NONCOMMUTATIVE ATIYAH-PATODI-SINGER BOUNDARY CONDITIONS AND INDEX PAIRINGS IN KK-THEORY

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Abstract

We investigate an extension of ideas of Atiyah-Patodi-Singer (APS) to a noncommutative geometry setting framed in terms of Kasparov modules. We use a mapping cone construction to relate odd index pairings to even index pairings with APS boundary conditions in the setting of KK-theory, generalising the commutative theory. We find that Cuntz-Kreiger systems provide a natural class of examples for our construction and the index pairings coming from APS boundary conditions yield complete K-theoretic information about certain graph C^*-algebras.

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1. Introduction

This paper is about a noncommutative analogue of APS index theory. We will focus on one aspect of generalising the APS theory. Namely we replace classical first order elliptic operators on a manifold with product metric near the boundary by a 'cylinder' of operators on a Kasparov module. We explain below how the classical theory provides an example of this more general framework. We also show in the last Section that there are many noncommutative examples as well. Our motivation is not simply that we are trying to understand noncommutative manifolds with boundary but is derived from the fact that the construction in this paper can be applied to many index problems in semifinite noncommutative geometry using [9] (which we plan to address elsewhere).

To explain our point of view let us recast a simple special case, using the language of later Sections, the connection between spectral flow and APS boundary conditions discussed in [2]. Let X be a closed Riemannian manifold, of odd dimension, and let $\mathcal{D}$ be a (self-adjoint) Dirac type operator on X. Then $\mathcal{D}$ determines an odd K-homology class $[\mathcal{D}]$ for the algebra $C(X)$ and we may pair $[\mathcal{D}]$ with the K-theory class of a unitary $u \in M_k(C(X))$ to obtain the integer

$$\mathrm{Index}(P_k u P_k) = sf(\mathcal{D}_k, u\mathcal{D}_k u^*).$$

Here P_k is the nonnegative spectral projection for $\mathcal{D}_k := \mathcal{D} \otimes Id_{\mathbf{C}^k}$ and the index of the 'Toeplitz operator' $P_k u P_k$ gives the spectral flow $sf(\mathcal{D}_k, u\mathcal{D}_k u^*)$ from $\mathcal{D}_k$ to $u\mathcal{D}_k u^*$.

We may also attach a semi-infinite cylinder to X, and consider the manifold-with-boundary $X \times \mathbf{R}_+$. If $\mathcal{D}$ acts on sections of some bundle $S \to X$, then $\mathcal{D}$ determines a self-adjoint operator on the L^2-sections of S, $\mathcal{H} = L^2(X, S)$, with respect to an appropriate measure constructed from the Riemannian metric and bundle inner products. We define

$$\hat{\mathcal{H}} = \begin{pmatrix} L^2(\mathbf{R}_+, \mathcal{H}) \\ L^2(\mathbf{R}_+, \mathcal{H}) \oplus \Phi_0\mathcal{H} \end{pmatrix}, \quad \hat{\mathcal{D}} = \begin{pmatrix} 0 & -\partial_t + \mathcal{D} \\ \partial_t + \mathcal{D} & 0 \end{pmatrix},$$

where Φ_0 is the projection onto the kernel of $\mathcal{D}$. It is necessary to single out the zero eigenvalue of $\mathcal{D}$ for special attention since it gives rise to 'extended L^2-solutions' which contribute to the index, [1]. We let $\hat{\mathcal{D}}$ act as zero on $\Phi_0\mathcal{H}$, and regard this subspace as being composed of values at infinity of extended solutions (more on this in the text).

We give $\hat{\mathcal{D}}$ APS boundary conditions. That is, we take the domain of $\partial_t + \mathcal{D}$ to be

$$\{\xi \in L^2(\mathbf{R}_+, \mathcal{H}) : (\partial_t + \mathcal{D})\xi \in L^2(\mathbf{R}_+, \mathcal{H}),\ P\xi(0) = 0\}$$

where again P is the nonnegative spectral projection for $\mathcal{D}$. The domain of $-\partial_t + \mathcal{D}$ is defined similarly using $1 - P$ in place of P. Then it can be shown, see for instance [1], that $\hat{\mathcal{D}}$ is an unbounded self-adjoint operator and for any $f \in C^\infty(X \times \mathbf{R}_+)$ which is of compact support and equal to a constant on the boundary, the product $f(1 + \hat{\mathcal{D}}^2)^{-1/2}$ is a compact operator on $\hat{\mathcal{H}}$.

Such functions lie in the mapping cone algebra for the inclusion $\mathbf{C} \hookrightarrow C(X)$. This is defined as

$$M(\mathbf{C}, C(X)) = \{f : \mathbf{R}_+ \to C(X) : f(0) \in \mathbf{C}1_X,\ \ f \text{ continuous and vanishes at } \infty\}.$$

We have an exact sequence

$$0 \to C(X) \otimes C_0((0, \infty)) \to M(\mathbf{C}, C(X)) \to \mathbf{C} \to 0$$

from which we get a six term sequence in K-theory. Since $K_1(\mathbf{C}) = 0$, this sequence simplifies to

$$0 \to K_1(C(X)) \to K_0(M(\mathbf{C}, C(X)) \to K_0(\mathbf{C}) \to K_0(C(X)) \to K_1(M(\mathbf{C}, C(X))) \to 0.$$

A careful analysis, which we present in greater generality in this paper, shows that the map $\mathbf{Z} = K_0(\mathbf{C}) \to K_0(C(X))$ takes n to the class of the trivial bundle of rank n on X, and so is injective. Thus we find that

$$K_1(C(X)) \cong K_0(M(\mathbf{C}, C(X))),$$

and the mapping cone algebra is providing a suspension of sorts. The relationship between the even index pairing for $\hat{\mathcal{D}}$ and the odd index pairing for $\mathcal{D}$ is then as follows. Let e_u be the projection over $M(\mathbf{C}, C(X))$ determined by the unitary u over $C(X)$, so that $[e_u] - [1] \in K_0(M(\mathbf{C}, C(X)))$. Then

$$\mathrm{Index}(e_u(\partial_t + \mathcal{D})e_u) - \mathrm{Index}(\partial_t + \mathcal{D}) = \langle [e_u] - [1], [\hat{\mathcal{D}}] \rangle = \langle [u], [\mathcal{D}] \rangle = sf(\mathcal{D}, u\mathcal{D}u^*).$$

The purpose of this paper is to present a noncommutative analogue of this picture. Our main result, Theorem 5.1, shows that the situation described above for the commutative case carries over to a class of Kasparov modules for noncommutative algebras. We exploit a paper of Putnam [17] on the K-theory of mapping cone algebras to give an APS type construction for a Kasparov module with boundary conditions that implies an equality between even and odd indices. Not only will we find a new version of this index equality, but we will see that it allows us to use APS boundary conditions to obtain interesting index pairings, and consequences, that were previously unknown. For instance we show that the complicated K-theory calculations of [14] can be given a simple functorial description.

A description of the organisation and main results of the paper now follows. We begin in the next Section with some preliminaries on Kasparov modules. In Section 3 we review [17], describing K_0 of mapping cone algebras, $M(F,A)$ where $F \subset A$ are certain C^*-algebras (replacing the pair $\mathbf{C} \subset C(X)$ in the classical setting above). We make some basic computations related to these groups and associated exact sequences.

The application of APS boundary conditions for Kasparov modules is done in Section 4. We show that certain odd Kasparov modules for algebras A, B with F a subalgebra of A, can be 'suspended' to obtain even Kasparov modules for the algebras $M(F,A)$, B, using APS boundary conditions. The proof is surprisingly complicated as there are substantial technical issues. Even self-adjointness of the abstract Dirac operator on the suspension with APS boundary conditions is not clear. We solve all of the difficulties using a careful construction in the noncommutative setting of a parametrix for our abstract Dirac operators on the even Kasparov module.

The main theorem (Theorem 5.1) shows that two index pairings – one from an odd Kasparov module and one from its even 'suspension' – with values in $K_0(B)$ are equal. Replacing $K_0(\mathbf{C}) = \mathbf{Z}$ with $K_0(B)$ gives us an analogue of the classical example above. The proof is quite difficult; solving differential equations in Hilbert C^*-modules is a more complex issue than in Hilbert space.

In Section 6 we explain one class of examples. There we calculate the K-groups of the mapping cone algebra $M(F,A)$ for the inclusion of the fixed point algebra F of the gauge action on certain graph C^*-algebras A. For these algebras, the application of Theorem 5.1 yields in Proposition 5.7 an isomorphism from $K_0(M(F,A))$ to $K_0(F)$, which leads to a functorial description of the calculations of $K_0(A)$, $K_1(A)$ in [14].

Readers familiar with [3] may be puzzled by the fact that we do not study the more general question of boundary conditions parametrised by a Grassmanian. In fact we make, in our main theorem, an assumption that classicially corresponds to assuming that we can work with a fixed APS boundary condition for all of the perturbed operators we study. We know that for classical index problems it is often the case that a more general operator can be homotopied to one that preserves the APS boundary conditions. In the noncommutative context of this paper we have not studied this homotopy argument. The examples in Section 6 illustrate that for many cases our restricted analysis suffices and provides complete information about the K-theory of the relevant algebras.

Acknowledgements. We thank Rsyzard Nest for advice on Section 5, David Pask, Aidan Sims and Iain Raeburn for enlightening conversations and Ian Putnam for bringing his work to the third author's attention. The first and second named authors acknowledge the financial assistance of the Australian Research Council and the Natural Sciences and Engineering Research Council of Canada while the third named author thanks Statens Naturvidenskabelige Forskningsråd, Denmark. All authors are grateful for the support of the Banff International Research Station where some of this research was undertaken.

2. Kasparov modules

The Kasparov modules considered in this subsection are for C^*-algebras with trivial grading.

Definition 2.1. *An* **odd Kasparov** A-B**-module** *consists of a countably generated ungraded right* B-C^**-module* E*, with* $\phi : A \to End_B(E)$ *a* $*$*-homomorphism, together with* $P \in End_B(E)$ *such that* $a(P-P^*)$*,* $a(P^2-P)$*,* $[P,a]$ *are all compact endomorphisms. Alternatively, for* $V = 2P-1$*,* $a(V-V^*)$*,* $a(V^2-1)$*,* $[V,a]$ *are all compact endomorphisms for all* $a \in A$*. One can modify* P *to* $\tilde{P}$ *so that* $\tilde{P}$ *is self-adjoint;* $\| \tilde{P} \| \leq 1$*;* $a(P-\tilde{P})$ *is compact for all* $a \in A$ *and the other conditions for* P *hold with* $\tilde{P}$ *in place of* P *without changing the module* E*. If* P *has a spectral gap about* 0 *(as happens in the cases of interest here) then we may and do assume that* $\tilde{P}$ *is in fact a projection without changing the module,* E*. (Note that by 17.6 of* [5] *we may assume that* P *is a projection by changing to a new module in the same class as* E*.)*

By [10], [Lemma 2, Section 7], the pair (ϕ, P) determines a $KK^1(A,B)$ class, and every class has such a representative. The equivalence relation on pairs (ϕ, P) that give KK^1 classes is generated by unitary equivalence $(\phi, P) \sim (U\phi U^*, UPU^*)$ and homology: $(\phi_1, P_1) \sim (\phi_2, P_2)$ if $P_1\phi_1(a) - P_2\phi_2(a)$ is a compact endomorphism for all $a \in A$, see also [10, Section 7]. Later we will also require **even**, or **graded**, Kasparov modules.

Definition 2.2. *An* **even Kasparov** A-B**-module** *has, in addition to the data of the previous definition, a grading by a self-adjoint endomorphism* Γ *with* $\Gamma^2 = 1$ *and* $\phi(a)\Gamma = \Gamma\phi(a)$*,* $V\Gamma + \Gamma V = 0$*.*

The next theorem presents a general result used in [15][Appendix] about the Kasparov product in the odd case.

Theorem 2.3. *Let* (Y,T) *be an odd Kasparov module for the* C^**-algebras* A, B*. Then (assuming that* T *has a spectral gap around* 0*) the Kasparov product of* $K_1(A)$ *with the class of* (Y,T) *is represented by*

$$\langle [u], [(Y,T)] \rangle = [\ker PuP] - [\mathrm{coker} PuP] \in K_0(B),$$

where P *is the non-negative spectral projection for the self-adjoint operator* T*.*

This pairing was studied in [15], as well as the relation to the semifinite local index formula in noncommutative geometry. It is also the starting point for this work. More detailed information about the KK-theory version of this can be found in [9].

In this paper we will employ unbounded representatives of KK-classes. The theory of unbounded operators on C^*-modules that we require is all contained in Lance's book, [12], [Chapters 9,10]. We quote the following definitions (adapted to our situation).

Definition 2.4. *Let* Y *be a right* C^*-B*-module. A densely defined unbounded operator* $\mathcal{D} : \mathrm{dom}\ \mathcal{D} \subset Y \to Y$ *is a* B*-linear operator defined on a dense* B*-submodule* $\mathrm{dom}\ \mathcal{D} \subset Y$*. The operator* $\mathcal{D}$ *is* **closed** *if the graph* $G(\mathcal{D}) = \{(x, \mathcal{D}x) : x \in \mathrm{dom}\ \mathcal{D}\}$ *is a closed submodule of* $Y \oplus Y$*.*

If $\mathcal{D} : \mathrm{dom}\ \mathcal{D} \subset Y \to Y$ is densely defined and unbounded, we define the domain of the **adjoint** of $\mathcal{D}$ to be the submodule:

$$\mathrm{dom}\ \mathcal{D}^* := \{y \in Y : \exists z \in Y \text{ such that } \forall x \in \mathrm{dom}\ \mathcal{D}, \langle \mathcal{D}x|y\rangle_R = \langle x|z\rangle_R\}.$$

Then for $y \in \mathrm{dom}\ \mathcal{D}^*$ define $\mathcal{D}^*y = z$. Given $y \in \mathrm{dom}\ \mathcal{D}^*$, the element z is unique, so $\mathcal{D}^* : \mathrm{dom}\mathcal{D}^* \to Y$, $\mathcal{D}^*y = z$ is well-defined, and moreover is closed.

Definition 2.5. *Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D} : \mathrm{dom}\ \mathcal{D} \subset Y \to Y$ is* **symmetric** *if for all $x, y \in \mathrm{dom}\ \mathcal{D}$*

$$\langle \mathcal{D}x|y\rangle_R = \langle x|\mathcal{D}y\rangle_R.$$

A symmetric operator $\mathcal{D}$ is **self-adjoint** *if $\mathrm{dom}\ \mathcal{D} = \mathrm{dom}\ \mathcal{D}^*$ (so $\mathcal{D}$ is closed). A densely defined operator $\mathcal{D}$ is* **regular** *if $\mathcal{D}$ is closed, $\mathcal{D}^*$ is densely defined, and $(1 + \mathcal{D}^*\mathcal{D})$ has dense range.*

The extra requirement of regularity is necessary in the C^*-module context for the continuous functional calculus, and is not automatic, [12],[Chapter 9].

Definition 2.6. *An* **odd unbounded Kasparov** A-B**-module** *consists of a countably generated ungraded right B-C^*-module E, with $\phi : A \to End_B(E)$ a $*$-homomorphism, together with an unbounded self-adjoint regular operator $\mathcal{D} : \mathrm{dom}\mathcal{D} \subset E \to E$ such that $[\mathcal{D}, a]$ is bounded for all a in a dense $*$-subalgebra of A and $a(1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism of E for all $a \in A$. An* **even unbounded Kasparov** A-B**-module** *has, in addition to the previous data, a $\mathbf{Z}_2$-grading with A even and $\mathcal{D}$ odd, as in Definition 2.2.*

3. K-Theory of the Mapping Cone Algebra and pairing with KK-theory

3.1. **The mapping cone.** Let $F \subset A$ be a C^*-subalgebra of a C^*-algebra A. Recall [17] that the mapping cone algebra is

$$M(F, A) = \{f : [0, 1] \to A : f \text{ is continuous},\ f(0) = 0,\ f(1) \in F\}.$$

The algebra operations are pointwise addition and multiplication and the norm is the uniform (sup) norm. There is a natural exact sequence

$$0 \to C_0(0, 1) \otimes A \xrightarrow{i} M(F, A) \xrightarrow{ev} F \to 0.$$

Here $ev(f) = f(1)$ and $i(g \otimes a)(t) \to g(t)a$. It is well known that when F is an ideal in the algebra A we have $K_*(M(F, A)) \cong K_*(A/F)$.

We will always be considering the situation where $K_1(F) = 0$, as is the case for graph C^*-algebras, though this is not strictly necessary. When $K_1(F) = 0$, the six term sequence in K-theory coming from this short exact sequence degenerates into

$$0 \to K_1(A) \to K_0(M(F, A)) \xrightarrow{ev_*} K_0(F) \xrightarrow{j_*} K_0(A) \to K_1(M(F, A)) \to 0. \tag{1}$$

We need to justify the notation j_*; namely we need to display the map j which induces j_*.

Lemma 3.1. *In the above exact sequence the map $j_* : K_0(F) \to K_0(A)$ is induced by minus the inclusion map $j : F \to A$ (up to Bott periodicity).*

Proof. The map we have denoted by j_* is actually a composite:

$$j_* : K_0(F) \xrightarrow{\partial} K_1(C_0(0, 1) \otimes A) \xrightarrow{\cong} K_0(A).$$

The isomorphism here is the inverse of the Bott map $Bott : K_0(A) \to K_1(C_0(0, 1) \otimes A)$, where $Bott([p]) = [e^{-2\pi it} \otimes p + 1 \otimes (1 - p)]$. The boundary map ∂ is defined as follows, [8, p 113]. For $[p] - [q] \in K_0(F)$, we choose representatives p, q over F, and then choose self-adjoint lifts x, y over $M(F, A)$. Then $e^{2\pi ix}, e^{2\pi iy}$ are unitaries over $C(S^1) \otimes A$ which are equal to the identity modulo $C_0(0, 1) \otimes A$. Then

$$\partial([p] - [q]) = [e^{2\pi ix}] - [e^{2\pi iy}] \in K_1(C_0(0, 1) \otimes A).$$

Now we choose the particular lifts over $M(F,A)$ given by $x(t) = tp$ and $y(t) = tq$ (in fact these are $t \otimes j(p)$ and $t \otimes j(q)$). Both these elements are self-adjoint, vanish at $t = 0$ and at $t = 1$ are in F. Now

$$\left[e^{2\pi i x}\right] - \left[e^{2\pi i y}\right] = \left[e^{2\pi i t \otimes p}\right] - \left[e^{2\pi i t \otimes q}\right] = -Bott([p] - [q]) \in K_1(C_0(0,1) \otimes A).$$

So modulo the isomorphism $Bott : K_0(A) \to K_1(C_0(0,1) \otimes A)$, $j_*([p] - [q]) = -([j(p)] - [j(q)])$. □

We now describe $K_0(M(F,A))$ [17]. Let $V_m(F,A)$ be the set of partial isometries $v \in M_m(A)$ such that $v^*v,\ vv^* \in M_m(F)$. Using the inclusion $V_m \hookrightarrow V_{m+1}$ given by $v \to v \oplus 0$ we can define

$$V(F,A) = \cup_m V_m(F,A).$$

Our aim, following [17], is to define a map $\kappa : V(F,A) \to K_0(M(F,A))$, and we proceed in steps. First, let $v \in V(F,A)$ and define a self-adjoint unitary v_1 via:

$$v_1 = \begin{pmatrix} 1 - vv^* & v \\ v^* & 1 - v^*v \end{pmatrix},$$

that is, $v_1^2 = 1,\ v_1 = v_1^*$. So, $v_1 = p_+ - p_-$ where $p_+ = \frac{1}{2}(v_1 + 1)$ and $p_- = \frac{1}{2}(1 - v_1)$ are the positive and negative spectral projections for v_1. Then for $t \in [0,1]$ define

$$v_2(t) = p_+ + e^{i\pi t} p_-$$

so that we have a continuous path of unitaries from the identity ($t = 0$) to v_1 ($t = 1$). Observe that $v_2(t)$ is unitary for all $t \in [0,1]$, $v_2 \in C([0,1]) \otimes M_{2m}(A)$, $v_2(0) = 1$ and $v_2(1) = v_1$. Now define

$$e_v(t) = v_2(t) e v_2(t)^*, \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $e_v(t)$ is a projection over the unitization $\tilde{M}(F,A)$ of $M(F,A)$ given by

$$\tilde{M}(F,A) = \{f : [0,1] \to \tilde{A} : f \text{ is continuous, } f(0) \in \mathbf{C}1,\ f(1) \in \tilde{F}\}.$$

Thus $[e_v] - [e]$ defines an element of $K_0(M(F,A))$. So with $\kappa(v) = [e_v] - [e]$ we find:

Lemma 3.2. [17, Lemmas 2.2,2.4,2.5]
1) $\kappa(v \oplus w) = \kappa(v) + \kappa(w)$
2) If $v, w \in V_m(F,A)$ *and* $\|v - w\| < (200)^{-1}$ *then* $\kappa(v) = \kappa(w)$
3) If $v \in V_m(F,A)$, $w_1, w_2 \in U_m(F)$ *then* $w_1 v w_2 \in V_m(F,A)$, $\kappa(w_1) = \kappa(w_2) = 0$, $\kappa(w_1 v w_2) = \kappa(v)$.
4) For $v \in M_m(F)$ *a partial isometry,* $\kappa(v) = 0$, *so, for* $p \in M_m(F)$ *a projection,* $\kappa(p) = 0$.
5) The map $\kappa : V(F,A) \to K_0(M(F,A))$ *is onto.*
6) Generate an equivalence relation $\sim$ *on* $V(F,A)$ *by*
(i) $v \sim v \oplus p$ *for* $v \in V(F,A)$, $p \in M_k(F)$
(ii) If $v(t),\ t \in [0,1]$ *is a continuous path in* $V(F,A)$ *then* $v(0) \sim v(1)$.
Then $\kappa : V(F,A)/\sim \to K_0(M(F,A))$ *is a well-defined bijection.*

Hence we may realise $K_0(M(F,A))$ as equivalence classes of partial isometries in $M_m(A)$ whose source and range projections lie in $M_m(F)$. Observe that when $K_1(F) = 0$, $K_1(A)$ embeds in $K_0(M(F,A))$ by regarding a unitary (possibly in a unitization of A) as a partial isometry. We add the following lemmas which we will need later.

Lemma 3.3. *Let* $v, w \in V_m(F,A)$ *have the same source projection, so* $v^*v = w^*w = p$, *say. Then* $[v \oplus w^*] = [v] + [w^*] = [v] - [w] = [vw^*]$.

Remark If $v = p$ we get a proof that $-[w] = [w^*]$.

Proof. The homotopy is given by

$$V_\theta = \begin{pmatrix} \cos^2(\theta)v + \sin^2(\theta)p & \cos(\theta)\sin(\theta)(w^* - vw^*) \\ \cos(\theta)\sin(\theta)(p - v) & \cos^2(\theta)w^* + \sin^2(\theta)vw^* \end{pmatrix}, \qquad \theta \in [0, \pi/2].$$

□

Lemma 3.4. *Suppose $v^*v = p + q$ with $p, q \in F$ projections, $p \perp q$. Then $v = vp + vq$, $vv^* = vpv^* + vqv^*$, $vpv^* \perp vqv^*$ and if we assume that $vpv^* \in F$ then $[v] = [vp \oplus vq] = [vp] + [vq]$.*

Proof. The first few statements are simple algebraic consequences of the hypothesis. The homotopy from $v \sim v \oplus 0$ to $vp \oplus vq$ is

$$V_\theta = \begin{pmatrix} vp + vq\cos^2(\theta) & vq\sin(\theta)\cos(\theta) \\ vq\sin(\theta)\cos(\theta) & vq\sin^2(\theta) \end{pmatrix}, \qquad \theta \in [0, \pi/2].$$

□

We will use the following equivalent definition of the mapping cone algebra, as it is more useful for our intended applications and agrees with the definition in the classical commutative case. We let

$$M(F, A) = \{f : \mathbf{R}_+ \to A : f \text{ continuous and vanishes at } \infty \text{ and } f(0) \in F\}.$$

This way of defining the mapping cone algebra gives an isomorphic C^*-algebra and we will take this as our definition from now on.

3.2. **The pairing in KK for the mapping cone.** Using the Kasparov product, $K_0(M(F, A))$ pairs with $KK^0(M(F, A), B)$ for any C^*-algebra B. However, $K_0(M(F, A))$ also pairs with odd A, B Kasparov modules (Y, V) such that the left action by $f \in F \subset A$ commutes with V. While all our constructions work for such A, B Kasparov modules, we will restrict in the sequel to A, F Kasparov modules. This will cause no loss of generality to those wishing to extend these results to the general case, but is the situation which arises naturally in examples.

Standing Assumptions (SA). For the rest of this Section, let $v \in A$ be a partial isometry with $v^*v, vv^* \in F$ (the same will work for matrix algebras over A, F). Let (Y, V) be an odd Kasparov module for A, F such that the left action of $f \in F \subset A$ commutes with $V = 2P - 1$ where P is the non-negative spectral projection of V.

Remarks. In all the calculations we do here, if $v \in M_k(A)$ then we use $P_k := P \otimes 1_k$ in place of P: we will **usually** suppress this inflation notation in the interests of avoiding notation inflation.

To define the pairing between the mapping cone and Kasparov modules satisfying **SA**, we need a preliminary result.

Lemma 3.5. *Let (Y, V) satisfy* **SA***. The two projections vv^*P and vPv^* differ by a compact endomorphism, and consequently $PvP : v^*vP(Y) \to vv^*P(Y)$ is Fredholm.*

Proof. It is a straightforward calculation that

$$vPv^* = vv^*P + v[P, v^*] = vv^*P + \frac{1}{2}v[V, v^*]$$

and, as $[V, v^*]$ is compact, vv^*P and vPv^* differ by a compact endomorphism. One easily checks that $Pv^*P : vv^*P(Y) \to v^*vP(Y)$ is a parametrix for PvP and the second statement follows. □

As PvP commutes with the right action of F, the kernel and cokernel are right F-modules. It follows from the detailed discussion in [7] that while it may not be the case that the kernel and cokernel are both finitely generated projective F-modules, the difference

$$[\ker PvP] - [\mathrm{coker} PvP]$$

makes sense as an element of $K_0(F)$.

Definition 3.6. *For $[v] \in K_0(M(F,A))$ and $(Y, 2P-1)$ satisfying* **SA***, define*

$$[v] \times (Y,V) = \mathrm{Index}(PvP : v^*vP(Y) \to vv^*P(Y)) = [\ker PvP] - [\mathrm{coker} PvP] \in K_0(F).$$

We make some general observations.

• If v is unitary over A, we recover the usual Kasparov pairing between $K_1(A)$ and $KK^1(A,F)$, [9], [15, Appendix]. Thus the pairing depends only on the class of $(Y, 2P-1)$ in $KK^1(A,F)$ for v unitary.

• In general the operator PvP does not have closed range. However the operator

$$\widetilde{PvP} := \begin{pmatrix} PvP & 0 \\ (1-P)vP & 0 \end{pmatrix} : \begin{pmatrix} v^*vP(Y) \\ v^*vP(Y) \end{pmatrix} \to \begin{pmatrix} vv^*P(Y) \\ vv^*(1-P)(Y) \end{pmatrix}$$

does have closed range, [7, Lemma 4.10], and the index is easily seen to be

$$\mathrm{Index}(\widetilde{PvP}) = \left[\begin{pmatrix} Pv^*(1-P)(Y) \\ (1-P)v^*P(Y) \end{pmatrix}\right] - \left[\begin{pmatrix} (1-P)v^*P(Y) \\ (1-P)v^*P(Y) \end{pmatrix}\right].$$

The index of PvP is in fact defined to be the index of any suitable 'amplification' like $\widetilde{PvP}$, [7], and we see that if the right F-module $Pv^*(1-P)(Y)$ is closed, then the 'correction' term $(1-P)v^*P(Y)$ arising from the amplification process cancels out. Since the K-theory class of the index does not in fact depend on the choice of amplification, we will ignore this subtlety from here on. That is, we assume without any loss of generality that the various Fredholm operators we consider satisfy the stronger condition of being **regular** in the sense of having a pseudoinverse [7][Definition 4.3]. Since we will be concerned only with showing that certain indices coincide, this will not affect our conclusions.

• The pairing depends only on the class of v in $K_0(M(F,A))$ with the module (Y,V) held fixed, in particular it vanishes if $v \in F$. These statements follow in the same way as the analogous statements for unitaries, cf [15, Appendix].

• Since addition in the "Putnam picture" of $K_0(M(F,A))$ is by direct sum as is addition in the usual picture of $K_0(M(F,A))$, it is easy to see that the pairing is additive in the $K_0(M(F,A))$ variable with the module (Y,V) held fixed. So with (Y,V) held fixed we have a well-defined group homomorphism:

$$\times(Y, 2P-1) : K_0(M(F,A)) \to K_0(F).$$

3.3. **Dependence of the pairing on the choice of** $(Y, 2P-1)$**.** The dependence on the Kasparov module $(Y, 2P-1)$ is not straightforward. For instance, we require that P commute with the left action of F, and so homotopy invariance is necessarily broken. We now fix $v \in V_m(F,A)$ and show that we can obtain an **even** Kasparov module (Y_v, R_v) for $(A_v, F) := (vv^*Avv^*, F)$ so that the two classes $[v] \times (Y, 2P-1)$ and $[1_{A_v}] \times [(Y_v, R_v)]$ are equal in $K_0(A)$, with the latter being a Kasparov product of *genuine* KK-classes.

The purpose in doing this is to understand the homotopy invariance properties of Index(PvP) by characterising it as a Kasparov product. In this subsection this is achieved by creating a 'smaller' Kasparov module, which depends on v. In our main theorem, Theorem 5.1, we associate to an odd unbounded Kasparov module $(X, \mathcal{D})$ a 'larger' even unbounded Kasparov module $(\hat{X}, \hat{\mathcal{D}})$. This latter module is independent of v and allows us to characterise, for all $[v] \in K_0(M(F,A))$, the class Index(PvP) as the Kasparov product $[v] \times [(\hat{X}, \hat{\mathcal{D}})]$.

Lemma 3.7. *With $v, (Y, 2P-1)$ as above, the pair*

$$(Y_v, R_v) := \left(\begin{pmatrix} vv^*(Y) \\ v^*v(Y) \end{pmatrix}, \begin{pmatrix} 0 & R_- \\ R_+ & 0 \end{pmatrix} \right) \quad \textit{where } R_- = (PvP - (1-P)v) \textit{ and } R_+ = R_-^*$$

is an even (vv^*Avv^*, F) Kasparov module for the representation

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & v^*av \end{pmatrix} \textit{ for } a \in vv^*Avv^*.$$

Proof. First observe that vv^*Avv^* is always unital, with unit $1_{A_v} = vv^*$, and that $\pi(a)$ leaves Y_v invariant for $a \in vv^*Avv^*$. Next, R_v is clearly self-adjoint and moreover, $R_-v^*v = vv^*R_-$. Taking adjoints we obtain $R_+vv^* = v^*vR_+$ so that R_v also leaves Y_v invariant. Now since v and v^* commute with P up to compacts we see that

$$R_- = (2P-1)v \ (\textit{mod compacts}) = v(2P-1) \ (\textit{mod compacts}) \ \textit{and} \tag{2}$$

$$R_+ = (2P-1)v^* \ (\textit{mod compacts}) = v^*(2P-1) \ (\textit{mod compacts}). \tag{3}$$

Hence,

$$R_v^2 = \begin{pmatrix} vv^* & 0 \\ 0 & v^*v \end{pmatrix} = 1_{Y_v} \ (\textit{mod compacts}).$$

The compactness of commutators $[R_v, \pi(a)]$ can be reduced by (2) and (3) to the equations:

$$a(2P-1)v = (2P-1)vv^*av \ \text{ and } \ v^*avv^*(2P-1) = v^*(2P-1)a \ (\textit{mod compacts}).$$

This completes the proof using $a = vv^*a = avv^*$ and $[P,a]$ compact. □

The following corollary is obvious once we note that

$$\pi(1_{A_v}) = \begin{pmatrix} vv^* & 0 \\ 0 & v^*v \end{pmatrix}$$

Corollary 3.8. *We have the equality in $K_0(F)$: $[v] \times (Y, 2P-1) = [1_{A_v}] \times [(Y_v, R_v)]$. Hence the pairing $[v] \times (Y, 2P-1)$ depends only on $[v] \in K_0(M(F,A))$ and the class $[(Y_v, R_v)] \in KK^0(vv^*Avv^*, F)$.*

Remarks. In the Kasparov module (Y_v, R_v) there is a dependence on v. This result also shows that we can pair with any subprojection of vv^* in F instead of $vv^* = 1_{vv^*Avv^*}$. The Kasparov module (Y_v, R_v) is formally reminiscent of the module obtained by a cap product of an odd module with a unitary. The remaining homotopy invariance is for homotopies of operators on Y_v, or operators on Y commuting with vv^*.

It should be clear by now that the mapping cone algebra provides a partial suspension, but mixes odd and even in a fascinating way. In the next section we relate the even index pairing for $M(F,A)$ to the odd index pairing described here.

4. APS Boundary Conditions and Kasparov Modules for the Mapping Cone

In this Section we begin the substantially new material by constructing an even Kasparov module for the mapping cone algebra $M(F,A)$ starting from an odd Kasparov F-module $(X, \mathcal{D})$ for A. In particular we are assuming that $\mathcal{D}$ is self-adjoint and regular on X, has discrete spectrum and the eigenspaces are closed F-submodules of X which sum to X. Our even module $\hat{X}$ is initially defined to be the direct sum of two copies of the C^*-module: $\mathcal{E} = L^2(\mathbf{R}_+) \otimes_{\mathbf{C}} X$ which is the completion of the algebraic tensor product in the tensor product C^*-module norm. That is, we take finite sums of elementary tensors which can naturally be regarded as functions $f : \mathbf{R}_+ \to X$. The inner product on such $f = \sum_i f_i \otimes x_i$, $g = \sum_j g_j \otimes y_j$ is defined to be

$$\langle f|g\rangle_{\mathcal{E}} = \sum_{i,j} \int_0^\infty \bar{f}_i(t) g_j(t) dt \, \langle x_i|y_j\rangle_X,$$

where we have written $\langle\cdot|\cdot\rangle_X$ for the inner product on X. Clearly the collection of all continuous compactly supported functions from $\mathbf{R}_+$ to X is naturally contained in the completion of this algebraic tensor product and for such functions f, g the inner product is given by:

$$\langle f|g\rangle_{\mathcal{E}} = \int_0^\infty \langle f(t)|g(t)\rangle_X dt.$$

The corresponding norm is

$$||f||_{\mathcal{E}} = ||\langle f|f\rangle_{\mathcal{E}}||^{1/2}.$$

Remarks. While many elements in the completion $\mathcal{E}$ can be realised as functions it may **not** be true that all of $\mathcal{E}$ consists of X-valued functions. We also note that the Banach space $L^2(\mathbf{R}_+, X)$ of functions f defined by square-integrability of $t \mapsto \|f(t)\|$ is **strictly** contained in $\mathcal{E}$. However, we shall show below that the domain of the operator $\partial_t \otimes 1$ on $\mathcal{E}$ (free boundary conditions) consists of X-valued functions which are square-integrable in the C^*-module sense above. We will define our operators using APS boundary conditions on the domains.

4.1. **Domains and APS boundary conditions.** Let P be the spectral projection for $\mathcal{D}$ corresponding to the nonnegative axis and let $T_\pm = \pm\partial_t \otimes 1 + 1 \otimes \mathcal{D}$ ($= \pm\partial_t + \mathcal{D}$ for brevity) with initial domain determined by Atiyah-Patodi-Singer type boundary conditions, namely

$$\mathrm{dom}\, T_\pm = \{f : \mathbf{R}_+ \to X_{\mathcal{D}} : \ f = \sum_{i=1}^n f_i \otimes x_i, \ f \text{ is smooth and compactly supported},$$

$$x_i \in X_{\mathcal{D}}, \ P(f(0)) = 0 \ (+ \text{ case}), \ (1 - P)(f(0)) = 0 \ (- \text{ case})\}. \tag{4}$$

By smooth we mean C^∞, using one-sided derivatives at $0 \in \mathbf{R}_+$. Then $T_\pm : \mathrm{dom}\, T_\pm \subset \mathcal{E} \to \mathcal{E}$. These are both densely defined, and so the operator

$$\hat{\mathcal{D}} = \begin{pmatrix} 0 & T_- \\ T_+ & 0 \end{pmatrix}$$

is densely defined on $\mathcal{E} \oplus \mathcal{E}$. An integration by parts (using the boundary conditions) shows that

$$(T_\pm f|g)_{\mathcal{E}} = (f|T_\mp g)_{\mathcal{E}}, \quad f \in \mathrm{dom}\, T_\pm, \ g \in \mathrm{dom}\, T_\mp.$$

Hence the adjoints are also densely defined, and so each of these operators is closable. This shows that $\hat{\mathcal{D}}$ is likewise closable, and symmetric.

The subtlety noted above, namely that the module $\mathcal{E}$ does not necessarily consist of functions, forces us to consider some seemingly circuitous arguments. Basically, to prove self-adjointness, we require knowledge about domains, and we must prove various properties of these domains without the benefit of a function representation of all elements of $\mathcal{E}$. However, we will prove below a function representation for elements in the natural domain of $\partial_t \otimes 1$, and therefore in the domains of the closures of $T_\pm$ because if $\{f_j\} \subset \mathrm{dom}\, T_\pm$ is a Cauchy sequence in the norm of $\mathcal{E}$ such that $\{T_\pm f_j\}$ is also Cauchy then as $T_\pm$ is closable, the limit f of the sequence f_j lies in the domain of the closure, and $\lim T_\pm f_j = \overline{T_\pm} f$.

Lemma 4.1. *For $f \in \mathrm{dom}\, T_\pm$, the initial domain, we have:*
(1) $\langle T_\pm f | T_\pm f\rangle = \langle (\partial_t \otimes 1)f | (\partial_t \otimes 1)f\rangle_{\mathcal{E}} + \langle (1\otimes \mathcal{D})f | (1 \otimes \mathcal{D})f\rangle_{\mathcal{E}} \mp \langle f(0) | \mathcal{D}(f(0))\rangle_X$, and
(2) $\mp\langle f(0)|\mathcal{D}(f(0))\rangle_X \geq 0$.

Proof. We do the case T_+; the proof for T_- is the same. With a little computation it suffices to see:

$$\langle (\partial_t \otimes 1)f | (1\otimes \mathcal{D})f\rangle_{\mathcal{E}} + \langle (1\otimes \mathcal{D})f | (\partial_t \otimes 1)f\rangle_{\mathcal{E}} = -\langle f(0)|\mathcal{D}(f(0))\rangle_X$$

for $f = \sum_i f_i \otimes x_i$ with f_i compactly supported and $f(0) \in \ker P$. Then, using integration by parts:

$$\begin{aligned} \langle (\partial_t \otimes 1)f | (1\otimes \mathcal{D})f\rangle_{\mathcal{E}} &= \sum_{i,j} \int_0^\infty (\frac{d}{dt}\overline{f_i(t)})(f_j(t))dt \cdot \langle x_i | \mathcal{D}x_j\rangle_X \\ &= -\sum_{i,j} \left\{ \overline{f_i(0)} f_j(0 + \int_0^\infty \overline{f_i(t)} \frac{d}{dt} f_j(t) dt \right\} \langle x_i|\mathcal{D}x_j\rangle_X \\ &= -\langle \sum_i f_i(0)x_i | \sum_j f_j(0)\mathcal{D}x_j\rangle_X - \langle \sum_i f_i \otimes x_i | \sum_j \partial_t f_j \otimes \mathcal{D}x_j\rangle_{\mathcal{E}} \\ &= -\langle f(0)|\mathcal{D}(f(0)\rangle_X - \langle f | (\partial_t \otimes \mathcal{D})f\rangle_{\mathcal{E}}. \end{aligned}$$

But, since $\mathcal{D}$ is self-adjoint and $1\otimes\mathcal{D}$ commutes with $\partial_t \otimes 1$ we have

$$\langle f | (\partial_t \otimes \mathcal{D})f\rangle_{\mathcal{E}} = \langle (1\otimes\mathcal{D})f | (\partial_t \otimes 1)f\rangle_{\mathcal{E}}$$

and item (1) follows. To see item (2), we have $(1-P)(f(0)) = f(0)$ where $(1-P) = \mathcal{X}_{(-\infty,0)}(\mathcal{D})$ so we see that $\mathcal{D}$ restricted to the range of $(1-P)$ is negative and therefore $-\langle f(0)|\mathcal{D}(f(0)\rangle_X \geq 0$ in our C^*-algebra. □

Corollary 4.2. *If $\{f_n\} \subseteq \mathrm{dom}\,(T_\pm)$ is a Cauchy sequence in the initial domain of $T_\pm$ and $\{T_\pm(f_n)\}$ is also a Cauchy sequence in $||\cdot||_{\mathcal{E}}$ norm then both $\{(\partial_t \otimes 1)(f_n)\}$ and $\{(1\otimes\mathcal{D})(f_n)\}$ are also Cauchy sequences in the $||\cdot||_{\mathcal{E}}$ norm. Therefore, the limit, f of $\{f_n\}$ in $\mathcal{E}$ which is in the domain of the closure of $T_\pm$, is also in the domain of the closures of both $(\partial_t \otimes 1)$ and $(1\otimes\mathcal{D})$.*

Proof. This follows from the lemma and the fact that if $A = B + C$ are all positive elements in a C^*-algebra, then $||A|| \geq ||B||$ and $||A|| \geq ||C||$. □

Lemma 4.3. (1) *If $g = \sum_i f_i \otimes x_i$ where the f_i are smooth and compactly supported then*

$$\langle (\partial_t \otimes 1)g|g\rangle_{\mathcal{E}} = -\langle g(0)|g(0)\rangle_X - \langle g|(\partial_t \otimes 1)g\rangle_{\mathcal{E}}.$$

(2) *With g as above*

$$||g(0)||_X^2 \leq 2||(\partial_t \otimes 1)g||_{\mathcal{E}} \cdot ||g||_{\mathcal{E}}.$$

Proof. Item (1) is an integration by parts similar to the previous computation and item (2) follows from item (1) by the triangle and Cauchy-Schwarz inequalities. □

4.2. Elements in $\mathrm{dom}(\partial_t \otimes 1)$ are functions.

Definition 4.4. *For each $t \in \mathbf{R}_+$, we define two shift operators S_t and T_t on $L^2(\mathbf{R}_+)$ via: $S_t(\xi)(s) = \xi(s+t)$ and $T_t = S_t^*$. Clearly both have norm 1 and $S_tT_t = 1$ and $T_tS_t = 1 - E_t$ where E_t is the projection, multiplication by $\mathcal{X}_{[0,t]}$. Hence, $S_t \otimes 1$, $T_t \otimes 1$, and $E_t \otimes 1$ are in $\mathcal{L}(\mathcal{E})$ and $E_t \otimes 1$ converges strongly to $1_{\mathcal{E}}$ as $t \to \infty$.*

Lemma 4.5. *Let $\partial_t \otimes 1$ denote the closed operator on $\mathcal{E}$ with free boundary condition at 0. That is, $\partial_t \otimes 1$ is the closure of $\partial_t \otimes 1$ defined on the initial domain $\mathrm{dom}'(\partial_t \otimes 1)$ consisting of finite sums of elementary tensors $f \otimes x$ where f is smooth and compactly supported. Then,*
(1) S_t leaves $\mathrm{dom}(\partial_t \otimes 1)$ invariant and commutes with $\partial_t \otimes 1$.
(2) If $g \in \mathrm{dom}'(\partial_t \otimes 1)$ then for each $t_0 \in \mathbf{R}_+$

$$||g(t_0)||_X^2 \leq 2||(\partial_t \otimes 1)g||_{\mathcal{E}}||g||_{\mathcal{E}}.$$

(3) If $g \in \mathrm{dom}(\partial_t \otimes 1)$ and $\{g_n\}$ is a sequence in $\mathrm{dom}'(\partial_t \otimes 1)$ with $g_n \to g$ in $\mathcal{E}$ and $(\partial_t \otimes 1)(g_n) \to (\partial_t \otimes 1)(g)$ in $\mathcal{E}$ then there is a continuous function $\hat{g} : \mathbf{R}_+ \to X$ so that $g_n \to \hat{g}$ uniformly on $\mathbf{R}_+$. Moreover $\hat{g} \in C_0(\mathbf{R}_+, X)$ and depends only on g, not on the particular sequence $\{g_n\}$.
(4) If $g \in \mathrm{dom}(\partial_t \otimes 1)$ and $\hat{g}$ is the function defined in item (3) then for all elements $h \in \mathcal{E}$ which are finite sums of elementary tensors of the form $f \otimes x$ where f is compactly supported and piecewise continuous we have:

$$(g|h)_{\mathcal{E}} = \int_0^\infty \langle \hat{g}(t)|h(t)\rangle_X dt.$$

(5) If $g \in \mathrm{dom}(\partial_t \otimes 1)$ then:

$$\langle g|g\rangle_{\mathcal{E}} = \lim_{M\to\infty} \int_0^M \langle \hat{g}(t)|\hat{g}(t)\rangle_X dt := \int_0^\infty \langle \hat{g}(t)|\hat{g}(t)\rangle_X dt.$$

Proof. To see item (1), one easily checks that $S_t \otimes 1$ leaves $\mathrm{dom}'(\partial_t \otimes 1)$ invariant and commutes with $\partial_t \otimes 1$ on this space. Since $\partial_t \otimes 1$ is the closure of its restriction to $\mathrm{dom}'(\partial_t \otimes 1)$ and $S_t \otimes 1$ is bounded the conclusion follows by an easy calculation.
To see item (2), we apply item (1) and the previous lemma:

$$\begin{aligned} ||g(t_0)||_X^2 &= ||(S_{t_0}g)(0)||_X^2 \leq 2||(\partial_t \otimes 1)S_{t_0}(g)||_{\mathcal{E}}||S_{t_0}(g)||_{\mathcal{E}} \\ &= 2||S_{t_0}(\partial_t \otimes 1)(g)||_{\mathcal{E}}||S_{t_0}(g)||_{\mathcal{E}} \\ &\leq 2||(\partial_t \otimes 1)(g)||_{\mathcal{E}}||g||_{\mathcal{E}}. \end{aligned}$$

To see item (3), apply item (2) to the sequence $\{(g_n - g_m)(t_0)\}$ to see that the sequence $\{g_n(t_0)\}$ in X is uniformly Cauchy for $t_0 \in \mathbf{R}_+$. Since we can intertwine two such sequences converging to g, we see that $\hat{g}$ is independent of the particular sequence. That $\hat{g}$ vanishes at ∞ follows immediately from the uniform convergence.
To see item (4), let $\{g_n\}$ be a sequence satisfying the conditions of item (3). Then for h supported on $[0, M]$ satisfying the conditions of item (4):

$$\begin{aligned} \langle g|h\rangle_{\mathcal{E}} &= \lim_{n\to\infty} \langle g_n|h\rangle_{\mathcal{E}} = \lim_{n\to\infty} \int_0^\infty \langle g_n(t)|h(t)\rangle_X dt \\ &= \lim_{n\to\infty} \int_0^M \langle g_n(t)|h(t)\rangle_X dt = \int_0^M \langle \hat{g}(t)|h(t)\rangle_X dt \\ &= \int_0^\infty \langle \hat{g}(t)|h(t)\rangle_X dt. \end{aligned}$$

To see item (5), fix $M > 0$ and use item (4):

$$\begin{aligned} \langle g|E_M(g)\rangle_{\mathcal{E}} &= \lim_{n\to\infty} \langle g|E_M(g_n)\rangle_{\mathcal{E}} = \lim_{n\to\infty} \int_0^\infty \langle \hat{g}(t)|E_M(g_n)(t)\rangle_X dt \\ &= \lim_{n\to\infty} \int_0^M \langle \hat{g}(t)|g_n(t)\rangle_X dt \\ &= \int_0^M \langle \hat{g}(t)|\hat{g}(t)\rangle_X dt. \end{aligned}$$

Taking the limit as $M \to \infty$ completes the proof. □

Corollary 4.6. (1) *If* $g \in \mathrm{dom}(\partial_t \otimes 1)^2$ *then* $(\partial_t \otimes 1)(g)$ *is also given by a continuous* X*-valued function as above.* (2) *If* $g \in \mathrm{dom}(\partial_t \otimes 1)^n$ *for all* $n \geq 1$ *then* $(\partial_t \otimes 1)^n(g)$ *is given by a continuous* X*-valued function for all* n.

Proposition 4.7. *(1) If* $g \in \mathrm{dom}\,(\overline{T_\pm})$ *the domain of the closure of* $T_\pm$ *on its initial domain then* $g \in \mathrm{dom}\,(\overline{\partial_t \otimes 1}) \cap \mathrm{dom}\,(\overline{1 \otimes \mathcal{D}})$*. Moreover,* $g(0)$ *is well-defined and* $P(g(0)) = 0$ *in the* T_+ *case while in the* T_- *case,* $(1 - P)(g(0)) = 0$*. Furthermore*

$$\overline{T_\pm} g = \pm \overline{(\partial_t \otimes 1)} g + \overline{(1 \otimes \mathcal{D})} g.$$

(2) If $g \in \mathrm{dom}\,(\overline{T_\pm})$ *as above, then* $g(0) \in \mathrm{dom}\,(|\mathcal{D}|^{1/2})$.

Proof. For the first item, by Corollary 4.2, $g \in \mathrm{dom}\,(\overline{\partial_t \otimes 1}) \cap \mathrm{dom}\,(\overline{1 \otimes \mathcal{D}})$. Then, by the previous Lemma $g(0)$ is defined. Since P is a bounded operator on X, $P(g(0)) = 0$ in the T_+ case and $(1 - P)(g(0)) = 0$ in the T_- case. To see item (2), we use part (2) of Lemma 4.1 to see that for $f \in \mathrm{dom}\,(T_\pm)$ we have:

$$\mp \langle f(0) \mid \mathcal{D}(f(0)) \rangle_X = \langle |\mathcal{D}|^{1/2}(f(0)) \mid |\mathcal{D}|^{1/2}(f(0)) \rangle_X.$$

If we apply this observation to $f = g_n - g_m$ where $\{g_n\}$ is a Cauchy sequence in $\mathrm{dom}\,(T_\pm)$ we get the conclusion of item (2). □

Remark. Note that evaluation at a point is continuous on $\mathrm{dom}(\partial_t \otimes 1)$ in the $\mathrm{dom}(\partial_t \otimes 1)$-norm, but **not** in the module norm.

4.3. **Self-adjointness of $\hat{\mathcal{D}}$ away from the kernel.** To show that $\hat{\mathcal{D}}$ is self-adjoint we will follow the basic strategy of [1] and display a parametrix which is (almost) an exact inverse. Note that we assume that $\mathcal{D}$ has discrete spectrum with eigenvalues r_k for $k \in \mathbf{Z}$ where the spectral projection of $\mathcal{D}$ corresponding to the eigenvalue r_k is denoted by Φ_k. We suppose that r_k is increasing with k and if $k > 0$ then $r_k > 0$, and conversely, so that the zero eigenvalue, if it exists, corresponds to the index $k = 0$. Moreover, the eigenspaces $X_k = \Phi_k(X)$ are F-bimodules which sum to X by hypothesis. We note that $X_0 = \Phi_0(X) = \ker\mathcal{D}$.

We observe that if f is any real-valued function defined (at least) on $\{r_k : k \in \mathbf{Z}\}$, the spectrum of $\mathcal{D}$, then $f(\mathcal{D})$ is the self-adjoint operator with domain:

$$\{x = \sum_k x_k \in X : \sum_k f(r_k) x_k \text{ converges in } X\},$$

and is defined on this domain by $f(\mathcal{D})x = \sum_k f(r_k)x_k$. The convergence condition on the domain is equivalent to $\sum_k |f(r_k)|^2 \langle x_k | x_k \rangle_X$ converges in F.

We further note that if $g : \mathbf{R}_+ \to X$ is continuous and compactly supported then for each $k \in \mathbf{Z}$, the function $g_k := \Phi_k \circ g : \mathbf{R}_+ \to X_k$ is continuous with $supp(g_k) \subseteq supp(g)$ and $g = \sum_k g_k$ converges in $\mathcal{E}$. Furthermore, if g is smooth then so is each g_k and $\partial_t(g_k) = (\partial_t(g))_k$ and by the previous sentence $\partial_t(g) = \partial_t(\sum_k g_k) = \sum_k \partial_t(g_k)$.

As both $\partial_t \otimes 1$ and $1 \otimes \mathcal{D}$ leave the subspaces $L^2(\mathbf{R}_+) \otimes X_k$ invariant, in order to construct parametrices Q_+ and Q_- for T_+ and T_- we can begin by considering homogeneous solutions f_k to the equation

$$T_{+,k} f_k = (\partial_t + r_k) f_k = g_k$$

where g_k is a smooth compactly supported function with values in X_k for each $k > 0$. Setting

$$f_k(t) = Q_{+,k}(g_k)(t) = \int_0^t e^{-r_k(t-s)} g_k(s)ds = \int_0^\infty H(t-s)e^{-r_k(t-s)}g_k(s)ds,$$

where $H = \mathcal{X}_{\mathbf{R}_+}$ (the characteristic function of $\mathbf{R}_+$) is the Heaviside function, we get a solution satisfying the boundary conditions, as the reader will readily confirm.

Observe that for these homogeneous solutions our parametrix is given by a convolution operator

$$f_k(t) = Q_{+,k}(g_k)(t) = (G_k * g_k)(t) := L_{G_k} g_k(t).$$

Here $G_k(s) = H(s)e^{-r_k s} \in L^1(\mathbf{R})$, and $\|G_k\|_1 = 1/r_k$. Since the operator norm of L_{G_k} on $L^2(\mathbf{R})$ is bounded by $\|G_k\|_1$, we have

$$\|Q_{+,k}\| = \|(L_{G_k} \otimes \Phi_k)\|_{End\mathcal{E}} \leq \|G_k\|_1 \leq 1/r_k.$$

For $k < 0$ we set

$$f_k(t) = Q_{+,k}(g_k)(t) = -\int_t^\infty e^{-r_k(t-s)}g_k(s)ds = -\int_{-\infty}^\infty \mathcal{X}_{(-\infty,0)}(t-s)e^{-r_k(t-s)}g_k(s)ds.$$

The verification that $T_+ f_k = g_k$ is again straightforward, and the solution is an L^2-function with values in $\Phi_k(X)$ since it is given by the convolution of an L^1 function and an L^2-function.

Later when we have defined $Q_{+,0}$ we will sum all the $Q_{+,k}$ to obtain the parametrix Q_+. At the moment we note that for a smooth compactly supported g we have:

$$\left[Q_+(1 \otimes (P - \Phi_0))(g)\right](t) := \left[\sum_{k>0} Q_{+,k}(1\otimes\Phi_k)(g)\right](t) = \left[\sum_{k>0} Q_{+,k}g_k\right](t) = \sum_{k>0}\int_0^t e^{-r_k(t-s)}g_k(s)ds.$$

If we formally interchange the sum and the integral we get the equation:

$$\left[Q_+(1 \otimes (P - \Phi_0))(g)\right](t) `=' \int_0^t \sum_{k>0} e^{-r_k(t-s)}\Phi_k(g)(s)ds = \int_0^t e^{-\mathcal{D}(t-s)}(P-\Phi_0)(g(s))ds.$$

It is not hard to see that this convolution on the right actually converges to the expression on the left in the norm of our module $L^2(\mathbf{R}_+) \otimes X$.

Similarly for the equation $T_{-,k} f_k = (-\partial_t + r_k) f_k = g_k$ we have the solutions

$$Q_{-,k}(g_k)(t) = \int_t^\infty e^{-r_k(s-t)}g_k(s)ds = \int_{-\infty}^\infty \mathcal{X}_{(-\infty,0)}(t-s)e^{r_k(t-s)}g_k(s)ds, \quad k > 0,$$

$$Q_{-,k}(g_k)(t) = -\int_0^t e^{r_k(t-s)}g_k(s)ds = -\int_{-\infty}^\infty H(t-s)e^{r_k(t-s)}g_k(s)ds, \quad k < 0.$$

Again this solution is given by a convolution, and in all cases $k \neq 0$ we get $\|Q_{\pm,k}(1\otimes\Phi_k)\| \leq 1/|r_k|$. We can get a similar operator convolution equation for $\sum_{k<0} Q_{+,k}g_k$.

Before proceeding we require a general lemma.

Lemma 4.8. *Let Y be a C^*-F-module and $Y_0 \subseteq Y$ a dense F-submodule. Let $T : Y_0 \to Y_0$ be closable as a module mapping on Y, with closure $\overline{T}$. Suppose there exists a bounded module mapping S on Y such that (1) $S(Y_0) \subset Y_0$, and (2) $ST = Id_{Y_0}$ and $TS|_{Y_0} = Id_{Y_0}$. Then S is one-to-one and $\overline{T} = S^{-1} : \mathrm{Image}\,(S) \to Y$, $\mathrm{dom}\,\overline{T} = \mathrm{Image}\,(S)$, $S \circ \overline{T} = Id_{\mathrm{dom}\,\overline{T}}$, and $\overline{T} \circ S = Id_Y$.*

Proof. This is essentially just a careful check of the definitions of the domains and closures in question. Let $y \in \mathrm{dom}\,(\overline{T})$ so there exists a sequence $\{y_n\} \subset Y_0$ converging to y and $Ty_n \to \overline{T}y$ also. Now, since S is bounded,

$$y_n = STy_n \to S(\overline{T}y) \text{ and } y_n \to y,$$

so $S(\overline{T}y) = y$ and $S \circ \overline{T} = Id_{\mathrm{dom}\,(\overline{T})}$. This also shows $\mathrm{dom}(\overline{T}) \subset \mathrm{Image}\,(S)$.

On the other hand, let $y = Sy' \in \mathrm{Image}\,(S)$. Then $y' = \lim z_n$, where $\{z_n\} \subset Y_0$, and so $y = Sy' = \lim Sz_n$. Since $S : Y_0 \to Y_0$, we see that $\{Sz_n\} \subset Y_0 \subset \mathrm{dom}\,(T)$, and so $z_n = TSz_n$ converges to $y' \in Y$. Hence $y \in \mathrm{dom}\,\overline{T}$ and $\overline{T}y = y'$. That is $\mathrm{Image}\,(S) \subset \mathrm{dom}\,(\overline{T})$, and so they are equal. Finally, $\overline{T}Sy' = \overline{T}y = y'$, and as $y' \in Y$ was arbitrary, $\overline{T}S = Id_Y$. Hence S is one-to-one, and $\overline{T} = S^{-1}$. □

Returning to the operators $T_\pm$ and $Q_\pm$ on the module $\mathcal{E} \ominus (1 \otimes \Phi_0)\mathcal{E}$, we have the following preliminary result. The proof is just a check of the hypotheses of the previous lemma.

Corollary 4.9. *For $k \neq 0$, let $\mathcal{E}_k = L^2(\mathbf{R}_+) \otimes X_k$ and $\mathcal{E}_{k,0} \subset \mathcal{E}_k$ be the algebraic tensor product of*

$$C_{00}^\infty(\mathbf{R}_+) := \{g \in C^\infty(\mathbf{R}_+) : g(0) = 0 \text{ and } supp(g) \text{ is compact}\}$$

with X_k. That is, $\mathcal{E}_{k,0} = C_{00}^\infty(\mathbf{R}_+) \odot X_k$. Then $T_{\pm,k}$, $Q_{\pm,k}$ map $\mathcal{E}_{k,0}$ to itself, and are mutual inverses there. Hence $\mathrm{dom}\,(\overline{T_{\pm,k}}) = \mathrm{Image}\,(Q_{\pm,k})$, $Q_{\pm,k}$ is one-to-one, and the operators $\overline{T_{\pm,k}}$ and $Q_{\pm,k}$ are mutually inverse (on appropriate subspaces).

We extend this result by another application of Lemma 4.8:

Corollary 4.10. *Let the algebraic direct sum of the $\mathcal{E}_{k,0}$ with $k \neq 0$ be denoted*

$$\mathcal{E}_{alg,0} := \sum_{alg,k\neq 0} \mathcal{E}_{k,0} = \sum_{alg,k\neq 0} C_{00}^\infty(\mathbf{R}_+) \odot X_k = C_{00}^\infty(\mathbf{R}_+) \odot \sum_{alg,k\neq 0} X_k$$

Define $Q_\pm$ on $\mathcal{E}_{alg,0}$ as the algebraic direct sum of the $Q_{\pm,k}$, and similarly for $T_\pm$. Then $Q_\pm$ extends to an operator on the completion, $\mathcal{E}_0$ where it is bounded and one-to-one. Moreover, $\overline{T_\pm} = Q_\pm^{-1}$: $\mathrm{Image}(Q_\pm) \to \mathcal{E}_0$ so that $Q_\pm \circ \overline{T_\pm} = Id_{\mathrm{dom}\overline{T_\pm}}$ and $\overline{T_\pm} \circ Q_\pm = Id_{\mathcal{E}_0}$. We observe that $\mathcal{E} = \mathcal{E}_0 \oplus (L^2(\mathbf{R}_+) \otimes X_0)$ as an internal orthogonal direct sum. That is, $\mathcal{E}_0^\perp = (L^2(\mathbf{R}_+) \otimes X_0)$.

4.4. **The adjoint on $L^2(\mathbf{R}_+) \otimes X_0$ and self-adjointness of $\hat{\mathcal{D}}$.** On $L^2(\mathbf{R}_+) \otimes X_0$ the operator $T_{+,0}$ becomes $\partial_t \otimes Id_{X_0}$ with boundary conditions $\xi(0) = 0$ while $T_{-,0} = -\partial_t \otimes Id_{X_0}$ with free boundary conditions, and it is well-known that these two operators are mutual adjoints, cf [12, page 116]. The parametrix $Q_{+,0}$ for $T_{+,0}$ is given by

$$Q_{+,0}(g)(t) = \int_0^t g(t)dt \text{ for } g \in range(\overline{T_{+,0}}),$$

while the parametrix $Q_{-,0}$ for $T_{-,0}$ is given by

$$Q_{-,0}(g)(t) = -\int_t^\infty g(t)dt \text{ for } g \in range(\overline{T_{-,0}}).$$

Of course, both $Q_{+,0}$ and $Q_{-,0}$ are unbounded operators and on $L^2(\mathbf{R}_+) \otimes X_0$ we have:

$$\overline{T_{\pm,0}}Q_{\pm,0} = Id_{range(\overline{T_{\pm,0}})} \text{ and } Q_{\pm,0}\overline{T_{\pm,0}} = Id_{dom(\overline{T_{\pm,0}})}.$$

Letting $Q_\pm$ denote the (closure of the) direct sum of all the $Q_{\pm,k}$ we get the parametrix for $\overline{T_\pm}$.

Proposition 4.11. *The adjoint of $\overline{T_\pm} : \mathrm{dom}\,(\overline{T_\pm}) \to \mathcal{E}$ is $\overline{T_\mp}$. Moreover,*

$$\overline{T_\pm}Q_\pm = Id_{range(\overline{T_\pm})} \text{ and } Q_\pm\overline{T_\pm} = Id_{dom(\overline{T_\pm})}.$$

Proof. In the following we write $T_\pm$ for the closure of $T_\pm$. We write $T_\pm = T_\pm(1\otimes\Phi_0)\oplus T_\pm(1_{\mathcal{E}}-(1\otimes\Phi_0))$ and observe from our last comments that $(T_\pm(1\otimes\Phi_0))^* = T_\mp(1\otimes\Phi_0)$.

Restricting to $(1_{\mathcal{E}}-(1\otimes\Phi_0))\mathcal{E} = \mathcal{E}_0$ we have $Q_\pm^* = Q_\mp$. To see this, recall that $Q_\pm$ is bounded, and so it suffices to check on the dense submodule $\mathcal{E}_{alg,0}$ of Corollary 4.10. For ξ, $\eta \in \mathcal{E}_{alg,0}$, there is ξ_0, $\eta_0 \in \mathcal{E}_{alg,0}$ such that $\xi = T_\pm\xi_0$ and $\eta = T_\mp\eta_0$ ($\xi_0 = Q_\pm\xi$ and similarly for η_0). Then

$$\begin{aligned}(Q_\pm\xi|\eta)_{\mathcal{E}} &= (Q_\pm(T_\pm\xi_0)|T_\mp\eta_0)_{\mathcal{E}} = (\xi_0|T_\mp\eta_0)_{\mathcal{E}}\\ &= (T_\pm\xi_0|\eta_0)_{\mathcal{E}} \quad \text{by symmetry}\\ &= (\xi|Q_\mp\eta)_{\mathcal{E}}.\end{aligned}$$

Hence $Q_\pm^* = Q_\mp$ on $(1_{\mathcal{E}}-(1\otimes\Phi_0))\mathcal{E} = \mathcal{E}_0$. In order to deduce from this a similar relation for the $T_\pm$ on $\mathcal{E}_0$ we need the following general considerations.

For a densely defined module map $T : \mathcal{E}_0 \to \mathcal{E}_0$ we have the relation between graphs

$$G(T^*) = [\nu(G(T))]^\perp = \nu[G(T)^\perp],$$

where $\nu : \mathcal{E}_0\oplus\mathcal{E}_0 \to \mathcal{E}_0\oplus\mathcal{E}_0$ is the unitary given by $\nu(x,y) = (y,-x)$, [12, page 95]. Also for one-to-one module maps Q, $G(Q^{-1}) = \theta(G(Q))$ where $\theta(x,y) = (y,x)$ and $\theta\nu = -\nu\theta$. So restricting $T_\pm$ to $\mathcal{E}_0$ we calculate:

$$\begin{aligned}G(T_+^*) &= [\nu(G(T_+))]^\perp = [\nu(G(Q_+^{-1}))]^\perp\\ &= [\nu(\theta(G(Q_+)))]^\perp = -[\theta(\nu(G(Q_+)))]^\perp = -\theta[\nu(G(Q_+))^\perp]\\ &= -\theta[G(Q_+^*)] = -\theta[G(Q_-)] = -[G(Q_-^{-1})] = -[G(T_-)]\\ &= G(T_-).\end{aligned}$$

The same proof works for T_-, and so $T_\pm^* = T_\mp$ on all of $\mathcal{E}$. □

The next step is to introduced the notion of **extended solutions**. In [1], the analogue of our module was introduced as a model of a (product) neighbourhood of the boundary for a manifold-with-boundary. Since the interest there, as here, was in the index of the operator on the whole manifold-with-boundary, it was necessary to modify the space of solutions considered to account for those functions on the boundary which extended to interior solutions in a non-trivial way. Such functions are not L^2 on this product description of the boundary, but are bounded. Nevertheless they contribute to the index, and so we make a definition.

Definition 4.12. *Let $(X,\mathcal{D})$ be an unbounded odd Kasparov $A-F$-module. Let $\mathcal{E} = L^2(\mathbf{R}_+)\otimes X$ be the $M(F,A)-F$-module defined above. As seen in Lemma 4.5, any element in the domain of the operator $\partial_t\otimes 1$ (free boundary conditions) is given by a uniformly continuous X-valued function g which vanishes at ∞ and the integral $\langle g|g\rangle_{\mathcal{E}} = \int_0^\infty \langle g(t)|g(t)\rangle_X dt$ converges in F^+. We enlarge $\mathcal{E}$ to a space $\hat{\mathcal{E}}$ consisting of formal sums, $f = g+x$ where $g\in\mathcal{E}$ and $x \in X_0$. For $g \in \mathrm{dom}(\partial_t\otimes 1)$, the element $f = g+x$ is naturally a function on $\mathbf{R}_+$ where $f(t) = g(t)+x$ and $\lim_{t\to\infty} f(t) = x \in X_0$. We call such an f an* **extended L^2-function** *and we may regard f as a function $f : \mathbf{R}_+ \to X$ with a limit: $\lim_{t\to\infty} f(t) := f(\infty)$ such that $f-f(\infty)$ is in $L^2(\mathbf{R}_+)\otimes X$ and $f(\infty) \in X_0$, that is, $\mathcal{D}f(\infty) = 0$. Note we reserve the terms* **extended L^2-function** *and* **extended solution** *to the case where $f(\infty) \neq 0$.*

So, we have a new module $\hat{\mathcal{E}} = \{f = g+x \mid g\in\mathcal{E} \text{ and } x\in X_0\}$. We let F act on the left and right of this extra copy of X_0 by its natural action. The F-valued inner product on $\hat{\mathcal{E}}$ is given by:

$$\langle f+x|h+y\rangle = \langle f(t)|h(t)\rangle_{\mathcal{E}} + \langle x|y\rangle_X.$$

The left action of $M(F,A)$ on the extra component X_0 is naturally defined to be **zero** since $M(F.A)$ consists of functions which vanish at ∞. However, when we extend the left action to the unitization of $M(F,A)$ the added identity will of course act as the identity on the extra copy of X_0. While $\mathcal{D}$ naturally acts as **zero** on this extra copy of X_0, functions $f(\mathcal{D})$ act as multiplication by $f(0)$ so that in particular, P acts as the identity operator on this copy of X_0 and the operator, ∂_t naturally extends here as the **zero** operator.

We now modify our earlier definition of $\hat{X}$ to include $\hat{\mathcal{E}}$ **only** in the second component. Hence, by definition:

$$\hat{X} = \begin{pmatrix} \mathcal{E} \\ \hat{\mathcal{E}} \end{pmatrix}.$$

For the first component any solution (i.e. element of the kernel of T_+) necessarily vanishes on the boundary, and classically cannot contribute to the index and the same situation persists in this non-commutative setting.

We extend the action of T_- to a map: $\hat{\mathcal{E}} \to \mathcal{E}$ via $T_-(f+x) = T_-(f)$. Similarly we extend the action of T_+ to a map: $\mathcal{E} \to \hat{\mathcal{E}}$ via $T_+(f) = T_+(f) + 0$ and we extend the definitions of the actions of Q_+ and Q_-. In order to emphasize the extension of T_- we use the somewhat clumsy notation:

$$\hat{\mathcal{D}} = \begin{pmatrix} 0 & T_- \oplus 0 \\ T_+ & 0 \end{pmatrix}.$$

The addition of the zero map does not affect the adjointness properties proved above, and so

$$(T_- \oplus 0)^* = T_+ \quad \text{and} \quad T_+^* = T_- \oplus 0.$$

Thus $\hat{\mathcal{D}}$ is self-adjoint. We summarise this lengthy discussion.

Proposition 4.13. *Let X be a right C^*-F-module, and $\mathcal{D} : \mathrm{dom}\mathcal{D} \subset X \to X$ be a self-adjoint regular operator with discrete spectrum. Then the operator*

$$\hat{\mathcal{D}} = \begin{pmatrix} 0 & (-\partial_t \otimes 1 + 1 \otimes \mathcal{D}) \oplus 0 \\ \partial_t \otimes 1 + 1 \otimes \mathcal{D} & 0 \end{pmatrix} \text{ defined on } \begin{pmatrix} \mathcal{E} \\ \hat{\mathcal{E}} \end{pmatrix}$$

satisfying APS boundary conditions, given in Equation (4) *and modified for extended solutions as above, is self-adjoint and regular on $\hat{X} = (\mathcal{E} \oplus \hat{\mathcal{E}})^T$.*

Proof. It remains only to show that $\hat{\mathcal{D}}$ is regular, namely $(1 + \hat{\mathcal{D}}^2)$ has dense range. We begin with $\hat{\mathcal{D}}$ restricted to $(\mathcal{E} \oplus \mathcal{E})^T$. We restrict ourselves further to the invariant subspace $(\mathcal{E}_0 \oplus \mathcal{E}_0)^T$. To this end let $R = Q_+Q_-$. This is a bounded, positive endomorphism on $\mathcal{E}_0$ which is injective and has dense range (both Q_+, Q_- are injective with dense range, and are mutual adjoints by Proposition 4.11). Hence the (unbounded) densely defined operator $R^{-1} = (Q_+Q_-)^{-1} = Q_-^{-1}Q_+^{-1} = T_-T_+$ on $\mathcal{E}_0$ is a one-to-one positive operator which is onto. As the operator $R+1$ is bounded, positive and (boundedly) invertible, it is surjective. Thus on $\mathrm{dom}\,(T_-T_+)$ consider the operator

$$(R+1)R^{-1} = 1 + R^{-1} = 1 + T_-T_+.$$

This is the composition of two surjective operators and so is surjective (on $\mathcal{E}_0$). Similar comments apply to $1 + T_+T_-$ (on $\mathcal{E}_0$). Thus $(1+\hat{\mathcal{D}}^2)$ restricted to (its domain in) $(\mathcal{E}_0 \oplus \mathcal{E}_0)^T$ maps onto $(\mathcal{E}_0 \oplus \mathcal{E}_0)^T$.

Next, inside $\mathcal{E}$, we have $\mathcal{E}_0^\perp = L^2(\mathbf{R}_+) \otimes X_0$ and $\hat{\mathcal{D}}$ on $(\mathcal{E}_0^\perp \oplus \mathcal{E}_0^\perp)^T$ is just $\begin{pmatrix} 0 & -\partial_t \\ \partial_t & 0 \end{pmatrix} \otimes 1_{X_0}$. As regularity is automatic on $(L^2(\mathbf{R}_+) \oplus L^2(\mathbf{R}_+))^T$, we have regularity on all of $(\mathcal{E} \oplus \mathcal{E})^T$. Now, on $X_0 \hookrightarrow \hat{\mathcal{E}}$, $\hat{\mathcal{D}}$ is defined as zero, so $(1 + \hat{\mathcal{D}}^2)|_{X_0} = 1_{X_0}$, which is surjective. Putting the pieces together, $1 + \hat{\mathcal{D}}^2$ is surjective on $\hat{X}$. $\square$

For use in the next proposition, we consider a more explicit discussion of regularity. So we consider the equation

$$\begin{pmatrix} 1+T_-T_+ & 0 \\ 0 & 1+T_+T_- \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1-\partial_t^2+\mathcal{D}^2 & 0 \\ 0 & 1-\partial_t^2+\mathcal{D}^2 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Here we initially suppose each of $(g_1,g_2)^T$ is in $C_{00}^\infty(\mathbf{R}_+)\odot\sum_{alg}X_k$. With the exception of the extra kernel term, such pairs are dense in $\hat{X}$. We need to find $f=(f_1,f_2)^T$ in the domain of $\hat{\mathcal{D}}^2$ satisfying this equation. In solving this equation we may therefore assume that all terms are homogeneous, meaning that the general solution is built from functions that map $\mathbf{R}_+$ to a single eigenspace for $\mathcal{D}$, corresponding to the eigenvalue r_k. Thus the equation we must solve, for given $(g_1,g_2)^T\in\hat{X}$, is

$$\begin{pmatrix} 1-\partial_t^2+r_k^2 & 0 \\ 0 & 1-\partial_t^2+r_k^2 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

The boundary conditions are

$$r_k\geq 0\quad \begin{cases} f_1(0)=0 \\ ((-\partial_t+r_k)f_2)(0)=0 \end{cases},\qquad r_k<0\quad \begin{cases} f_2(0)=0 \\ ((\partial_t+r_k)f_1)(0)=0 \end{cases}$$

We use the notation $\widehat{r_k}:=(1+r_k^2)^{1/2}$ as this term appears so often. The solution for f_1 is

$$f_1(t)=(2\widehat{r_k})^{-1}\left(\int_t^\infty e^{\widehat{r_k}(t-w)}g_1(w)dw+\int_0^t e^{-\widehat{r_k}(t-w)}g_1(w)dw\right)+Ae^{-\widehat{r_k}t},$$

where for

$$r_k\geq 0,\quad A=\frac{-1}{2\widehat{r_k}}\int_0^\infty e^{-w\widehat{r_k}}g_1(w)dw,\ \text{ and for } r_k<0,\quad A=\frac{1}{2\widehat{r_k}}\frac{\widehat{r_k}+r_k}{\widehat{r_k}-r_k}\int_0^\infty e^{-w\widehat{r_k}}g_1(w)dw.$$

Observe that in terms of the Heaviside function H:

$$f_1(t)=\frac{1}{2\widehat{r_k}}\left(\int_{-\infty}^\infty H^\perp(t-w)e^{\widehat{r_k}(t-w)}g_1(w)dw\right.$$
$$\left.+\int_{-\infty}^\infty H(t-w)e^{\widehat{r_k}(t-w)}g_1(w)dw+\begin{cases} -\langle e^{-\widehat{r_k}\cdot},g_1(\cdot)\rangle e^{-\widehat{r_k}t} & r_k\geq 0 \\ +\frac{\widehat{r_k}+r_k}{\widehat{r_k}-r_k}\langle e^{-\widehat{r_k}\cdot},g_1(\cdot)\rangle e^{-\widehat{r_k}t} & r_k<0 \end{cases}\right).$$

The point of this observation is that it displays the integral as a convolution by an L^1-function, plus a rank one operator, namely a multiple of the projection onto span$\{e^{-\widehat{r_k}t}\}$. Thus f_1 is an L^2-function.

For f_2 the situation is analogous. We have

$$f_2(t)=(2\widehat{r_k})^{-1}\left(\int_t^\infty e^{\widehat{r_k}(t-w)}g_2(w)dw+\int_0^t e^{-\widehat{r_k}(t-w)}g_2(w)dw\right)+Be^{-\widehat{r_k}t},$$

where for

$$r_k<0,\quad B=\frac{1}{2\widehat{r_k}}\int_0^\infty e^{-w\widehat{r_k}}g_2(w)dw,\ \text{ and for } r_k\geq 0,\quad B=\frac{1}{2\widehat{r_k}}\frac{\widehat{r_k}-r_k}{\widehat{r_k}+r_k}\int_0^\infty e^{-w\widehat{r_k}}g_2(w)dw.$$

Now we consider elements of $\hat{X}$ which only have a nonzero component in X_0. For such elements $(0,0+x)^T$ we have

$$(1-\partial_t^2+\mathcal{D}^2)x=(1-0+0)x=x,$$

so we have surjectivity for such elements. Now write a general $g=(g_1,g_2+x)^T\in\hat{X}$ as

$$g=\begin{pmatrix} g_1 \\ g_2+0 \end{pmatrix}+\begin{pmatrix} 0 \\ 0+x \end{pmatrix}.$$

Then the above solutions show that for any g in a dense subspace of $\hat{X}$, we can find $f\in\operatorname{dom}\hat{\mathcal{D}}^2$ with $(1+\hat{\mathcal{D}}^2)f=g$. Hence, we have a second proof that $\hat{\mathcal{D}}$ is regular which we now exploit.

In the next result APS boundary conditions mean that $\hat{\mathcal{D}}$ is defined on those $\xi = (\xi_1 \oplus \xi_2)^T$ in $(\mathcal{E} \oplus \hat{\mathcal{E}})^T$ such that $\hat{\mathcal{D}}\xi \in \hat{X}$, $P\xi_1(0) = 0$, $(1-P)\xi_2(0) = 0$. This is all well defined thanks to Lemma 4.5.

Proposition 4.14. *Let $(X, \mathcal{D})$ be an ungraded unbounded Kasparov module for C^*-algebras A, F with $F \subset A$ a subalgebra satisfying $\overline{A \cdot F} = A$. Suppose that $\mathcal{D}$ also commutes with the left action of $F \subset A$, and that $\mathcal{D}$ has discrete spectrum. Then there is an unbounded graded Kasparov module*

$$(\hat{X}, \hat{\mathcal{D}}) = \left(\begin{pmatrix} \mathcal{E} \\ \hat{\mathcal{E}} \end{pmatrix}, \begin{pmatrix} 0 & T_- \\ T_+ & 0 \end{pmatrix} \right) = \left(\begin{pmatrix} L^2(\mathbf{R}_+) \otimes X \\ \widehat{L^2(\mathbf{R}_+) \otimes X} \end{pmatrix}, \begin{pmatrix} 0 & -\partial_t + \mathcal{D} \\ \partial_t + \mathcal{D} & 0 \end{pmatrix} \right)$$

(with APS boundary conditions, Equation (4)) for the mapping cone algebra $M(F, A)$.

Proof. The most important observation is that the left action of $M(F, A)$ on $\hat{X}$ preserves the APS boundary condition, and therefore the domain of $\hat{\mathcal{D}}$ because for every $f \in M(F, A)$, $f(0) \in F$ and hence commutes with the spectral projections defining the boundary conditions. We note that to see that the action of $M(F, A)$ on $\hat{X}$ is by **bounded** module maps requires the strong boundedness property of all adjointable mappings [12] Proposition 1.2. We let $\mathcal{A} \subset A$ be the $*$-subalgebra of A such that for all $a \in \mathcal{A}$, $[\mathcal{D}, a]$ is bounded (on X) and $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism of X. We define the algebra

$$\mathcal{M}(F, \mathcal{A}) = \{f : \mathbf{R}_+ \to \mathcal{A} : \ f(0) \in F \text{ and } f \in C_0^\infty(\mathbf{R}_+) \ \text{ and } \ [\hat{\mathcal{D}}, f] \ \text{ is bounded}\}.$$

We observe that the *-algebra of finite sums:

$$\{\sum_i f_i \otimes a_i : f_i \in C^\infty(\mathbf{R}_+) \ \text{ and } \ f_i(0) = 0 \ \text{ if } \ a_i \notin F\}$$

is dense in $M(F, A)$ and is a *-subalgebra of $\mathcal{M}(F, \mathcal{A})$.

By Proposition 4.13, the operator $\hat{\mathcal{D}}$ is regular and self-adjoint, so we may employ the continuous functional calculus [12], to prove that $f(1+\hat{\mathcal{D}}^2)^{-1/2}$ is a compact endomorphism. It suffices to show that $f(1+\hat{\mathcal{D}}^2)^{-1}$ is compact. To see this, observe that $f(1+\hat{\mathcal{D}}^2)^{-1/2}$ is compact if and only if

$$f(1+\hat{\mathcal{D}}^2)^{-1} f^* = f(1+\hat{\mathcal{D}}^2)^{-1/2}(1+\hat{\mathcal{D}}^2)^{-1/2} f^*$$

is compact and this follows if $f(1+\hat{\mathcal{D}}^2)^{-1}$ is compact. The latter follows by observing that from our second proof of Proposition 4.13 we have that each diagonal entry of

$$f(1+\hat{\mathcal{D}}^2)^{-1} \begin{pmatrix} (1 \otimes \Phi_k) & 0 \\ 0 & (1 \otimes \Phi_k) \end{pmatrix} := f(1+\hat{\mathcal{D}}^2)^{-1}((1 \otimes \Phi_k) \otimes 1_2)$$

can be expressed as a finite sum of terms of the form $f(L_{g_k} \otimes \Phi_k) + f(R_k \otimes \Phi_k)$ where L_{g_k} is convolution by an L^1-function and R_k is a rank one operator. We consider a single elementary tensor in the above subalgebra of $\mathcal{M}(F, \mathcal{A})$: $f = h \otimes a$, where $a = a_1 \cdot b$, where $b \in F$ and $a_1 \in A$. For such an elementary tensor the diagonal entry is $(h \cdot L_{g_k} + h \cdot R_k) \otimes a_1 \cdot b\Phi_k$. Since g_k is in $\mathcal{L}^1$, the product $h \cdot L_{g_k}$ is a compact operator on $L^2(\mathbf{R}_+)$, and of course hR_k is compact. Since $b(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism on X, it is straightforward to check that $b\Phi_k$ is a compact endomorphism. So as $End_F^0(L^2(\mathbf{R}_+) \otimes X) = End_{\mathbf{C}}^0(L^2(\mathbf{R}_+)) \otimes End_F^0(X)$, [18][Corollary 3.38], the endomorphism

$$B_k := f((L_{g_k} + R_k) \otimes \Phi_k) = (1 \otimes a_1)(h(L_{g_k} + R_k) \otimes b\Phi_k) = (1 \otimes a_1) C_k$$

is compact: indeed each C_k is compact on $L^2(\mathbf{R}_+) \otimes X_k$. The importance of this description is that $f(1+\hat{\mathcal{D}}^2)^{-1} = (1 \otimes a_1)(\oplus_k C_k)$ is a *direct sum* of compacts on $\oplus_k(L^2(\mathbf{R}_+) \otimes X_k)$ times the bounded operator $(1 \otimes a_1)$.

The operator norm of L_{g_k} on $L^2(\mathbf{R}_+)$ is bounded by the L^1-norm of g_k, and so

$$\|L_{g_k}\|_{op} \leq \|g_k\|_1 = (1+r_k^2)^{-1/2}.$$

The norm of the rank one operator R_k on $L^2(\mathbf{R}_+)$ is given by Cauchy-Schwarz as

$$\|R_k\|_{op} \leq (2(1+r_k^2))^{-1}.$$

(This inequality is unaffected by multiplication by $(\widehat{r_k}+|r_k|)/(\widehat{r_k}-|r_k|)$, so can be applied to both $r_k<0$ and $r_k \geq 0$). Hence

$$\begin{aligned}\|C_k\|_{op} &\leq \|h\|_{op}\|L_{g_k}\|_{op}\|b\|_{op} + \|h\|_{op}\|R_k\|_{op}\|b\|_{op} \\ &\leq \|h\|_{op}\|b\|_{op}((1+r_k^2)^{-1/2} + (2(1+r_k^2))^{-1}).\end{aligned}$$

Since $1+r_k^2 \to \infty$ as $|k| \to \infty$, the sequence of compact endomorphisms $\{(1\otimes a_1)\sum_{-N}^{N} C_k\}$ converges in norm to $f(1+\hat{\mathcal{D}}^2)^{-1}$, which is therefore compact. Since an arbitrary $f \in \mathcal{M}(F,\mathcal{A})$ is the norm limit of finite sums $\sum f_j \otimes a_j$ we see that $f(1+\hat{\mathcal{D}}^2)^{-1}$ is compact for general f in the mapping cone algebra.

We can now show that we do indeed obtain a Kasparov module. First $V = \hat{\mathcal{D}}(1+\hat{\mathcal{D}}^2)^{-1/2}$ is self-adjoint. Also $f(1-V^2) = f(1+\hat{\mathcal{D}}^2)^{-1}$ is a compact endomorphism for $f \in \mathcal{M}(F,\mathcal{A})$. Since V clearly anticommutes with the grading operator $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we need only show that $[V,f]$ is compact for all $f \in M(F,A)$. For f a sum of elementary tensors (using smooth functions), we may write this commutator as

$$[V,f] = [\hat{\mathcal{D}},f](1+\hat{\mathcal{D}}^2)^{-1/2} + \hat{\mathcal{D}}[(1+\hat{\mathcal{D}}^2)^{-1/2},f]$$

Now for an elementary tensor $f\otimes a$, we get $[\hat{\mathcal{D}}, f\otimes a] = \partial f \otimes a + f\otimes[\mathcal{D},a]$ and so the first term in the above equation is compact. In the proof of Proposition 2.4 of [6] we have the formula:

$$\begin{aligned}&\hat{\mathcal{D}}[(1+\hat{\mathcal{D}}^2)^{-1/2},f] \\ =\ & \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\{\hat{\mathcal{D}}(1+\hat{\mathcal{D}}^2+\lambda)^{-1/2}\left\{(1+\hat{\mathcal{D}}^2+\lambda)^{-1/2}[f,\hat{\mathcal{D}}](1+\hat{\mathcal{D}}^2+\lambda)^{-1/2}\right\}\hat{\mathcal{D}}(1+\hat{\mathcal{D}}^2+\lambda)^{-1/2} \\ +\ & \hat{\mathcal{D}}^2(1+\hat{\mathcal{D}}^2+\lambda)^{-1}[f,\hat{\mathcal{D}}](1+\hat{\mathcal{D}}^2+\lambda)^{-1}\}d\lambda.\end{aligned}$$

where the integral converges in operator norm and we have grouped the terms in the integrand so that they are clearly compact by the discussion above. It follows that $[V,f]$ is a compact endomorphism for f a sum of elementary tensors. Since these are norm dense in $M(F,A)$ and V is bounded, $[V,f]$ is compact for all $f \in M(F,A)$. So we have an even Kasparov module for $(M(F,A),F)$ with an unbounded representative for $(\mathcal{M}(F,\mathcal{A}),F)$. □

Remark. It should be noted that in this context, discreteness of the spectrum of $\mathcal{D}$ does NOT imply that $(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism. We are assuming that we have a Kasparov module, so that for all $a \in A$ $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism, but these two compactness conditions are not equivalent unless A is unital. Kasparov modules corresponding to infinite graphs provide examples of this phenomenon, [15].

5. Equality of the index pairings from the Kasparov modules.

We formulate our main theorem in this Section demonstrating how even and odd Kasparov modules give equal index pairings.

We recall that given a partial isometry $v \in A$ with range and source projections in F (observe this includes unitaries in A), we defined $v_1 = \begin{pmatrix} 1 - vv^* & v \\ v^* & 1 - v^*v \end{pmatrix}$. This is a self-adjoint unitary in $M_2(\tilde{A})$, and hence there exists a norm continuous path of self-adjoint unitaries in $M_2(\tilde{A})$ from v_1 to the identity. We choose the path

$$v_1(t) = \frac{1}{2}(e^{2i\tan^{-1}(t)}(v_1 - 1_2) + (v_1 + 1_2)),$$

so that $v_1(0) = v_1$ and $v_1(\infty) = 1_2$. Now define a projection $e_v(t)$ over $\tilde{M}(F, A)$ by

$$e_v(t) = v_1(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_1(t)^* = \begin{pmatrix} 1 - \frac{1}{1+t^2} vv^* & \frac{-it}{1+t^2} v \\ \frac{it}{1+t^2} v^* & \frac{1}{1+t^2} v^*v \end{pmatrix},$$

where we have used some elementary trigonometry to simplify the expressions. It is important to observe that this is a finite sum of elementary tensors $\sum f_j \otimes a_j$ with f_j smooth and square integrable or $f_j - f_j(\infty)$ smooth and square integrable. As such it maps $(\hat{\mathcal{E}} \oplus \hat{\mathcal{E}})^T$ to itself and leaves $(\mathcal{E} \oplus \mathcal{E})^T$ invariant.

The difference of classes

$$[e_v(t)] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

lies in $K_0(M(F, A))$: see Lemma 3.2 and the discussion preceding it, as well as [17]. Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, a constant function, then the index pairing of $[v] \in K_0(M(F, A))$ with $[(\hat{X}, \hat{\mathcal{D}})]$ is

$$\langle [e_v] - [e], [(\hat{X}, \hat{\mathcal{D}})] \rangle := \text{Index}(e_v(\hat{\mathcal{D}} \otimes 1_2)e_v) - \text{Index}(e(\hat{\mathcal{D}} \otimes 1_2)e) \in K_0(F).$$

Remarks. To explain this notation we review **even** index theory. On $\begin{pmatrix} \mathcal{E} \\ \hat{\mathcal{E}} \end{pmatrix}$, $\hat{\mathcal{D}} = \begin{pmatrix} 0 & T_- \\ T_+ & 0 \end{pmatrix}$ while the grading operator $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. That is $\hat{\mathcal{D}}$ is **odd** while the action of $M(F, A)$ is **even**, i.e., diagonal. Then, on $\begin{pmatrix} \mathcal{E} \otimes \mathbf{C}^2 \\ \hat{\mathcal{E}} \otimes \mathbf{C}^2 \end{pmatrix}$ we have: $\hat{\mathcal{D}} \otimes 1_2 = \begin{pmatrix} \hat{\mathcal{D}} & 0 \\ 0 & \hat{\mathcal{D}} \end{pmatrix}$ and $\Gamma \otimes 1_2 = \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix}$ while $e_v = \begin{pmatrix} f & g \\ h & k \end{pmatrix} \in M_2(M(F, A))$ acts as $\begin{pmatrix} f \otimes 1_2 & g \otimes 1_2 \\ h \otimes 1_2 & k \otimes 1_2 \end{pmatrix}$. Let $([\mathcal{E} \oplus \hat{\mathcal{E}}] \oplus [\mathcal{E} \oplus \hat{\mathcal{E}}])^T \cong ([\mathcal{E} \oplus \mathcal{E}] \oplus [\hat{\mathcal{E}} \oplus \hat{\mathcal{E}}])^T$ be the obvious unitary equivalence. Under this equivalence $\hat{\mathcal{D}} \otimes 1_2$ becomes $\begin{pmatrix} 0 & T_- \otimes 1_2 \\ T_+ \otimes 1_2 & 0 \end{pmatrix}$, while $e_v = \begin{pmatrix} f & g \\ h & k \end{pmatrix} \in M_2(M(F, A))$ acts as $\begin{pmatrix} e_v & 0 \\ 0 & e_v \end{pmatrix}$. Also, $\text{Index}(e_v(\hat{\mathcal{D}} \otimes 1_2)e_v)$ really means the index of the lower corner operator of $\begin{pmatrix} e_v & 0 \\ 0 & e_v \end{pmatrix} (\hat{\mathcal{D}} \otimes 1_2) \begin{pmatrix} e_v & 0 \\ 0 & e_v \end{pmatrix} = \begin{pmatrix} 0 & e_v(T_- \otimes 1_2)e_v \\ e_v(T_+ \otimes 1_2)e_v & 0 \end{pmatrix}$:

$$e_v \begin{pmatrix} T_+ & 0 \\ 0 & T_+ \end{pmatrix} e_v : \text{ as a mapping } e_v \begin{pmatrix} \mathcal{E} \\ \mathcal{E} \end{pmatrix} \to e_v \begin{pmatrix} \hat{\mathcal{E}} \\ \hat{\mathcal{E}} \end{pmatrix}.$$

That is we must compute both:

$$\ker(e_v(T_+ \otimes 1_2)e_v) \subseteq e_v \begin{pmatrix} \mathcal{E} \\ \mathcal{E} \end{pmatrix} \quad \text{and} \quad \ker(e_v(T_- \otimes 1_2)e_v) \subseteq e_v \begin{pmatrix} \hat{\mathcal{E}} \\ \hat{\mathcal{E}} \end{pmatrix} \subseteq \begin{pmatrix} \hat{\mathcal{E}} \\ \mathcal{E} \end{pmatrix}.$$

Similarly, $\text{Index}(e(\hat{\mathcal{D}} \otimes 1_2)e)$ means the index of the lower corner operator: $e \begin{pmatrix} T_+ & 0 \\ 0 & T_+ \end{pmatrix} e$, that is, T_+ as a mapping from $\mathcal{E} \to \hat{\mathcal{E}}$, which we will write as $\text{Index}(\hat{\mathcal{D}})$. With this reminder, and the convention that if T is an operator on the module Y, we write T_k for $T\otimes 1_k$ on the module $Y\otimes \mathbf{C}^k$, we now state our key result.

Theorem 5.1. *Let $(X,\mathcal{D})$ be an ungraded unbounded Kasparov module for the (pre-) C^*-algebras $\mathcal{A}\subset A, F$ with $F\subset A$ a subalgebra satisfying $\overline{A\cdot F}=A$. Suppose that $\mathcal{D}$ also commutes with the left action of $F\subset A$, and that $\mathcal{D}$ has discrete spectrum. Let $(\hat{X},\hat{\mathcal{D}})$ be the unbounded Kasparov $M(F,A),F$ module of Proposition 4.14. Then for any unitary $u\in M_k(A)$ such that P_k and $(\Phi_0)_k$ both commute with $u\mathcal{D}_ku^*$ and $u^*\mathcal{D}_ku$ we have the following equality of index pairings with values in $K_0(F)$:*

$$\begin{aligned}\langle[u],[(X,\mathcal{D})]\rangle &:= \mathrm{Index}(P_ku^*P_k)=\mathrm{Index}(e_u(\hat{\mathcal{D}}_k\otimes 1_2)e_u)-\mathrm{Index}(\hat{\mathcal{D}}_k)\\ &=: \langle[e_u]-\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right],[(\hat{X},\hat{\mathcal{D}})]\rangle\in K_0(F).\end{aligned}$$

Moreover, if v is a partial isometry, $v\in M_k(\mathcal{A})$, with $vv^*,v^*v\in M_k(F)$ and such that P_k and $(\Phi_0)_k$ both commute with $v\mathcal{D}_kv^*$ and $v^*\mathcal{D}_kv$ we have

$$\begin{aligned}\langle[e_v]-\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right],[(\hat{X},\hat{\mathcal{D}})]\rangle &= -\mathrm{Index}(PvP:v^*vP(X)\to vv^*P(X))\in K_0(F)\\ &= \mathrm{Index}(Pv^*P:vv^*P(X)\to v^*vP(X))\in K_0(F).\end{aligned}\tag{5}$$

Remarks. (1) In the last statement we really are taking a Kasparov product when we consider

$$K_0(M(F,A))\times KK^0(M(F,A),F)\to K_0(F).$$

Hence the index is well-defined, depends only on the class of $[e_v]-[1]=[v]$ and the class of the 'APS Kasparov module'.
(2) We note that our hypothesis that P and Φ_0 commute with $v^*\mathcal{D}v$ is equivalent to P and Φ_0 commuting with v^*dv since P, Φ_0 commute with $\mathcal{D}$ and with v^*v. Thus P, Φ_0 commute with all functions of $v^*\mathcal{D}v$, and in particular with each spectral projection $v^*\Phi_kv$. Similarly, the first set of commutation relations imply that $\mathcal{D}$ and all of $\mathcal{D}$'s spectral projections commute with v^*Pv and $v^*\Phi_0v$.
(3) Whether every class $[v]\in K_0(M(F,A))$ possesses a representative satisfying the hypotheses of the theorem is unknown to us in general. Just as with the issues of regularity, it *may* be that one can always homotopy v and/or $(X,\mathcal{D})$ so that the hypotheses are satisfied. We leave this issue for future work, noting that for the applications we have in mind the hypotheses are satisfied.
(4) With regards to the regularity of PvP (in the sense of having a pseudoinverse [7, Definition 4.3,]), we observe that since P commutes with v^*Pv, the operator PvP is regular as an operator from $v^*vP(X)$ to $vv^*P(X)$, where the pseudoinverse of PvP is provided by Pv^*P. That is, $(PvP)(Pv^*P)(PvP)=PvP$ and $(Pv^*P)(PvP)(Pv^*P)=Pv^*P$. Thus our hypotheses guarantee the regularity of PvP, and the independence of the index of PvP on which regular 'amplification' we take gives some evidence that the hypotheses may be relaxed.

The proof of Theorem 5.1 will occupy the rest of the Section.

5.1. **Preliminaries.** As is usual for an index calculation such as this, we will assume without loss of generality (by replacing $\mathcal{A}$ by $M_k(\mathcal{A})$ if necessary) that the partial isometry v lies in $\mathcal{A}$. To begin the proof it is helpful to write e_v as an orthogonal sum of subprojections in $\mathcal{L}(\hat{X}\oplus\hat{X})$ which, of course, commute with e_v:

$$e_v=\begin{pmatrix}\frac{t^2}{1+t^2}vv^* & \frac{-it}{1+t^2}v\\ \frac{it}{1+t^2}v^* & \frac{1}{1+t^2}v^*v\end{pmatrix}+\begin{pmatrix}1-vv^* & 0\\ 0 & 0\end{pmatrix}:=\widehat{e_v}+e_v^0.\tag{6}$$

Note that to prove the Theorem it suffices to demonstrate the equality in Equation (5), and that is what we shall do. Using the decomposition of e_v into orthogonal subprojections in (6) an elementary calculation now gives:

Lemma 5.2. (1) *Let* $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \begin{pmatrix} \mathcal{E} \\ \mathcal{E} \end{pmatrix}$. *Then* $\xi \in e_v \begin{pmatrix} \mathcal{E} \\ \mathcal{E} \end{pmatrix}$ *if and only if* $v^*v\xi_2 = \xi_2$ *and* $vv^*\xi_1 = -itv\xi_2$. *In this case by Equation (6) we get an orthogonal decomposition:*

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = e_v \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \widehat{e_v} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + e_v^0 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ 0 \end{pmatrix}, \tag{7}$$

where $\eta_1 = vv^*\xi_1 = -itv\xi_2$ *and* $\zeta_1 = (1 - vv^*)\xi_1$*; and* **both** $\begin{pmatrix} \eta_1 \\ \xi_2 \end{pmatrix}$ *and* $\begin{pmatrix} \zeta_1 \\ 0 \end{pmatrix}$ *lie in* $e_v \begin{pmatrix} \mathcal{E} \\ \mathcal{E} \end{pmatrix}$.

(2) *The same statement (mutatis mutandis) holds for* $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \begin{pmatrix} \hat{\mathcal{E}} \\ \hat{\mathcal{E}} \end{pmatrix}$

In order to solve the differential equations to find the index in the Theorem we need the commutation relations recorded in the following lemma.

Lemma 5.3. *The operators* $v^*\mathcal{D}v$, $v^*v\mathcal{D}$ *and* v^*dv *preserve the subspaces of* $v^*v(X)$ *(intersected with the appropriate domains where necessary) given by* $v^*QvP(X)$, $v^*Qv(1-P)(X)$, *where* Q *is any of the projections* P, $P - \Phi_0$, $1 - P$, $1 - P + \Phi_0$, Φ_0.

Proof. In the remarks after the statement of Theorem 5.1, we noted that all spectral projections of $v^*v\mathcal{D}$ commute with the projections v^*Qv with Q. As $v^*v\mathcal{D}$ also commutes with P and $1 - P$, $v^*v\mathcal{D}$ preserves these subspaces. Likewise, $v^*\mathcal{D}v$ commutes with $v^*Q'v$ for *any* spectral projection Q' of $\mathcal{D}$, and by the hypotheses on v, $v^*\mathcal{D}v$ commutes with P and so $1 - P$. Thus $v^*\mathcal{D}v$ preserves all these subspaces. The result for $v^*dv = v^*\mathcal{D}v - v^*v\mathcal{D}$ follows immediately. ☐

5.2. **Simplifying the equations.** The main consequence of Lemma 5.2 is that we can consider two orthogonal subspaces of solutions separately and this greatly reduces the complexity of our task. In this subsection we will cover the T_+ case: $\ker(e_v(T_+ \otimes 1_2)e_v)$.

We observe that $(\partial_t + \mathcal{D}) \otimes 1_2$ commutes with the projection $\begin{pmatrix} 1 - vv^* & 0 \\ 0 & 0 \end{pmatrix}$ (which is $\leq e_v$). Thus with Q_+ the parametrix for $T_+ = \partial_t + \mathcal{D}$ constructed earlier we have

$$\begin{aligned} &\begin{pmatrix} (1 - vv^*) & 0 \\ 0 & 0 \end{pmatrix} (Q_+ \otimes 1_2) \begin{pmatrix} (1 - vv^*) & 0 \\ 0 & 0 \end{pmatrix} e_v((\partial_t + \mathcal{D}) \otimes 1_2)e_v \begin{pmatrix} 1 - vv^* & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (1 - vv^*) & 0 \\ 0 & 0 \end{pmatrix} (Q_+ \otimes 1_2)((\partial_t + \mathcal{D}) \otimes 1_2) \begin{pmatrix} 1 - vv^* & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (1 - vv^*) & 0 \\ 0 & 0 \end{pmatrix} (Id \otimes 1_2) \begin{pmatrix} 1 - vv^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 - vv^* & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Thus the kernel is $\{0\}$ on this subspace, and so we need only calculate the kernel on the range of $\widehat{e_v}$. Using the notation $da := [\mathcal{D}, a]$ and recalling that vv^* and v^*v commute with $\mathcal{D}$, so that $v^*vdv^* = dv^*$

and $vv^*dv = dv$ we now obtain:

$$\begin{aligned}
&\widehat{e_v}[(\partial_t+\mathcal{D})\otimes 1_2]\widehat{e_v}\\
=&\begin{pmatrix} \frac{t}{(1+t^2)^2}vv^* + \frac{t^2}{1+t^2}vv^*(\partial_t+\mathcal{D}) + \frac{t^2}{(1+t^2)^2}vdv^* & \frac{it^2}{(1+t^2)^2}v + \frac{-it}{1+t^2}v(\partial_t+\mathcal{D}) + \frac{-it^3}{(1+t^2)^2}dv \\ \frac{i}{(1+t^2)^2}v^* + \frac{it}{1+t^2}v^*(\partial_t+\mathcal{D}) + \frac{it}{(1+t^2)^2}dv^* & \frac{-t}{(1+t^2)^2}v^*v + \frac{1}{1+t^2}v^*v(\partial_t+\mathcal{D}) + \frac{t^2}{(1+t^2)^2}v^*dv \end{pmatrix}\\
=&\frac{1}{1+t^2}\begin{pmatrix} t^2vv^*(\partial_t+\mathcal{D}) & -itv(\partial_t+\mathcal{D}) \\ itv^*(\partial_t+\mathcal{D}) & v^*v(\partial_t+\mathcal{D})\end{pmatrix} + \frac{1}{(1+t^2)^2}\begin{pmatrix} tvv^*+t^2vdv^* & it^2v - it^3dv \\ iv^* + itdv^* & -tv^*v + t^2v^*dv\end{pmatrix}.
\end{aligned}$$

Using this formula, we obtain

$$\widehat{e_v}\begin{pmatrix} (\partial_t+\mathcal{D}) & 0 \\ 0 & (\partial_t+\mathcal{D})\end{pmatrix}\widehat{e_v}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix} -itv(\partial_t+\mathcal{D})\xi_2 - \frac{it^2}{1+t^2}v\xi_2 - \frac{it^3}{1+t^2}dv\xi_2 \\ (\partial_t+\mathcal{D})\xi_2 + \frac{t}{1+t^2}\xi_2 + \frac{t^2}{1+t^2}v^*dv\xi_2\end{pmatrix}.$$

Since this vector is also in the range of $\widehat{e_v}$ we check that the first coordinate is $-itv$ times the second coordinate as required by Lemma 5.2. We may rewrite the second coordinate in the preceding equation:

$$\rho_2(t) = (\partial_t+\mathcal{D})\xi_2 + \frac{t}{1+t^2}\xi_2 + v^*dv\xi_2 - \frac{v^*dv}{1+t^2}\xi_2$$

using $\xi_2 = v^*v(\xi_2)$, and $1 - 1/(1+t^2) = t^2/(1+t^2)$ as:

$$\rho_2(t) = \left(\frac{1}{\sqrt{1+t^2}}\partial_t \circ \sqrt{1+t^2} + v^*v\mathcal{D} + \frac{t^2v^*dv}{1+t^2}\right)\xi_2 =: (\tilde{\mathcal{D}}_v + V)\xi_2$$

where $\tilde{\mathcal{D}}_v = \left(\frac{1}{\sqrt{1+t^2}}\partial_t \circ \sqrt{1+t^2} + v^*v\mathcal{D}\right)$ and $V = \frac{t^2}{1+t^2}\otimes(v^*dv) := V_0\otimes(v^*dv)$. So in order to compute the kernel of $\widehat{e_v}[(\partial_t+\mathcal{D})\otimes 1_2]\widehat{e_v}$ acting on the range of $\widehat{e_v}$, it suffices to compute the kernel of $\tilde{\mathcal{D}}_v + V$ acting on vectors $\xi_2 \in \mathrm{dom}(\tilde{\mathcal{D}})$ satisfying $v^*v(\xi_2) = \xi_2$ and $t\xi_2 \in L^2(\mathbf{R}_+)\otimes X$. **In the T_+ case only,** such vectors are precisely those ξ_2 in $\mathrm{dom}(\tilde{\mathcal{D}})$ which lie in $L^2(\mathbf{R}_+, (1+t^2)dt)\otimes v^*v(X)$. We make the important observation that $\tilde{\mathcal{D}}_v$ is naturally a densely defined closed operator on $L^2(\mathbf{R}_+, (1+t^2)dt)\otimes v^*v(X)$ completely analogous to the operator $T_+ = \partial_t + \mathcal{D}$ of Section 4 which acts on $L^2(\mathbf{R}_+)\otimes X$.

Now we consider boundary values. For the equation $e_v((\partial_t+\mathcal{D})\otimes 1_2)e_v\xi = 0$ we want to impose the boundary condition $e_v(0)(P\otimes 1_2)e_v(0)\xi(0) = 0$ where P is the non-negative spectral projection for $\mathcal{D}$. This projection is

$$e_v(0)\begin{pmatrix} P & 0 \\ 0 & P\end{pmatrix}e_v(0) = \begin{pmatrix}(1-vv^*)P & 0 \\ 0 & v^*vP\end{pmatrix}.$$

Observe that our boundary projection is also the non-negative spectral projection of $e_v(0)(\mathcal{D}\otimes 1_2)e_v(0)$. As noted above, the only solution which lies in the range of $e_v^0(P\otimes 1_2)e_v^0 = \begin{pmatrix}(1-vv^*)P & 0 \\ 0 & 0\end{pmatrix}$ is the zero solution, for which this condition is automatically satisfied. Hence, we need not concern ourselves any further with this subcase.

5.3. **Solutions, integral kernels and parametrices.** In the following we make some notational simplifications. **We replace $v^*v\mathcal{D}$ by $\mathcal{D}$**, and similarly for other operators, since everything commutes with v^*v and we will always be working on the subspace $v^*v(X)$. In the notation of the previous subsection we aim to find the solutions of $(\tilde{\mathcal{D}}_v + V)\rho = 0$ on $L^2(\mathbf{R}_+, (1+t^2)dt)\otimes v^*v(X)$.

We will break our space up into orthogonal pieces preserved by $\tilde{\mathcal{D}}_v + V$. We first split our space as the image of $1\otimes P$ and $1\otimes(1-P)$. On the image of $1\otimes P$ we define a two parameter family of bounded operators which will be the integral kernel of a local left inverse for $\tilde{\mathcal{D}}_v + V$ on this space. The reason for our notation $\tilde{\mathcal{D}}_v + V$ is that we regard V as a (time dependent) perturbation, and we will define our integral kernels using a variant of the Dyson expansion for time dependent Hamiltonians, [19, X.12].

So for $t \geq s \geq 0$ define an operator on $Pv^*v(X)$ by

$$U(t,s) = e^{-(t-s)P\mathcal{D}} + \sum_{n=1}^{\infty}(-1)^n \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} e^{-(t-t_1)P\mathcal{D}} V(t_1) e^{-(t_1-t_2)P\mathcal{D}} \cdots V(t_n) e^{-(t_n-s)P\mathcal{D}} \mathbf{dt},$$

where we write: $dt_n \cdots dt_2 dt_1 = \mathbf{dt}$, and where $P\mathcal{D}$ really means $\mathcal{D}$ restricted to $Pv^*v(X)$.

Lemma 5.4. *For all $t \geq s \geq 0$ the integrals and the infinite sum defining $U(t,s)$ converge absolutely in the operator norm on the space $Pv^*v(X)$. For all $t \geq s \geq 0$ we have*

$$\|U(t,s)\| \leq \|e^{-(t-s)P\mathcal{D}}\|\, e^{(t-s)\|v^*dv\|}.$$

Moreover $U(t,s)$ satisfies the differential equations

$$\frac{d}{dt}U(t,s) = -(\mathcal{D} + V(t))U(t,s) \quad \text{and} \quad \frac{d}{ds}U(t,s) = U(t,s)(\mathcal{D} + V(s)).$$

Proof. To see the convergence and the norm inequality, we use the crude estimate $\|V(t)\| \leq \|v^*dv\|$ together with the equalities:

$$\|e^{-(t_k-t_{k+1})P\mathcal{D}}\| = \|e^{-P\mathcal{D}}\|^{(t_k-t_{k+1})} \quad \text{and} \quad \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} \mathbf{1}\, dt_n \cdots dt_1 = (t-s)^n/n!,$$

to obtain the inequality:

$$\|U(t,s)\| \leq \|e^{-(t-s)P\mathcal{D}}\| \sum_{n-0}^{\infty} \frac{(t-s)^n \|v^*dv\|^n}{n!} = \|e^{-(t-s)P\mathcal{D}}\|\, e^{(t-s)\|v^*dv\|}.$$

Differentiating formally yields the two differential equations but to see that the difference quotients converge in operator norm to the formal derivative takes a little effort. For example, using the mean value theorem and the functional calculus for unbounded self-adjoint operators, one shows that for any $f \in C^{(2)}(\mathbf{R}_+)$ which satisfies $x^2|f''(x)| \leq C$ for all $x \in \mathbf{R}_+$ we have: $\frac{d}{dt}(f((at+b)\mathcal{D})) = a\mathcal{D}f'((at+b)\mathcal{D})$ when $(at+b) > 0$ with norm convergence of the difference quotient. Applying this to $f(x) = e^{-x}$ for $(t-s) > 0$ we get $\frac{d}{dt}e^{-(t-s)\mathcal{D}} = -\mathcal{D}e^{-(t-s)\mathcal{D}}$, and $\frac{d}{ds}e^{-(t-s)\mathcal{D}} = \mathcal{D}e^{-(t-s)\mathcal{D}}$.

As for differentiating the integral terms, formally one uses a product rule which technically is invalid as one term is unbounded; however, by using the product rule trick of adding in a term and subtracting it out, one shows the formal calculation works. Since the original series and the series for the derivatives converge uniformly and absolutely, we are done. □

Using these results we now construct a (local) left inverse for $(\tilde{\mathcal{D}}_v + V)(1 \otimes P)$. We define for any $t \geq 0$ and continuous function $\rho \in (L^2(\mathbf{R}_+, (1+t^2)dt) \otimes Pv^*v(X))$,

$$(\tilde{Q}\rho)(t) := \frac{1}{\sqrt{1+t^2}} \int_0^t U(t,s)\sqrt{1+s^2}\rho(s)ds.$$

Observe that $(\tilde{Q}\rho)(0) = 0$, and is differentiable. First we need an elementary operator-theoretic lemma.

Lemma 5.5. *Let T be a closed densely defined operator on a Banach space B and let $\mathcal{S} \subseteq$ dom(T) be a dense subspace of* dom(T) *in the domain norm. Let A : dom$(T) \to B$ be a bounded operator in the* dom(T) *norm, and let Q be a densely defined closable linear operator whose domain contains $T(\mathcal{S})$ and such that $QT = 1_{\mathcal{S}} + A_{|\mathcal{S}}$. Then,* range$(T) \subseteq$ dom$(\overline{Q})$ *and $\overline{Q}T = 1_{\text{dom}(T)} + A$.*

Proof. Let $Tx \in \text{range}(T)$, so there exists a sequence $\{x_n\}$ in $\mathcal{S}$ with $x_n \to x$ and $Tx_n \to Tx$. But then, the fact that $\lim_n Tx_n = Tx$ and $\lim_n Q(Tx_n) = \lim_n(x_n + A(x_n)) = x + A(x)$ implies that $Tx \in \text{dom}(\overline{Q})$ and $\overline{Q}(Tx) = x + A(x)$. □

Lemma 5.6. *The equation $(\tilde{\mathcal{D}}_v + V)\rho = 0$ has no nonzero solutions in*

$$(L^2(\mathbf{R}_+, (1+t^2)dt) \otimes Pv^*v(X)).$$

Proof. Fix $M > 0$ and let E_M be the orthogonal projection of $L^2(\mathbf{R}_+, (1+t^2)dt)$ onto the subspace $L^2([0, M], (1+t^2)dt)$. Then $E_M \otimes 1$ is the orthogonal projection of $(L^2(\mathbf{R}_+, (1+t^2)dt) \otimes Pv^*v(X))$ onto the subspace $(L^2([0, M], (1+t^2)dt) \otimes Pv^*v(X))$. Now, we see that $\tilde{Q}$ defines a linear operator on the dense subspace of $(L^2([0, M], (1+t^2)dt) \otimes Pv^*v(X))$ consisting of continuous functions, call it $\tilde{Q}_M$. This operator has a densely defined adjoint defined on the same subspace, $\tilde{Q}_M^{\#}$, given by the formula:

$$(\tilde{Q}_M^{\#}\rho)(t) := \frac{1}{\sqrt{1+t^2}} \int_t^M U(s,t)^* \sqrt{1+s^2}\rho(s)ds.$$

Thus, $\tilde{Q}_M$ is not only densely defined, but also closable on $(L^2([0, M], (1+t^2)dt) \otimes Pv^*v(X))$.

The smooth functions ρ in the domain of $(\tilde{\mathcal{D}}_v + V)(1 \otimes P)$ form a domain-dense subspace and

$$(E_M \otimes 1)(\tilde{\mathcal{D}}_v + V)(\rho) = (\tilde{\mathcal{D}}_v + V)(E_M \otimes 1)(\rho) \in \text{dom}(\tilde{Q}_M).$$

Let $\rho_M = (E_M \otimes 1)(\rho)$, fix $t \in [0, M]$ and calculate:

$$\begin{aligned}
(\tilde{Q}_M(\tilde{\mathcal{D}}_v + V)\rho_M)(t) &= (\tilde{Q}_M(\tilde{\mathcal{D}}_v + V)\rho)(t) = (\tilde{Q}(\tilde{\mathcal{D}}_v + V)\rho)(t) \\
&= \frac{1}{\sqrt{1+t^2}} \int_0^t U(t,s) \left(\partial_s(\sqrt{1+s^2}\rho(s)) + \sqrt{1+s^2}(\mathcal{D} + V(s))\rho(s)\right) ds \\
&= \frac{1}{\sqrt{1+t^2}} \int_0^t \partial_s(U(t,s)\sqrt{1+s^2}\rho(s))ds - \frac{1}{\sqrt{1+t^2}} \int_0^t (\partial_s U(t,s))\sqrt{1+s^2}\rho(s)ds \\
&\qquad + \frac{1}{\sqrt{1+t^2}} \int_0^t U(t,s)\sqrt{1+s^2}(\mathcal{D} + V(s))\rho(s)ds \\
&= \rho(t) - \frac{1}{\sqrt{1+t^2}} U(t,0)\rho(0) = \rho(t) = \rho_M(t),
\end{aligned}$$

As $\rho(0) = P(\rho(0)) = 0$ the previous lemma implies that $(\tilde{\mathcal{D}}_v + V)(E_M \otimes P)$ is injective and $(\tilde{\mathcal{D}}_v + V)\rho = 0$ has no nonzero local solutions on $[0, M]$ for any $M > 0$. Hence, $(\tilde{\mathcal{D}}_v + V)\rho = 0$ has no nonzero global solutions in $(L^2(\mathbf{R}_+, (1+t^2)dt) \otimes Pv^*v(X))$. □

Next we split the range of $1 \otimes (1 - P)$ into two pieces, namely

$$1 \otimes (1-P) = 1 \otimes v^*(1 - P + \Phi_0)v(1-P) \,\oplus\, 1 \otimes v^*(P - \Phi_0)v(1-P).$$

Lemma 5.7. *The equation $(\tilde{\mathcal{D}}_v + V)\rho = 0$ has no nonzero solutions in the subspace*

$$L^2(\mathbf{R}_+, (1+t^2)dt) \otimes v^*(1 - P + \Phi_0)v(1-P)(X).$$

Proof. Suppose we did have a solution $\rho \in L^2(\mathbf{R}_+, (1+t^2)dt) \otimes v^*(1 - P + \Phi_0)v(1-P)(X)$. We write $\rho(t) = \frac{1}{\sqrt{1+t^2}}\sigma(t)$, where σ is now an (ordinary) L^2 function with values in $v^*(1 - P + \Phi_0)v(1-P)(X)$. A brief calculation shows that

$$\begin{aligned}
\frac{1}{1+t^2}\frac{d}{dt}\langle\sigma(t)|\sigma(t)\rangle_X &= \frac{1}{\sqrt{1+t^2}}\frac{d}{dt}\sqrt{1+t^2}\langle\rho(t)|\rho(t)\rangle_X \\
&= \langle -(\mathcal{D} + V(t))\rho(t)|\rho(t)\rangle_X + \langle\rho(t)| - (\mathcal{D} + V(t))\rho(t)\rangle_X\,.
\end{aligned}$$

Since $v^*\mathcal{D}v$ is non-positive and $\mathcal{D}$ strictly negative on $v^*(1-P+\Phi_0)v(1-P)(X)$, we have the estimate

$$\mathcal{D}+V(t)=(1+t^2)^{-1}(t^2v^*\mathcal{D}v+\mathcal{D})<-c1/(1+t^2),$$

where $c>0$ and $0<c<|r_{-1}|$ where r_{-1} is the first negative eigenvalue of $\mathcal{D}$ on this subspace. Thus

$$\frac{1}{1+t^2}\frac{d}{dt}\langle\sigma(t)|\sigma(t)\rangle_X\geq\frac{2c}{1+t^2}\langle\rho(t)|\rho(t)\rangle_X\,.$$

Multiplying by $1+t^2$ and integrating from 0 to s gives (this is an integral of a continuous function into the positive cone of the C^*-algebra F)

$$\int_0^s\frac{d}{dt}\langle\sigma(t)|\sigma(t)\rangle_X dt=\langle\sigma(s)|\sigma(s)\rangle_X-\langle\sigma(0)|\sigma(0)\rangle_X\geq 2c\int_0^s\langle\rho(t)|\rho(t)\rangle_X dt.$$

The right hand side is a nondecreasing function of s, and if ρ is nonzero, this function is eventually positive. Hence $\langle\sigma(s)|\sigma(s)\rangle_X$ is a continuous non-decreasing function of s in F^+, and so can not be integrable as can be seen by evaluating on a state of F. Hence σ is not an element of L^2 and there are no nonzero solutions ρ of $(\tilde{\mathcal{D}}_v+V)(\rho)=0$ in the space $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(1-P+\Phi_0)v(1-P)(X)$. □

Finally, we come to the subspace $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)v(1-P)(X)$. On this subspace we will define a parametrix which is a right inverse, but is not a left inverse, instead providing solutions to our equation. Thus, for $t\geq s\geq 0$ define an operator $H(t,s)$ on the space $v^*(P-\Phi_0)v(1-P)(X)$ by:

$$e^{-(t-s)v^*\mathcal{D}v}+\sum_{n=1}^{\infty}\int_s^t\int_s^{t_1}\cdots\int_s^{t_{n-1}}((1+t_1^2)\cdots(1+t_n^2))^{-1}e^{-(t-t_1)v^*\mathcal{D}v}\,v^*dv\cdots v^*dv\,e^{-(t_n-s)v^*\mathcal{D}v}\mathbf{dt}.$$

where $v^*\mathcal{D}v$ means $v^*\mathcal{D}v$ restricted to the subspace $v^*(P-\Phi_0)v(1-P)(X)$.

Lemma 5.8. *For all $t\geq s\geq 0$ the integrals and the infinite sum defining $H(t,s)$ converge absolutely in norm. For $t\geq s\geq 0$, $H(t,s)$ is an endomorphism of the module $v^*(P-\Phi_0)v(1-P)(X)$ with norm*

$$\|H(t,s)\|\leq\|e^{-(t-s)v^*\mathcal{D}v}\|\,e^{\tan^{-1}(t)\|v^*dv\|}\leq e^{-(t-s)r_1}e^{\tan^{-1}(t)\|v^*dv\|},$$

where r_1 is the smallest positive eigenvalue of $v^*\mathcal{D}v$ on this subspace. The family of endomorphisms $H(t,s)$ satisfies the differential equations

$$\frac{d}{dt}H(t,s)=-(\mathcal{D}+V(t))H(t,s),\qquad\frac{d}{ds}H(t,s)=H(t,s)(\mathcal{D}+V(s)).$$

Proof. Except for the final estimate the proof of this is similar to the proof of Lemma 5.4. Now, the norm of $H(t,s)$ (on $v^*(P-\Phi_0)v(1-P)(X)$) can be estimated as follows:

$$\begin{aligned}\|H(t,s)\|&\leq\|e^{-(t-s)v^*\mathcal{D}v}\|\left(1+\sum_{n=1}^{\infty}\|v^*dv\|^n\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}((1+t_1^2)\cdots(1+t_n^2))^{-1}\mathbf{dt}\right)\\&=\|e^{-(t-s)v^*\mathcal{D}v}\|\left(1+\sum_{n=1}^{\infty}\frac{\|v^*dv\|^n}{n!}(\tan^{-1}(t))^n\right)\\&=\|e^{-(t-s)v^*\mathcal{D}v}\|e^{\tan^{-1}(t)\|v^*dv\|}\leq e^{-(t-s)r_1}e^{\tan^{-1}(t)\|v^*dv\|},\end{aligned}$$

where r_1 is the smallest positive eigenvalue of $v^*\mathcal{D}v$ on the subspace. □

We now define a local parametrix on the space $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)v(1-P)(X)$. Let ρ be given by a continuous function in $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)v(1-P)(X)$ and let $t\geq 0$. Define

$$(\tilde{R}\rho)(t) := \frac{1}{\sqrt{1+t^2}}\int_0^t H(t,s)\sqrt{1+s^2}\rho(s)ds.$$

As in the proof of Lemma 5.6 $\tilde{R}$ defines a closable linear mapping locally on $[0,M]$ on it's initial dense domain of continuous functions. We note that $\tilde{R}(\rho)$ is differentiable.

Lemma 5.9. *For every vector x in the subspace $v^*(P-\Phi_0)v(1-P)(X)$ there exists a unique element $\rho\in L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)v(1-P)(X)$ with $\rho(0)=x$ and $(\tilde{\mathcal{D}}_v+V)\rho=0$. Moreover, these are the only solutions in the space $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)v(1-P)(X)$.*

Proof. As in the proof of Lemma 5.6 we work locally with t in the interval $[0,M]$, however, we suppress the local notations ρ_M, etc. Take ρ a continuous function in $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)v(1-P)(X)$ with values in $\mathrm{dom}(\mathcal{D})$ and compute using the differential equations from Lemma 5.8.

$$\frac{1}{\sqrt{1+t^2}}\partial_t\sqrt{1+t^2}(\tilde{R}\rho)(t) = \frac{1}{\sqrt{1+t^2}}\partial_t\left(\int_0^t H(t,s)\sqrt{1+s^2}\rho(s)ds\right) = \rho(t)-(\mathcal{D}+V(t))(\tilde{R}\rho)(t),$$

Thus $(\tilde{\mathcal{D}}_v+V(t))(\tilde{R}\rho)(t)=\rho(t)$ and $\tilde{R}$ is injective. The injectivity is first proved locally on $[0,M]$ by using Lemma 5.5 which easily implies global injectivity. On the other hand if ρ is smooth and lies in the domain of $\tilde{\mathcal{D}}_v+V$ then $(\tilde{\mathcal{D}}_v+V)(\rho)$ is continuous and so locally we get:

$$\begin{aligned}(\tilde{R}(\tilde{\mathcal{D}}_v+V)\rho)(t) &= \frac{1}{\sqrt{1+t^2}}\int_0^t H(t,s)\left(\partial_s(\sqrt{1+s^2}\rho(s))+\sqrt{1+s^2}(\mathcal{D}+V(s))\rho(s)\right)ds\\ &= \frac{1}{\sqrt{1+t^2}}\int_0^t \partial_s(H(t,s)\sqrt{1+s^2}\rho(s))ds - \frac{1}{\sqrt{1+t^2}}\int_0^t(\partial_s H(t,s))\sqrt{1+s^2}\rho(s)ds\\ &\qquad + \frac{1}{\sqrt{1+t^2}}\int_0^t H(t,s)\sqrt{1+s^2}(\mathcal{D}+V(s))\rho(s)ds\\ &= \rho(t)-\frac{1}{\sqrt{1+t^2}}H(t,0)\rho(0),\end{aligned}$$

where we have again used the differential equations from Lemma 5.8. Applying Lemma 5.5 we obtain this equation for all $\rho\in\mathrm{dom}(\tilde{\mathcal{D}}_v+V)$. By the estimate on $\|H(t,0)\|$ in the previous lemma, the function $\frac{1}{\sqrt{1+t^2}}H(t,0)\rho(0)$ is in $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)v(1-P)(X)$, and so if ρ is in the kernel of $\tilde{\mathcal{D}}_v+V$ we have locally and hence globally:

$$\rho(t) = (1+t^2)^{-1/2}H(t,0)\rho(0). \tag{8}$$

Conversely, with $x=\rho(0)\in v^*(P-\Phi_0)v(1-P)(X)$, Eq. (8) defines a solution as $\tilde{R}$ is injective. □

Putting together Lemmas 5.6, 5.7, 5.9, we have the following preliminary result.

Corollary 5.10. *The kernel of $\tilde{\mathcal{D}}_v+V$ on $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*v(X)$ is isomorphic to the right F-module $v^*(P-\Phi_0)v(1-P)(X)$. Consequently*

$$\ker(e_v((\partial_t+\mathcal{D})\otimes 1_2)e_v) = \ker(\hat{e}_v((\partial_t+\mathcal{D})\otimes 1_2)\hat{e}_v)\cong\ker(\tilde{\mathcal{D}}_v+V)\cong v^*(P-\Phi_0)v(1-P)(X).$$

Thus we have part of the Index of $(e_v(\hat{\mathcal{D}}\otimes 1_2)e_v)$. To complete the calculation, we compute the kernel of the adjoint operator $e_v(-\partial_t+\mathcal{D})e_v$. We follow an essentially similar path, but must take a little more care with the extended L^2-space $\hat{\mathcal{E}}$.

5.4. **The kernel of the adjoint.** As explained above, we must compute the kernel of the operator $e_v\begin{pmatrix} -\partial_t+\mathcal{D} & 0 \\ 0 & -\partial_t+\mathcal{D}\end{pmatrix}e_v$ as a map from $e_v\begin{pmatrix}\hat{\mathcal{E}}\\ \hat{\mathcal{E}}\end{pmatrix}$ to $e_v\begin{pmatrix}\mathcal{E}\\ \mathcal{E}\end{pmatrix}$. Recall that $M(F,A)$ acts as **zero** on the constant X_0-valued functions but the added unit element acts as the identity. Thus for a pair of constant functions $\begin{pmatrix}x_1\\ x_2\end{pmatrix}\in\begin{pmatrix}\hat{\mathcal{E}}\\ \hat{\mathcal{E}}\end{pmatrix}$ we have $e_v\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}x_1\\ 0\end{pmatrix}$. Hence $e_v\begin{pmatrix}\hat{\mathcal{E}}\\ \hat{\mathcal{E}}\end{pmatrix}\subseteq\begin{pmatrix}\hat{\mathcal{E}}\\ \mathcal{E}\end{pmatrix}$. For $\xi\in e_v\begin{pmatrix}\hat{\mathcal{E}}\\ \hat{\mathcal{E}}\end{pmatrix}$ to be in the domain of $e_v((-\partial_t+\mathcal{D})\otimes 1_2)e_v$ we impose the boundary condition:

$$\begin{pmatrix}(1-vv^*)(1-P) & 0\\ 0 & v^*v(1-P)\end{pmatrix}\xi(0)=0.$$

For the constant function $\begin{pmatrix}x_1\\ 0\end{pmatrix}\in e_v\begin{pmatrix}\hat{\mathcal{E}}\\ \hat{\mathcal{E}}\end{pmatrix}$ to be in the domain this means that x_1 must satisfy $(1-vv^*)(1-P)(x_1)=0$. However, this is automatic as $x_1\in X_0$ so that $(1-P)(x_1)=0$. Thus the domain of $e_v((-\partial_t+\mathcal{D})\otimes 1_2)e_v$ extended to the constant X_0-valued functions $(X_0\oplus X_0)^T$ is $(X_0\oplus 0)^T$. Of course, the extended operator $e_v((-\partial_t+\mathcal{D})\otimes 1_2)e_v$ is identically 0 here. **It is important to note that:** $\mathrm{dom}(e_v((-\partial_t+\mathcal{D})\otimes 1_2)e_v)\subseteq e_v(\hat{\mathcal{E}}\oplus\mathcal{E})^T$.

As before we use the orthogonal decomposition of e_v to enable separate analysis of the two subspaces:

$$\widehat{e_v}\begin{pmatrix}\hat{\mathcal{E}}\\ \hat{\mathcal{E}}\end{pmatrix}\to\widehat{e_v}\begin{pmatrix}\mathcal{E}\\ \mathcal{E}\end{pmatrix}\quad\text{and}\quad e_v^0\begin{pmatrix}\hat{\mathcal{E}}\\ \hat{\mathcal{E}}\end{pmatrix}\to e_v^0\begin{pmatrix}\mathcal{E}\\ \mathcal{E}\end{pmatrix}.$$

Now,

$$e_v^0\begin{pmatrix}\hat{\mathcal{E}}\\ \hat{\mathcal{E}}\end{pmatrix}=e_v^0\begin{pmatrix}\mathcal{E}\\ \mathcal{E}\end{pmatrix}\oplus\begin{pmatrix}(1-vv^*)(X_0)\\ 0\end{pmatrix}.$$

As in the case of $e_v((\partial_t+\mathcal{D})\otimes 1_2)e_v$ we have $e_v((-\partial_t+\mathcal{D})\otimes 1_2)e_v$ is one-to-one on $e_v^0(\mathcal{E}\oplus\mathcal{E})^T$ and so the kernel there is 0. Since $e_v((-\partial_t+\mathcal{D})\otimes 1_2)e_v$ is identically 0 on $e_v^0((1-vv^*)(X_0)\oplus 0)^T\cong(1-vv^*)(X_0)$, we have the following result.

Proposition 5.11. *The kernel of* $e_v((-\partial_t+\mathcal{D})\otimes 1_2)e_v$ *restricted to* $e_v^0\begin{pmatrix}\hat{\mathcal{E}}\\ \hat{\mathcal{E}}\end{pmatrix}$ *is isomorphic to the right F-module* $(1-vv^*)(X_0)$.

These solutions are a rather trivial type of **extended solution** to the adjoint equation. Next:

$$\widehat{e_v}\begin{pmatrix}(-\partial_t+\mathcal{D}) & 0\\ 0 & (-\partial_t+\mathcal{D})\end{pmatrix}\widehat{e_v}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}-itv(-\partial_t+\mathcal{D})\xi_2+\frac{it^2}{1+t^2}v\xi_2-\frac{it^3}{1+t^2}dv\xi_2\\ (-\partial_t+\mathcal{D})\xi_2-\frac{t}{1+t^2}\xi_2+\frac{t^2}{1+t^2}v^*dv\xi_2\end{pmatrix}.$$

That is, any vector $(\rho_1,\rho_2)^T$ in the range of $\widehat{e_v}[(\partial_t+\mathcal{D})\otimes 1_2]\widehat{e_v}$ satisfies $\rho_1(t)=-itv(\rho_2)(t)$ and as before, after simplifying,

$$\rho_2(t)=\left(\frac{-1}{\sqrt{1+t^2}}\partial_t\circ\sqrt{1+t^2}+v^*v\mathcal{D}+\frac{t^2v^*dv}{1+t^2}\right)\xi_2=:(\widehat{\mathcal{D}}_v+V)\xi_2$$

$$\text{where}\quad\widehat{\mathcal{D}}_v=\left(\frac{-1}{\sqrt{1+t^2}}\partial_t\circ\sqrt{1+t^2}+v^*v\mathcal{D}\right)\quad\text{and}\quad V=\frac{t^2}{1+t^2}\otimes(v^*dv):=V_0\otimes(v^*dv).$$

So in order to compute the kernel of $\widehat{e_v}[(-\partial_t+\mathcal{D})\otimes 1_2]\widehat{e_v}$ acting on the range of $\widehat{e_v}$, it suffices to compute the kernel of $\widehat{\mathcal{D}}_v+V$ acting on vectors $\xi_2\in\mathcal{E}$ satisfying $v^*v(\xi_2)=\xi_2$ and $-itv(\xi_2)\in\hat{\mathcal{E}}$. **As opposed to the T_+ case,** such vectors ξ_2 need only lie in the larger space $L^2(\mathbf{R}_+)\otimes v^*v(X)$, while $\xi_1(t) = -itv(\xi_2(t))$ **may** have a nonzero limit at ∞ in X_0 subject to the boundary conditions $P(\xi_2(0)) = \xi_2(0)$.

Again we split $L^2(\mathbf{R}_+) \otimes v^*v(X)$ into the range of $1 \otimes P$ and $1 \otimes (1-P)$. On the image of $1 \otimes (1-P)$ we define a two parameter family of bounded operators which will be the integral kernel of a local parametrix for $\widehat{\mathcal{D}}_v + V$ on this space. Thus with $\mathcal{D}$ standing for $(1-P)\mathcal{D}$ and for $t \geq s \geq 0$, define an operator on $(1-P)v^*v(X)$ by

$$W(t,s) = e^{(t-s)\mathcal{D}} + \sum_{n=1}^{\infty}(-1)^n \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} e^{(t-t_1)\mathcal{D}} V(t_1) e^{(t_1-t_2)\mathcal{D}} V(t_2) \cdots V(t_n) e^{(t_n-s)\mathcal{D}} \mathbf{dt}.$$

Lemma 5.12. *For all $t \geq s \geq 0$ the integrals and the infinite sum defining $W(t,s)$ converge absolutely in norm. For all $t \geq s \geq 0$ we have (in the operator norm for endomorphisms of $v^*v(X)$)*

$$\|W(t,s)\| \leq \|e^{(t-s)\mathcal{D}}\| \, e^{(t-s)\|v^*dv\|}.$$

Moreover $W(t,s)$ satisfies the differential equations

$$\frac{d}{dt}W(t,s) = (\mathcal{D} + V(t))W(t,s), \qquad \frac{d}{ds}W(t,s) = -W(t,s)(\mathcal{D} + V(s)).$$

Proof. This is very similar to the proof of Lemma 5.4 so we omit the details. □

Using these results we construct a local parametrix for $(\widehat{\mathcal{D}}_v + V)(1 \otimes (1-P))$. For ρ a continuous function in $L^2(\mathbf{R}_+) \otimes (1-P)v^*v(X)$ define

$$(\widehat{Q}\rho)(t) := -(1+t^2)^{-1/2} \int_0^t W(t,s)\sqrt{1+s^2}\rho(s)ds.$$

Observe that $(\widehat{Q}\rho)(0) = 0$, and is differentiable, and so if ρ has range in $\mathrm{dom}(\mathcal{D})$ then $\widehat{Q}(\rho)$ is locally in the domain of $\widehat{\mathcal{D}}_v + V$. As in the proof of Lemma 5.6, $\widehat{Q}$ defines a closable linear mapping locally on $[0, M]$ on its initial dense domain of continuous functions. All our calculations below are local as in Lemma 5.6.

Lemma 5.13. *In the space $L^2(\mathbf{R}_+) \otimes (1-P)v^*v(X)$ the equation $(\widehat{\mathcal{D}}_v + V)\rho = 0$ has no nonzero solutions and therefore it has no nonzero solutions in the subspace $L^2(\mathbf{R}_+, (1+t^2)dt) \otimes (1-P)v^*v(X)$.*

Proof. Let ρ be a smooth function in the domain of $(\widehat{\mathcal{D}}_v + V)(1-P)$:

$$\begin{aligned}
(\widehat{Q}(\widehat{\mathcal{D}}_v + V)\rho)(t) &= \frac{-1}{\sqrt{1+t^2}} \int_0^t W(t,s)\left(-\partial_s(\sqrt{1+s^2}\rho(s)) + \sqrt{1+s^2}(\mathcal{D} + V(s))\rho(s)\right) ds \\
&= \frac{1}{\sqrt{1+t^2}} \int_0^t \partial_s(W(t,s)\sqrt{1+s^2}\rho(s))ds - \frac{1}{\sqrt{1+t^2}} \int_0^t (\partial_s W(t,s))\sqrt{1+s^2}\rho(s)ds \\
&\qquad - \frac{1}{\sqrt{1+t^2}} \int_0^t W(t,s)\sqrt{1+s^2}(\mathcal{D} + V(s))\rho(s)ds \\
&= \rho(t) - (1+t^2)^{-1/2}W(t,0)\rho(0) = \rho(t),
\end{aligned}$$

where, as ρ has values in the range of $(1-P)$, we have $\rho(0) = 0$. Arguing as in the proof of Lemma 5.6 this implies that $(\widehat{\mathcal{D}}_v + V)(1-P)$ is injective on its whole domain. Hence, $(\widehat{\mathcal{D}}_v + V)\rho = 0$ has no nonzero solutions in $L^2(\mathbf{R}_+) \otimes (1-P)v^*v(X)$. □

Next we split the range of $1\otimes P$ into three pieces, namely

$$1\otimes P=[1\otimes v^*(P-\Phi_0)vP]\,\oplus\,[1\otimes v^*(1-P)vP]\,\oplus\,[1\otimes v^*\Phi_0vP].$$

Lemma 5.14. *In the subspace $L^2(\mathbf{R}_+)\otimes v^*(P-\Phi_0)vPv^*v(X)$ the equation $(\widehat{\mathcal{D}}_v+V)\rho=0$ has no nonzero solutions and therefore has no nonzero solutions in $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)vPv^*v(X)$.*

Proof. First, suppose we have a solution ρ with $\rho(t)\in v^*(P-\Phi_0)vPv^*v(X)$ for all $t\geq 0$, and $\rho\in L^2(\mathbf{R}_+)\otimes v^*(P-\Phi_0)vPv^*v(X)$. Then $-itv(\rho(t))\in(P-\Phi_0)vPv^*v(X)$ and so if this has a limit at ∞ in $\Phi_0(X)$, the limit must be 0. That is, $-itv(\rho)\in L^2(\mathbf{R}_+)\otimes(P-\Phi_0)vPv^*v(X)$ and so our solution ρ actually lies in the smaller space:

$$L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)vPv^*v(X).$$

Arguing as in Lemma 5.7 write $\rho(t)=(1+t^2)^{-1/2}\sigma(t)$, where σ is now an (ordinary) L^2 function with values in $v^*(P-\Phi_0)vPv^*v(X)$:

$$(1+t^2)^{-1}\frac{d}{dt}\langle\sigma(t)|\sigma(t)\rangle_X\frac{d}{dt}\sqrt{1+t^2}\langle\rho(t)|\rho(t)\rangle_X=\langle(\mathcal{D}+V(t))\rho(t)|\rho(t)\rangle_X+\langle\rho(t)|(\mathcal{D}+V(t))\rho(t)\rangle_X\,.$$

Since $v^*\mathcal{D}v$ is strictly positive and $\mathcal{D}$ is non-negative on $v^*(P-\Phi_0)vPv^*v(X)$, we have the estimate

$$\mathcal{D}+V(t)=(1+t^2)^{-1}(t^2v^*\mathcal{D}v+\mathcal{D})>r_1t^2/(1+t^2),$$

where r_1 is the first positive eigenvalue of $v^*\mathcal{D}v$ on this subspace and therefore

$$\frac{1}{1+t^2}\frac{d}{dt}\langle\sigma(t)|\sigma(t)\rangle_X\geq\frac{2r_1t^2}{1+t^2}\,\langle\rho(t)|\rho(t)\rangle_X\,.$$

Multiplying by $1+t^2$ and integrating from 0 to s gives

$$\int_0^s\frac{d}{dt}\langle\sigma(t)|\sigma(t)\rangle_Xdt=\langle\sigma(s)|\sigma(s)\rangle_X-\langle\sigma(0)|\sigma(0)\rangle_X\geq 2r_1\int_0^s t^2\langle\rho(t)|\rho(t)\rangle_Xdt.$$

The right hand side is a nondecreasing function of s, and if ρ is nonzero, this function is eventually positive. Thus arguing further as in Lemma 5.7 there are no nonzero solutions ρ of $(\widehat{\mathcal{D}}_v+V)(\rho)=0$ in $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(P-\Phi_0)vPv^*v(X)$, and hence none in $L^2(\mathbf{R}_+)\otimes v^*(P-\Phi_0)vPv^*v(X)$. □

Next, we come to the subspace $L^2(\mathbf{R}_+)\otimes v^*(1-P)vPv^*v(X)$. On this subspace we will define a local parametrix which is a right inverse, but is not a left inverse, instead providing solutions to our equation. So for $t\geq s\geq 0$ define $G(t,s)$ (on the module $v^*(1-P)vPv^*v(X)$) by

$$e^{(t-s)v^*\mathcal{D}v}+\sum_{n=1}^{\infty}(-1)^n\int_s^t\int_s^{t_1}\cdots\int_s^{t_{n-1}}((1+t_1^2)\cdots(1+t_n^2))^{-1}\,e^{(t-t_1)v^*\mathcal{D}v}\,v^*dv\cdots v^*dv\,e^{(t_n-s)v^*\mathcal{D}v}\mathbf{dt}.$$

Lemma 5.15. *For all $t\geq s\geq 0$ the integrals and the infinite sum defining $G(t,s)$ converge absolutely in norm. For $t\geq s\geq 0$, $G(t,s)$ is a bounded endomorphism of the module $v^*(1-P)vPv^*v(X)$ with norm bounded by*

$$\|G(t,s)\|\leq\|e^{(t-s)v^*\mathcal{D}v}\|\,e^{(t-s)\|v^*dv\|}\leq e^{(t-s)r_{-1}}e^{(t-s)\|v^*dv\|},$$

where r_{-1} is the largest negative eigenvalue of $\mathcal{D}$. The family of endomorphisms $G(t,s)$ satisfies the differential equations

$$\frac{d}{dt}G(t,s)=(\mathcal{D}+V(t))G(t,s),\qquad\frac{d}{ds}G(t,s)=-G(t,s)(\mathcal{D}+V(s)).$$

Proof. The proof of this is very similar to the proof of Lemma 5.8. □

Now define a local parametrix on continuous functions ρ by

$$(\widehat{R}\rho)(t) := (1+t^2)^{-1/2}\int_0^t G(t,s)\sqrt{1+s^2}\rho(s)ds, \quad \rho \in L^2(\mathbf{R}_+)\otimes v^*(1-P)vPv^*v(X).$$

As in the proof of Lemma 5.6 $\widehat{R}$ defines a closable linear mapping locally on $[0, M]$ on the initial dense domain of continuous functions. We note that $\widehat{R}(\rho)$ is differentiable.

Lemma 5.16. *For every vector x in the space $v^*(1-P)vPv^*v(X)$ there exists a unique element $\rho \in L^2(\mathbf{R}_+)\otimes v^*(1-P)vPv^*v(X)$ with $\rho(0) = x$ and $(\widehat{\mathcal{D}}_v + V)\rho = 0$. Moreover, these are the only solutions in the subspace $L^2(\mathbf{R}_+)\otimes v^*(1-P)vPv^*v(X)$. In fact these solutions ρ clearly lie in $L^2(\mathbf{R}_+,(1+t^2)dt)\otimes v^*(1-P)vPv^*v(X)$ and satisfy $\lim_{t\to\infty} -itv(\rho(t)) = 0$.*

Proof. We work locally as in the proofs of Lemmas 5.6 and 5.9. Take ρ a continuous function in $L^2(\mathbf{R}_+)\otimes v^*(1-P)vPv^*v(X)$ with values in $\mathrm{dom}(\mathcal{D})$ and compute

$$\frac{-1}{\sqrt{1+t^2}}\partial_t\sqrt{1+t^2}(\widehat{R}\rho)(t) = \frac{1}{\sqrt{1+t^2}}\partial_t\left(\int_0^t G(t,s)\sqrt{1+s^2}\rho(s)ds\right) = \rho(t) - (\mathcal{D}+V(t))(\widehat{R}\rho)(t),$$

where we have used the computations from Lemma 5.15. Thus $(\widehat{\mathcal{D}}_v + V(t))(\widehat{R}\rho)(t) = \rho(t)$ and $\widehat{R}$ is injective. The injectivity is first proved locally on $[0, M]$ by using Lemma 5.5 which easily implies global injectivity. On the other hand if ρ is smooth and lies in the domain of $\widehat{\mathcal{D}}_v + V$ then $(\widehat{\mathcal{D}}_v + V)(\rho)$ is continuous and so locally we get:

$$\begin{aligned}
(\widehat{R}(\widehat{\mathcal{D}}_v + V)\rho)(t) &= \frac{-1}{\sqrt{1+t^2}}\int_0^t G(t,s)\left(-\partial_s(\sqrt{1+s^2}\rho(s)) + \sqrt{1+s^2}(\mathcal{D}+V(s))\rho(s)\right)ds \\
&= \frac{1}{\sqrt{1+t^2}}\int_0^t \partial_s(G(t,s)\sqrt{1+s^2}\rho(s))ds - \frac{1}{\sqrt{1+t^2}}\int_0^t (\partial_s G(t,s))\sqrt{1+s^2}\rho(s)ds \\
&\qquad - \frac{1}{\sqrt{1+t^2}}\int_0^t G(t,s)\sqrt{1+s^2}(\mathcal{D}+V(s))\rho(s)ds \\
&= \rho(t) - (1+t^2)^{-1/2}G(t,0)\rho(0),
\end{aligned}$$

where we have again used the derivative computations from Lemma 5.15. Applying Lemma 5.5 we get this formula for all $\rho \in \mathrm{dom}(\widehat{\mathcal{D}}_v + V)$.

Now if ρ is in the kernel of $\widehat{\mathcal{D}}_v + V$ we have locally and hence globally

$$\rho(t) = (1+t^2)^{-1/2}G(t,0)\rho(0), \tag{9}$$

and this lies in $L^2(\mathbf{R}_+)\otimes v^*(1-P)vPv^*v(X)$ by the estimate:

$$\|G(t,0)\| \le e^{tr_{-1}}e^{\tan^{-1}(t)\|v^*dv\|}$$

where r_{-1} is the largest negative eigenvalue of $\mathcal{D}$ on the subspace. Conversely, given any vector $\rho(0) \in v^*(1-P)vPv^*v(X)$, Equation (9) defines a solution since $\widehat{R}$ is injective. □

Finally we need to consider the subspace $L^2(\mathbf{R}_+)\otimes v^*\Phi_0 vP(X)$. This subspace gives rise to extended solutions. That is, the solutions we seek here are the second components ξ_2 of a solution $\xi = (\xi_1, \xi_2)^T$ in $e_v(\hat{\mathcal{E}}\oplus\hat{\mathcal{E}})^T \subseteq (\hat{\mathcal{E}}\oplus\mathcal{E})^T$ to the equation $e_v(-\partial_t + \mathcal{D})e_v\xi = 0$, where $\xi_1 \in \hat{\mathcal{E}}$ satisfies $\xi_1 = -ivt\xi_2$. Hence, a true extended solution (one where $\xi_1 \notin \mathcal{E}$) comes from those ξ_2 which behave like $(1+t^2)^{-1/2}$ as $t\to\infty$. With this reminder, we have

Lemma 5.17. *For every vector $x \in v^*(\Phi_0)vP(X)$ there exists a unique solution to the equation $(\widehat{\mathcal{D}}_v + V)\rho = 0$ in the space $L^2(\mathbf{R}_+) \otimes v^*(\Phi_0)vP(X)$ with $\rho(0) = x$. Moreover, every solution in this space is of the form $\rho(t) = (1+t^2)^{-1/2} e^{-(\tan^{-1}(t))v^*dv}\rho(0)$ and*

$$(1)\ \lim_{t\to\infty} -itv(\rho(t)) = -ive^{-\pi/2v^*dv}(\rho(0)) \in \Phi_0(X) \quad \text{and}$$

$$(2)\ \ t \mapsto \left(-ivt\rho(t) + ive^{-(\pi/2)v^*dv}(\rho(0)\right) \quad \text{is in} \quad L^2(\mathbf{R}_+) \otimes v^*(\Phi_0)vP(X).$$

Proof. We define a local parametrix for ρ a continuous function in $L^2(\mathbf{R}_+) \otimes v^*(\Phi_0)vP(X)$ by

$$(\widehat{E}\rho)(t) := \frac{-1}{\sqrt{1+t^2}} e^{-(\tan^{-1}(t))v^*dv} \int_0^t e^{(\tan^{-1}(s))v^*dv}\sqrt{1+s^2}\rho(s)ds.$$

We observe that $(\widehat{E}\rho)$ is differentiable and satisfies $(\widehat{E}\rho)(0) = 0$. To show that this a parametrix, first use $v^*dv = v^*\mathcal{D}v - v^*v\mathcal{D}$ to rewrite

$$\widehat{\mathcal{D}}_v + V = \frac{-1}{\sqrt{1+t^2}}\partial_t\sqrt{1+t^2} + v^*\mathcal{D}v - \frac{1}{1+t^2}v^*dv.$$

As $v^*\mathcal{D}v$ acts as zero on $v^*\Phi_0 v(X)$, this reduces on $v^*\Phi_0 v(X)$ to

$$\widehat{\mathcal{D}}_v + V = \frac{-1}{\sqrt{1+t^2}}\partial_t\sqrt{1+t^2} - \frac{1}{1+t^2}v^*dv.$$

Applying $\frac{-1}{\sqrt{1+t^2}}\partial_t\sqrt{1+t^2}$ to $\widehat{E}(\rho)$ and using the product rule gives

$$\begin{aligned}\frac{-1}{\sqrt{1+t^2}}\partial_t\sqrt{1+t^2}(\widehat{E}\rho)(t) &= \frac{1}{\sqrt{1+t^2}}\partial_t\left(e^{-(\tan^{-1}(t))v^*dv}\int_0^t e^{(\tan^{-1}(s))v^*dv}\sqrt{1+s^2}\rho(s)ds.\right)\\ &= \rho(t) - \frac{v^*dv}{1+t^2}(\widehat{E}\rho)(t).\end{aligned}$$

Thus $(\widehat{\mathcal{D}}_v + V)(\widehat{E}\rho) = \rho$ locally for continuous functions As in previous cases $\widehat{E}$ is locally a closable operator and so by Lemma 5.5 we get that $(\widehat{\mathcal{D}}_v + V)(\widehat{E}\rho) = \rho$ locally for all ρ in the domain of $\widehat{E}$. Hence, $\widehat{E}$ is globally injective. Integration by parts for smooth ρ in the domain gives

$$(\widehat{E}(\widehat{\mathcal{D}}_v + V)\rho)(t) = \rho(t) - (1+t^2)^{-1/2}e^{-\tan^{-1}(t)v^*dv}\rho(0).$$

Applying Lemma 5.5, we get this equation for all $\rho \in \mathrm{dom}(\widehat{\mathcal{D}}_v + V)$. Hence if $\rho \in \ker(\widehat{\mathcal{D}}_v + V)$ we have

$$\rho(t) = (1+t^2)^{-1/2}e^{-\tan^{-1}(t)v^*dv}\rho(0).$$

On the other hand if $x \in v^*(\Phi_0)vP(X)$ and we define ρ by this equation with $\rho(0) = x$ then we have a solution of $(\widehat{\mathcal{D}}_v + V)(\rho) = 0$ in the space $\rho \in L^2(\mathbf{R}_+) \otimes v^*(\Phi_0)vP(X)$. Since for each $t \geq 0$ we have $\rho(t) \in v^*\Phi_0 v(X)$, we also have $-itv(\rho(t)) \in \Phi_0 v(X) \subseteq \Phi_0(X)$, and therefore:

$$\lim_{t\to\infty} -itv(\rho(t)) = -ive^{-\pi/2v^*dv}(\rho(0)) \in \Phi_0(X).$$

It is an exercise to check that $t \mapsto \left(-ivt\rho(t) + ive^{-(\pi/2)v^*dv}(\rho(0)\right)$ is in $L^2(\mathbf{R}_+) \otimes v^*(\Phi_0)vP(X)$. □

Putting together Proposition 5.11 and Lemmas 5.13, 5.14, 5.16, 5.17, we have the following.

Corollary 5.18. *The kernel of $(\widehat{\mathcal{D}}_v + V)$ on $L^2(\mathbf{R}_+) \otimes v^*v(X)$ is isomorphic to the right F-module*

$$\ker(\widehat{\mathcal{D}}_v + V) \cong [v^*(1-P)vP(X)] \oplus [v^*\Phi_0 vP(X)],$$

where the first summand consists of ordinary solutions in $L^2(\mathbf{R}_+, (1+t^2)dt) \otimes v^*v(X)$, while the second summand consists of extended solutions whose second component is in $L^2(\mathbf{R}_+) \otimes v^*v(X)$. Consequently, *taking into account the (trivial) extended solutions of Proposition 5.11,* $(1 - vv^*)\Phi_0(X)$ *we have the full kernel*

$$\ker(e_v((-\partial_t + \mathcal{D}) \otimes 1_2)e_v) \cong [v^*(1-P)vP(X)] \oplus [v^*\Phi_0 vP(X)] \oplus [(1-vv^*)\Phi_0(X)].$$

5.5. **Completing the proof of Theorem 5.1.** Consider the pairing of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ with $\partial_t + \mathcal{D}$. Examining our earlier parametrix computations shows that $\partial_t + \mathcal{D}$ with boundary condition P has no kernel, while $-\partial_t + \mathcal{D}$ with boundary condition $1 - P$ has extended solutions: the constant functions with value in X_0. The projection onto these extended solutions is Φ_0 and $\mathrm{Index}(\partial_t + \mathcal{D}) = -[X_0]$. Since the mapping cone algebra is nonunital, we can not just pair with the class of e_v, but must pair with $[e_v] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$. We have computed the pairing of $(\hat{X}, \hat{\mathcal{D}})$ with both these terms, and so we have the following intermediate result:

Proposition 5.19. *The pairing of* $[e_v] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$ *with* $(\hat{X}, \hat{\mathcal{D}})$ *is given by*

$$\begin{aligned} &\mathrm{Index}(e_v(\partial_t + \mathcal{D})e_v) - \mathrm{Index}(\partial_t + \mathcal{D}) = \mathrm{Index}(e_v(\partial_t + \mathcal{D})e_v) + [X_0] \\ = \ &[v^*(P - \Phi_0)v(1-P)(X)] - [v^*(1-P)vP(X)] - [v^*\Phi_0 vP(X)] - [(1-vv^*)(X_0)] + [X_0] \\ = \ &[v^*Pv(1-P)(X)] - [v^*\Phi_0 v(1-P)(X)] - [v^*(1-P)vP(X)] - [v^*\Phi_0 vP(X)] + [vv^*(X)_0] \\ = \ &[v^*Pv(1-P)(X)] - [v^*(1-P)vP(X)] - [v^*\Phi_0 v(X)] + [vv^*(X_0)] \\ = \ &[v^*Pv(1-P)(X)] - [v^*(1-P)vP(X)]. \end{aligned}$$

The last line follows because $w = v^*\Phi_0$ is a partial isometry with $ww^* = v^*\Phi_0 v$ and $w^*w = vv^*\Phi_0$, showing that the modules defined by these projections are isomorphic.

Now we can finalise the proof of the Theorem by computing the index of

$$PvP : v^*vP(X) \to vv^*P(X),$$

where P is the non-negative spectral projection for $\mathcal{D}$. The kernel of PvP is given by the set

$$\{\xi \in v^*vP(X) : v\xi \in vv^*(1-P)(X) = (1-P)v(X)\} = Pv^*(1-P)v(X),$$

while the cokernel is given by

$$\{\xi \in vv^*P(X) = Pv(X) : \xi = v\eta, \ \ \eta \in v^*v(1-P)(X)\} = (1-P)v^*Pv(X).$$

Thus

$$\mathrm{Index}(PvP) = [Pv^*(1-P)v(X)] - [(1-P)v^*Pv(X)] \in K_0(F).$$

Hence

$$\mathrm{Index}(PvP : v^*vP(X) \to vv^*P(X)) = -\left(\mathrm{Index}(e_v(\partial_t + \mathcal{D})e_v) - \mathrm{Index}(\partial_t + \mathcal{D})\right),$$

and the proof of Theorem 5.1 is complete.

****

Remark When $[\mathcal{D}, v^*dv] = 0$, enormous simplifications occur in the preceeding analysis. In this case one can verify that for the equation $\tilde{\mathcal{D}}_v + V$ in $v^*v\mathcal{E}$, a solution of $\rho = (\tilde{\mathcal{D}}_v + V)\xi$ vanishing at zero is given by

$$\xi(t) = \frac{e^{v^*dv\tan^{-1}(t)}}{\sqrt{1+t^2}} \int_0^t e^{-v^*\mathcal{D}v(t-s)} \sqrt{1+s^2} e^{-v^*dv\tan^{-1}(s)} \rho(s) ds,$$

and we require $\rho \in v^*(P - \Phi_0)v\mathcal{E}$. This formula can be obtained by performing the sums and integrals in the definition of our more general parametrix. Similar comments apply to the other cases.

In the next section we apply Theorem 5.1 to graph algebras and the Kasparov module constructed from the gauge action in [15]. We will see that in this case we can always assume that v^*dv commutes with $\mathcal{D}$, so that we are in the simplest situation described above.

6. Applications to certain Cuntz-Krieger systems

For a detailed introduction to Cuntz-Krieger systems as graph algebras see [20]. A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. **We will always assume that the graph is locally-finite** which means that each vertex emits at most finitely many edges and each vertex receives at most finitely many edges. We write E^n for the set of paths $\mu = \mu_1\mu_2 \cdots \mu_n$ of length $|\mu| := n$; that is, sequences of edges μ_i such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \leq i < n$. The maps r, s extend to $E^* := \bigcup_{n\geq 0} E^n$ in an obvious way. A **sink** is a vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$, a **source** is a vertex $w \in E^0$ with $r^{-1}(w) = \emptyset$ however we will always assume there are **no sources**.

A **Cuntz-Krieger E-family** in a C^*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying the **Cuntz-Krieger relations**

$$S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \quad \text{and} \quad p_v = \sum_{\{e:s(e)=v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

There is a universal C^*-algebra $C^*(E)$ generated by a non-zero Cuntz-Krieger E-family $\{S_e, p_v\}$ [11, Theorem 1.2] . A product $S_\mu := S_{\mu_1} S_{\mu_2} \dots S_{\mu_n}$ is non-zero precisely when $\mu = \mu_1\mu_2 \cdots \mu_n$ is a path in E^n. The Cuntz-Krieger relations imply that words in $\{S_e, S_f^*\}$ collapse to products of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$ satisfying $r(\mu) = r(\nu)$ and we have

$$C^*(E) = \overline{\text{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}. \tag{10}$$

There is a canonical gauge action of $\mathbf{T}$ on $A := C^*(E)$ determined on the generators via: $\gamma_z(p_v) = p_v$ and $\gamma_z(S_e) = zS_e$. Because $\mathbf{T}$ is compact, averaging over γ with respect to normalised Haar measure gives a faithful expectation Φ from A onto the fixed-point algebra $F = A^\gamma$:

$$\Phi(a) := \frac{1}{2\pi} \int_{\mathbf{T}} \gamma_z(a)\, d\theta \quad \text{for} \quad a \in C^*(E), \quad z = e^{i\theta}.$$

As described in [15], right multiplication by F makes A into a right (pre-Hilbert) F-module with inner product: $(a|b)_R := \Phi(a^*b)$. Then X denotes the Hilbert F-module completion of A in the norm

$$\|a\|_X^2 := \|(a|a)_R\|_F = \|\Phi(a^*a)\|_F.$$

For each $k \in \mathbf{Z}$, the projection Φ_k onto the k-th spectral subspace of the gauge action is defined by

$$\Phi_k(x) = \frac{1}{2\pi} \int_{\mathbf{T}} z^{-k} \gamma_z(x) d\theta, \quad z = e^{i\theta}, \quad x \in X.$$

The generator of the gauge action on X, $\mathcal{D} = \sum_{k\in\mathbf{Z}} k\Phi_k$, is determined on the generators of $A = C^*(E)$ by the formula

$$\mathcal{D}(S_\alpha S_\beta^*) = (|\alpha| - |\beta|) S_\alpha S_\beta^*.$$

The following result is proved in [15].

Proposition 6.1. *Let A be the graph C^*-algebra of a directed graph with no sources. Then $(X,\mathcal{D})$ is an odd unbounded Kasparov A-F-module. The operator $\mathcal{D}$ has discrete spectrum, and commutes with left multiplication by $F \subset A$. Set $V = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then (X,V) defines a class in $KK^1(A,F)$.*

We are going to investigate relations in $K_0(M(F,A))$. As graph algebras are generated by partial isometries in A with range and source in F, so $K_0(M(F,A))$ contains a lot of information about A and the underlying graph. The main result of Section 5 will give us more information.

Proposition 6.2. *Let A be the graph C^*-algebra of a locally finite directed graph. Let $\alpha = \alpha_1\alpha_2\cdots\alpha_{|\alpha|}$ be a path in the graph, and S_α the corresponding partial isometry in A. If μ is also a path let $P_\mu = S_\mu S_\mu^*$. Then in $K_0(M(F,A))$ we have the relations*

$$[S_\alpha P_\mu] = \sum_{j=1}^{|\alpha|-1} [S_{\alpha_j} S_{\alpha_{j+1}} S_{\alpha_{j+2}} \cdots S_{\alpha_n} P_\mu S_{\alpha_n}^* \cdots S_{\alpha_{j+2}}^* S_{\alpha_{j+1}}^*] \; + [S_{\alpha_{|\alpha|}} P_\mu],$$

$$[S_\alpha S_\beta^*] = [S_\alpha] - [S_\beta], \quad \alpha,\, \beta \;\text{ paths.}$$

Proof. This proceeds by induction on $|\alpha|$. If $|\alpha| = 0$ then $[S_\alpha] = [p_{r(\alpha)}] = 0$ and if $|\alpha| = 1$, there is nothing to prove. So suppose the relation is true for all α with $|\alpha| < n$. Let α be a path with $|\alpha| = n$ and write $\alpha = \underline{\alpha}\alpha_n$ where $|\underline{\alpha}| = n-1$. Then

$$\begin{aligned}[S_\alpha P_\mu] &= [S_{\underline{\alpha}} S_{\alpha_n} P_\mu] = [S_{\underline{\alpha}} S_{\alpha_n} P_\mu S_{\alpha_n}^* S_{\alpha_n} P_\mu] = [S_{\underline{\alpha}} S_{\alpha_n} P_\mu S_{\alpha_n}^*] + [S_{\alpha_n} P_\mu] \;\text{ by Lemma 3.3}\\ &= \sum^{|\alpha|-2} [S_{\alpha_j} S_{\alpha_{j+1}} S_{\alpha_{j+2}} \cdots S_{\alpha_n} P_\mu S_{\alpha_n}^* \cdots S_{\alpha_{j+2}}^* S_{\alpha_{j+1}}^*] \; + [S_{\alpha_{|\alpha|-1}} S_{\alpha_n} P_\mu S_{\alpha_n}^*] + [S_{\alpha_n} P_\mu],\end{aligned}$$

the last line following by induction. The application of Lemma 3.3 requires

$$(S_{\underline{\alpha}} S_{\alpha_n} P_\mu S_{\alpha_n}^*)^* (S_{\underline{\alpha}} S_{\alpha_n} P_\mu S_{\alpha_n}^*) = S_{\alpha_n} P_\mu S_{\alpha_n}^* = (S_{\alpha_n} P_\mu)(S_{\alpha_n} P_\mu)^*.$$

The second relation follows from Lemma 3.3 also, since $S_\alpha^* S_\alpha = p_{r(\alpha)} = S_\beta^* S_\beta$. □

Lemma 6.3. *Let A be the graph C^*-algebra of a locally finite directed graph E with no sources. Then for all edges $e \in E^1$, the class $[S_e] \in K_0(M(F,A))$ is not zero. Similarly if $r(e) = s(\alpha)$ then $[S_e P_\alpha] \neq 0$.*

Proof. The assumptions on the graph ensure the existence of the Kasparov module $(X,\mathcal{D})$ constructed from the gauge action. The pairing $\langle [S_e P_\alpha], [(X,\mathcal{D})]\rangle$ is given by $[S_e P_\alpha S_e^* \Phi_0] = [S_e P_\alpha S_e^*] \in K_0(F)$, where Φ_0 is the kernel projection of $\mathcal{D}$, whose range is the trivial F-module F. This class is nonzero since F is an AF algebra, and so satisfies cancellation. □

Remark. The hypothesis of 'no sources' was introduced so that we could use the nonzero index pairing to infer nonvanishing of the class $[S_e P_\alpha]$. This restriction may be loosened provided we use other ways of deducing the nonvanishing. For instance, if the class $[P_\alpha] - [S_e P_\alpha S_e^*] = ev_*([S_e P_\alpha]) \neq 0$ in $K_0(F)$, then the class $[S_e P_\alpha]$ cannot be zero. On the other hand, if $[P_\alpha] = [S_e P_\alpha S_e^*]$ in $K_0(F)$, then since F is AF, there exists a partial isometry $v \in F$ such that $S_e P_\alpha S_e^* = vv^*$ and $P_\alpha = v^*v$. Then $u = 1 - P_\alpha + v^* S_e P_\alpha$ is a unitary, and so defines a class in $K_1(A)$. Since the map $K_1(A) \to K_0(M(F,A))$ is an injection, and takes $[u]$ to $[S_e P_\alpha]$, we would know that $[S_e P_\alpha] \neq 0$ if we knew that $[u] \neq 0$.

Corollary 6.4. *Let A be the graph C^*-algebra of a locally finite connected directed graph with no sources. Two nonzero classes $[S_e P_\alpha]$, $[S_f P_\alpha]$, with e, f edges in the graph and α an arbitrary path, are equal if and only if $r(e) = r(f)$. Two nonzero classes $[S_e]$, $[S_f]$, $r(e)$ a sink, are equal, $[S_e] = [S_f]$, if and only if $r(e) = r(f)$.*

Proof. Suppose that $r(e) = r(f)$, and that $[S_eP_\alpha] \neq 0$ (otherwise there is nothing to prove). Then as $S_eP_\alpha S_f^* \in F$ we have

$$0 = [S_eP_\alpha S_f^*] = [S_eP_\alpha] - [S_fP_\alpha],$$

by Lemma 3.3. Conversely, if $r(e) \neq r(f)$ at least one of these classes is zero.

For the second statement we observe that if $r(e) = r(f)$ then $S_eS_f^*$ is nonzero, and then $[S_e] = [S_eS_f^*] + [S_f] = [S_f]$ by Lemma 3.3. If $r(e) \neq r(f)$, we suppose $[S_e] = [S_f]$, for a contradiction, and compute the index pairing with the Kasparov module $(X, \mathcal{D})$ constructed from the gauge action. The pairing is given by

$$\langle [S_e], [(X, \mathcal{D})] \rangle = -[S_eS_e^*] = -[S_fS_f^*] = \langle [S_f], [(X, \mathcal{D})] \rangle.$$

Hence the class of $S_eS_e^*$ in $K_0(F)$ (F is the fixed point algebra) coincides with the class of $S_fS_f^*$. Since F is an AF algebra, there exists a partial isometry $v \in \text{span}\{S_\mu S_\nu^* : |\mu| = |\nu|\}$ such that $S_eS_e^* = vS_fS_f^*v^*$. Thus

$$p_{r(e)} = S_e^*vS_f\, S_f^*v^*S_e = \sum_j c_j\overline{c_k} S_e^*S_{\mu_j}S_{\nu_j}^*S_f\, S_f^*S_{\nu_k}S_{\mu_k}^*S_e.$$

Here the paths μ_j start from $s(e)$ and end at some vertex v_j, while the corresponding path ν_j starts from $s(f)$ and ends at the same vertex v_j. Moreover there is at least one path μ_j with $S_e^*S_{\mu_j} \neq 0$ so $\mu_j = e\mu_{j_2} \cdots \mu_{j_k}$, where $|\mu_j| = k$. However, $r(e)$ is a sink, so any such path is of the form $\mu_j = e$. This forces the length of the corresponding ν_j to be 1, and $\nu_j = f$. The only way the product $S_{\mu_j}S_{\nu_j}^* = S_eS_f^*$ can now be non-zero is if $r(e) = r(f)$, contradicting our assumption. □

Corollary 6.5. *Let A be the graph C^*-algebra of a locally finite connected directed graph with no sources. Then if two partial isometries of the form $[S_e], [S_f]$ satisfy $[S_e] = [S_f] \in K_0(M(F, A))$ then there exists a partial isometry ρ in F such that $\rho S_e = S_f$ and $\rho^*\rho S_e = S_e = \rho^*S_f$.*

Proof. The required partial isometry ρ is $S_fS_e^*$. The remaining statements are immediate. □

Lemma 6.6. *Let E be a row-finite directed graph. Then the group $K_0(M(C^*(E)^\gamma, C^*(E)))$ is generated by the classes $[S_eP_\alpha]$, where e is an edge and α is a path.*

Proof. Let $[v] \in K_0(M(C^*(E)^\gamma, C^*(E)))$ and consider

$$ev_*[v] = [v^*v] - [vv^*] \in K_0(C^*(E)^\gamma).$$

Now $K_0(C^*(E)^\gamma)$ is generated by the classes $[p_\mu]$, $p_\mu = S_\mu S_\mu^*$, where $\mu \in E^*$ is a path, [14]. As $C^*(E)^\gamma$ is an AF algebra, there are partial isometries W, Z over $C^*(E)^\gamma$ such that

$$W^*W = v^*v, \quad WW^* = \sum_j p_{\mu_j}, \qquad ZZ^* = vv^*, \quad Z^*Z = \sum_k p_{\nu_k}, \tag{11}$$

and $[v] = [Z^*vW^*]$. The latter follows because Z, W are partial isometries over F and so represent zero, while $[Z^*vW^*] = [Z^*] + [v] + [W^*]$. In Equation (11) the sums are necessarily orthogonal, and may be in a matrix algebra over $C^*(E)^\gamma$, and some zeroes (place-holders to make the matrix dimensions equal) may have been omitted from the sums. Observe that $ev_*[Z^*vW^*] = \sum_k [p_{\nu_k}] - \sum_j [p_{\mu_j}]$. By considering $p_{\nu_k}Z^*vW^*p_{\mu_j}$ we may suppose without loss of generality that we have only one summand so that $WW^* = p_\mu$ and $Z^*Z = p_\nu$. Then

$$ev_*[Z^*vW^*S_\mu S_\nu^*] = [p_\nu] - [p_\nu] = 0.$$

Hence $[v] = [Z^*vW^*] = [S_\nu S_\mu^*]$ modulo the image of i_*, and Lemma 6.2 completes the proof for $[v] \notin \text{Image}(i_*)$. Observe that $S_\nu S_\mu^* \neq 0$ (and so $r(\mu) = r(\nu)$) is a consequence.

In the case $ev_*[v] = 0$, so that $[v] \in \text{Image}(i_*)$ we observe that there is a partial isometry X over $C^*(E)^\gamma$ such that $X^*X = v^*v$ and $XX^* = vv^*$ so that $1 - v^*v + X^*v$ is unitary. Then, again since all partial isometries are over F,

$$[v] = [WX^*vW^*] = [WX^*ZZ^*vW^*] = [WX^*ZS_\nu S_\mu^*] = i_*[1 - p_\mu + WX^*ZS_\nu S_\mu^*]$$

gives a unitary representative of v. Since $i_*[1-p_\mu + WX^*ZS_\nu S_\mu^*] = [S_\nu S_\mu^*]$, Lemma 6.2 completes the proof. □

The structure of $K_1(M(F, A))$ is even simpler.

Lemma 6.7. *If E is a row-finite directed graph, $A = C^*(E)$ and $F = C^*(E)^\gamma$, then $K_1(M(F, A)) = 0$.*

Proof. The exact sequence $0 \to A \otimes C_0(0, 1) \to M(F, A) \to F \to 0$ and $K_1(F) = 0$ yields

$$0 \to K_1(A) \to K_0(M(F, A)) \stackrel{ev_*}{\to} K_0(F) \to K_0(A) \to K_1(M(F, A)) \to 0. \tag{12}$$

By Lemma 3.1, the map $K_0(F) \to K_0(A)$ is induced (up to sign and Bott periodicity) by inclusion $j : F \to A$. This map is surjective on K_0 by [14][Lemma 4.2.2], and so $K_1(M(F, A)) = 0$. □

In [14], the K-theory of a graph algebra $C^*(E)$, where E has no sources or sinks, was computed as the kernel (K_1) and cokernel (K_0) of the map given by the vertex matrix on $\mathbf{Z}^{E^0}$ (there are subtleties when sinks are involved). The proof of this result involves the dual of the gauge action and the Pimsner-Voiculescu exact sequence for crossed products. In Equation (12), we see the K-theory again expressed as the kernel and cokernel of a map, but this time it arises with no serious effort. The difference of course is that the groups $K_0(M(F, A))$ and $K_0(F)$ are in general harder to compute.

While the map $ev_* : K_0(M(F, A)) \to K_0(F)$ is neither one-to-one nor onto in general, we can deduce that the two groups $K_0(M(F, A))$ and $K_0(F)$ are in fact isomorphic in a wide range of examples. We let $(\hat{X}, \hat{\mathcal{D}})$ be the APS Kasparov module arising from the Kasparov module $(X, \mathcal{D})$.

Proposition 6.8. *Let A be the graph C^*-algebra of a locally finite connected directed graph with no sources and no sinks. Then the map $\text{Index}_{\hat{\mathcal{D}}} : K_0(M(F, A)) \to K_0(F)$ given by the Kasparov product with the Kasparov module of the gauge action is an isomorphism.*

Proof. First the index map is a well-defined homomorphism, [10]. We begin by showing that the index map is one-to-one. So suppose that we have edges e, g and paths α, β in our graph (with no range a sink), and suppose that $\text{Index}_{\hat{\mathcal{D}}}([S_eP_\alpha]) = \text{Index}_{\hat{\mathcal{D}}}([S_gP_\beta])$. A simple computation using Theorem 5.1 yields

$$\text{Index}_{\hat{\mathcal{D}}}([S_eP_\alpha]) = [S_eP_\alpha S_e^*] = [S_gP_\beta S_g^*] = \text{Index}_{\hat{\mathcal{D}}}([S_gP_\beta]).$$

As F is an AF algebra, we can find a partial isometry v in F such that

$$S_eP_\alpha S_e^* = vS_gP_\beta S_g^* v^*.$$

Then setting $w = P_\alpha S_e^* v S_g P_\beta \neq 0$ we have

$$P_\alpha = ww^* = wP_\beta w^* \quad \text{and} \quad P_\beta = w^*w = w^*P_\alpha w.$$

We will use Lemma 3.3 below and need to check that some partial isometries have the same source projections. First observe that $(S_eP_\alpha wP_\beta)^*(S_eP_\alpha wP_\beta) = P_\beta = w^*w$, so

$$[S_eP_\alpha] = [S_eP_\alpha wP_\beta w^*] = [S_eP_\alpha wP_\beta] + [w^*] = [S_eP_\alpha wP_\beta] = [S_eP_\alpha S_e^* v S_g P_\beta],$$

the second last last equality following since w is a partial isometry in F. Now since $(S_gP_\beta)(S_gP_\beta)^* = S_gP_\beta S_g^*$ and $(S_eP_\alpha S_e^* v)^*(S_eP_\alpha S_e^* v) = S_gP_\beta S_g^*$, we can apply Lemma 3.3 again to find

$$[S_eP_\alpha] = [S_eP_\alpha S_e^* v S_g P_\beta] = [S_eP_\alpha S_e^* v] + [S_gP_\beta] = [S_gP_\beta].$$

Thus $\mathrm{Index}_{\hat{\mathcal{D}}}$ is one-to-one. Now supposing that our graph has no sinks, every class in $K_0(F)$ is a sum of classes $[p_\mu] = [S_\mu S_\mu^*]$, where μ is a path in the graph of length at least one. For a given $\mu = \mu_1 \cdots \mu_{|\mu|}$, define $\overline{\mu} = \mu_2 \cdots \mu_{|\mu|}$. Then it is straightforward to check that

$$\mathrm{Index}_{\hat{\mathcal{D}}}([S_\mu S_{\overline{\mu}}^*]) = [p_\mu].$$

Hence the index map is onto and we are done. □

Observe that this does not mean that the K-theory of the graph algebra is zero! The evaluation map and the index map are very different. For the Cuntz algebra O_n, $n \geq 2$, for example, the fixed point algebra has K-theory $K_0(F) \cong \mathbf{Z}[1/n]$ and so we have

$$ev_*([S_\mu]) = [1] - [S_\mu S_\mu^*] \sim 1 - \frac{1}{n^{|\mu|}} = (n^{|\mu|} - 1)\frac{1}{n^{|\mu|}},$$

with $\ker(ev_*) \cong K_1(O_n) = 0$ and $\mathrm{coker}(ev_*) \cong K_0(O_n) = \mathbf{Z}_{n-1}$. The index map gives us

$$\mathrm{Index}_{\hat{\mathcal{D}}}([S_\mu]) = \sum_{j=0}^{|\mu|-1} [S_\mu S_\mu^* \Phi_j].$$

This equality follows from Theorem 5.1, and to determine the right hand side more explicitly, set $\overline{\mu} = \mu_{j+1} \cdots \mu_{|\mu|}$ and define the partial isometry $W = S_\mu S_{\overline{\mu}}^* \Phi_0$. Then $WW^* = S_\mu S_\mu^* \Phi_j$ and $W^*W = S_{\overline{\mu}} S_{\overline{\mu}}^* \Phi_0$. Thus in $K_0(F)$ we have

$$\mathrm{Index}_{\hat{\mathcal{D}}}([S_\mu]) = \sum_{j=0}^{|\mu|-1} [S_\mu S_\mu^* \Phi_j] = \sum_{j=0}^{|\mu|-1} [S_{\overline{\mu}} S_{\overline{\mu}}^* \Phi_0] = \sum_{j=0}^{|\mu|-1} [S_{\overline{\mu}} S_{\overline{\mu}}^*] \sim \sum_{j=0}^{|\mu|-1} n^{-(|\mu|-j)} = \left(\frac{n^{|\mu|} - 1}{n-1}\right) \frac{1}{n^{|\mu|}}.$$

The evaluation map and the mapping cone exact sequence gives us $K_0(M(O_n^\gamma, O_n)) \cong (n-1)\mathbf{Z}[1/n]$ (those polynomials all of whose coefficients have a factor of $n-1$) which is of course isomorphic to $\mathbf{Z}[1/n] \cong K_0(F)$ as an additive group.

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