation for rank one operators:

$$
\begin{aligned}
\tilde{\tau}(\Theta^R_{w,z}\Theta^R_{x,y}) &= \tilde{\tau}(\Theta^R_{w(z|x),y}) = \tau \circ \Phi(y^*w(z|x)) = \tau((y|w(z|x))) \\
&= \tau((y|w)(z|x)) = \tau((z|x)(y|w)) \\
&= \tilde{\tau}(\Theta^R_{x(y|w),z}) = \tilde{\tau}(\Theta^R_{x,y}\Theta^R_{w,z}).
\end{aligned}
$$

Next we must show that $\mathcal{D}$ is affiliated to $\mathcal{N}$. However, we have already noted that the spectral projections of $\mathcal{D}$ are finite sums of rank one endomorphisms of X_c, in the paragraph immediately preceding Theorem 2.8. This proves the claim. That A_c embeds in $\mathcal{N}$ follows from Lemma 2.7 and the fact that the Φ_k sum to the identity. Since A is the unique C^*-completion of A_c we see that π embeds A in $\mathcal{N}$. $\qquad\square$

Unfortunately, in contrast to the situation in [PR], this trace is not what we need for defining summability. This can be seen from the following calculations. For $k \geq 0$

$$
\tilde{\tau}(\Phi_k) = \tilde{\tau}(\sum_{|\rho|=k} \Theta^R_{S_\rho,S_\rho}) = \tau(\sum_{|\rho|=k}(S_\rho|S_\rho)) = \tau(\sum_{|\rho|=k} S_\rho^*S_\rho) = \sum_{|\rho|=k} 1 = n^k.
$$

Similarly, for $k < 0$ we have $\tilde{\tau}(\Phi_k) = n^k$. Hence with respect to this trace we cannot expect $\mathcal{D}$ to satisfy any summability criterion.

Definition 3.7. *We define a new weight on $\mathcal{N}^+$: let $T \in \mathcal{N}^+$ then $\tau_\Delta(T) := \sup_N \tilde{\tau}(\Delta_N T)$ where $\Delta_N = \Delta(\sum_{|k|\leq N} \Phi_k)$.*

Remarks. Since Δ_N is $\tilde{\tau}$-trace-class, we see that $T \mapsto \tilde{\tau}(\Delta_N T)$ is a normal positive linear functional on $\mathcal{N}$ and hence τ_Δ is a normal weight on $\mathcal{N}^+$ which is easily seen to be faithful and semifinite.

We now give another way to define τ_Δ which is not only conceptually useful but also makes a number of important properties straightforward to verify.

Notation. Let $\mathcal{M}$ be the relative commutant in $\mathcal{N}$ of the operator Δ. Equivalently, $\mathcal{M}$ is the relative commutant of the set of spectral projections $\{\Phi_k | k \in \mathbf{Z}\}$ Clearly, $\mathcal{M} = \sum_{k\in\mathbf{Z}} \Phi_k\mathcal{N}\Phi_k$.

Definition 3.8. *As $\tilde{\tau}$ restricted to each $\Phi_k\mathcal{N}\Phi_k$ is a faithful finite trace with $\tilde{\tau}(\Phi_k) = n^k$ we define $\hat{\tau}_k$ on $\Phi_k\mathcal{N}\Phi_k$ to be n^{-k} times the restriction of $\tilde{\tau}$. Then, $\hat{\tau} := \sum_k \hat{\tau}_k$ on $\mathcal{M} = \sum_{k\in\mathbf{Z}} \Phi_k\mathcal{N}\Phi_k$ is a faithful normal semifinite trace $\hat{\tau}$ with $\hat{\tau}(\Phi_k) = 1$ for all k.*

We use $\hat{\tau}$ to give an alternative expression for τ_Δ below This alternative might be avoidable but at the expense of a detailed use of [PT]. However, (see the bottom of page 61 of [PT]), **the semifiniteness** of τ_Δ restricted to $\mathcal{M}$ **depends on** the existence of a normal τ_Δ-invariant projection (such as Ψ defined below) from $\mathcal{N}$ onto $\mathcal{M}$.

Lemma 3.9. *An element $m \in \mathcal{N}$ is in $\mathcal{M}$ if and only if it is in the fixed point algebra of the action, $\sigma_t^{\tau_\Delta}$ on $\mathcal{N}$ defined for $T \in \mathcal{N}$ by $\sigma_t^{\tau_\Delta}(T) = \Delta^{it}T\Delta^{-it}$. Both $\pi(F)$ and the projections Φ_k belong to $\mathcal{M}$. The map $\Psi : \mathcal{N} \to \mathcal{M}$ defined by $\Psi(T) = \sum_k \Phi_k T\Phi_k$ is a conditional expectation onto $\mathcal{M}$ and $\tau_\Delta(T) = \hat{\tau}(\Psi(T))$ for all $T \in \mathcal{N}^+$. That is, $\tau_\Delta = \hat{\tau} \circ \Psi$ so that $\hat{\tau}(T) = \tau_\Delta(T)$ for all $T \in \mathcal{M}^+$. Finally, if one of $A, B \in \mathcal{M}$ is $\hat{\tau}$-trace-class and $T \in \mathcal{N}$ then $\tau_\Delta(ATB) = \tau_\Delta(A\Psi(T)B) = \hat{\tau}(A\Psi(T)B)$.*

Proof. The first two statements are immediate. Also, the fact that Ψ is a unital norm one projection of $\mathcal{N}$ onto $\mathcal{M}$ (and hence a normal conditional expectation by Tomiyama's theorem [T]) is clear. Only

the last assertions of the Lemma need proof. To this end let $T \in \mathcal{N}^+$, then

$$
\begin{aligned}
\tau_\Delta(T) &= \sup_N \tilde{\tau}(\Delta_N T) = \sup_N \tilde{\tau}(\Delta(\sum_{|k|\leq N} \Phi_k)T) = \sup_N \tilde{\tau}(\sum_{|k|\leq N} \Delta\Phi_k T) \\
&= \sup_N \tilde{\tau}(\sum_{|k|\leq N} n^{-k}\Phi_k T) = \sup_N \sum_{|k|\leq N} n^{-k}\tilde{\tau}(\Phi_k T\Phi_k) = \sum_{k\in\mathbf{Z}} n^{-k}\tilde{\tau}(\Phi_k T\Phi_k) = \hat{\tau}(\Psi(T)).
\end{aligned}
$$

Hence if $T \in \mathcal{M}$ then $\hat{\tau}(T) = \hat{\tau}(\Psi(T)) = \tau_\Delta(T)$. Finally the last statement follows from the fact that $\Psi(ATB) = A\Psi(T)B$ by Tomiyama's Theorem [T]. $\square$

Lemma 3.10. *The modular automorphism group $\sigma_t^{\tau_\Delta}$ of τ_Δ is inner and given by $\sigma_t^{\tau_\Delta}(T) = \Delta^{it}T\Delta^{-it}$. The weight τ_Δ is a KMS weight for the group $\sigma_t^{\tau_\Delta}$, and $\sigma_t^{\tau_\Delta}|_{O_n} = \sigma_t^{\tau\circ\Phi}$.*

Proof. This follows from: [KR, Thm 9.2.38], which gives us the KMS properties of τ_Δ: the modular group is inner since Δ is affiliated to $\mathcal{N}$. The final statement about the restriction of the modular group to O_n is clear. $\square$

The reward for having sacrificed a trace on $\mathcal{N}$ for a trace on $\mathcal{M}$ is the following.

Lemma 3.11. *Suppose g is a function on $\mathbf{R}$ such that $g(\mathcal{D})$ is τ_Δ trace-class in $\mathcal{M}$, then for all $f \in F$ we have*

$$
\tau_\Delta(\pi(f)g(\mathcal{D})) = \tau_\Delta(g(D))\tau(f) = \tau(f)\sum_{k\in\mathbf{Z}} g(k).
$$

Proof. First note that $\tau_\Delta(g(\mathcal{D})) = \hat{\tau}(\sum_{k\in\mathbf{Z}} g(k)\Phi_k) = \sum_{k\in\mathbf{Z}} g(k)\hat{\tau}(\Phi_k) = \sum_{k\in\mathbf{Z}} g(k)$. Now,

$$
\tau_\Delta(\pi(f)g(\mathcal{D})) = \hat{\tau}(\pi(f)\sum_{k\in\mathbf{Z}} g(k)\Phi_k) = \sum_{k\in\mathbf{Z}} g(k)\hat{\tau}(\pi(f)\Phi_k)
$$

$$
= \sum_{k\in\mathbf{Z}} g(k)\hat{\tau}_k(\pi(f)\Phi_k) = \sum_{k\in\mathbf{Z}} g(k)n^{-k}\tilde{\tau}(\pi(f)\Phi_k).
$$

So it suffices to see for each $k \in \mathbf{Z}$, we have $\tilde{\tau}(\pi(f)\Phi_k) = n^k\tau(f)$.

For all $f \in F$, f is a norm limit of finite sums of terms like $S_\alpha S_\beta^*$, $|\alpha| = |\beta| = r$. So we compute for $f = S_\alpha S_\beta^*$. Recall that we have the formulae

$$
\Phi_k = \sum_{|\mu|=k} \Theta_{S_\mu, S_\mu}^R, \quad k > 0, \qquad \Phi_k = n^{-k}\sum_{|\mu|=|k|} \Theta_{S_\mu^*, S_\mu^*}^R, \quad k < 0. \qquad \text{and } \Phi_0 = \Theta_{1,1}^R
$$

where, with μ the path of length zero, we are using the notation $1 = S_\mu$.

First for $k \geq 0$

$$
\tilde{\tau}(\pi(f)\Phi_k) = \tilde{\tau}(\pi(f)\sum_{|\mu|=k} \Theta_{S_\mu, S_\mu}^R) = \tilde{\tau}(\sum_{|\mu|=k} \Theta_{fS_\mu, S_\mu}^R) = \sum_{|\mu|=k} \tau\circ\Phi(S_\mu^* f S_\mu)
$$

$$
= \sum_{|\mu|=k} \tau(S_\mu^* S_\alpha S_\beta^* S_\mu) = n^{k-r}\delta_{\alpha,\beta} = n^k\frac{1}{n^{|\alpha|}}\delta_{\alpha,\beta} = n^k\tau(S_\alpha S_\beta^*) = n^k\tau(f).
$$

A similar calculation holds for $k < 0$ using the other formula for Φ_k in this case. Since all $f \in F_c$ are linear combinations $S_\alpha S_\beta^*$, $|\alpha| = |\beta|$, we get for all $f \in F_c$, the formula

$$
\tau_\Delta(\pi(f)g(\mathcal{D})) = \tau_\Delta(g(D))\tau(f) = \sum_{k\in\mathbf{Z}} g(k)\tau(f).
$$

Now, the right hand side is a norm-continuous function of f. To see that the left side is norm-continuous we do it in more generality. Let $T \in \mathcal{N}$, then since $\widehat{\tau}$ is a trace on $\mathcal{M}$ we get:

$$|\tau_\Delta(Tg(\mathcal{D}))| = |\widehat{\tau}(\Psi(Tg(\mathcal{D}))| = |\widehat{\tau}(\Psi(T)g(\mathcal{D}))| \leq \|\Psi(T)\|\widehat{\tau}(|g(\mathcal{D})|) \leq \|T\|\widehat{\tau}((|g(\mathcal{D})|) = \|T\|\tau_\Delta(|g(\mathcal{D})|).$$

That is the left hand side is norm-continuous in T and so we have the formula:

$$\tau_\Delta(\pi(f)g(\mathcal{D})) = \tau_\Delta(g(\mathcal{D}))\tau(f) = \sum_{k \in \mathbf{Z}} g(k)\tau(f)$$

for all $f \in F$. □

Remarks. The inequality above clearly holds in more generality. That is, if $T \in \mathcal{N}$ and $B \in \mathcal{L}^1(\mathcal{M}, \tau_\Delta)$ then:

(7) $$|\tau_\Delta(TB)| \leq \|T\|_\infty \tau_\Delta(|B|) = \|T\|_\infty \|B\|_1.$$

Proposition 3.12. *We have* $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \tau_\Delta)$. *That is,* $\tau_\Delta((1 + \mathcal{D}^2)^{-s/2}) < \infty$ *for all* $s > 1$. *Moreover, for all* $f \in F$

$$\lim_{s \to 1^+} (s - 1)\tau_\Delta(\pi(f)(1 + \mathcal{D}^2)^{-s/2}) = 2\tau(f)$$

so that $\pi(f)(1 + \mathcal{D}^2)^{-1/2}$ *is a measurable operator in the sense of* [C].

Proof. Let $s > 1$. Then $\tau_\Delta((1 + \mathcal{D}^2)^{-s/2}) = \widehat{\tau}(\sum_{k \in \mathbf{Z}}(1 + k^2)^{-s/2}\Phi_k) = \sum_{k \in \mathbf{Z}}(1 + k^2)^{-s/2}$. Hence, $(1 + \mathcal{D}^2)^{-s/2}$ is τ_Δ-trace-class in $\mathcal{M}$ for all $Re(s) > 1$ and

$$\lim_{s \to 1^+} (s - 1)\tau_\Delta((1 + \mathcal{D}^2)^{-s/2}) = 2.$$

By Lemma 3.11 we have the equality:

$$\tau_\Delta(\pi(f)(1 + \mathcal{D}^2)^{-s/2}) = \sum_{k \in \mathbf{Z}}(1 + k^2)^{-s/2}\tau(f)$$

for all $f \in F$. Hence,

$$\lim_{s \to 1^+} (s - 1)\tau_\Delta(\pi(f)(1 + \mathcal{D}^2)^{-s/2}) = 2\tau(f)$$

and $\pi(f)(1 + \mathcal{D}^2)^{-1/2}$ is measurable, for all $f \in F$. □

We wish to extend our conclusions about τ_Δ and $\lim_{s \to 1^+}(s - 1)\tau_\Delta(T(1 + \mathcal{D}^2)^{-s/2})$ to the whole von Neumann algebra $\mathcal{N}$. Unfortunately, these limits do not exist for general $T \in \mathcal{N}$ and we are forced to consider generalised limits as in the Dixmier trace theory.

Definition 3.13. *Let* $\tilde{\omega}$ *be a state on* $\mathcal{L}^\infty(\mathbf{R}_+)$ *which satisfies the condition that if* $g \in \mathcal{L}^\infty(\mathbf{R}_+)$ *is real-valued then*

$$\liminf_{t \to \infty} g(t) \leq \tilde{\omega}(g) \leq \limsup_{t \to \infty} g(t).$$

Clearly any such state is identically 0 on $C_0(\mathbf{R}_+)$ *and also on any function which is essentially compactly supported. Moreover, if* g *has a limit at* ∞ *then* $\tilde{\omega}(g) = \lim_{t \to \infty} g(t)$. *We define*

$$\tilde{\omega} - \lim_{t \to \infty} g(t) := \tilde{\omega}(g).$$

The existence of such states (with even more properties) can be found in [CPS2, Corollary 1.6]. In order to evaluate such states $\tilde{\omega}$ on functions g of the form $s \mapsto (s - 1)\tau_\Delta(T(1 + \mathcal{D}^2)^{-s/2})$ for $s > 1$ we need to do a translation: let $s = 1 + 1/r$ then letting $s \to 1^+$ is the same as letting $r \to \infty$. And we consider

$$(s - 1)\tau_\Delta(T(1 + \mathcal{D}^2)^{-s/2}) = \frac{1}{r}\tau_\Delta\left(T\left((1 + \mathcal{D}^2)^{-1/2}\right)^{1+1/r}\right).$$

Of course, the limit of the left hand side of this equation exists as $s \to 1^+$ if and only if the limit of the right hand side exists as $r \to \infty$ and in this case they are equal.

Abuse of notation:

$$\tilde{\omega} - \lim_{r \to \infty} \frac{1}{r} \tau_\Delta \left(T \left((1 + \mathcal{D}^2)^{-1/2} \right)^{1+1/r} \right) \quad \textbf{becomes} \quad \tilde{\omega} - \lim_{s \to 1^+} (s-1) \tau_\Delta(T(1+\mathcal{D}^2)^{-s/2}).$$

Remarks. Since $\tau_\Delta(T(1+\mathcal{D}^2)^{-s/2}) = \widehat{\tau}(\Psi(T)(1+\mathcal{D}^2)^{-s/2})$ these generalised traces are taking place completely inside $\mathcal{M}$ with respect to the trace $\widehat{\tau}$. That is, we are in the now well-understood semifinite situation.

Proposition 3.14. *Let $\tilde{\omega}$ be a state on $\mathcal{L}^\infty(\mathbf{R}_+)$ which satisfies the condition above. The functional $\widehat{\tau}_\omega$ on $\mathcal{N}$ defined by*

$$\widehat{\tau}_\omega(T) = \frac{1}{2} \tilde{\omega} - \lim_{s \to 1^+} (s-1) \tau_\Delta(T(1+\mathcal{D}^2)^{-s/2})$$

is a state. For $T = \pi(a) \in \pi(O_n) \subset \mathcal{N}$ the following (ordinary) limit exists and

$$\widehat{\tau}_\omega(\pi(a)) = \frac{1}{2} \lim_{s \to 1^+} (s-1) \tau_\Delta(\pi(a)(1+\mathcal{D}^2)^{-s/2}) = \tau \circ \Phi(a),$$

the original KMS state $\psi = \tau \circ \Phi$ on O_n.

Proof. First we observe that $\tau_\Delta(T(1+\mathcal{D}^2)^{-s/2})$ is finite for $s > 1$ for all $T \in \mathcal{N}$, since we showed in the proof of the previous Proposition that:

$$|\tau_\Delta(T(1+\mathcal{D}^2)^{-s/2})| \le \|T\| \tau_\Delta((1+\mathcal{D}^2)^{-s/2}).$$

Therefore, $(s-1)\tau_\Delta(T(1+\mathcal{D}^2)^{-s/2})$ is uniformly bounded and so the generalised limit exists as $s \to 1^+$. It is easy to see that this functional is positive on $\mathcal{N}^+$ and by the previous proposition $\widehat{\tau}_\omega(1) = 1$, so that $\widehat{\tau}_\omega$ is a state on $\mathcal{N}$.

Now, one easily checks by calculating on generators that for $\pi(a) \in \pi(O_{nc})$, $\Psi(\pi(a)) = \pi(\Phi(a)) \in \pi(F_c)$ and since Ψ is norm continuous we have that $\Psi(\pi(a)) = \pi(\Phi(a)) \in \pi(F)$ for all $a \in O_n$. Thus by Proposition 3.12, for $a \in O_n$ (letting $f = \Phi(a)$) we have

$$\tau \circ \Phi(a) = \frac{1}{2} \lim_{s \to 1^+} (s-1) \tau_\Delta(\pi(\Phi(a))(1+\mathcal{D}^2)^{-s/2})$$

$$= \frac{1}{2} \lim_{s \to 1^+} (s-1) \tau_\Delta(\Psi(\pi(a))(1+\mathcal{D}^2)^{-s/2}) = \widehat{\tau}_\omega(\pi(a)).$$

Of course $\widehat{\tau}_\omega$ is a true Dixmier-trace since for $T \in \mathcal{N}$ with $T \ge 0$, we have $\Psi(T) \in \mathcal{M}$, $\Psi(T) \ge 0$, and $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \widehat{\tau})$. Thus

$$\tilde{\omega} - \lim_{r \to \infty} \frac{1}{r} \tau_\Delta(T((1+\mathcal{D}^2)^{-1/2})^{1+1/r}) = \tilde{\omega} - \lim_{r \to \infty} \frac{1}{r} \widehat{\tau}(\Psi(T)((1+\mathcal{D}^2)^{-1/2})^{1+1/r})$$

and the right hand side is a true Dixmier-trace on the semifinite algebra $\mathcal{M}$ provided we choose $\tilde{\omega}$ as in [CPS2, Theorem 3.1]. $\qquad \square$

We summarise our construction to date.

0) We have a $*$-subalgebra $\mathcal{A} = O_{nc}$ of the Cuntz algebra faithfully represented in $\mathcal{N}$ with the latter acting on the Hilbert space $\mathcal{H} = \mathcal{L}^2(O_n, \psi)$,

1) there is a faithful normal semifinite weight τ_Δ on $\mathcal{N}$ such that the modular automorphism group of τ_Δ is an inner automorphism group $\tilde{\sigma}$ of $\mathcal{N}$ with $\tilde{\sigma}|_{\mathcal{A}} = \sigma$,

2) τ_Δ restricts to a faithful semifinite trace $\widehat{\tau}$ on $\mathcal{M} = \mathcal{N}^\sigma$, with a faithful normal projection $\Psi : \mathcal{N} \to \mathcal{M}$ satisfying $\tau_\Delta = \widehat{\tau} \circ \Psi$ on $\mathcal{N}$.

3) With $\mathcal{D}$ the generator of the one parameter group implementing σ on $\mathcal{H}$ we have:
$[\mathcal{D}, \pi(a)]$ extends to a bounded operator (in $\mathcal{N}$) for all $a \in \mathcal{A}$ and for λ in the resolvent set of $\mathcal{D}$, $(\lambda - \mathcal{D})^{-1} \in \mathcal{K}(\mathcal{M}, \tau_\Delta)$, where $\mathcal{K}(\mathcal{M}, \tau_\Delta)$ is the ideal of compact operators in $\mathcal{M}$ relative to τ_Δ. In particular, $\mathcal{D}$ is affiliated to $\mathcal{M}$.

Terminology/Definition. The triple $(O_{nc}, \mathcal{H}, \mathcal{D})$ along with $\mathcal{N}$, τ_Δ constructed in this section satisfying properties (0) to (3) above we will refer to as a (unital) **modular spectral triple**. For matrix algebras $\mathcal{A} = O_{nc} \otimes M_k$ over O_{nc}, $(O_{nc} \otimes M_k, \mathcal{H} \otimes M_k, \mathcal{D} \otimes Id_k)$ is also a modular spectral triple in the obvious fashion. In work in progress we have found that this structure arises in other examples and appears to be a quite general phenomenon.

We need some technical lemmas for the discussion in the next Section. A function f from a complex domain Ω into a Banach space X is called **holomorphic** if it is complex differentiable in norm on Ω.

Lemma 3.15. *(1) Let $\mathcal{B}$ be a C^*-algebra and let $T \in \mathcal{B}^+$. The mapping $z \mapsto T^z$ is holomorphic (in operator norm) in the half-plane $Re(z) > 0$.*
(2) Let $\mathcal{B}$ be a von Neumann algebra with faithful normal semifinite trace ϕ and let $T \in \mathcal{B}^+$ be in $\mathcal{L}^{(1,\infty)}(\mathcal{B}, \phi)$. Then, the mapping $z \mapsto T^z$ is holomorphic (in trace norm) in the half-plane $Re(z) > 1$.
(3) Let $\mathcal{B}$, and T be as in item (2) and let $A \in \mathcal{B}$ then the mapping $z \mapsto \phi(AT^z)$ is holomorphic for $Re(z) > 1$.

Proof. To see item (1) we assume without loss of generality that $||T|| \leq 1$. We fix $z_0 \in \mathbf{C}$ with $Re(z_0) > 0$, and fix $R > 0$ with $R < Re(z_0)$ so that the circle $C : z = z_0 + Re^{i\theta}$ for $\theta \in [0, 2\pi]$ lies in the half-plane $Re(z) > 0$. Temporarily we fix $t \neq 0$ in the spectrum of T so that $t \in (0, 1]$. Now with $|z - z_0| < (1/2)R$ we apply the complex version of Taylor's theorem to the function $z \mapsto t^z$ (see [Ahl, Theorem 8, pp125-6]) and get:

$$\frac{t^z - t^{z_0}}{z - z_0} - t^{z_0} Log(t) = f_2(z)(z - z_0) \quad \text{where} \quad f_2(z) = \frac{1}{2\pi i} \int_C \frac{t^w dw}{(w - z_0)^2 (w - z)}.$$

So with $|z - z_0| < (1/2)R$ we get the estimate:

$$|f_2(z)| \leq \frac{\max_C |t^w| \cdot R}{R^2 \cdot (1/2)R} \leq \frac{2t^{(Re(z_0)+R)}}{R^2} \leq \frac{2}{R^2}.$$

Therefore,

$$\left| \frac{t^z - t^{z_0}}{z - z_0} - t^{z_0} Log(t) \right| \leq \frac{2}{R^2} |z - z_0|.$$

Since this is true for all nonzero t in the spectrum of T we have:

$$\left\| \frac{1}{z - z_0}(T^z - T^{z_0}) - T^{z_0} Log(T) \right\|_\infty \leq \frac{2}{R^2} |z - z_0|.$$

That is $d/dz(T^z) = T^z Log(T)$ for $Re(z) > 0$ with the limit existing in operator norm.

To see item (2) we fix z_0 with $Re(z_0) > 1$ and then fix ϵ sufficiently small so that $Re(z_0 - (1 + \epsilon)) = Re(z_0) - (1 + \epsilon) > 0$. Then $T^{(1+\epsilon)}$ is trace-class, and this factor converts the operator norm limits

below into trace norm limits:

$$|| \cdot ||_1 \lim_{z \to z_0} \frac{1}{z - z_0}(T^z - T^{z_0}) = || \cdot ||_1 \lim_{z \to z_0} T^{(1+\epsilon)} \frac{1}{z - z_0}(T^{z-(1+\epsilon)} - T^{z_0-(1+\epsilon)})$$

$$= T^{(1+\epsilon)}\left(|| \cdot ||_\infty \lim_{z \to z_0} \frac{1}{z - z_0}(T^{z-(1+\epsilon)} - T^{z_0-(1+\epsilon)})\right)$$

$$= T^{(1+\epsilon)}\left(|| \cdot ||_\infty \lim_{z \to z_0} \frac{1}{(z - (1+\epsilon)) - (z_0 - (1+\epsilon))}(T^{z-(1+\epsilon)} - T^{z_0-(1+\epsilon)})\right)$$

$$= T^{(1+\epsilon)}(T^{z_0-(1+\epsilon)}Log(T) = T^{z_0}Log(T).$$

Item (3) follows from item (2) and inequality (7): $|\phi(AB)| \leq ||A||_\infty ||B||_1$ if B is ϕ-trace-class. $\qquad\square$

Lemma 3.16. *In these modular spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for matrices over the Cuntz algebras we have $(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{M}, \tau_\Delta)$ for all $s > 1$ and for $x \in \mathcal{N}$, $\tau_\Delta(x(1 + \mathcal{D}^2)^{-r/2})$ is holomorphic for $Re(r) > 1$ and we have for $a \in O_{nc}$ $\tau_\Delta([\mathcal{D}, \pi(a)](1 + \mathcal{D}^2)^{-r/2}) = 0$, for $Re(r) > 1$.*

Proof. Since the eigenvalues for $\mathcal{D}$ are precisely the set of integers, and the projection Φ_k on the eigenspace with eigenvalue k satisfies $\tau_\Delta(\Phi_k) = 1$, it is clear that $(1 + \mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{M}, \tau_\Delta)$. Now, $\tau_\Delta(x(1 + \mathcal{D}^2)^{-r/2}) = \widehat{\tau}(\Psi(x)(1 + \mathcal{D}^2)^{-r/2})$ is holomorphic for $Re(r) > 1$ by item (3) of the previous lemma.

To see the last statement, we observe that $\tau_\Delta([\mathcal{D}, \pi(a)](1 + \mathcal{D}^2)^{-r/2}) = \tau_\Delta(\Psi([\mathcal{D}, \pi(a)])(1 + \mathcal{D}^2)^{-r/2})$, so it suffices to see that $\Psi([\mathcal{D}, \pi(a)]) = 0$ for $a \in \mathcal{A} = O_{nc}$. To this end, let $a = S_\alpha S_\beta^*$ be one of the linear generators of O_{nc}. Then by calculating the action of the operator $\mathcal{D}\pi(S_\alpha S_\beta^*)$ on the linear generators $S_\mu S_\nu^*$ of the Hilbert space, $\mathcal{H}$, we obtain:

$$\mathcal{D}\pi(S_\alpha S_\beta^*) = (|\alpha| - |\beta|)\pi(S_\alpha S_\beta^*) + \pi(S_\alpha S_\beta^*)\mathcal{D} \quad \text{that is} \quad [\mathcal{D}, \pi(S_\alpha S_\beta^*)] = (|\alpha| - |\beta|)\pi(S_\alpha S_\beta^*).$$

More generally,

$$[\mathcal{D}, \pi(\sum_{i=1}^m c_i S_{\alpha_i} S_{\beta_i}^*)] = \sum_{i=1}^m c_i(|\alpha_i| - |\beta_i|)\pi(S_{\alpha_i} S_{\beta_i}^*).$$

If we apply Ψ to this equation, we see that $\Psi(\pi(S_{\alpha_i} S_{\beta_i}^*)) = \pi(\Phi(S_{\alpha_i} S_{\beta_i}^*)) = 0$ whenever $(|\alpha| - |\beta|) \neq 0$, and so the whole sum is 0. We also observe that $[\mathcal{D}, \pi(a)] \in \pi(O_{nc})$ for all $a \in O_{nc}$. This is not too surprising since $\mathcal{D}$ is the generator of the action γ of $\mathbb{T}$ on O_n. $\qquad\square$

In the remainder of this paper we will shed some light on the cohomological significance of these modular spectral triples. Just as ordinary $\mathcal{B}(\mathcal{H})$ spectral triples represent K-homology classes, [C, CPRS1], and semifinite spectral triples represent KK-classes, [KNR], modular spectral triples provide analytic representatives of some K-theoretic type data which we now describe.

4. MODULAR K_1

In this Section we introduce elements of $\mathcal{A}$ that will have a well defined pairing with our Dixmier functional $\widehat{\tau}_\omega$. Following [HR] we say that a unitary (invertible, projection,...) **in** $M_n(A)$ for some n **is** a unitary (invertible, projection,...) **over** A.

Definition 4.1. *Let $\mathcal{A}$ be a unital $*$-algebra and $\sigma : \mathcal{A} \to \mathcal{A}$ an algebra automorphism such that $\sigma(a)^* = \sigma^{-1}(a^*)$. We say that σ is a **regular automorphism**, [KMT].*

Remark. The automorphism $\sigma(a) := \Delta^{-1}(a)$ of a modular spectral triple is regular. This follows from AXIOM IV of Lemma 3.2:

$$(\sigma(a))^* = (\Delta^{-1}(a))^* = \mathcal{S}(\Delta^{-1}(a)) = \Delta(\mathcal{S}(a)) = \Delta(a^*) = \sigma^{-1}(a^*).$$

Definition 4.2. *Let u be a unitary over the $*$-algebra $\mathcal{A}$, and $\sigma : \mathcal{A} \to \mathcal{A}$ a regular automorphism with fixed point algebra $F = \mathcal{A}^\sigma$. We say that u satisfies the* **modular condition** *with respect to σ if both the operators $u\sigma(u^*)$ and $u^*\sigma(u)$ are matrices over the algebra F. We denote by U_σ the set of* **modular unitaries**. *Of course, any unitary over F is a modular unitary.*

Here we are thinking of the case $\sigma(a) = \Delta^{-1}(a)$, where Δ is the modular operator for some weight on A. Again, to avoid confusion, we remind the reader that as operators we have:

$$\pi(\sigma(a)) = \pi(\Delta^{-1}(a)) = \Delta^{-1}\pi(a)\Delta.$$

Hence the terminology modular unitaries. For unitaries in matrix algebras over $\mathcal{A}$ we use the regular automorphism $\sigma \otimes Id_n$ to state the modular condition, where Id_n is the identity automorphism of $M_n(\mathbf{C})$.

Example. For $S_\mu \in O_{nc}$ we write $P_\mu = S_\mu S_\mu^*$. Then for each μ, ν we have a unitary

$$u_{\mu,\nu} = \begin{pmatrix} 1 - P_\mu & S_\mu S_\nu^* \\ S_\nu S_\mu^* & 1 - P_\nu \end{pmatrix}.$$

It is simple to check that this a self-adjoint unitary satisfying the modular condition.

Definition 4.3. *Let u_t be a continuous path of modular unitaries in the $*$-subalgebra $\mathcal{A}$ such that $u_t\sigma(u_t^*)$ and $u_t^*\sigma(u_t)$ are also* **continuous** *paths in F (this is not guaranteed since σ is not generally bounded). Then we say that u_t is a* **modular homotopy**, *and say that u_0 and u_1 are* **modular homotopic**. *If u and v are modular unitaries, we say that u is* **equivalent** *to v if there exist $k, m \geq 0$ so that $u \oplus 1_k$ is modular homotopic to $v \oplus 1_m$.*

Lemma 4.4. *The relation defined above is an equivalence relation. Moreover, if u is a modular unitary and $k \geq 0$ then $1_k \oplus u$ is modular homotopic to $u \oplus 1_k$. The binary operation on equivalence classes in U_σ, given by $[u] + [v] := [u \oplus v]$ is well-defined and abelian.*

Proof. It is straightforward to show that this is an equivalence relation. To see that $1_k \oplus u$ is modular homotopic to $u \oplus 1_k$ it suffices to do this for $k = 1$. If $u \in M_m(\mathcal{A})$ is a modular unitary then let $x_0 \in M_{m+1}(\mathbf{C})$ be the (backward) shift matrix whose action on the standard basis of $\mathbf{C}^{m+1}$ is given by $x_0(\bar{e}_k) = \bar{e}_{k-1}(mod)(m+1)$. Then, $x_0(1 \oplus u)x_0^* = (u \oplus 1)$. Let $\{x_t\}$ be a coninuous path of scalar unitaries from x_0 to $x_1 = 1_{m+1}$. Of course each $x_t \in M_{m+1}(F)$ as well. Since $\sigma(x_t) = x_t$, one easily checks that $\{x_t(1 \oplus u)x_t^*\}$ is a modular homotopy from $u \oplus 1$ to $1 \oplus u$.

To see that addition is well-defined, we must show that $u \oplus v$ is equivalent to $(u \oplus 1_k) \oplus (v \oplus 1_m)$. But this equals $u \oplus (1_k \oplus v) \oplus 1_m$. By the previous argument this is equivalent to $u \oplus (v \oplus 1_k) \oplus 1_m$ which equals $(u \oplus v) \oplus 1_{k+m}$ which is equivalent to $(u \oplus v)$.

To see that addition of classes is abelian let u, v be modular unitaries. By adding on copies of the identity, we can assume that u and v are both the same size matrices. Hence, it suffices to show that $u \oplus v$ is modular homotopic to $v \oplus u$. To this end, let

$$R_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

for $t \in [0, \pi/2]$. Let $w_t = R_t(u \oplus v)R_t^*$. Then we have

$$w_t = \begin{pmatrix} \cos^2(t)u + \sin^2(t)v & \cos(t)\sin(t)(v - u) \\ \cos(t)\sin(t)(v - u) & \cos^2(t)v + \sin^2(t)u \end{pmatrix}.$$

Observe that at $t = 0$ we have $u \oplus v$ and at $t = \pi/2$ we have $v \oplus u$. We need to show that $w_t \sigma(w_t^*) \in M_2(F)$ for all $t \in [0, \pi/2]$. Write $\hat{u}$ for $\sigma(u^*)$ and similarly for v. Then we compute

$$
\begin{aligned}
&w_t \sigma(w_t^*) \\
&= \begin{pmatrix} \cos^2(t)u + \sin^2(t)v & \cos(t)\sin(t)(v-u) \\ \cos(t)\sin(t)(v-u) & \cos^2(t)v + \sin^2(t)u \end{pmatrix} \begin{pmatrix} \cos^2(t)\hat{u} + \sin^2(t)\hat{v} & \cos(t)\sin(t)(\hat{v}-\hat{u}) \\ \cos(t)\sin(t)(\hat{v}-\hat{u}) & \cos^2(t)\hat{v} + \sin^2(t)\hat{u} \end{pmatrix} \\
&= \begin{pmatrix} \cos^2(t)u\hat{u} + \sin^2(t)v\hat{v} & \cos(t)\sin(t)(v\hat{v}-u\hat{u}) \\ \cos(t)\sin(t)(v\hat{v}-u\hat{u}) & \cos^2(t)v\hat{v} + \sin^2(t)u\hat{u} \end{pmatrix}
\end{aligned}
$$

and since both $u\hat{u}$ and $v\hat{v}$ lie in F, this is in $M_2(F)$. The other half of the modular condition follows by replacing u, v by u^*, v^*. $\qquad\square$

We can now also see why the usual proof that the inverse of u is u^* in $K_1(A)$ is not available to us. This usual proof is as follows. In the K_1 setting one uses: $u \oplus v = (u \oplus 1)(1 \oplus v) \sim (1 \oplus u)(1 \oplus v) = (1 \oplus uv)$, so that addition in K_1 arises from multiplication of unitaries, and hence $[u] + [u^*] = [uu^*] = [1] = 0$. However, in the modular setting, while the homotopy from $u \oplus 1$ to $1 \oplus u$ is a modular homotopy in U_σ by the last Lemma, the homotopy from $(u \oplus 1)(1 \oplus v)$ to $(1 \oplus u)(1 \oplus v)$ is not in general. The multiplication on the right by $(1 \oplus v)$ breaks the modular condition. In particular, the product of two modular unitaries **need not be a modular unitary**.

Lemma 4.5. *If $u \in M_k(F)$ is unitary then $u \oplus u^* \sim 1$.*

Proof. There is a path w_t from $u \oplus u^*$ to 1 through unitaries in $M_k(F)$ and so $w_t \sigma(w_t^*) = 1$ for all t and hence we find $u \oplus u^* \sim 1$. $\qquad\square$

We now formalise the above discussion. Compare the following with [HR, Definition 4.8.1]

Definition 4.6. *Let $K_1(\mathcal{A}, \sigma)$ be the abelian semigroup of equivalence classes of modular unitaries u over $\mathcal{A}$ under the equivalence relation u is **equivalent** to v if there exist $k, m \geq 0$ so that $u \oplus 1_k$ is modular homotopic to $v \oplus 1_m$. The following relations hold in $K_1(\mathcal{A}, \sigma)$*

1) $[1] = 0$,

2) $[u] + [v] = [u \oplus v]$,

3)   *If u_t, $t \in [0, 1]$ is a continuous paths of unitaries in $M_k(\mathcal{A})$ with $u_t \sigma(u_t^*)$ and $u_t^* \sigma(u_t)$ continuous over F then $[u_0] = [u_1]$.*

Corollary 4.7. *If $u \in M_k(F)$ then $-[u] = [u^*]$ in $K_1(A, \sigma)$.*

We can make $K_1(A, \sigma)$ a group by the Grothendieck construction, but this is not needed here. The following lemma is a clear departure from the situation in [PR] (it implies that the 'obvious' map from $K_0(M(F, A))$ to $K_1(A, \sigma)$ is not well-defined).

Lemma 4.8. *Recall, for all paths μ, ν with $P_\mu = S_\mu S_\mu^*$ we have a modular unitary*

$$
u_{\mu,\nu} = \begin{pmatrix} 1 - P_\mu & S_\mu S_\nu^* \\ S_\nu S_\mu^* & 1 - P_\nu \end{pmatrix}.
$$

Then there is a modular homotopy $u_{\mu,\nu} \sim u_{\nu,\mu}$.

Proof. We do the homotopy in two steps. The first is given by conjugating $u_{\mu,\nu}$ by the scalar unitary matrix

$$
\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, \pi/2],
$$

which takes us to

$$\begin{pmatrix} 1 - P_\nu & -S_\nu S_\mu^* \\ -S_\mu S_\nu^* & 1 - P_\mu \end{pmatrix}.$$

Then for $\theta \in [0, \pi]$ we consider

$$\begin{pmatrix} 1 - P_\nu & e^{i\theta} S_\nu S_\mu^* \\ e^{-i\theta} S_\mu S_\nu^* & 1 - P_\mu \end{pmatrix}.$$

The reader will readily confirm that these two homotopies are modular. $\square$

Example. More generally, if σ is a regular automorphism of a unital $*$-algebra $\mathcal{A}$ with fixed point algebra F, $v \in \mathcal{A}$ is a partial isometry with range and source projections in F, and furthermore **if** $v\sigma(v^*)$, $v^*\sigma(v)$ lie in F, then

$$u_v = \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix}$$

is a modular unitary over $\mathcal{A}$, as the reader may check. The proof of Lemma 4.8 applies to these unitaries to show that $u_v \sim u_{v^*}$.

Lemma 4.9. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be our modular spectral triple relative to $(\mathcal{N}, \tau_\Delta)$ and set $F = \mathcal{A}^\sigma$ and $\sigma : \mathcal{A} \to \mathcal{A}$. Let $L^\infty(\Delta) = L^\infty(\mathcal{D})$ be the von Neumann algebra generated by the spectral projections of Δ then $L^\infty(\Delta) \subset \mathcal{Z}(\mathcal{M})$. Let $u \in \mathcal{A}$ be a unitary, then $\pi(u)Q\pi(u^*) \in \mathcal{M}$ and $\pi(u^*)Q\pi(u) \in \mathcal{M}$ for all spectral projections Q of $\mathcal{D}$, if and only if u is modular. That is, $\pi(u)\Delta\pi(u^*)$ and $\pi(u^*)\Delta\pi(u)$ (or $\pi(u)\mathcal{D}\pi(u^*)$ and $\pi(u^*)\mathcal{D}\pi(u)$) are both affiliated to $\mathcal{M}$ if and only if u is modular.*

Proof. First, $L^\infty(\Delta)$ is an abelian algebra. By Lemma 3.9 all the Φ_k are in $\mathcal{M}$ and since the Φ_k are also the spectral projections of Δ, we have $L^\infty(\Delta)$ is contained in the centre. (Note that this extends the fact that $\mathcal{D}$ commutes with $\pi(F) = \pi(\mathcal{A}^\sigma)$). Next we observe that $\pi(u)Q\pi(u^*)$ is a projection in $\mathcal{N}$. For one direction, suppose u is modular, then we have

$$\begin{aligned} \Delta^{-1}\pi(u)Q\pi(u^*)\Delta &= \Delta^{-1}\pi(u)\Delta\Delta^{-1}Q\Delta\Delta^{-1}\pi(u^*)\Delta \qquad Q \in \mathcal{M} \\ &= \pi(\sigma(u))Q\pi(\sigma(u^*)) \\ &= \pi(u)\pi(u^*\sigma(u))Q\pi(\sigma(u^*)) \\ &= \pi(u)Q\pi(u^*\sigma(u)\sigma(u^*)), \qquad u^*\sigma(u) \in F \\ &= \pi(u)Q\pi(u^*). \end{aligned}$$

Hence $\pi(u)Q\pi(u^*)$ commutes with Δ, and so is in $\mathcal{M}$. Similarly, $u\sigma(u^*) \in F$ implies that $\pi(u^*)Q\pi(u) \in \mathcal{M}$. On the other hand if $\pi(u)Q\pi(u^*) \in \mathcal{M}$ then

$$\pi(u)Q\pi(u^*) = \Delta^{-1}\pi(u)Q\pi(u^*)\Delta = \pi(\sigma(u))Q\pi(\sigma(u^*))$$

and so we have

$$Q = \pi(u^*\sigma(u))Q\pi(\sigma(u^*)u) = Q + [\pi(u^*\sigma(u)), Q]\pi(\sigma(u^*)u).$$

As $\sigma(u^*)u$ is invertible, we see that $[\pi(u^*\sigma(u)), Q] = 0$. Since $\pi(u^*\sigma(u)) \in \pi(\mathcal{A})$, and commutes with all Q, we have $\pi(u^*\sigma(u)) \in \mathcal{M}$ and so lies in $\pi(F) = \mathcal{M} \cap \pi(\mathcal{A})$. That is, $u^*\sigma(u) \in F$. Similarly, $\pi(u^*)Q\pi(u) \in \mathcal{M}$ implies that $u\sigma(u^*) \in F$. $\square$

The fundamental aspect of the last lemma is that modular unitaries conjugate Δ to an operator affiliated to $\mathcal{M}$, and so $u\Delta u^*$ commutes with Δ (and $u\mathcal{D}u^*$ commutes with $\mathcal{D}$). We will next show that there is a pairing between (part of) modular K_1 and modular spectral triples. To do this, we are going to use the analytic formulae for spectral flow in [CP2].

5. An $\mathcal{L}^{(1,\infty)}$ local index formula

In this Section we will couch our results in terms of the notion of a modular spectral triple. That is we will assume properties (0) to (3) listed in Section 3 apply. Of course at this time the only examples we have presented are the matrix algebras over the smooth subalgebra O_{nc} of the Cuntz algebra. However, we know from work in progress that there are other examples and hence it is worth arguing directly from the general properties and avoiding the explicit formulae of the Cuntz example.

5.1. **The spectral flow formula: correction terms.** The spectral flow formula of [CP2] is, a priori, complicated in our setting. This is because we are computing the spectral flow between two operators which **are not** unitarily equivalent via a unitary **in** $\mathcal{M}$. Thus we must consider η-type correction terms. We will also recognise that the spectral flow we are calculating depends on the choice of trace ϕ on $\mathcal{M}$ and use the notation sf_ϕ. We now quote [CP2, Corollary 8.11].

Proposition 5.1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}_0)$ be an odd unbounded θ-summable semifinite spectral triple relative to $(\mathcal{M}, \phi)$. For any $\epsilon > 0$ we define a one-form α^ϵ on $\mathcal{M}_0 = \mathcal{D}_0 + \mathcal{M}_{sa}$ by*

$$\alpha^\epsilon(A) = \sqrt{\frac{\epsilon}{\pi}} \phi(A e^{-\epsilon \mathcal{D}^2})$$

for $\mathcal{D} \in \mathcal{M}_0$ and $A \in T_{\mathcal{D}}(\mathcal{M}_0) = \mathcal{M}_{sa}$. Then the integral of α^ϵ is independent of the piecewise C^1 path in $\mathcal{M}_0$ and if $\{\mathcal{D}_t\}_{t \in [a,b]}$ is any piecewise C^1 path in $\mathcal{M}_0$ then

$$sf_\phi(D_a, D_b) = \sqrt{\frac{\epsilon}{\pi}} \int_a^b \phi(\mathcal{D}'_t e^{-\epsilon \mathcal{D}_t^2}) dt + \frac{1}{2}\eta_\epsilon(\mathcal{D}_b) - \frac{1}{2}\eta_\epsilon(\mathcal{D}_a) + \frac{1}{2}\phi\left([\ker(\mathcal{D}_b)] - [\ker(\mathcal{D}_a)]\right).$$

where the following integral converges for all $\epsilon > 0$

$$\eta_\epsilon(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \phi(\mathcal{D} e^{-t\mathcal{D}^2}) t^{-1/2} dt.$$

We note that the η terms are measures of ϕ-spectral asymmetry. We will show that for the pair $\mathcal{D}, \tau_\Delta$ we use on the Cuntz algebra, and the kinds of perturbations we consider, these η terms vanish. Moreover we will show that the τ_Δ-dimension of the kernel of $\mathcal{D}$ is unchanged by the particular type of perturbations we consider, so these correction terms will cancel. First we must show that we are actually working with the right kinds of perturbations, that is, elements in $\mathcal{M}_{sa}$.

Notation. We denote the densely defined spatial homomorphism on $\mathcal{N}$, $T \mapsto \Delta^{-1}T\Delta$ by $\sigma_i(T)$, so that for $a \in \mathcal{A}$ we have $\pi(\sigma(a)) = \sigma_i(\pi(a))$. We observe that $\mathcal{M}$, and $\pi(\mathcal{A})$ are in the domain of σ_i, and that $\mathcal{M}$ is exactly the fixed point subalgebra of σ_i.

Lemma 5.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a modular spectral triple. If u is a modular unitary, then $\pi(u)[\mathcal{D}, \pi(u^*)] \in \mathcal{M}_{sa}$. This is a **key fact** which allows us to directly use results about semifinite spectral flow in $(\mathcal{M}, \tau_\Delta)$ from [CP2].*

Proof. We just compute the action of σ_i on $\pi(u)[\mathcal{D}, \pi(u^*)]$. As observed above the operator $[\mathcal{D}, \pi(u^*)] \in \pi(\mathcal{A})$, and we easily calculate:

$$\sigma_i(\pi(u)[\mathcal{D}, \pi(u^*)]) = \pi(\sigma(u))[\mathcal{D}, \pi(\sigma(u^*))] = \pi(uu^*\sigma(u))[\mathcal{D}, \pi(\sigma(u^*))] = \pi(u)[\mathcal{D}, \pi(u^*)].$$

$\square$

Remarks. In the following few pages we will sometimes abuse notation and write a in place of $\pi(a)$ for $a \in \mathcal{A}$ in order to make our formulae more readable. Whenever we do this, however, we will use $\sigma_i(\cdot) = \Delta^{-1}(\cdot)\Delta$ the spatial version of the algebra homomorphism, σ. We will generally use the spatial version σ_i when in the presence of operators not in $\pi(\mathcal{A})$.

Lemma 5.3. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be our modular spectral triple for the Cuntz algebra and let u be a modular unitary. Then*

$$\tau_\Delta([\ker(\mathcal{D})] - [\ker(u\mathcal{D}u^*)]) = \tau_\Delta((1 - \sigma_i(u^*)u)[\ker(\mathcal{D})]) = \tau(1 - \sigma(u^*)u),$$

and for all $\epsilon > 0$, $\eta_\epsilon(u\mathcal{D}u^*) = \tau(\sigma(u^*)u)\eta_\epsilon(\mathcal{D})$.

Proof. We show the second equality first. By the σ_i-invariance of τ_Δ and the fact that $\sigma_i(u^*)u \in \mathcal{M}$ we have, using Lemma 3.11 in the last equality:

$$\tau_\Delta(u\mathcal{D}u^* e^{-t(u\mathcal{D}u^*)^2}) = \tau_\Delta(u\mathcal{D}e^{-t\mathcal{D}^2}u^*) = \tau_\Delta(\sigma_i(u)\mathcal{D}e^{-t\mathcal{D}^2}\sigma_i(u^*)) = \tilde\tau(u\Delta\mathcal{D}e^{-t\mathcal{D}^2}\sigma_i(u^*))$$

$$= \tilde\tau(\Delta\mathcal{D}e^{-t\mathcal{D}^2}\sigma_i(u^*)u) = \tilde\tau(\Delta\sigma_i(u^*)u\mathcal{D}e^{-t\mathcal{D}^2}) = \tau_\Delta(\sigma_i(u^*)u\mathcal{D}e^{-t\mathcal{D}^2}) = \tau(\sigma(u^*)u)\tau_\Delta(\mathcal{D}e^{-t\mathcal{D}^2}).$$

Thus, we have

$$\eta_\epsilon(u\mathcal{D}u^*) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \tau(\sigma(u^*)u)\tau_\Delta(\mathcal{D}e^{-t\mathcal{D}^2})t^{-1/2}dt = \tau(\sigma(u^*)u)\eta_\epsilon(\mathcal{D}),$$

as was to be shown. For the kernel we simply observe that $[\ker(u\mathcal{D}u^*)] = u[\ker(\mathcal{D})]u^* \in \mathcal{N}$, so that

$$\tau_\Delta([\ker(u\mathcal{D}u^*)]) = \tilde\tau(\Delta u[\ker(\mathcal{D})]u^*) = \tilde\tau(u[\ker(\mathcal{D})]\Delta\Delta^{-1}u^*\Delta) = \tau_\Delta(\sigma_i(u^*)u[\ker(\mathcal{D})]).$$

Then, by Lemma 3.11, $\tau_\Delta(\sigma_i(u^*)u[\ker(\mathcal{D})]) = \tau(\sigma(u^*)u)\tau_\Delta([\ker(\mathcal{D})]) = \tau(\sigma(u^*)u) \cdot 1$. $\qquad\square$

If we have a modular unitary for which we have both $\eta_\epsilon(u_v\mathcal{D}u_v) - \eta_\epsilon(\mathcal{D}) = 0$ and $\phi([\ker\mathcal{D}]) - \phi([\ker u\mathcal{D}u^*]) = 0$, we may apply the Laplace transform technique discussed in [CP2, Section 9] to reduce the θ-summable formula to the finitely summable formula. For $r > 0$ this gives us

$$(8) \qquad sf_\phi(\mathcal{D}, u\mathcal{D}u^*) = \frac{1}{C_{1/2+r}} \int_0^1 \phi(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-1/2-r})dt.$$

We are now in a position to apply the methods employed in the proof of the semifinite local index formula, [CPS2] or [CPRS2], to compute an index pairing.

5.2. **A local index formula for the Cuntz algebras.**

Lemma 5.4 (cf [CP2]). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be our $(1, \infty)$-summable modular spectral for the Cuntz algebra triple for a matrix algebra $\mathcal{A}$ over O_{nc}. Let $\mathcal{M} = \mathcal{N}^\sigma$ be the fixed point algebra for the modular automorphism group. The functional α defined on the self adjoint elements $\mathcal{M}_{sa}$ of $\mathcal{M}$ by*

$$\alpha_S(T) = \hat\tau(T(1 + (\mathcal{D} + S)^2)^{-s/2}), \quad T \in \mathcal{M}_{sa}$$

for $s > 1$ is an exact one form on the tangent space to the affine space $\mathcal{M}_0 = \mathcal{M}_{sa} + \mathcal{D}$ of $\mathcal{M}_{sa}$ perturbations of $\mathcal{D}$.

This fact is all that we need to calculate $\hat\tau$-spectral flow along paths in the affine space $\mathcal{M}_0$. We will be interested in $\hat\tau$-spectral flow along the linear path joining $\mathcal{D}$ to $\mathcal{D} + u[\mathcal{D}, u^*]$ where u is a unitary in $End_F(X)$ such that $u[\mathcal{D}, u^*] \in \mathcal{M}_{sa}$. Since modular unitaries, u satisfy these requirements, we can now produce a formula for spectral flow which is analogous to the local index formula in noncommutative geometry. We remind the reader that $\tau_\Delta = \hat\tau \circ \Psi$ where $\Psi : \mathcal{N} \to \mathcal{M}$ is the canonical expectation, so that τ_Δ **restricted to $\mathcal{M}$ is $\hat\tau$**.

Theorem 5.5. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the $(1, \infty)$-summable, modular spectral triple for the Cuntz algebra we have constructed previously. Then for any modular unitary such that the difference of eta terms $\eta_\epsilon(u\mathcal{D}u^*) - \eta_\epsilon(\mathcal{D})$ and $\widehat{\tau}([\ker \mathcal{D}]) - \widehat{\tau}([\ker u\mathcal{D}u^*])$ vanishes, and for any Dixmier trace $\widehat{\tau}_{\widetilde{\omega}}$ associated to $\widehat{\tau}$, we have spectral flow as an actual limit*

$$sf_{\widehat{\tau}}(\mathcal{D}, u\mathcal{D}u^*) = \frac{1}{2} \lim_{s \to 1+} (s-1)\widehat{\tau}(u[\mathcal{D}, u^*](1 + \mathcal{D}^2)^{-s/2}) = \frac{1}{2}\widehat{\tau}_{\widetilde{\omega}}(u[\mathcal{D}, u^*](1 + \mathcal{D}^2)^{-1/2}) = \tau \circ \Phi(u[\mathcal{D}, u^*]).$$

The functional on $\mathcal{A} \otimes \mathcal{A}$ defined by $a_0 \otimes a_1 \mapsto \frac{1}{2}\lim_{s\to 1+}(s-1)\tau_\Delta(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-s/2})$ is a σ-twisted b, B-cocycle (see the proof below for the definition).

Proof. First we observe that by [CPS2, Lemma 6.1], the difference

$$(1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-s/2} - (1 + \mathcal{D}^2)^{-s/2}$$

is uniformly bounded in trace class norm for $t \in [0, 1]$ and $s \in (1, 4/3)$. Hence in the spectral flow formula (8), by the simple change of variable $r = 1/2(s-1)$, we may write

$$C_{s/2}\, sf_{\widehat{\tau}}(\mathcal{D}, u\mathcal{D}u^*) = \widehat{\tau}(u[\mathcal{D}, u^*](1 + \mathcal{D}^2)^{-s/2}) + \text{remainder}.$$

Where the remainder is bounded as $s \to 1^+$. Multiplying this equation by $(s-1)/2$ and taking the limit as $s \to 1^+$ recalling that $C_{s/2} = \frac{\Gamma((s-1)/2)\Gamma(1/2)}{\Gamma(s/2)}$ so that $(s-1)/2\, C_{s/2} \to 1$, we get:

$$sf_{\widehat{\tau}}(\mathcal{D}, u\mathcal{D}u^*) = \frac{1}{2} \lim_{s \to 1+} (s-1)\widehat{\tau}(u[\mathcal{D}, u^*](1 + \mathcal{D}^2)^{-s/2}).$$

Now by the proof of Lemma 3.16, $u[\mathcal{D}, u^*]$ is in $\mathcal{A} = O_{nc}$ and since it is also in $\mathcal{M}$ it is in F_c and so by Proposition 3.14 this last limit equals $\tau \circ \Phi(u[\mathcal{D}, u^*])$ as claimed.

To see that we obtain a σ-twisted cocycle, we denote by θ the functional

$$\theta(a_0, a_1) = \frac{1}{2} \lim_{s \to 1+} (s-1)\tau_\Delta(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-s/2}),$$

and observe that by the proof of Lemma 3.16 the elements $[\mathcal{D}, a_1]$ are in $\mathcal{A}$ and so by Proposition 3.14 we see that not only do these limits exist, but in fact,

$$\theta(a_0, a_1) = \tau \circ \Phi(a_0[\mathcal{D}, a_1]).$$

By definition $B^\sigma \theta(a_0) = \theta(1, a_0)$, and so by Lemma 3.16

$$(B^\sigma \theta)(a_0) = \lim_{s \to 1+} (s-1)\tau_\Delta([\mathcal{D}, a_0](1 + \mathcal{D}^2)^{-1/2-r}) = 0$$

By definition,

$$\begin{aligned} b^\sigma \theta(a_0, a_1, a_2) &= \theta(a_0 a_1, a_2) - \theta(a_0, a_1 a_2) + \theta(\sigma(a_2)a_0, a_1) \\ &= -\tau \circ \Phi(a_0[\mathcal{D}, a_1]a_2) + \tau \circ \Phi(\sigma(a_2)a_0[\mathcal{D}, a_1]) \end{aligned}$$

This is 0 by the KMS condition (see the Remark prior to Corollary 3.4) for the state $\psi = \tau \circ \Phi$. Thus, both $b^\sigma \theta = 0$ and $B^\sigma \theta = 0$, and we're done. $\square$

Remark. Spectral flow in this setting is independent of the path joining the endpoints of unbounded self adjoint operators affiliated to $\mathcal{M}$ however it is not obvious that this is enough to show that it is constant on homotopy classes of modular unitaries. This latter fact is true but the proof is lengthy and we defer it until we have a fuller understanding of the structure of the modular unitaries.

Theorem 5.6. *We let* $(O_{nc} \otimes M_2, \mathcal{H} \otimes \mathbf{C}^2, \mathcal{D} \otimes 1_2)$ *be the modular spectral triple of* $(O_{nc} \otimes M_2)$ *and* u *a modular unitary of the form*

$$u_{\mu,\nu} = \begin{pmatrix} 1 - P_\mu & S_\mu S_\nu^* \\ S_\nu S_\mu^* & 1 - P_\nu \end{pmatrix}.$$

Then the spectral flow is positive being given by

$$sf_{\tau_\Delta}(\mathcal{D}, u\mathcal{D}u^*) \quad = \quad (|\mu| - |\nu|)(n^{-|\nu|} - n^{-|\mu|}) \in (n-1)\mathbf{Z}[1/n]$$

Proof. Once we have verified that the difference of eta terms and the difference of kernel corrections vanish, this is just a computation. In fact, by Lemma 5.3,

$$\eta_\epsilon(u\mathcal{D}u^*) = \tau(\sigma(u^*)u)\eta_\epsilon(\mathcal{D}) = \tau(\sigma(u^*)u) \int_\epsilon^\infty \left(\sum_{k \in \mathbf{Z}} ke^{-tk^2} \right) dt = 0 = \int_\epsilon^\infty \left(\sum_{k \in \mathbf{Z}} ke^{-tk^2} \right) dt = \eta_\epsilon(\mathcal{D}).$$

For the kernel corrections we use Lemma 5.3 and first compute $1 - \sigma(u_v^*)u_v$, noting that

$$\sigma(v)(1 - v^*v) = \sigma(v)\sigma(1 - v^*v) = \sigma(v - vv^*v) = 0.$$

$$1 - \sigma(u_v^*)u_v = 1 - \sigma(u_v)u_v = \begin{pmatrix} v^*v - \sigma(v^*)v & 0 \\ 0 & vv^* - \sigma(v)v^* \end{pmatrix}.$$

For $\tau(1 - \sigma(u_v^*)u_v)$ we use the KMS property of $\psi = \tau \circ \Phi$:

$$\tau(1 - \sigma(u_v^*)u_v) = \tau(v^*v - \sigma(v^*)v) + \tau(vv^* - \sigma(v)v^*) = \tau(v^*v - vv^*) + \tau(vv^* - v^*v) = 0.$$

Hence both the eta terms and kernel corrections vanish, and the spectral flow can be computed from the integral of the exact one form of Lemma 5.4.

For the computation we use a calculation in the proof of Lemma 3.16 to get

$$u_{\mu,\nu}[\mathcal{D} \otimes 1_2, u_{\mu,\nu}] = \begin{pmatrix} 1 - P_\mu & S_\mu S_\nu^* \\ S_\nu S_\mu^* & 1 - P_\nu \end{pmatrix} \begin{pmatrix} 0 & [\mathcal{D}, S_\mu S_\nu^*] \\ [\mathcal{D}, S_\nu S_\mu^*] & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - P_\mu & S_\mu S_\nu^* \\ S_\nu S_\mu^* & 1 - P_\nu \end{pmatrix} \begin{pmatrix} 0 & (|\mu| - |\nu|)S_\mu S_\nu^* \\ (|\nu| - |\mu|)S_\nu S_\mu^* & 0 \end{pmatrix} = (|\mu| - |\nu|) \begin{pmatrix} -P_\mu & 0 \\ 0 & P_\nu \end{pmatrix}.$$

So using Theorem 5.5 and our previous computation of the Dixmier trace, Proposition 3.12, we have

$$sf_{\tau_\Delta}(\mathcal{D}, u_{\mu,\nu}\mathcal{D}u_{\mu,\nu}) = (|\mu| - |\nu|)\tau(P_\nu - P_\mu) = (|\mu| - |\nu|)(n^{-|\nu|} - n^{-|\mu|}).$$

This number is always positive as the reader may check, and is contained in $(n-1)\mathbf{Z}[1/n]$, the integer polynomials in $1/n$ all of whose coeffficients have a factor of $(n-1)$. $\square$

Remarks. We observe that since this unitary $u_{\mu,\nu}$ is self-adjoint the spectral flow **cannot** be interpreted simply as the index of the Toeplitz compression of $u_{\mu,\nu}$ by the non-negative spectral projection of $\mathcal{D} \otimes 1_2$: for one thing this "Toeplitz compression" is not in $\mathcal{M}$ and if it were in $\mathcal{M}$ its index would have to be 0. Next we use the viewpoint provided by the noncommutative APS theory of [CPR]. This gives a partial explanation of the numerical values of the spectral flow obtained for the Cuntz algebras.

Corollary 5.7. *Let* $(O_{nc} \otimes M_2, \mathcal{H} \otimes \mathbf{C}^2, \mathcal{D} \otimes 1_2)$ *be the modular spectral triple of the theorem and* u *a modular unitary of the form* u_v, *where* $v = S_\mu S_\nu^*$ *so that* $v^*v = S_\nu S_\nu^*$ *and* $vv^* = S_\mu S_\mu^*$ *are both in F. Let* $(X, \mathcal{D} \otimes 1_2)$ *be the Kasparov module for* $O_n \otimes M_2, F \otimes M_2$ *described earlier. Then from the pairing*

$$K_0(M(F \otimes M_2, O_n \otimes M_2)) \times (X, \mathcal{D} \otimes 1_2) \to K_0(F)$$

we have the classes of the projections

$$\text{Index}(PvPv^* : vPv^*(X) \to vv^*P(X)) \quad \text{and} \quad \text{Index}(Pv^*Pv : v^*Pv(X) \to v^*vP(X)) \in K_0(F).$$

These two classes are negatives of each other in $K_0(F)$, but

$$sf_{\tau_\Delta}(\mathcal{D}, u_v \mathcal{D} u_v) = \tau_\Delta(\text{Index}(PvPv^* : vPv^*X \to vv^*PX)) + \tau_\Delta(\text{Index}(Pv^*Pv : v^*PvX \to v^*vPX)),$$

where here we apply τ_Δ to the difference of projections defining the index as a difference of F-modules.

Proof. In [CPR] Lemma 3.5 and Theorem 5.1, the authors used the following operators and indices:

$$\text{Index}(PvP : v^*vP(X) \to vv^*P(X)) \quad \text{and} \quad \text{Index}(Pv^*P : vv^*P(X) \to v^*vPX).$$

Each index is the **exact** negative of the other, and so if we evaluate both with τ_Δ and add we get exactly 0. The $K_0(F)$ elements given by the indices of these two operators are the same as the ones considered in this Corollary. However, the point of view of this Corollary is to consider mappings from say the non-negative spectral subspace of $v\mathcal{D}v^*$ (i.e., $vPv^*(X)$) to the non-negative spectral subspace of $vv^*\mathcal{D}$ (i.e., $vv^*P(X)$). Here we get quite a different answer.

Let $v = S_\mu S_\nu^*$, and $m = |\mu| - |\nu|$, $m > 0$; so that $vv^* = S_\mu S_\mu^*$. A simple computation on monomials $S_\alpha S_\beta^*$ gives us the **key fact** that : $v\Phi_k v^* = vv^*\Phi_{k+m}$ for all $k \in \mathbf{Z}$. This easily implies that $vPv^* = vv^*(\sum_{k \geq m} \Phi_k) \leq vv^*P$ so that $(PvPv^*)vPv^* = vPv^*$ and so $\ker(PvPv^*) = \{0\}$. This also shows that:

$$\text{cokernel}(PvPv^* : vPv^*(X) \to vv^*P(X)) = vv^*P(X) \ominus vv^*(\sum_{k \geq m} \Phi_k)(X) = \sum_{k=0}^{m-1} vv^*\Phi_k(X).$$

Similarly, $v^*Pv = \sum_{k \geq -m} v^*v\Phi_k \geq v^*vP$ so that $(Pv^*Pv)v^*Pv = v^*vP$ and so Pv^*Pv is **onto** $v^*vP(X)$. That is, $\text{cokernel}(Pv^*Pv) = \{0\}$. This also shows that

$$\text{kernel}(Pv^*Pv : v^*Pv(X) \to v^*vP(X)) = \sum_{k \geq -m} v^*v\Phi_k(X) \ominus v^*vP(X) = \sum_{k=-m}^{-1} v^*v\Phi_k(X).$$

To see that these indices are negatives in $K_0(F)$ it suffices to see the equivalence between the two projections $\sum_{k=0}^{m-1} vv^*\Phi_k$ and $\sum_{k=-m}^{-1} v^*v\Phi_k$. This is obtained from our **key fact** above:

$$(v\Phi_k)(\Phi_k v^*) = vv^*\Phi_{k+m}, \quad (\Phi_k v^*)(v\Phi_k) = v^*v\Phi_k.$$

This is of course the Murray-von Neumann equivalence which τ_Δ does not respect.

Assume then that $m > 0$. Applying τ_Δ we have

$$\tau_\Delta(\text{Index}(PvPv^*)) = -m\tau(vv^*) = -\frac{m}{n^{|\mu|}},$$

while

$$\tau_\Delta(\text{Index}(Pv^*Pv)) = m\tau(v^*v) = \frac{m}{n^{|\nu|}}.$$

The case $m < 0$ is similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. This Corollary makes it clear that our new index pairings are non-trivial precisely because τ_Δ does not induce a map on $K_0(End_F^0(X))$. Of course $\widehat{\tau}_\omega$ just becomes the trace on elements of F, but τ_Δ is a weight on $\mathcal{N}$ and so on $End_F^0(X)$, which is Morita equivalent to F. However, since τ_Δ is not a trace on $\mathcal{N}$ it does not respect all Murray-von Neumann equivalences in $\mathcal{N}$, and so does not give a well-defined map on K-theory. So we may think of the spectral flow invariant associated to $u_{\mu,\nu}$ as a measure of the failure of τ_Δ to respect the Murray-von Neumann equivalence between $\text{Index}(PvPv^*)$ and $-\text{Index}(Pv^*Pv)$.

More generally we have

$$sf_{\tau_\Delta}(\mathcal{D}_2, u_v \mathcal{D}_2 u_v)$$
$$= \text{Index}_{\tau_\Delta}(P_2 u_v P_2 u_v) = \text{Index}_{\tau_\Delta}\left(\left(\begin{array}{cc} (1-vv^*)P + PvPv^* & 0 \\ 0 & (1-v^*v)P + Pv^*Pv \end{array}\right)\right).$$

Since $P(1-vv^*) = (1-vv^*)P$ is an isomorphism from $(1-vv^*)PX$ to itself, and similarly for $(1-v^*v)P$, we see that the index is precisely the sum of the indices of $PvPv^*$ from vPv^*X to vv^*PX, and Pv^*Pv from v^*PvX to v^*vPX. Hence the spectral flow for modular unitaries of the form u_v arises precisely because τ_Δ does not induce a homomorphism on $K_0(End_F^0(X))$.

Our arguments here rely on the vanishing of the difference of eta terms and kernel corrections. In the general case these eta and kernel terms contribute and may have cohomological significance. We will return to this more general set up in a future work.

6. Concluding Remarks

6.1. **Relative entropy.** In this subsection we give a physical interpretation of our index. Let u be a modular unitary over O_n. Recall that ψ is the state on O_n defined by $\psi = \tau \circ \Phi$. Let ψ_u be the state $\psi \circ Adu$ on O_n defined by $\psi_u(a) = \psi(u^*au), a \in A$. The modular group for ψ_u is $t \to u\Delta^{it}u^*$ $t \in \mathbf{R}$. The relative entropy of a pair of KMS states on a von Neumann algebra was introduced by Araki [Ar] (it uses explicitly a cyclic and separating vector). The Hilbert space $\mathcal{H} = \mathcal{L}^2(O_n, \psi)$ has a cyclic and separating vector for the action of O_n. In fact this vector remains cyclic and separating for the weak closure $\pi(O_n)''$ in $\mathcal{N}$ of $\pi(O_n)$. It may be thought of as the identity element in O_n but we will use the notation Ω because of the potential for confusion.

For $a \in O_n$, $\psi(a) = \langle \Omega, \pi(a)\Omega \rangle$ so that we may write $\psi(T) = \langle \Omega, T\Omega \rangle$ for all $T \in \pi(O_n)''$. So we can regard ψ_u and ψ as a pair of KMS states on $\pi(O_n)''$. Then the relative entropy of ψ_u and ψ is [Ar]

$$S(\psi_u, \psi) = -\langle \Omega, \log(u\Delta u^*)\Omega \rangle.$$

This can be written as

$$S(\psi_u, \psi) := -\psi(u(\log\Delta)u^* - \log\Delta)$$

This is because $\Delta\Omega = \Omega$ implies that $(\log\Delta)(\Omega) = 0$. Now we can relate the relative entropy for this pair of KMS states on the weak closure of $\pi(O_n)$ to spectral flow for the Cuntz algebra example when we have a modular unitary u. We just use the formula $\log\Delta = -(\log n)\mathcal{D}$ and then by Theorem 5.5 we see that this relative entropy is just

$$(\log n)\psi(u\mathcal{D}u^* - \mathcal{D}) = (\log n)\psi(u[\mathcal{D}, u^*]) = (\log n)\tau \circ \Phi(u[\mathcal{D}, u^*]) = (\log n)sf(\mathcal{D}, u\mathcal{D}u^*).$$

That is, the relative entropy is just $\log n$ times the spectral flow from $\mathcal{D}$ to $u\mathcal{D}u^*$. We remark that the relative entropy is always positive [Ar].

6.2. **Manifold structures.** In [PRS2] it was shown that many of the (tracial) examples of semifinite spectral triples constructed for graph and k-graph algebras satisfied natural generalisations of Connes' axioms for noncommutative manifolds, [C1].

Much of the discussion of [PRS2] can be applied verbatim to the triple $(O_{nc}, \mathcal{H}, \mathcal{D})$ constructed here. For instance the axiom of finiteness is obvious, as is Morita equivalence (spin^c), first order condition, regularity (or QC^∞), and irreducibility. The reality, or spin, condition can be proved as in [PRS2], and we have proven the closedness condition in Lemma 3.16.

The chief differences come from the summability/dimension/absolute continuity and crucial orientability conditions. We have a version of summability satisfied since $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \widehat{\tau})$, and for $a \in O_{nc}$ nonzero and positive,

$$\lim_{s \to 1^+} (s-1)\tau_\Delta(a(1+\mathcal{D}^2)^{-s/2}) = \lim_{s \to 1^+}(s-1)\widehat{\tau}(\Psi(a)(1+\mathcal{D}^2)^{-s/2}) = 2\psi(a) > 0.$$

Moreover, we have a twisted Hochschild cycle satisfying the (twisted) orientability condition, and moreover it is given *by the same formula* as in the tracial case. This cocycle is

$$c = \frac{1}{n}\sum_{j=1}^n S_j^* \otimes S_j.$$

We have two properties to check: that it is indeed a cocycle, and that it is represented by the identity operator on $\mathcal{H}$. Applying the twisted Hochschild boundary gives

$$b^\sigma c = \frac{1}{n}\sum_{j=1}^n (S_j^* S_j - \sigma(S_j)S_j^*) = \frac{1}{n}\sum_{j=1}^n (1 - nS_j S_j^*) = 1 - \sum_{j=1}^n S_j S_j^* = 0.$$

This Hochschild cycle is represented on $\mathcal{H}$ by

$$\pi(c) = \frac{1}{n}\sum_{j=1}^n S_j^*[\mathcal{D}, S_j] = \frac{1}{n}\sum_{j=1}^n S_j^* S_j = \frac{1}{n}\sum_{j=1}^n 1 = 1.$$

Hence c has the required representation properties, and the replacement of the Hochschild theory with its twisted analogue has provided us with an orientation cycle for the 'modular spectral triple' of the Cuntz algebra. Thus Cuntz algebras may be a prototype for 'type III noncommutative one dimensional manifolds'.

6.3. Outlook.
There are many unresolved issues raised by these examples of an index theory for the KMS state on the Cuntz algebra. The main point is to understand the nature of the invariant being computed by our spectral flow formula for the modular unitaries. Just as semifinite spectral triples give rise to KK-classes, modular spectral triples also give rise to KK-classes. This follows in the same way as the semifinite case, [KNR]. However, the relationship to the KK-index pairing is obviously very different and we are investigating this now. At this time we do not see a relationship to the viewpoint of Connes and Moscovici [CoM].

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