

SPECTRAL FLOW INVARIANTS AND TWISTED CYCLIC THEORY FROM THE HAAR STATE ON $SU_q(2)$

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Abstract

In [CPR2], we presented a K -theoretic approach to finding invariants of algebras with no non-trivial traces. This paper presents a new example that is more typical of the generic situation. This is the case of an algebra that admits only non-faithful traces, namely $SU_q(2)$ and also KMS states. Our main results are index theorems (which calculate spectral flow), one using ordinary cyclic cohomology and the other using twisted cyclic cohomology, where the twisting comes from the generator of the modular group of the Haar state. In contrast to the Cuntz algebras studied in [CPR2], the computations are considerably more complex and interesting, because there are nontrivial ‘eta’ contributions to this index.

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1. INTRODUCTION

Motivated by our study of semifinite spectral triples and Kasparov modules for graph algebras, [CPR1, PR], we have found a new way of extracting invariants from algebras using non-tracial states. The basic constructions and first examples are in [CPR2], where we studied the Cuntz algebras using their unique KMS states for the canonical gauge action. However, we have found that this example, though illuminating, is not generic. Here we study a more generic situation, the example of $SU_q(2)$ using the Haar state, which is KMS for a certain circle action (which is not the gauge action).

The approach of [CPR2] yields a local index formula in twisted cyclic cohomology, where the twisting comes from the generator of the modular group of the KMS state. The chief drawback of twisted cyclic theory, from the point of view of index theory, is that it does not pair with K -theory. However, in [CPR2] we showed that there was an abelian group, called modular K_1 , which pairs with twisted cyclic cohomology. It seems that in general we must understand algebras that admit both traces (not necessarily faithful) and KMS states. The former situation, described in detail in Section 7 for $SU_q(2)$, exploits semifinite index theory, using untwisted cyclic cohomology. The latter, in Section 8, uses the Haar state and the associated twisted cyclic theory, and leads to a pairing with modular K_1 . In both cases the pairing is given by computing spectral flow in the semifinite sense as described in [CP2]. This example points to the existence of a rich interplay between the tracial and KMS index theories.

Many of the constructions of this paper mirror those of [CPR1, CPR2, PR], and we give precise references to those papers for more information about our constructions and, where necessary, for proofs. We focus here on the new aspects of these constructions that $SU_q(2)$ throws up.

There are two motivations for this study. First there is the observation [CLM] that there is a way to associate directed graphs to Mumford curves and that from the corresponding graph C^* -algebras we might extract topological information about the curves. It eventuates that $SU_q(2)$ is an example of a graph algebra that shares with the Mumford curve graph algebras the property that it does not admit faithful traces but does admit faithful KMS states for nontrivial circle actions. If we are going to be able to exploit graph algebras to study invariants of Mumford curves then we need to demonstrate that it is possible to actually calculate numerical invariants explicitly. We find in Sections 7 and 8 that we are able to obtain not only abstract formulae but also the numbers produced by these formulae for particular unitaries in matrix algebras over $SU_q(2)$.

The second motivation comes from the general formula obtained in the main result in Section 8, namely Theorem 8.2. This Theorem shows that there are two contributions to spectral flow in the twisted cocycle one of which comes from truncated eta type correction terms. This example points to the existence of a ‘twisted eta cocycle’, a matter we plan to investigate further in another place.

The novel feature of our approach is to make use of the structure of $SU_q(2)$ as a graph C^* -algebra and a small part of its Hopf algebra structure via the Haar state. We find that this is best described by using an intermediate presentation of the algebra in terms of generators which are functions of the graph algebra generators. The Haar state provides us with a natural faithful KMS state on $SU_q(2)$ which we want to use because any trace on $SU_q(2)$ cannot be faithful. This has the consequence that any Dixmier type trace on $SU_q(2)$ will only see ‘part’ of the algebra.

Nevertheless for non-faithful traces we can calculate what the odd semifinite local index formula in noncommutative geometry [CPRS2] tells us. More specifically we construct a particular $(1, \infty)$ summable semifinite spectral triple for $SU_q(2)$ in Section 7. Then in subsection 7.5 we use ideas from [CPR1] to give some explicit computations which are actually the result of pairings with the K -theory of a mapping cone algebra constructed from $SU_q(2)$. These calculations use general formulae for spectral flow in von Neumann algebras found in [CP2, CPS2]. These pairings yield rational functions of the deformation parameter q , in fact q -numbers, which are naturally interpreted as q -winding numbers.

Then, in Section 8, we turn to the question of what information we can extract from the Haar state or ‘twisted’ situation (where ‘twist’ refers to twisted cyclic cohomology). We constructed a twisted cocycle and the pairing with what we termed ‘modular K -theory’ in [CPR2]. When applied to particular unitaries in matrix algebras over $SU_q(2)$ the pairing of the twisted cocycle with modular K_1 gives a spectral flow invariant that is a polynomial function of the deformation parameter q . These functions are distinct from those obtained in the tracial case although they depend on the same variables. We believe that the mapping cone plays a role here as well but much further investigation needs to be done to prove this.

We remark that our aims are different from those of [ChP1, ChP2, DLSSV] where the quantum group structure of the algebra plays the main role through the construction of equivariant spectral

triples, some satisfying conditions related to the axioms for a noncommutative spin manifold [Co2]. In this paper we ignore the quantum group structure of $SU_q(2)$ in order to explain by example an ‘index theory for KMS states’.

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2. THE C^* -ALGEBRA OF $SU_q(2)$

In this Section we will describe the relationship between two descriptions of the algebra $C(SU_q(2))$; the ‘traditional’ q -deformation picture [W], and the graph algebra picture, [HS].

2.1. The q deformation picture of $SU_q(2)$. We recall the construction of the algebra $A = SU_q(2)$.

Proposition 1. *Given any $q \in [0, 1)$ there is a unital C^* -algebra A with elements a, b , b normal, satisfying the relations*

$$(1) \quad a^*a + b^*b = 1, aa^* + q^2bb^* = 1, ab = qba, ab^* = qb^*a,$$

with the following universal property. Let A_{cc} be the algebra $\mathbb{C}[a, b]$ modulo the above relations. Then every $$ -algebra homomorphism from A_{cc} to a C^* -algebra B extends to a unique C^* -algebra homomorphism from A to B . In particular A_{cc} is dense in A .*

Proof. This is a restatement of [W, Theorem 1.1]. □

For the rest of this paper, $0 \leq q < 1$. However we observe that the Haar state is not faithful for $q = 0$, see Lemma 3 below, and so any statement relying on the faithfulness of the Haar state requires $0 < q < 1$. We will use the following \mathbb{Z}^2 -grading of A_{cc} in many places below.

Proposition 2. *The algebra A_{cc} has a \mathbb{Z}^2 -grading so that $A_{cc} = \bigoplus_{(m,n) \in \mathbb{Z}^2} A_{cc}[m, n]$. With respect to this grading we have $\deg(a) = (-1, 0)$, $\deg(b) = (0, 1)$.*

It is well known, (we will amplify below), that for all $0 \leq q < 1$, the C^* -algebras of $C(SU_q(2))$ are all isomorphic. What changes with q is the quantum group structure, and this is captured, in part, by the Haar state.

Proposition 3. *For each $q \in [0, 1)$, there is a state h , the Haar state, such that for all $x \in A_{cc}[m, n]$, we have $h(x) = 0$ unless $m = n = 0$ and on $A_{cc}[0, 0]$ the state h is given by*

$$(2) \quad h(b^{*n}b^n) = (1 - q^2)/(1 - q^{2n+2}).$$

Proof. The existence of a unique invariant (with respect to the coproduct) state is given in [W, chapter 1]. This coproduct restricts to the algebraic coproduct on A_{cc} and so the restriction of this state to A_{cc} is invariant with respect to the algebraic coproduct. But the existence of, and the above formula for, a unique invariant functional on A_{cc} is given in [KS, section 4.3.2]. □

We now extend Propositions 2 and 3 to the C^* -algebra A .

Proposition 4. *The \mathbb{Z}^2 -grading of A_{cc} extends to a \mathbb{Z}^2 -grading of A and the Haar state satisfies $h(x) = 0$ for all $x \in A[m, n]$ with $(m, n) \neq (0, 0)$.*

Proof. We first define a function $\gamma : \mathbb{T}^2 \times A \rightarrow A$ by

$$(3) \quad \gamma_{z,w}(a) = z^{-1}a, \quad \gamma_{z,w}(b) = wb$$

and extend $\gamma_{z,w}$ as a $*$ -homomorphism. It is clear that for all $(z, w) \in \mathbb{T}^2$ the elements $z^{-1}a, wb$ satisfy the same relations as a, b , and so the above defines $\gamma_{z,w}$ uniquely as an algebra homomorphism from A_{cc} to A . Using Proposition 1, $\gamma_{z,w}$ is uniquely defined as a C^* -homomorphism from A to itself.

It is easy to check that γ is a group action. To show it is strongly continuous, first note that by definition γ acts on A_{cc} strongly continuously. Let $x \in A$ and $\epsilon > 0$. Since A_{cc} is dense in A , we can choose some $y \in A_{cc}$ such that $\|x - y\| < \epsilon/3$. Since γ acts by C^* -algebra homomorphisms, this implies $\|\gamma_{z,w}(x) - \gamma_{z,w}(y)\| < \epsilon/3$, for all $(z, w) \in \mathbb{T}^2$. Then for (z, w) sufficiently close to $(0, 0)$, we have $\|y - \gamma_{z,w}(y)\| < \epsilon/3$. Combined with the other two inequalities this gives,

$$\|x - \gamma_{z,w}(x)\| \leq \|x - y\| + \|y - \gamma_{z,w}(y)\| + \|\gamma_{z,w}(x) - \gamma_{z,w}(y)\| < \epsilon.$$

Since \mathbb{T}^2 is compact, we can construct the following operators $\Phi_{m,n} : A \rightarrow A$,

$$(4) \quad \Phi_{m,n}(x) := (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} z^{-m} w^{-n} \gamma_{z,w}(x) d\theta d\phi, \quad z = e^{i\theta}, w = e^{i\phi},$$

for $(m, n) \in \mathbb{Z}^2$. First note that for all m, n the map Φ_{mn} is continuous since

$$\|\Phi_{mn}(x)\| \leq (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} \|z^{-m} w^{-n} \gamma_{z,w}(x)\| d\theta d\phi = \|x\|$$

since γ acts by isometries. Furthermore, by a change of variables in the defining integral we have

$$\gamma_{z,w} \Phi_{m,n}(x) = z^m w^n \Phi_{m,n}(x).$$

From this it follows that $\Phi_{m,n} \Phi_{m',n'} = \delta_{m,m'} \delta_{n,n'} \Phi_{m,n}$. Since the $\Phi_{m,n}$ are continuous projections, $A[m, n] := \Phi_{m,n} A$ is a closed subspace of A . Further, we have $A = \bigoplus A[m, n]$ because A_{cc} is dense in A , while every element of A_{cc} is a finite sum of elements of pure degree. Finally, given $x \in A[m, n]$ and $y \in A[m', n']$ we have $\gamma_{z,w}(xy) = z^{m+m'} w^{n+n'} xy$. This implies $xy \in A[m+m', n+n']$. Therefore $A = \bigoplus A[m, n]$ is an algebra grading of A . \square

2.2. $SU_q(2)$ as a graph algebra. By a directed graph we mean a quadruple $E = (E^0, E^1, r, s)$ where E^0 and E^1 are countable sets which we call the vertices and edges of E and r, s are maps from E^1 to E^0 which we call the range and source maps respectively. We call a vertex v a sink if $s^{-1}(v)$ is empty.

A Cuntz-Krieger E -family in C^* -algebra B is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{S_e : e \in E^1\}$ satisfying the Cuntz-Krieger relations:

$$(5) \quad S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{s(e)=v} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

There is a universal C^* algebra $C^*(E)$ generated by a non-zero Cuntz-Krieger E -family $\{S_e, p_v\}$, see for instance [KPR, Theorem 1.2] or [R]. More precisely, we have

Proposition 5. *For every row finite directed graph E , there is a C^* -algebra $C^*(E)$ containing a Cuntz-Kreiger E -family $\{S_e, p_v\}$, with the property that for every C^* -algebra A containing a Cuntz-Kreiger E -family $\{S'_e, p'_v\}$, there is a unique C^* -algebra homomorphism from $C^*(E)$ to A that maps $\{S_e, p_v\}$ to $\{S'_e, p'_v\}$.*

Denote by E^* the set of finite directed paths in E . We can extend the range and source maps from E^1 to E^* . Given a path $\rho = e_1 e_2 \cdots e_k$, we denote by $S_\rho \in C^*(E)$ the partial isometry $S_{e_1} S_{e_2} \cdots S_{e_k}$. Some results we will use from the theory of graph C^* -algebras are

Proposition 6 ([KPR, R]). *The algebra $C^*(E)$ is densely spanned by the monomials of the form $S_\rho S_\sigma^*$ where ρ, σ are paths in E^* with $r(\rho) = r(\sigma)$ which form a subalgebra denoted by A_c .*

Proposition 7 ([KPR, R]). *If ρ, σ are paths of equal length then*

$$(6) \quad S_\rho^* S_\sigma = \delta_{\rho, \sigma} p_{r(\rho)}.$$

We now specialise to the graph algebra presentation of the C^* -algebra of $SU_q(2)$, $0 \leq q < 1$ which is row-finite, that is the set $s^{-1}(v) := \{e : s(e) = v\}$ is finite for all $v \in E^0$.

Definition 1. *Let B be the Cuntz-Kreiger algebra associated to the graph given in Figure 1. The vertex set E^0 is $\{v, w\}$ and the edge set is $\{\mu, \nu, \xi\}$.*

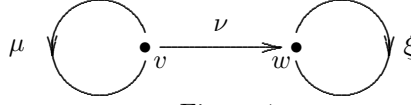


Figure 1

The particular Cuntz-Krieger relations here are

$$(7) \quad S_\mu^* S_\mu = S_\nu S_\nu^* + S_\mu S_\mu^* = p_v, \quad S_\xi^* S_\xi = S_\nu^* S_\nu = S_\xi S_\xi^* = p_w.$$

Proposition 8 ([HS]). *There is an isomorphism of C^* -algebras $\phi : A \rightarrow B$ satisfying*

$$(8) \quad \phi(a) = \sum_{k=0}^{\infty} \left(\sqrt{1 - q^{2(k+1)}} - \sqrt{1 - q^{2k}} \right) (S_\mu + S_\nu)^k (S_\mu^* + S_\nu^*)^{k+1}$$

$$(9) \quad \phi(b) = \sum_{k=0}^{\infty} q^k (S_\mu + S_\nu)^k S_\xi (S_\mu^* + S_\nu^*)^k$$

By this isomorphism we identify A and B from now on, and use the letter A for both. We keep A_{cc} for the polynomials in a, b (the generators in the $SU_q(2)$ picture) and A_c for the dense subalgebra of polynomials in the graph algebra generators. Note that elements of A_c are not in general polynomials in the generators a, b of $C(SU_q(2))$, and conversely. Next we prove A_c is a graded subalgebra of A .

Proposition 9. *The elements S_μ, S_ν have pure degree $(1, 0)$ while S_ξ has pure degree $(0, 1)$.*

Proof. Let $\tilde{\gamma}$ be the action of \mathbb{T}^2 on A given by

$$(10) \quad \tilde{\gamma}_{z,w}(S_\mu) = zS_\mu, \tilde{\gamma}_{z,w}(S_\nu) = zS_\nu, \tilde{\gamma}_{z,w}(S_\xi) = wS_\xi.$$

By the universal property of graph C^* -algebras, this defines $\tilde{\gamma}_{z,w}$ uniquely as a C^* -algebra homomorphism from A to A . Equations (8) and (9) imply $\tilde{\gamma}_{z,w}(a) = z^{-1}a$ and $\tilde{\gamma}_{z,w}(b) = wb$. But this is exactly how γ was uniquely defined, and so the two actions are equal. Thus we can combine equations (10) and (4) to obtain the gradings of S_μ, S_ν and S_ξ . \square

Before proceeding further we establish some notation for dealing with this algebra. This notation represents a specialisation of the graph algebra notation for this particular graph.

Definition 2. *Given any non-negative integer m , and $n \in \mathbb{Z}$, let*

$$(11) \quad T_k = \begin{cases} p_v & k = 0 \\ S_\mu^k & k \geq 1 \end{cases}, \quad \tilde{T}_k = \begin{cases} p_w & k = 0 \\ S_\nu & k = 1 \\ S_\mu^{k-1} S_\nu & k \geq 2 \end{cases}, \quad U_n = \begin{cases} S_\xi^{*|n|} & n \leq -1 \\ p_w & n = 0 \\ S_\xi^n & n \geq 1 \end{cases}.$$

Lemma 1. *The following hold in A , for all $k, l \in \mathbb{N}_0$ and $n, n' \in \mathbb{Z}$.*

$$(12) \quad \tilde{T}_k = \tilde{T}_k p_w,$$

$$(13) \quad T_k = p_v T_k p_v,$$

$$(14) \quad \tilde{T}_k^* \tilde{T}_l = \delta_{kl} p_w, \quad T_k^* T_k = p_v$$

$$(15) \quad \tilde{T}_{k+1} \tilde{T}_{l+1}^* = T_k T_l^* - T_{k+1} T_{l+1}^*,$$

$$(16) \quad T_k T_k^* \tilde{T}_l = \begin{cases} \tilde{T}_l & k \leq l-1, \\ 0 & k > l-1. \end{cases}$$

$$(17) \quad U_n U_{n'} = U_{n+n'}$$

$$(18) \quad T_k T_l = T_{k+l}$$

Proof. Equations (12) and (13) are trivial. Equations (14) and (16) follow from (6). Equations (17) and (18) follow from (7). From (18) and (7) we have

$$T_{k+1} T_{l+1}^* = T_k S_\mu S_\mu^* T_l = T_k p_v T_l^* - T_k S_\nu S_\nu^* T_l$$

From the definition we have $T_k S_\nu = \tilde{T}_{k+1}$, and along with (13), we obtain (15). \square

Lemma 2. *The algebra A_c is generated, as a vector space, by elements of the form $T_k T_l^*$ for $k, l \in \mathbb{N}_0$ and $\tilde{T}_k U_n \tilde{T}_l^*$ for $k, l \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. Hence elements of this type densely span A .*

Proof. By Proposition 6, we need only show that for all $\rho, \sigma \in E^*$ with $r(\rho) = r(\sigma)$, the monomial $S_\rho S_\sigma^*$ can be written in one of the above forms. First we consider the case $r(\rho) = r(\sigma) = v$. Since there is no directed path from w to v , every path in E ending in v must have zero length or be of the form μ^k for some $k \geq 1$. Hence we have $S_\rho S_\sigma^* = T_k T_l^*$ for some $k, l \in \mathbb{N}_0$, since the cases $k = 0$ and $l = 0$ deal with the cases when ρ and σ are respectively of zero length.

Next we have the case $r(\rho) = r(\sigma) = w$. By similar reasoning, S_ρ and S_σ both take one of the following forms

$$S_\mu^k S_\nu S_\xi^l, \quad S_k S_\nu, \quad S_\nu S_\xi^l, \quad S_\nu, \quad S_\xi^l p_w.$$

Note that all of these can be written in the form $\tilde{T}_k U_n$ where $k, n \in \mathbb{N}_0$. Hence $S_\rho S_\sigma^*$ is of the form $\tilde{T}_k U_{n-n'} \tilde{T}_l^*$ by Equation (17). \square

Lemma 3. *We have the following formulae for the Haar state on generators:*

$$(19) \quad h(\tilde{T}_k U_n \tilde{T}_l^*) = \delta_{n,0} \delta_{k,l} q^{2k} (1 - q^2),$$

$$(20) \quad h(T_k T_l^*) = \delta_{k,l} q^{2k+2}.$$

Proof. By considering the grading of the terms on the left hand side and applying Proposition 4 we are reduced to the case $k = l$ and $n = 0$. We first simplify Equation (9). Since $S_\nu S_\mu = S_\nu S_\nu = 0$, when we expand $(S_\mu + S_\nu)^k$ for $k \geq 1$ we obtain exactly two terms, \tilde{T}_k and T_k . Note that $T_k U_1 = 0$ by Equation (13). Hence for $k > 0$,

$$(S_\mu + S_\nu)^k S_\xi (S_\mu + S_\nu)^{*k} = (\tilde{T}_k + T_k) U_1 (\tilde{T}_k^* + T_k^*) = \tilde{T}_k U_1 \tilde{T}_k^*.$$

For $k = 0$ the corresponding formula is $U_1 = p_w U_1 p_w = \tilde{T}_0 U_1 \tilde{T}_0^*$. Now \tilde{T}_k is a partial isometry so $\|\tilde{T}_k\| = \|\tilde{T}_k^*\| = 1$. Since U_1 is also a partial isometry, $\tilde{T}_k U_1 \tilde{T}_k^*$ has norm at most 1, and so the series

$$(21) \quad b = \sum_{k=0}^{\infty} q^k \tilde{T}_k U_1 \tilde{T}_k^*$$

converges absolutely. Therefore we have

$$\begin{aligned} bb^* &= \sum_{k,k'=0}^{\infty} q^{k+k'} \tilde{T}_k U_1 \tilde{T}_k^* \tilde{T}_{k'} U_{-1} \tilde{T}_{k'}^* = \sum_{k,k'=0}^{\infty} \delta_{k,k'} q^{k+k'} \tilde{T}_k U_1 p_w U_{-1} \tilde{T}_k^* \quad \text{by (14)} \\ (22) \quad &= \sum_{k,k'=0}^{\infty} \delta_{k,k'} q^{k+k'} \tilde{T}_k \tilde{T}_{k'}^* = \sum_{k=0}^{\infty} q^{2k} \tilde{T}_k \tilde{T}_k^*. \end{aligned}$$

Raising the above to the n -th power and again applying Equation (14) we obtain

$$\begin{aligned} b^n b^{*n} &= \sum_{k_1, k_2, \dots, k_n=0}^{\infty} q^{2(k_1+k_2+\dots+k_n)} \tilde{T}_{k_1} \tilde{T}_{k_1}^* \tilde{T}_{k_2} \tilde{T}_{k_2}^* \dots \tilde{T}_{k_n} \tilde{T}_{k_n}^* \\ &= \sum_{k_1, k_2, \dots, k_n=0}^{\infty} \delta_{k_1, k_2} \delta_{k_2, k_3} \dots \delta_{k_{n-1}, k_n} q^{2(k_1+k_2+\dots+k_n)} \tilde{T}_{k_1} \tilde{T}_{k_n}^* = \sum_{k=0}^{\infty} q^{2kn} \tilde{T}_k \tilde{T}_k^* \end{aligned}$$

Evaluating the Haar state on both sides, Equation (2) gives

$$(23) \quad \sum_{k=0}^{\infty} q^{2kn} h(\tilde{T}_k \tilde{T}_k^*) = \frac{1 - q^2}{1 - q^{2(n+1)}}.$$

From this we wish to calculate $h(\tilde{T}_k \tilde{T}_k^*)$ for all $k \geq 0$. First we have shown the norm of $\tilde{T}_k \tilde{T}_k^*$ is at most 1. Therefore as $|h(x)| \leq \|x\|$,

$$(24) \quad \left| \sum_{k=l+1}^{\infty} q^{2kn} h(\tilde{T}_k \tilde{T}_k^*) \right| \leq \sum_{k=l+1}^{\infty} q^{2kn} |h(\tilde{T}_k \tilde{T}_k^*)| \leq \sum_{k=l+1}^{\infty} q^{2kn} = \frac{q^{2(l+1)n}}{1 - q^{2n}},$$

for all $l \in \mathbb{N}$. We now prove $h(\tilde{T}_l \tilde{T}_l^*) = q^{2l}(1 - q^2)$ by induction. For the case $l = 0$, we take the limit of Equation (23) as $n \rightarrow \infty$. Then by Equation (24), the left hand side converges to $h(\tilde{T}_0 \tilde{T}_0^*)$ while the right hand side converges to $1 - q^2$. Given $l > 0$, the inductive hypothesis and Equation (23) yield

$$(25) \quad \left(\sum_{k=0}^{l-1} q^{2k(n+1)}(1 - q^2) \right) + q^{2ln} h(\tilde{T}_l \tilde{T}_l^*) + \sum_{k=l+1}^{\infty} q^{2kn} h(\tilde{T}_l \tilde{T}_l^*) = \frac{1 - q^2}{1 - q^{2(n+1)}}.$$

Summing the first sum we obtain

$$\frac{(1 - q^2)(1 - q^{2l(n+1)})}{1 - q^{2(n+1)}} + q^{2ln} h(\tilde{T}_l \tilde{T}_l^*) + \sum_{k=l+1}^{\infty} q^{2kn} h(\tilde{T}_l \tilde{T}_l^*) = \frac{1 - q^2}{1 - q^{2(n+1)}}.$$

Cancelling and moving the remaining sum to the right hand side gives

$$-\frac{(1 - q^2)q^{2l(n+1)}}{1 - q^{2(n+1)}} + q^{2ln} h(\tilde{T}_l \tilde{T}_l^*) = -\sum_{k=l+1}^{\infty} q^{2kn} h(\tilde{T}_l \tilde{T}_l^*)$$

Taking the absolute value of both sides, and applying the estimate (24) we obtain

$$q^{2ln} \left| h(\tilde{T}_l \tilde{T}_l^*) - \frac{(1 - q^2)q^{2l}}{1 - q^{2(n+1)}} \right| \leq \frac{q^{2(l+1)n}}{1 - q^{2n}}$$

Cancelling the q^{2ln} on both sides and taking the limit as $n \rightarrow \infty$, we obtain Equation (??). Equation (20) follows inductively from this, Equation (15) and $h(1) = 1$. \square

Proposition 10. *The Haar state h on A is KMS (for $\beta = 1$) with respect to the action $\sigma : \mathbb{R} \times A \rightarrow A$ defined by*

$$\sigma_t(\tilde{T}_k) = q^{it2k} \tilde{T}_k, \quad \sigma_t(T_k) = q^{it2k} T_k, \quad \sigma_t(U_n) = U_n, \quad k \in \mathbb{N}_0, \quad n \in \mathbb{Z}.$$

Proof. Using the formulae for the Haar state on generators, it follows that $h(ab) = h(\sigma_i(b)a)$ for $a, b \in A_c$. For $a, b \in \{T_k, \tilde{T}_k\}$ we have a holomorphic function

$$F_{a,b}(z) = q^{-Im(z)2k} h(\sigma_{Re(z)}(b)a) = h(\sigma_z(b)a)$$

in the strip $0 \leq Im(z) \leq 1$, with boundary values given by

$$F_{a,b}(t + 0i) = h(\sigma_t(b)a), \quad F_{a,b}(t + i) = q^{-2k} h(\sigma_t(b)a) = h(\sigma_{t+i}(b)a) = h(a\sigma_t(b)).$$

The proposition now follows by standard theory see [BR1, Ped]. \square

3. THE GNS REPRESENTATION FOR THE HAAR STATE

As usual, since h is a state we may form the inner product on A given by $\langle a, b \rangle := h(a^*b)$. This gives a norm $\|a\|_{\mathcal{H}_h} := \langle a, a \rangle^{1/2}$ and

$$(26) \quad \|a\|_{\mathcal{H}_h}^2 = h(a^*a) \leq h(\|a^*a\|1) = \|a^*a\|.$$

We denote by \mathcal{H}_h the completion of A with respect to this norm and we denote the extension of this norm to \mathcal{H}_h by $\|\cdot\|_{\mathcal{H}_h}$. By construction we can consider A to be a subspace of \mathcal{H}_h and as an A -module \mathcal{H}_h has a cyclic and separating vector, namely 1.

Proposition 11 ([KR, Proposition 9.2.3]). *There is a self-adjoint unbounded operator H on \mathcal{H}_h and conjugate linear isometry J such that $S = JH^{1/2}$ where S is defined by $Sx \cdot 1 = x^* \cdot 1$, for all $x \in A$. The operator H generates a one parameter group that implements, on the GNS space, the modular automorphism group.*

Lemma 4. *The set $\{e_{k,n,l}\}_{n \in \mathbb{Z}, k, l \in \mathbb{N}}$ where $e_{k,n,l} := (1 - q^2)^{-1/2} q^{-l} \tilde{T}_k U_n \tilde{T}_l^*$, is an orthonormal basis for \mathcal{H}_h . For all $n \in \mathbb{Z}$ and $k, l \in \mathbb{N}$, $e_{k,n,l}$ is an eigenvector for the H with eigenvalue $q^{2(k-l)}$.*

Proof. First we show these elements are mutually orthogonal. By Equation (14) we have

$$\langle \tilde{T}_{k'} U_{n'} \tilde{T}_{l'}^*, \tilde{T}_k U_n \tilde{T}_l^* \rangle = h(\tilde{T}_{l'} U_{n'}^* \tilde{T}_{k'}^* \tilde{T}_k U_n \tilde{T}_l^*) = \delta_{k,k'} h(\tilde{T}_{l'} U_{n-n'} \tilde{T}_l^*).$$

The operator $\tilde{T}_{l'} U_{n-n'} \tilde{T}_l^*$ has grading $(l' - l, n - n')$ and so by Proposition 3, the above becomes

$$\langle \tilde{T}_{k'} U_{n'} \tilde{T}_{l'}^*, \tilde{T}_k U_n \tilde{T}_l^* \rangle = \delta_{k,k'} \delta_{l,l'} \delta_{n,n'} h(\tilde{T}_l \tilde{T}_l^*).$$

Then the correct normalization follows from Equation (19). Thus the $e_{k,n,l}$ are orthonormal, and it remains to show that they span \mathcal{H}_h .

Lemma 2 shows that the linear span of the elements $T_k T_l^*, \tilde{T}_{l'} U_n \tilde{T}_k'^*$ with $l, l', k, k' \in \mathbb{N}_0, n \in \mathbb{Z}$, is dense in A . By Equation (26) this set is also dense in \mathcal{H}_h . Therefore it suffices to show that for each k, l we can approximate $T_k T_l^*$ by elements of our basis. Indeed, by Equation (15) we have

$$(27) \quad \sum_{j=0}^{N-1} \tilde{T}_{k+j+1} \tilde{T}_{l+j+1}^* = \sum_{j=0}^{N-1} (T_{k+j} T_{l+j}^* - T_{k+j+1} T_{l+j+1}^*) = T_k T_l^* - T_{k+N} T_{l+N}^*.$$

Now as $N \rightarrow \infty$ we have

$$\|T_{k+N} T_{l+N}^*\|_{\mathcal{H}_h}^2 = \tau(T_{l+N} T_{k+N}^* T_{k+N} T_{l+N}^*) = \tau(T_{l+N} T_{l+N}^*) = q^{2(l+N)+2} \rightarrow 0$$

where we have used Equations (14), (12) and (20) above. Therefore taking the limit in \mathcal{H}_h of both sides of Equation (27) as $N \rightarrow \infty$ we obtain

$$T_k T_l^* = \sum_{j=0}^{\infty} \tilde{T}_{k+j+1} \tilde{T}_{l+j+1}^* = \sum_{j=1}^{\infty} \tilde{T}_{k+j} \tilde{T}_{l+j}^*.$$

Note that this does not hold in the norm topology on A .

For any $x, y \in A$, we have

$$(28) \quad \langle x^*, y^* \rangle = \langle Sx, Sy \rangle = \langle JH^{1/2}x, JH^{1/2}y \rangle = \langle Hx, y \rangle$$

Now note that

$$e_{k,n,l}^* = (1 - q^2)^{-1/2} q^{-l} (\tilde{T}_k U_n \tilde{T}_l^*)^* = (1 - q^2)^{-1/2} q^{-l} \tilde{T}_l U_{-n} \tilde{T}_k^* = q^{k-l} e_{l,-n,k}$$

Substituting this into Equation (28) we obtain

$$q^{2(k-l)} \langle e_{l,-n,k}, e_{l',-n',k'} \rangle = \langle H e_{k,n,l}, e_{k',n',l'} \rangle$$

But we have already shown that

$$\langle e_{l,-n,k}, e_{l',-n',k'} \rangle = \langle e_{k,n,l}, e_{k',n',l'} \rangle = \delta_{k,k'} \delta_{n,n'} \delta_{l,l'},$$

so we obtain

$$\langle H e_{k,n,l}, e_{k',n',l'} \rangle = q^{2(k-l)} \langle e_{k,n,l}, e_{k',n',l'} \rangle.$$

Since $e_{k',n',l'}$ is an orthonormal basis of \mathcal{H}_h , we have $H e_{k,n,l} = q^{2(k-l)} e_{k,n,l}$. □

In fact the linear span of the eigenvectors $e_{k,n,l}$ form a core for H by [RS, Theorem VIII.11].

Lemma 5. *There is a unique strongly continuous group action σ of \mathbb{T} on A such that for all $a \in A$, $z \in \mathbb{T}$ and $\xi \in \mathcal{H}_h$,*

$$(29) \quad \sigma_z(a)\xi = H^{it} a H^{-it} \xi,$$

where $z = e^{it \log q^2} = q^{2it}$. This action satisfies

$$(30) \quad \sigma_z(\tilde{T}_k U_n \tilde{T}_l^*) = z^{k-l} \tilde{T}_k U_n \tilde{T}_l^*$$

$$(31) \quad \sigma_z(T_k T_l^*) = z^{k-l} T_k T_l^*, \quad n \in \mathbb{Z}, \quad k, l \in \mathbb{N}_0$$

Proof. We define $\sigma_z = \gamma_{z,1}$ where γ is the action of \mathbb{T}^2 on A defined in the proof of Proposition 4. Since γ is a strongly continuous group action by C^* -algebra isomorphisms, so is σ . Comparing Equations (30) and (31) with Lemma 4, it is clear that Equation (29) holds whenever $a \in A_c$ and $\xi \in \mathcal{H}_h$. The usual $\epsilon/3$ proof now shows that Equation (29) holds on all of A . Uniqueness follows from the faithfulness of the Haar state. That is, $A \rightarrow B(\mathcal{H}_h)$ is injective. This follows from considering the action on $1 \in \mathcal{H}_h$. Therefore Equation (29) determines σ_z uniquely. \square

Observation If ρ is a path in E^* then we denote by $|\rho|'$ the number of edges in ρ counting only the edges μ and ν . Then the action σ is analogous to the gauge action in [PR] while $|\cdot|'$ is analogous to path length in [PR]. As a result many of the proofs in [PR] also apply here with only these changes allowing us to avoid repetition of these arguments.

4. THE KASPAROV MODULE

We denote by $F = A^\sigma$ the fixed point algebra for the KMS action of Lemma 5. Since this action of the reals factors through the circle, we can define a positive faithful expectation $\Phi : A \rightarrow F$ by

$$(32) \quad \Phi(x) = \frac{-\log q^2}{2\pi} \int_0^{(2\pi)/-\log q^2} \sigma_z(x) dt, \quad z = e^{it \log q^2}.$$

We form the norm $\|\cdot\|_X$ on A by setting $\|a\|_X^2 := \|\Phi(a^*a)\|_F$. Note that this norm is always greater than the GNS norm $h(a^*a)$ since

$$h(a^*a) = h(\Phi(a^*a)) \leq \|\Phi(a^*a)\| h(1) = \|\Phi(a^*a)\|$$

Therefore we can consider the completion X of A with respect to $\|\cdot\|_X$ to lie in \mathcal{H}_h . We denote by X_c the image of A_c in X . For each $z \in \mathbb{T}$, σ_z is a norm continuous map from A to A . We also have

$$\|\sigma_z(x)\|_X = \|\Phi(\sigma_z(x)^* \sigma_z(x))\|_F = \|\Phi(\sigma_z(x^*x))\|_F = \|\Phi(x^*x)\|_F = \|x\|_X$$

for all $x \in A$. Hence for all $z \in \mathbb{T}$, σ_z is an isometry with respect to the norm $\|\cdot\|_X$, and so extends to an isometry from X to X . This defines a strongly continuous action of \mathbb{T} on X . We define an F -valued inner product on X by $(x|y)_F = \Phi(x^*y)$.

Given $m \in \mathbb{Z}$ and $x \in X$ define the map $\Phi_m : X \rightarrow X$ by

$$(33) \quad \Phi_m(x) = \frac{-\log q^2}{2\pi} \int_0^{2\pi/-\log q^2} z^{-m} \sigma_z(x) dt, \quad z = e^{it \log q^2}.$$

Lemma 6. *The operators Φ_m restrict to continuous operators from A to A and as such they are the projections onto the space $\oplus_{n \in \mathbb{Z}} A[m, n]$.*

Proof. The first statement follows from the definition of Φ_m , and the fact that σ_z is a strongly continuous action on A . Since the Φ_m and the projections $\oplus_n \Phi_{m,n}$ are both norm continuous maps on A , it suffices to show that they coincide on the monomials of Lemma 2. This follows from Equations (30) and (31) and the definition of the algebra grading. \square

The proofs of the next three statements are minor reworkings of the proofs of the analogous statements in [PR], as the reader may check.

Lemma 7. *The operators Φ_m are adjointable endomorphisms of the F -module X such that $\Phi_m^* = \Phi_m$ and $\Phi_l \Phi_m = \delta_{l,m} \Phi_l$. The sum $\sum_{m \in \mathbb{Z}} \Phi_m$ converges strictly to the identity operator in X .*

Corollary 1. *For all $x \in X$, the sum $\sum_{m \in \mathbb{Z}} x_m$ where $x_m = \Phi_m x$, converges in X to x .*

We may now define an unbounded self-adjoint regular operator \mathcal{D} on X as in [PR].

Proposition 12. *Let $X_{\mathcal{D}}$ be the set of all $x \in X$ such that $\|\sum_{m \in \mathbb{Z}} m^2 (x_m | x_m)_F\| < \infty$ where $x_m = \Phi_m x$. Define the operator $\mathcal{D} : X_{\mathcal{D}} \rightarrow X$ by $\mathcal{D}x = \sum_{m \in \mathbb{Z}} m x_m$. Then \mathcal{D} is a self-adjoint regular unbounded operator on X .*

To show that the pair (X, \mathcal{D}) gives us a Kasparov module, we need to analyse the spectral projections of \mathcal{D} as endomorphisms of the module X . To do this, we recall the following

Definition 3. Rank 1-operators on the right C^* - F -module X , $\Theta_{x,y}$ for some $x, y \in X$, are given by

$$(34) \quad \Theta_{x,y}(z) = x(y|z)_F, \quad x, y, z \in X.$$

Denote by $\text{End}_F^{00}(X)$ the linear span of the rank one operators, which we call the finite rank operators. Denote by $\text{End}_F^0(X)$ the closure of $\text{End}_F^{00}(X)$ with respect to the operator norm $\|\cdot\|_{\text{End}}$ on $\text{End}_F(X)$. We call elements of $\text{End}_F^0(X)$ compact endomorphisms.

Lemma 8. For all $a \in A$ and $m \in \mathbb{Z}$ the operator $a\Phi_m$ belongs to $\text{End}_F^{00}(X)$.

Proof. For any rank 1 operator $\Theta_{x,y}$ we have $a\Theta_{x,y} = \Theta_{ax,y}$ so it suffices to show that for each $m \in \mathbb{Z}$ we have $\Phi_m \in \text{End}_F^{00}(X)$. Indeed, we show the following;

$$(35) \quad \Phi_m = \begin{cases} \Theta_{T_m, T_m} + \Theta_{\tilde{T}_m, \tilde{T}_m}, & m \geq 1, \\ \Theta_{1,1}, & m = 0, \\ \Theta_{T_m^*, T_m^*} + \Theta_{\tilde{T}_m^*, \tilde{T}_m^*}, & m \leq -1. \end{cases}$$

Recall that we can write any x in the form $x = \sum_{l \in \mathbb{Z}} x_l$ where $x_l = \Phi_l x$. Therefore if $y \in \Phi_m X$, we have

$$\Theta_{y,y}x = \Theta_{y,y} \sum_{l \in \mathbb{Z}} x_l = \sum_{l \in \mathbb{Z}} \Theta_{y,y}x_l = \sum_{l \in \mathbb{Z}} y\Phi(y^*x_l).$$

Now by Lemma 6 we have $\Phi(y^*x_l) = \delta_{l,m}y^*x_l$ and so the above implies that $\Theta_{y,y}x = yy^*\Phi_m x$.

Thus it suffices to prove that whenever $x \in \Phi_m X$ we have

$$(36) \quad x = \begin{cases} T_m T_m^* x + \tilde{T}_m \tilde{T}_m^* x, & m \geq 1, \\ x, & m = 0, \\ T_m^* T_m x + \tilde{T}_m^* \tilde{T}_m x, & m \leq -1. \end{cases}$$

By continuity it suffices to consider the case where $x \in X_c$. By linearity we are then reduced to the case where x is one of the monomials of Lemma 2.

When $m = 0$ there is nothing to prove. When $m \geq 1$, x has the form $\tilde{T}_{m+k} U_n \tilde{T}_k^*$ for some $k \geq 0$ and $n \in \mathbb{Z}$ or $T_{m+k} T_k^*$ for some $k \geq 0$. Now $\tilde{T}_m \tilde{T}_m^* + T_m T_m^* = T_{m-1} T_{m-1}^*$ for $m > 1$ and $\tilde{T}_1 \tilde{T}_1^* + T_1 T_1^* = p_v$ by the definitions. So for $m \geq 1$, Lemma 1 gives us

$$(\tilde{T}_m \tilde{T}_m^* + T_m T_m^*)x = T_{m-1} T_{m-1}^* x = x.$$

When $m \leq -1$ note that $T_m^* T_m = p_v$ and $\tilde{T}_m^* \tilde{T}_m = p_w$, and $p_v + p_w = 1_A$ acts as the identity of the C^* -module X , so we are done. \square

Lemma 8 underlies the proof of the following result which is analogous to [PR].

Lemma 9. The operator $a(1 + \mathcal{D}^2)^{-1/2}$ is a compact operator for all $a \in A$. Let $V = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$. Then (X, V) is a Kasparov A, F module, and so represents a class in $KK^1(A, F)$.

We shall refer to this Kasparov module as the Haar module for A, F , since (X, V) only depends on the algebra A , the action σ_t and the Haar state.

5. K-THEORY

Recall the well known results $K_0(A) = \mathbb{Z}$ and $K_1(A) = \mathbb{Z}$ from for example [HS]. The generators of these groups are $[1]$ and $[p_v + U_1]$ respectively. Now we need to examine the algebra F .

Lemma 10. The fixed point algebra F is the minimal unitization of the algebra $\oplus_{i=0}^{\infty} C(S^1)$.

Proof. Given any $x \in F$ choose a sequence $(y_n) \subset A_{cc}$ with $y_n \rightarrow x$. Then $\Phi y_n \rightarrow x$ also. Now by [W, Theorem 1.2], A_{cc} is spanned by monomials of the form $a^{i_1} b^{i_2} b^{*i_3}$ and $a^{*i_1} b^{i_2} b^{*i_3}$. The expectation Φ acts on A_{cc} is zero except on the monomials $b^{i_2} b^{*i_3}$, on which it acts as the identity. Thus F is densely spanned by powers of b and b^* . Since b is normal, by the Stone-Weierstrass Theorem we have $F = C(\text{spec}(b))$.

Now we show that $\text{spec}(b^*b) = \{0, \dots, q^{2k}, \dots, q^4, q^2, 1\}$. First recall that

$$\text{spec}(a^*a) \cup \{0\} = \text{spec}(aa^*) \cup \{0\}.$$

Applying the relations in Equation (1) and the spectral mapping theorem, the above implies

$$(1 - \text{spec}(b^*b)) \cup \{0\} = (1 - q^2 \text{spec}(b^*b)) \cup \{0\}.$$

This implies

$$\text{spec}(b^*b) \cup \{1\} = (q^2 \text{spec}(b^*b)) \cup \{1\}.$$

The only sets that satisfy the above are $\{0\}$ and the hypothesised spectrum of $\text{spec}(b^*b)$. But we know that $\text{spec}(b^*b)$ is not $\{0\}$ because this would imply b was zero.

By the spectral theorem, using the map $z \mapsto z\bar{z}$, we know that $\text{spec}(b)$ is a subset of the closure of the union of circles $\{z : \|z\| = q^m, m \geq 0\}$, and must contain at least one point in each circle. The action $\gamma_{1,w}$ sends b to wb , where γ is the action of \mathbb{T} on A defined in the proof of Proposition 4. However since it is an isomorphism it preserves the spectrum of b . Therefore $\text{spec}(b)$ contains the union of circles $\{z : \|z\| = q^m, m \geq 0\}$. Since it is closed it also contains 0. Hence it is exactly this set. This is also the one point compactification of the disjoint union of countably many circles, so F is the minimal unitization of $\oplus_{i=0}^{\infty} C(S^1)$. \square

Corollary 2. *The group $K_0(F)$ is given by $K_0(F) = \mathbb{Z} \oplus \bigoplus_{i=0}^{\infty} \mathbb{Z}$ and is freely generated by 1 and $\tilde{T}_k \tilde{T}_k^*$ for $k \in \mathbb{N}_0$. The group $K_1(F) = \bigoplus_{i=0}^{\infty} \mathbb{Z}$ and has generators $1 - \tilde{T}_k \tilde{T}_k^* + \tilde{T}_k U_1 \tilde{T}_k^*$ for $k \in \mathbb{N}_0$.*

Proof. The K -theory of F is generated by the projections onto each of the circles (connected components) in $\text{spec } b$, and 1. Thus we need to show the spectral projection onto the circle with radius q^k is $\tilde{T}_k \tilde{T}_k^*$. As each circle is connected, the spectral projection of b corresponding to each circle has no non-zero proper sub-projections, and by the spectral theorem is also the spectral projection onto the point q^{2k} in the spectrum of bb^* . Therefore it suffices, since $\tilde{T}_k \tilde{T}_k^*$ is nonzero by the universality of the graph C^* -algebra, to show that $\tilde{T}_k \tilde{T}_k^*$ satisfies $bb^* \tilde{T}_k \tilde{T}_k^* = q^{2k} \tilde{T}_k \tilde{T}_k^*$. This follows from the formula for bb^* , Equation (22), by the following calculation

$$\begin{aligned} bb^* \tilde{T}_k \tilde{T}_k^* &= \left(\sum_{l=0}^{\infty} q^{2l} \tilde{T}_l \tilde{T}_l^* \right) \tilde{T}_k \tilde{T}_k^* \\ &= \sum_{l=0}^{\infty} q^{2l} \delta_{l,k} \tilde{T}_l p_w \tilde{T}_k^* \quad \text{by Equation (14)} \\ &= q^{2k} \tilde{T}_k \tilde{T}_k^* \quad \text{by Equation (12)}. \end{aligned}$$

We can use the trace $h|_F$ to map $K_0(F)$ to the real numbers. By Lemma 3 we obtain

$$h_*(K_0(F)) = \mathbb{Z} + \sum_{i=0}^{\infty} (1 - q^2) q^{2i} \mathbb{Z} = \mathbb{Z}[q^2].$$

Here the first copy of \mathbb{Z} is generated by $h(1) = 1$, while the other terms come from $h(\tilde{T}_k \tilde{T}_k^*) = (1 - q^2) q^{2k}$. From these we may generate any polynomial in q^2 . As all the generators of $K_0(F)$ are clearly independent, we obtain the whole polynomial group $\mathbb{Z}[q^2]$. The generators of $K_1(F) = \bigoplus_{i=0}^{\infty} K_1(C(S^1))$ are given by $[1 - \tilde{T}_k \tilde{T}_k^* + q^{-k} b \tilde{T}_k \tilde{T}_k^*]$. In order to write these in terms of the graph algebra generators we first expand b according to Equation (21), and then apply Equation (14);

$$q^{-k} b \tilde{T}_k \tilde{T}_k^* = q^{-k} \left(\sum_{l=0}^{\infty} q^l \tilde{T}_l U_1 \tilde{T}_l^* \right) \tilde{T}_k \tilde{T}_k^* = \tilde{T}_k U_1 \tilde{T}_k^*.$$

□

We also require the K -theory of the mapping cone algebra $M(F, A)$ for the inclusion of F in A . Recall that the mapping cone is the C^* -algebra

$$M(F, A) = \{f : [0, 1] \rightarrow A : f \text{ is continuous, } f(1) = 0, f(0) \in F\}.$$

The even K -theory group of the mapping cone algebra can be described as homotopy classes of partial isometries $v \in M_\infty(A)$ with $vv^*, v^*v \in M_\infty(F)$ [P].

Lemma 11. *The group $K_0(M(F, A))$ is generated by the classes of partial isometries $[T_1]$ and $[\tilde{T}_k]$ for $k \in \mathbb{N}$. An alternative generating set is $[T_k]$, $k \in \mathbb{N}$ along with $[\tilde{T}_1]$.*

Proof. From the exact sequence $0 \rightarrow C_0(0, 1) \otimes A \rightarrow M(F, A) \rightarrow F \rightarrow 0$ we obtain the exact sequence in K -theory

$$\begin{array}{ccccc} K_1(A) & \rightarrow & K_0(M(F, A)) & \xrightarrow{ev_*} & K_0(F) \\ \uparrow & & & & \downarrow j_* \\ K_1(F) & \leftarrow & K_1(M(F, A)) & \leftarrow & K_0(A) \end{array}$$

where ev is evaluation at 1 and $j : F \rightarrow A$ is the inclusion map. By an explicit construction (see [CPR1, P]), a partial isometry $v \in M_\infty(A)$ satisfying $vv^*, v^*v \in M_\infty(F)$ gives a projection p_v in the matrices over the unitization of $M(F, A)$, and $[p_v] - [1] \in K_0(M(F, A))$. In particular the evaluation at 1 of this matrix is given by

$$(37) \quad \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

One easily checks that this gives the class $[vv^*] - [v^*v] \in K_0(F)$. On the other hand the map from $K_1(A)$ to $K_0(M(F, A))$ takes a unitary over A to itself, considered as a partial isometry with range and source 1. To see this, observe that the isomorphism between homotopy classes of partial isometries and $K_0(M(F, A))$ given in [P] takes a class $[u] \in K_1(A)$ to the class $[p_u] - [1] \in K_0(M(F, A))$. The projection p_u defining the class $[p_u] - [1]$ is the same as the projection used to define the map which identifies $K_1(A)$ and $K_0(C_0(0, 1) \otimes A)$, [HR].

Now we need the map from $K_1(F)$ to $K_1(A)$. We can compute this map from the other boundary map using Bott periodicity. In particular applying the same reasoning to the same mapping cone exact sequence tensored by $C_0(0, 1)$, we find that this map is given by the composition

$$K_1(F) \xrightarrow{\simeq} K_0(C_0(0, 1) \otimes F) \xrightarrow{j_*} K_0(C_0(0, 1) \otimes A) \xrightarrow{\simeq} K_1(A).$$

We only need to know the image of this map, and since the generator $[p_v + U_1]$ of $K_1(A)$ is also a generator of $K_1(F)$, this map is surjective. Therefore the map from $K_1(A)$ to $K_0(M(F, A))$ is the zero map. Therefore the map from $K_0(M(F, A))$ to $K_0(F)$ is injective, and we have $K_0(M(F, A)) = ev_*^{-1}(\ker j_*)$.

To calculate $\ker j_*$ on $K_0(F)$, first note the following Murray-von Neumann equivalences in A ;

$$(38) \quad \tilde{T}_k \tilde{T}_k^* \sim \tilde{T}_k^* \tilde{T}_k = p_w,$$

$$(39) \quad \begin{aligned} p_v &= S_\mu S_\mu^* + S_\nu S_\nu^* = (S_\mu + S_\nu)(S_\mu + S_\nu)^* \\ &\sim (S_\mu + S_\nu)^*(S_\mu + S_\nu) = S_\mu^* S_\mu + S_\nu^* S_\nu = p_v + p_w. \end{aligned}$$

Together these imply $[\tilde{T}_k \tilde{T}_k^*] = [0]$ in $K_0(A)$ for all $k \in \mathbb{N}_0$. The other generator $[1]$ of $K_0(F)$ is the generator of $K_0(A)$. Therefore $\ker j_*$ is generated by $[\tilde{T}_k \tilde{T}_k^*]$ for $k \in \mathbb{N}_0$.

We may also then say that $\ker j_*$ is generated by the elements $[\tilde{T}_0 \tilde{T}_0^*] = [p_w]$ and $[\tilde{T}_k \tilde{T}_k^*] - [p_w]$ for $k \in \mathbb{N}$. We can invert these elements under the map ev_* as follows

$$(40) \quad [p_w] = [(S_\mu^* + S_\nu^*)(S_\mu^* + S_\nu^*)^*] - [(S_\mu^* + S_\nu^*)^*(S_\mu^* + S_\nu^*)] = ev_*[S_\mu + S_\nu]$$

$$(41) \quad [\tilde{T}_k \tilde{T}_k^*] - [p_w] = [\tilde{T}_k \tilde{T}_k^*] - [\tilde{T}_k^* \tilde{T}_k] = ev_*([\tilde{T}_k])$$

Therefore $K_0(M(F, A))$ is generated by $[S_\mu + S_\nu]$ and $[\tilde{T}_k]$ for $k \in \mathbb{N}$. Since S_μ and S_ν have orthogonal ranges, by [CPR1, Lemma 3.4] we have

$$[S_\mu + S_\nu] = [S_\mu] + [S_\nu] = [\tilde{T}_1] + [T_1].$$

Thus, in the notation we prefer, we may say that $K_0(M(F, A))$ is generated by the classes $[T_1]$ and $[\tilde{T}_k]$ for $k \geq 1$. To prove the claim about the other generating set, we use [CPR1, Lemmas 3.3, 3.4] again to show that

$$\begin{aligned} [T_k] &= [S_\mu^k] = [S_\mu^k S_\mu S_\mu^*] + [S_\mu^k S_\nu S_\nu^*] \\ &= [S_\mu^{k+1}] - [S_\mu] + [S_\mu^k S_\nu] - [S_\nu] \\ &= [T_{k+1}] - [T_1] + [\tilde{T}_{k+1}] - [\tilde{T}_1] \end{aligned}$$

This is enough to give our other generating set. \square

6. THE INDEX PAIRING FOR THE MAPPING CONE

We are interested in the odd pairing in KK -theory. So let u be a unitary in $M_k(A)$, and $(Y, 2P - 1)$, P a projection, an odd Kasparov module for the algebras A, F , see [K] for more information. The pairing in KK -theory between $[u] \in K_1(A)$ and $[(Y, 2P - 1)] \in KK^1(A, F)$ is given, [PR], by the map

$$\begin{aligned} H : K_1(A) \times KK^1(A, B) &\rightarrow K_0(B), \\ H([u], [(Y, 2P - 1)]) &:= [\ker(P_k u P_k)] - [\operatorname{coker}(P_k u P_k)], \end{aligned}$$

where $P_k = P \otimes Id_k$, where Id_k is the identity of $M_k(\mathbb{C})$, and we are computing the index of the map $P_k u P_k : P_k Y^k \rightarrow P_k Y^k$. However, the generator of $K_1(SU_q(2))$ is (the class of) $p_v + U_1$, which commutes with \mathcal{D} , and so with the nonnegative spectral projection of \mathcal{D} . Hence the pairing of our Kasparov module for $SU_q(2)$ with K -theory is zero.

The index pairing of the following definition was introduced in [CPR1]. To show that it is well-defined requires extending an odd Kasparov module for A, F (with $F \subset A$ a subalgebra) to an even Kasparov module for $M(A, F), F$, where $M(F, A)$ is the mapping cone algebra for the inclusion of F into A .

Definition 4 ([CPR1]). For $[v] \in K_0(M(F, A))$ and $(Y, 2P - 1)$ an odd (A, F) -Kasparov module with P commuting with $F \subset A$ acting on the left, define

$$\begin{aligned} (42) \quad \langle [v], (Y, V) \rangle &:= \operatorname{Index}(PvP : v^*vPY \rightarrow vv^*PY) \\ &= [\ker(PvP)] - [\operatorname{coker}(PvP)] \in K_0(F). \end{aligned}$$

Proposition 13. Let (X, \mathcal{D}) be the Haar module of $A = C(SU_q(2))$, and $P = \chi_{[0, \infty]}(\mathcal{D})$. The pairing of Equation (42) for (X, \mathcal{D}) is determined by the following pairings on generators of $K_0(M(F, A))$:

$$\langle [T_k], [(X, \mathcal{D})] \rangle = - \sum_{l=0}^{k-1} [T_k T_k^* \Phi_l], \quad \langle [\tilde{T}_k], [(X, \mathcal{D})] \rangle = - \sum_{l=0}^{k-1} [\tilde{T}_k \tilde{T}_k^* \Phi_l].$$

The pairing for the adjoints is of course given by the negatives of these classes.

Proof. We first calculate the kernel and cokernel of the map $PvP : v^*vPX \rightarrow vv^*PX$ when v is given by one of $\tilde{T}_k, \tilde{T}_k^*, T_k, T_k^*$. By Lemma 6, $T_k \Phi_m = \Phi_{m+k} T_k$. This implies

$$(43) \quad PT_k = T_k \chi_{[-k, \infty)}(\mathcal{D}), \quad T_k P = \chi_{[k, \infty)}(\mathcal{D}) T_k.$$

where $\chi_{[-k, \infty)}$ is the characteristic function of the interval $[-k, \infty)$. Therefore we can rewrite the operator in question as

$$T_k \chi_{[-k, \infty)}(\mathcal{D}) P : T_k^* T_k P X \rightarrow T_k T_k^* P X.$$

This is the same as the operator

$$(44) \quad T_k P : T_k^* T_k P X \rightarrow T_k T_k^* P X.$$

Now the range and source projections of T_k lie in F and hence commute with all the spectral projections of \mathcal{D} . Therefore we can use Equation (43) to restrict the isomorphism $T_k : T_k^* T_k X \rightarrow T_k T_k^* X$ to obtain an isomorphism

$$T_k : T_k^* T_k P X \rightarrow T_k T_k^* \chi_{[k, \infty)}(\mathcal{D}) X.$$

From this it is evident that the kernel of the operator in Equation (44) is zero and the cokernel is $T_k T_k^* \chi_{[0, k-1]}(\mathcal{D}) X$. Similarly for the partial isometry T_k^* we may write the operator $P T_k^* P$ as

$$(45) \quad T_k^* \chi_{[k, \infty)}(\mathcal{D}) : T_k T_k^* P X \rightarrow T_k^* T_k P X.$$

By the same methods as before we obtain the isomorphism

$$T_k^* : T_k T_k^* \chi_{[k, \infty)}(\mathcal{D}) X \rightarrow T_k^* T_k P X.$$

From this we can see that the cokernel of the operator in Equation (45) is empty while the kernel is $T_k T_k^* \chi_{[0, k-1]}(\mathcal{D}) X$. Exactly the same reasoning gives the result for the cases \tilde{T}_k and \tilde{T}_k^* . \square

We will relate this $K_0(F)$ -valued index to two different numerical indices in the next two Sections.

7. THE SEMIFINITE SPECTRAL TRIPLE FOR $SU_q(2)$

We now wish to consider the Hilbert space \mathcal{H}_h again, this time to construct a von Neumann algebra which has a semifinite trace induced by h . From this we can construct a semifinite spectral triple, which we use to compute the index pairing using the spectral flow formula of [CP2].

7.1. Semifinite spectral triples. We use the viewpoint of [CPRS2] on semifinite spectral triples. Given a von Neumann algebra \mathcal{N} with a faithful, normal, semifinite trace τ , there is a norm closed ideal $\mathcal{K}_{\mathcal{N}}$ generated by the projections $E \in \mathcal{N}$ with $\tau(E) < \infty$.

Definition 5. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a Hilbert space \mathcal{H} , a $*$ -algebra $\mathcal{A} \subset \mathcal{N}$ where \mathcal{N} is a semifinite von Neumann algebra acting on \mathcal{H} , and a densely defined unbounded self-adjoint operator \mathcal{D} affiliated to \mathcal{N} such that $[D, a]$ is densely defined and extends to a bounded operator for all $a \in \mathcal{A}$ and $a(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_{\mathcal{N}}$ for all $\lambda \notin \mathbb{R}$ and all $a \in \mathcal{A}$. The triple is said to be even if there is some $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$ and $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Otherwise it is odd.

We note that if $T \in \mathcal{N}$ and $[\mathcal{D}, T]$ is bounded, then $[\mathcal{D}, T] \in \mathcal{N}$.

Definition 6. A $*$ -algebra \mathcal{A} is smooth if it is Fréchet and $*$ -isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C^* -algebra A which is stable under the holomorphic functional calculus.

Asking for $i(\mathcal{A})$ to be a proper dense subalgebra of A immediately implies that the Fréchet topology of \mathcal{A} is finer than the C^* -topology of A (since Fréchet means locally convex, metrizable and complete.) We will write $\bar{\mathcal{A}} = A$, as \mathcal{A} will be represented on a Hilbert space and the notation $\bar{\mathcal{A}}$ is unambiguous.

It has been shown that if \mathcal{A} is smooth in A then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [GVF, S]. This ensures that the K -theories of the two algebras are isomorphic, the isomorphism being induced by the inclusion map i . This definition ensures that a smooth algebra is a ‘good’ algebra, [GVF], so these algebras have a sensible spectral theory which agrees with that defined using the C^* -closure, and the group of invertibles is open.

The following Lemma, proved in [R1], explains one method of constructing smooth spectral triples.

Lemma 12. If the algebra \mathcal{A} in $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is in the domain δ^n for $n = 1, 2, 3, \dots$ where δ is the partial derivation $\delta = \text{ad}(|\mathcal{D}|)$ then the completion of \mathcal{A} in the locally convex topology determined by the seminorms

$$q_{n,i}(a) = \|\delta^n d^i(a)\|, \quad n \geq 0, \quad i = 0, 1,$$

where $d(a) = [\mathcal{D}, a]$ is a smooth algebra.

We call the topology on \mathcal{A} determined by the seminorms q_{ni} of Lemma 12 the δ -topology.

7.2. Summability. In the following, let \mathcal{N} be a semifinite von Neumann algebra with faithful normal trace τ . Recall from [FK] that if $S \in \mathcal{N}$, the t^{th} *generalized singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{\|SE\| : E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - E) \leq t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N})$ consists of those operators $T \in \mathcal{N}$ such that $\|T\|_1 := \tau(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will simply write \mathcal{L}^1 for this ideal in order to simplify the notation, and denote the norm on \mathcal{L}^1 by $\|\cdot\|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1$ if $\|T\|_1 := \int_0^\infty \mu_t(T) dt < \infty$.

Note that in the case where $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, \mathcal{L}^1 is not complete in this norm but it is complete in the norm $\|\cdot\|_1 + \|\cdot\|_\infty$. (where $\|\cdot\|_\infty$ is the uniform norm). Another important ideal for us is the domain of the Dixmier trace:

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} : \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds < \infty \right\}.$$

We will suppress the (\mathcal{N}) in our notation for these ideals, as \mathcal{N} will always be clear from context. The reader should note that $\mathcal{L}^{(1,\infty)}$ is often taken to mean an ideal in the algebra $\tilde{\mathcal{N}}$ of τ -measurable operators affiliated to \mathcal{N} . Our notation is however consistent with that of [C] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of τ -compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\tilde{\mathcal{N}}$) such that $\mu_\infty(T) := \lim_{t \rightarrow \infty} \mu_t(T) = 0$.

Definition 7. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, with \mathcal{A} a unital algebra, is $(1, \infty)$ -summable if $(\mathcal{D} - \lambda)^{-1} \in \mathcal{L}^{(1,\infty)}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

We need to briefly discuss the Dixmier trace (for more information on semifinite Dixmier traces, see [CPS2]). For $T \in \mathcal{L}^{(1,\infty)}$, $T \geq 0$, the function

$$F_T : t \mapsto \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds$$

is bounded. For certain functionals $\omega \in L^\infty(\mathbb{R}_*^+)^*$ (called Dixmier functionals in [CPS2]), we obtain a positive functional on $\mathcal{L}^{(1,\infty)}$ by setting $\tau_\omega(T) = \omega(F_T)$. This is the Dixmier trace associated to the semifinite normal trace τ , denoted τ_ω , and we extend it to all of $\mathcal{L}^{(1,\infty)}$ by linearity, where of course it is a trace. The Dixmier trace τ_ω vanishes on the ideal of trace class operators. Whenever the function F_T has a limit α at infinity then for all Dixmier functionals $\omega(F_T) = \alpha$.

The following result (see [C] for the original statement) relates measurability and residues in the semifinite case. We state the result for the $(1, \infty)$ -summable case.

Proposition 14 ([CPS2, Theorem 3.8]). *Let $A \in \mathcal{N}$, $T \geq 0$, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N})$ and suppose that $\lim_{s \rightarrow 1^+} (s-1)\tau(AT^s)$ exists, then it is equal to $\tau_\omega(AT)$ for any Dixmier functional ω .*

7.3. The spectral flow formula. Once we have constructed our semifinite spectral triple for $SU_q(2)$, we will want to examine the pairing with K -theory, just as for the $K_0(F)$ -valued pairing in KK -theory. It will turn out that our construction has zero pairing with $K_1(SU_q(2))$ (for the same reasons as in the KK -construction), but has nonzero pairing with the mapping cone algebra of the inclusion $F \hookrightarrow A$. As this involves pairing with partial isometries (at least in the odd formulation of the problem; see [CPR1]), the spectral flow formula is a priori more complicated and given by [CP2, Corollary 8.11].

Proposition 15. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}_0)$ be an odd unbounded θ -summable semifinite spectral triple relative to (\mathcal{M}, ϕ) . For any $\epsilon > 0$ we define a one-form α^ϵ on the affine space $\mathcal{M}_0 = \mathcal{D}_0 + \mathcal{M}_{sa}$ by*

$$\alpha^\epsilon(A) = \sqrt{\frac{\epsilon}{\pi}} \phi(Ae^{-\epsilon \mathcal{D}^2})$$

for $\mathcal{D} \in \mathcal{M}_0$ and $A \in T_{\mathcal{D}}(\mathcal{M}_0) = \mathcal{M}_{sa}$. Then the integral of α^ϵ is independent of the piecewise C^1 path in \mathcal{M}_0 and if $\{\mathcal{D}_t = \mathcal{D}_a + A_t\}_{t \in [a,b]}$ is any piecewise C^1 path in \mathcal{M}_0 joining \mathcal{D}_a and \mathcal{D}_b then

$$sf(\mathcal{D}_a, \mathcal{D}_b) = \sqrt{\frac{\epsilon}{\pi}} \int_a^b \phi(\mathcal{D}'_t e^{-\epsilon \mathcal{D}_t^2}) dt + \frac{1}{2} \eta_\epsilon(\mathcal{D}_b) - \frac{1}{2} \eta_\epsilon(\mathcal{D}_a) + \frac{1}{2} \phi([\ker(\mathcal{D}_b)] - [\ker(\mathcal{D}_a)]).$$

Here the truncated eta is given by $\eta_\epsilon(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \phi(\mathcal{D} e^{-t \mathcal{D}^2}) t^{-1/2} dt$, and the integral converges for any $\epsilon > 0$.

We want to employ this formula in a finitely summable setting, so we need to Laplace transform the various terms appearing in the formula. We introduce the notation $C_r := \frac{\sqrt{\pi} \Gamma(r-1/2)}{\Gamma(r)}$.

Lemma 13. *Let \mathcal{D} be a self-adjoint operator on the Hilbert space \mathcal{H} , affiliated to the semifinite von Neumann algebra \mathcal{M} . Suppose that for a fixed faithful, normal, semifinite trace ϕ on \mathcal{M} we have $(1 + \mathcal{D}^2)^{-r/2} \in \mathcal{L}^1(\mathcal{M}, \phi)$ for all $\text{Re}(r) > 1$. Then the Laplace transform of the truncated eta function of \mathcal{D} is given by*

$$\frac{1}{C_r} \eta_{\mathcal{D}}(r) = \frac{1}{C_r} \int_1^\infty \phi(\mathcal{D}(1 + s\mathcal{D}^2)^{-r}) s^{-1/2} ds, \quad \text{Re}(r) > 1.$$

Proof. To Laplace transform the ‘ θ summable formula’ for the truncated η we write it as

$$\eta_\epsilon(\mathcal{D}) = \sqrt{\frac{\epsilon}{\pi}} \int_1^\infty \phi(\mathcal{D} e^{-\epsilon s \mathcal{D}^2}) s^{-1/2} ds.$$

Now for $\text{Re}(r) > 1$, the Laplace transform is

$$\begin{aligned} \frac{1}{C_r} \eta_{\mathcal{D}}(r) &= \frac{1}{\sqrt{\pi} \Gamma(r-1/2)} \int_0^\infty \epsilon^{r-1} e^{-\epsilon} \int_1^\infty \phi(\mathcal{D} e^{-\epsilon s \mathcal{D}^2}) s^{-1/2} ds d\epsilon \\ &= \frac{1}{\sqrt{\pi} \Gamma(r-1/2)} \int_1^\infty s^{-1/2} \phi(\mathcal{D} \int_0^\infty \epsilon^{r-1} e^{-\epsilon(1+s\mathcal{D}^2)} d\epsilon) ds \\ (46) \quad &= \frac{\Gamma(r)}{\sqrt{\pi} \Gamma(r-1/2)} \int_1^\infty s^{-1/2} \phi(\mathcal{D}(1 + s\mathcal{D}^2)^{-r}) ds. \end{aligned}$$

□

Proposition 16. *Let \mathcal{D}_a be a self-adjoint densely defined unbounded operator on the Hilbert space \mathcal{H} , affiliated to the semifinite von Neumann algebra \mathcal{M} . Suppose that for a fixed faithful, normal, semifinite trace ϕ on \mathcal{M} we have for $\text{Re}(r) > 1$, $(1 + \mathcal{D}_a^2)^{-r/2} \in \mathcal{L}^1(\mathcal{M}, \phi)$. Let \mathcal{D}_b differ from \mathcal{D}_a by a bounded self adjoint operator in \mathcal{M} . Then for any piecewise C^1 path $\{\mathcal{D}_t = \mathcal{D}_a + A_t\}$; $t \in [a, b]$ joining \mathcal{D}_a and \mathcal{D}_b , the spectral flow is given by the formula*

$$\begin{aligned} sf(\mathcal{D}_a, \mathcal{D}_b) &= \frac{1}{C_r} \int_a^b \phi(\dot{\mathcal{D}}_t (1 + \mathcal{D}_t^2)^{-r}) dt + \frac{1}{2C_r} (\eta_{\mathcal{D}_b}(r) - \eta_{\mathcal{D}_a}(r)) \\ (47) \quad &+ \frac{1}{2} (\phi(P_{\ker \mathcal{D}_b}) - \phi(P_{\ker \mathcal{D}_a})), \quad \text{Re}(r) > 1. \end{aligned}$$

Proof. We apply the Laplace transform to the general spectral flow formula. The computation of the Laplace transform of the eta invariants is above, and the Laplace transform of the other integral is in [CP2], Section 9. □

We now obtain a residue formula for the spectral flow. The importance of such a formula is the drastic simplification of computations in the next few subsections, as we may throw away terms that are holomorphic in a neighbourhood of the critical point $r = 1/2$.

Proposition 17. *Let \mathcal{D}_a be a self-adjoint densely defined unbounded operator on the Hilbert space \mathcal{H} , affiliated to the semifinite von Neumann algebra \mathcal{M} . Suppose that for a fixed faithful, normal, semifinite trace ϕ on \mathcal{M} we have for $\text{Re}(r) > 1$, $(1 + \mathcal{D}_a^2)^{-r/2} \in \mathcal{L}^1(\mathcal{M}, \phi)$. Let \mathcal{D}_b differ from \mathcal{D}_a by*

a bounded self adjoint operator in \mathcal{M} . Then for any piecewise C^1 path $\{\mathcal{D}_t = \mathcal{D}_a + A_t\}$, $t \in [a, b]$ in \mathcal{M}_0 joining \mathcal{D}_a and \mathcal{D}_b , the spectral flow is given by the formula

$$(48) \quad sf(\mathcal{D}_a, \mathcal{D}_b) = \text{Res}_{r=1/2} C_r sf(\mathcal{D}_a, \mathcal{D}_b) = \text{Res}_{r=1/2} \left(\int_a^b \phi(\dot{\mathcal{D}}_t (1 + \mathcal{D}_t^2)^{-r}) dt + \frac{1}{2} (\eta_{\mathcal{D}_b}(r) - \eta_{\mathcal{D}_a}(r)) \right) + \frac{1}{2} (\phi(P_{\ker \mathcal{D}_b}) - \phi(P_{\ker \mathcal{D}_a}))$$

and in particular the sum in large brackets extends to a meromorphic function of r with a simple pole at $r = 1/2$.

7.4. The $SU_q(2)$ spectral triple. We now return to our task of building a semifinite spectral triple for $SU_q(2)$. We recall the unbounded Kasparov module (X, \mathcal{D}) from Section 4. The following basic results are proved in [PR].

Lemma 14. *Any endomorphism of X leaving X_c invariant extends uniquely to a bounded linear operator on \mathcal{H}_h . In particular, the operators Φ_m extend to operators on \mathcal{H}_h . The maps Φ_m are mutually orthogonal projections that sum strongly to the identity. The operator \mathcal{D} extends to an unbounded self-adjoint operator on \mathcal{H}_h .*

By Lemma 4 the following holds in $B(\mathcal{H}_h)$;

$$(49) \quad \Phi_m H = H \Phi_m = q^{2m} \Phi_m.$$

Lemma 15. *The algebra A_c is contained in the smooth domain of the derivation δ where for $T \in B(\mathcal{H}_h)$, $\delta(T) = [|\mathcal{D}|, T]$. That is $A_c \subset \cap_{n \geq 0} \text{dom} \delta^n$.*

Definition 8. *Define the $*$ -algebra $\mathcal{A} \subset A$ to be the completion of A_c in the δ -topology, so \mathcal{A} is Fréchet and stable under the holomorphic functional calculus.*

From this data we will construct a $(1, \infty)$ -summable spectral triple by constructing a trace \tilde{h} using the same methods as in [PR]. However, later we will see that we may twist \tilde{h} with the modular operator as in [CPR2] to obtain a normal weight $h_{\mathcal{D}}$. In [CPR2] it was necessary to twist with the modular operator in order to obtain finite summability. For $SU_q(2)$, both the twisted and untwisted traces give the same summability. However, the Dixmier trace we obtain from \tilde{h} is highly degenerate, while the ‘Dixmier weight’ obtained from $h_{\mathcal{D}}$ recovers the (faithful) Haar state.

In the remainder of this Section we will define a semifinite von Neumann algebra \mathcal{N} , and a faithful, semifinite, normal trace \tilde{h} on \mathcal{N} which enable us to prove the following theorem.

Theorem 1. *The triple $(\mathcal{A}, \mathcal{H}_h, \mathcal{D})$ is a QC^∞ , $(1, \infty)$ -summable, odd, local, semifinite spectral triple (relative to (\mathcal{N}, \tilde{h})). The operator $(1 + \mathcal{D}^2)^{-1/2}$ is not trace class, and*

$$(50) \quad \tilde{h}_\omega(f(1 + \mathcal{D}^2)^{-1/2}) = \begin{cases} \neq 0 & 0 < f \in C^*(T_k T_k^*, k \geq 0) \\ 0 & \text{otherwise} \end{cases},$$

where \tilde{h}_ω is any Dixmier trace associated to \tilde{h} .

We have a number of computations to make with finite rank endomorphisms, defined not on X but on the dense submodule $A_c \subset X$.

Definition 9. *Let $\text{End}_F^{00}(A_c)$ be the finite rank endomorphisms of the pre- C^* -module $A_c \subset X$. By Lemma 14, these endomorphisms act as bounded operators on \mathcal{H}_h , and we let $\mathcal{N} := (\text{End}_F^{00}(X_c))''$.*

Lemma 16. *There exists a faithful, semifinite, normal trace \tilde{h} on the algebra $\mathcal{N} = (\text{End}_F^{00}(X_c))''$. Moreover,*

$$\text{End}_F^{00}(X_c) \subset \mathcal{N}_{\tilde{h}} := \text{span}\{T \in \mathcal{N}_+ : \tilde{h}(T) < \infty\},$$

the domain of definition of \tilde{h} , and on the rank 1 operators this trace is given by

$$(51) \quad \tilde{h}(\Theta_{x,y}) = \langle y, x \rangle = h(y^* x).$$

Proof. This is a simple adaptation of [PR, Prop 5.11] using the following definition of the trace \tilde{h} . We define vector states ω_m for $m \in \mathbb{Z}$ by setting, for $V \in \mathcal{N}$, $\omega_0(V) = \langle 1, V1 \rangle$ and

$$\begin{aligned}\omega_m(V) &= \langle T_m, VT_m \rangle + \langle \tilde{T}_m, V\tilde{T}_m \rangle, \quad m > 0, \\ \omega_m(V) &= \langle T_m^*, VT_m^* \rangle + \langle \tilde{T}_m^*, V\tilde{T}_m^* \rangle, \quad m < 0.\end{aligned}$$

Then we define

$$\tilde{h}(V) = \lim_{L \nearrow} \sum_{m \in L \subset \mathbb{Z}} \omega_m(V),$$

where L ranges over the finite subsets of \mathbb{Z} . Then \tilde{h} is by definition normal. The rest of the claim is proved just as in [PR]. \square

Lemma 17. *The operator \mathcal{D} acting on \mathcal{H} is $(1, \infty)$ -summable, i.e. $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1, \infty)}(\mathcal{N}, \tilde{h})$. For any Dixmier trace \tilde{h}_ω associated to \tilde{h} we have*

$$\tilde{h}_\omega(f(1 + \mathcal{D}^2)^{-1/2}) = \begin{cases} 1 & f = T_k T_k^*, \quad k \geq 0 \\ 0 & f \notin C^*(T_k T_k^*, k \geq 0) \end{cases}.$$

The functional on A defined by $a \rightarrow \tilde{h}_\omega(a(1 + \mathcal{D}^2)^{-1/2})$ is continuous, and supported on $C^*(T_k T_k^*)$.

Proof. It is relatively simple to check that for $n \neq 0$ we have $\tilde{h}(\tilde{T}_k U_n \tilde{T}_k^* \Phi_m) = 0$ for all $k \geq 0$ and $m \in \mathbb{Z}$. For $n = 0$ we have, using Lemma 16 and the description of Φ_m as a sum of rank one projections given by Equation (56),

$$\begin{aligned}\tilde{h}(\tilde{T}_k \tilde{T}_k^* \Phi_m) &= \begin{cases} h(T_m^* \tilde{T}_k \tilde{T}_k^* T_m + \tilde{T}_m^* \tilde{T}_k \tilde{T}_k^* \tilde{T}_m) & m \geq 1 \\ h(\tilde{T}_k \tilde{T}_k^*) & m = 0 \\ h(T_{|m|} \tilde{T}_k \tilde{T}_k^* T_{|m|}^* + \tilde{T}_{|m|} \tilde{T}_k \tilde{T}_k^* \tilde{T}_{|m|}^*) & m \leq -1 \end{cases} \\ &= \begin{cases} 0 & m \geq k+1 \\ h(p_w) & m = k \\ h(\tilde{T}_{k-m} \tilde{T}_{k-m}^*) & 1 \leq m \leq k-1 \\ h(\tilde{T}_k \tilde{T}_k^*) & m = 0 \\ h(T_{|m|+k-1} T_{|m|+k-1}^* - T_{|m|+k} T_{|m|+k}^*) & m \leq -1 \end{cases} \\ &= \begin{cases} 0 & m \geq k+1 \\ q^{2(k-m)}(1 - q^2) & 0 \leq m \leq k \\ q^{2(|m|+k)}(1 - q^2) & m \leq -1 \end{cases}.\end{aligned}$$

(These formulae may lead the reader to doubt the faithfulness of \tilde{h} ; however the operators above on which \tilde{h} is zero are themselves the zero operator.) We have used the formulae of Lemma 1 several times here. Since $q < 1$ it is now easy to check that $\tilde{h}(\tilde{T}_k \tilde{T}_k^* (1 + \mathcal{D}^2)^{-1/2}) < \infty$, and so $\tilde{h}_\omega(\tilde{T}_k \tilde{T}_k^* (1 + \mathcal{D}^2)^{-1/2}) = 0$. For the generators $T_k T_k^*$ we have an entirely analogous calculation which yields

$$\tilde{h}(T_k T_k^* \Phi_m) = \begin{cases} 1 & m \geq k+1 \\ q^{2(k-m+1)} & 0 \leq m \leq k \\ q^{2(k+|m|+1)} & m \leq -1 \end{cases}.$$

Here the situation is different, as the result is constant for $m \geq k$ and so $\tilde{h}_\omega(T_k T_k^* (1 + \mathcal{D}^2)^{-1/2}) = 1$. Finally, we can use gauge invariance to show that \tilde{h} of $T_k T_l^*$ or $\tilde{T}_k \tilde{T}_l^*$ is zero unless $k = l$. Since $(1 + \mathcal{D}^2)^{-1/2}$ has finite Dixmier trace, $a \rightarrow \tilde{h}_\omega(a(1 + \mathcal{D}^2)^{-1/2})$ defines a continuous linear functional on A and so we can extend these computations from monomials, to the finite span, and so to the C^* -completion. \square

Remarks. Elements of A regarded as operators in \mathcal{N} are not in the domain of \tilde{h} . In addition, observing that $a \rightarrow \text{Res}_{s=0} \tilde{h}(a(1 + \mathcal{D}^2)^{-1/2-s})$ is *not* a faithful trace (since A has no faithful trace), this means that the residue/Dixmier trace cannot detect parts of the algebra. We observe that the following pathological behaviour lies behind the vanishing of the residue trace on some elements:

the operator $\tilde{T}_k \tilde{T}_k^* \mathcal{D}$ is *trace class* for all k . In the graph algebra picture we can then say, somewhat loosely, that as soon as a path leaves the loop μ , it becomes invisible to the residue trace.

7.5. The mapping cone pairing and spectral flow. We want to compute the spectral flow from $vv^* \mathcal{D}$ to $v \mathcal{D} v^*$ for partial isometries $v \in A$ satisfying $v^* v, vv^* \in F$. By Lemma 11 it suffices to consider the pairing on the generators $v = T_k, \tilde{T}_k$. We first compute the various terms.

Lemma 18. *For $v \in A$ a partial isometry with $vv^*, v^*v \in F$, we have*

$$\tilde{h}(v \Phi_0 v^*) - \tilde{h}(vv^* \Phi_0) = \tilde{h}((v^*v - vv^*) \Phi_0) = h(v^*v - vv^*).$$

For the generators of $K_0(M(F, A))$ we have

$$\tilde{h}((v^*v - vv^*) \Phi_0) = \begin{cases} q^2(1 - q^{2k}) & v = T_k \\ (1 - q^2)(1 - q^{2k}) & v = \tilde{T}_k \end{cases}.$$

Proof. This result follows because $\Phi_0 = \Theta_{1,1}$, so $\tilde{h}(a \Phi_0) = \tilde{h}(\Theta_{a,1}) = h(a)$, the last equality following from Equation (51). The values of these traces is computed using Lemma 3. \square

Lemma 19. *Modulo functions of r holomorphic in a neighbourhood of $r = 1/2$, the difference*

$$\eta_r(v \mathcal{D} v^*) - \eta_r(vv^* \mathcal{D}) = \int_1^\infty \tilde{h}((v^*v - vv^*) \mathcal{D}(1 + s \mathcal{D}^2)^{-r}) s^{-1/2} ds$$

is given by

$$\eta_r(v \mathcal{D} v^*) - \eta_r(vv^* \mathcal{D}) = C_r \times \begin{cases} k - q^2(1 + q^2)^{\frac{1-q^{2k}}{1-q^2}} & v = T_k \\ -(1 + q^2)(1 - q^{2k}) & v = \tilde{T}_k \end{cases}$$

Proof. The first equality comes from the functional calculus, the trace property and the fact that vv^* commutes with \mathcal{D} :

$$\tilde{h}((v \mathcal{D} v^*)(1 + (v \mathcal{D} v^*)^2)^{-r}) = \tilde{h}(v \mathcal{D}(1 + \mathcal{D}^2)^{-r} v^*) = \tilde{h}(v^* v \mathcal{D}(1 + \mathcal{D}^2)^{-r}),$$

and similarly for $vv^* \mathcal{D}$. Using Lemma 17, we find that

$$\begin{aligned} \tilde{h}((T_k^* T_k - T_k T_k^*) \Phi_m) &= \begin{cases} 0 & m > k \\ 1 - q^{2(k-m+1)} & 0 < m \leq k \\ q^{2(|m|+1)}(1 - q^{2k}) & m \leq 0 \end{cases}, \\ \tilde{h}((\tilde{T}_k^* \tilde{T}_k - \tilde{T}_k \tilde{T}_k^*) \Phi_m) &= \begin{cases} 0 & m > k \\ -q^{2(k-m)}(1 - q^2) & 1 \leq m \leq k \\ q^{2|m|}(1 - q^2)(1 - q^{2k}) & m \leq 0 \end{cases}. \end{aligned}$$

Since $q < 1$, this means that $\sum_m m \tilde{h}((v^*v - vv^*) \Phi_m) < \infty$ for $v = T_k, \tilde{T}_k$. This allows us to interchange the order of the summation and integral in

$$\eta_r(v \mathcal{D} v^*) - \eta_r(vv^* \mathcal{D}) = \int_1^\infty \sum_{m \neq 0} m(1 + sm^2)^{-r} s^{-1/2} \tilde{h}((v^*v - vv^*) \Phi_m) ds.$$

Since

$$r \rightarrow \int_0^1 \sum_m m(1 + sm^2)^{-r} \tilde{h}((v^*v - vv^*) \Phi_m) s^{-1/2} ds$$

defines a function of r holomorphic at $r = 1/2$, we have

$$\begin{aligned} \text{Res}_{r=1/2} \int_1^\infty \sum_{m \neq 0} m(1 + sm^2)^{-r} s^{-1/2} \tilde{h}((v^*v - vv^*) \Phi_m) ds \\ = \text{Res}_{r=1/2} \int_0^\infty \sum_{m \neq 0} m(1 + sm^2)^{-r} s^{-1/2} \tilde{h}((v^*v - vv^*) \Phi_m) ds. \end{aligned}$$

Performing the interchange yields the integral $\int_0^\infty (1 + sm^2)^{-r} s^{-1/2} ds = C_r/|m|$. So

$$\text{Res}_{r=1/2}(\eta_r(v\mathcal{D}v) - \eta_r(vv^*\mathcal{D})) = \sum_{m \neq 0} \frac{m}{|m|} \tilde{h}((v^*v - vv^*)\Phi_m).$$

The sums involved are just geometric series (with finitely many terms for $m > 0$ and infinitely many terms for $m < 0$), which are easily summed to give the stated results. \square

The last contribution to the index we require is from the integral along the path. Using Section 6 of [CPR2], we may show that modulo functions of r holomorphic at $r = 1/2$

$$\int_0^1 \tilde{h}(v[\mathcal{D}, v^*](1 + (\mathcal{D} + tv[\mathcal{D}, v^*])^2)^{-r}) dt = \tilde{h}(v[\mathcal{D}, v^*](1 + \mathcal{D}^2)^{-r}).$$

Using the formulae $\tilde{T}_k[\mathcal{D}, \tilde{T}_k^*] = -k\tilde{T}_k\tilde{T}_k^*$, $T_k[\mathcal{D}, T_k^*] = -kT_kT_k^*$, Lemma 17 gives us

$$(52) \quad \text{Res}_{r=1/2} \tilde{h}(v[\mathcal{D}, v^*](1 + \mathcal{D}^2)^{-r}) = \frac{1}{2} \text{Res}_{r'=0} \tilde{h}(v[\mathcal{D}, v^*](1 + \mathcal{D}^2)^{-1/2-r'}) = \begin{cases} -k/2 & v = T_k \\ 0 & v = \tilde{T}_k \end{cases}$$

Putting the pieces together yields

$$sf(vv^*\mathcal{D}, v\mathcal{D}v^*) = \begin{cases} -q^4 \frac{1-q^{2k}}{1-q^2} & v = T_k \\ -q^2(1-q^{2k}) & v = \tilde{T}_k \end{cases} = \begin{cases} -q^4[k]_q & v = T_k \\ -q^2(1-q^2)[k]_q & v = \tilde{T}_k \end{cases},$$

where $[k]_q$ is the q -integer given by the definition $[k]_q := (1 - q^{2k})/(1 - q^2)$. These computations prove the following Proposition, which shows that the analytic pairing of our spectral triple with $K_0(M(F, A))$ is simply related to the KK -pairing.

Proposition 18. *The trace \tilde{h} induces a homomorphism on $K_0(F)$ by choosing as representative of each class $x \in K_0(F)$ a projection $Q \in \text{End}_F^0(X)$, and defining $\tilde{h}_*(x) := \tilde{h}(Q)$. Let P be the non-negative spectral projection of \mathcal{D} , as an operator on X , and for v a partial isometry in A with range and source in F , let $\text{Index}(PvP) \in K_0(F)$ be the class obtained from the KK index pairing of (X, \mathcal{D}) and v . Then*

$$\tilde{h}_*(\text{Index}(PvP)) = sf(vv^*\mathcal{D}, v\mathcal{D}v^*).$$

Proof. We need to compute $\tilde{h}_*(\text{Index}(PvP))$ for the generating partial isometries of $K_0(M(F, A))$. By Proposition 13 and Lemma 17 we have

$$\begin{aligned} \tilde{h}_*([T_k], [(X, \mathcal{D})]) &= - \sum_{j=0}^{k-1} \tilde{h}(T_k T_k^* \Phi_j) = - \sum_{j=0}^{k-1} q^{2(k-j)} q^2 = -q^4 \frac{1-q^{2k}}{1-q^2}; \\ \tilde{h}_*([\tilde{T}_k], [(X, \mathcal{D})]) &= - \sum_{j=0}^{k-1} \tilde{h}(\tilde{T}_k \tilde{T}_k^* \Phi_j) = - \sum_{j=0}^{k-1} q^{2(k-j)} (1-q^2) = -q^2(1-q^{2k}); \end{aligned}$$

Thus the values obtained from the spectral flow formula and the map \tilde{h}_* agree. \square

Remarks. (i) This is a special case of a result in [KNR] (other special cases appeared in [PR, PRS]).

(ii) The formulae for the pairing of T_k, \tilde{T}_k have three factors: an overall q^2 ; then either q^2 or $(1-q^2)$, which is $h(v^*v)$ for $v = T_k, \tilde{T}_k$ respectively, and the q -number $[k]_q$. We view this formula as giving a kind of weighted q -winding number, though this is heuristic.

8. THE MODULAR SPECTRAL TRIPLE FOR $SU_q(2)$

Our aim in this Section is to construct a ‘modular spectral triple’ for $SU_q(2)$. The only real difference between the semifinite triple constructed already and the modular triple, is the replacement of $T \rightarrow \tilde{h}(T)$ by $T \rightarrow \tilde{h}(HT)$. This changes not just the analytic behaviour but also the homological behaviour. These modular triples do not pair with ordinary K -theory, but with modular K -theory which we describe next.

8.1. Modular K -theory. The unitary $U_1 + p_v$ generating $K_1(A)$ commutes with our operator \mathcal{D} , so there is no pairing between the Haar module and $K_1(A)$. Nonetheless, we have many self-adjoint unitaries of the form

$$u_v := \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix},$$

where v is a partial isometry with range and source projection in F . Whilst such unitaries are self-adjoint and so give the trivial class in $K_1(A)$, we showed in [CPR2] that they give rise to nontrivial index pairings in twisted cyclic cohomology. We summarise the key ideas from [CPR2].

Definition 10. Let A be a $*$ -algebra and $\sigma : A \rightarrow A$ an algebra automorphism such that $\sigma(a)^* = \sigma^{-1}(a^*)$ then we say that σ is a regular automorphism, [KMT].

Definition 11. Let u be a unitary over A (respectively matrix algebra over \mathcal{A}), and $\sigma : A \rightarrow A$ a regular automorphism with fixed point algebra $F = A^\sigma$. We say that u satisfies the **modular condition** with respect to σ if both the operators $u\sigma(u^*)$ and $u^*\sigma(u)$ are in (resp. a matrix algebra over) the algebra F . We denote by U_σ the set of modular unitaries.

Remarks. (i) We are of course thinking of the case $\sigma(a) = H^{-1}aH$, where H implements the modular group for some weight on A . Hence the terminology modular unitaries.

(ii) For unitaries in matrix algebras over A we use $\sigma \otimes Id_n$ to state the modular condition, where Id_n is the identity of $M_n(\mathbb{C})$.

Example. If σ is a regular automorphism of an algebra A with fixed point algebra F , and $v \in A$ is a partial isometry with range and source projections in F , and moreover has $v\sigma(v^*), v^*\sigma(v)$ in F , then

$$u_v = \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix}$$

is a modular unitary, and $u_v \sim u_{v^*}$. These statements are proved in [CPR2].

The following definitions and results are also from [CPR2].

Definition 12. Let u_t be a continuous path of modular unitaries such that $u_t\sigma(u_t^*)$ and $u_t^*\sigma(u_t)$ are also continuous paths in F . Then we say that u_t is a modular homotopy, and that u_0 and u_1 are modular homotopic.

Lemma 20. The binary operation on modular homotopy classes in U_σ $[u] + [v] := [u \oplus v]$ is abelian.

Definition 13. Let σ be a regular automorphism of the $*$ -algebra A . Define $K_1(A, \sigma)$ to be the abelian semigroup with one generator $[u]$ for each unitary $u \in M_l(A)$ satisfying the modular condition and with the following relations:

- 1) $[1] = 0$,
- 2) $[u] + [v] = [u \oplus v]$,
- 3) If u_t , $t \in [0, 1]$ is a continuous paths of unitaries in $M_l(A)$ satisfying the modular condition then $[u_0] = [u_1]$.

The modular K_1 group does not pair with ordinary K -homology, or KK -theory.

On the GNS Hilbert space of the Haar state τ we are going to construct a semifinite Neumann algebra \mathcal{N} with the $*$ -algebra \mathcal{A} faithfully represented in \mathcal{N} and having the following properties:

- 1) there is a faithful normal semifinite weight ϕ on \mathcal{N} such that the modular automorphism group of ϕ is an inner automorphism group $\tilde{\sigma}$ of \mathcal{N} with $\tilde{\sigma}|_{\mathcal{A}} = \sigma$,
- 2) ϕ restricts to a faithful semifinite trace on $\mathcal{M} = \mathcal{N}^\sigma$,
- 3) If \mathcal{D} is the generator of the one parameter group which implements the modular automorphism group of (\mathcal{N}, ϕ) then $[\mathcal{D}, a]$ extends to a bounded operator (in \mathcal{N}) for all $a \in \mathcal{A}$ and for λ in the resolvent set of \mathcal{D} we have $f(\lambda - \mathcal{D})^{-1} \in K(\mathcal{M}, \phi|_{\mathcal{M}})$, where $f \in \mathcal{A}^\sigma$, and $K(\mathcal{M}, \phi|_{\mathcal{M}})$ is the ideal of compact operators in \mathcal{M} relative to $\phi|_{\mathcal{M}}$. In particular, \mathcal{D} is affiliated to \mathcal{M} .

For ease of reference we will follow the practice of [CPR2] and refer to any triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ carrying the extra data (1), (2), (3) above as a modular spectral triple. For modular spectral triples there is also a residue type formula for the spectral flow, which is a Corollary of Proposition 17.

Theorem 2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ , $(1, \infty)$ -summable, modular spectral triple relative to (\mathcal{N}, ϕ) , such that \mathcal{D} commutes with $F = \mathcal{A}^\sigma$. Then for any modular unitary u and any Dixmier trace ϕ_ω associated to ϕ (restricted to \mathcal{M}) we have $sf_\phi(\mathcal{D}, u\mathcal{D}u^*)$ given by the residue at $r = 1/2$ of the analytic continuation of*

$$\phi(u[\mathcal{D}, u^*](1 + \mathcal{D}^2)^{-r}) + \frac{1}{2} \int_1^\infty \phi((\sigma(u^*)u - 1)\mathcal{D}(1 + s\mathcal{D}^2)^{-r}) s^{-1/2} ds.$$

Proof. The cancellation of the kernel corrections we will prove later. The methods of Section 6 of [CPS2] imply the formula for the first term and some elementary algebraic manipulations produce the eta term. \square

Remarks. The two functionals $\mathcal{A} \otimes \mathcal{A} \ni a_0 \otimes a_1 \rightarrow \phi(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-r})$, and

$$\mathcal{A} \otimes \mathcal{A} \ni a_0 \otimes a_1 \rightarrow \frac{1}{2} \int_1^\infty \phi((\sigma(u^*)u - 1)\mathcal{D}(1 + s\mathcal{D}^2)^{-r}) s^{-1/2} ds,$$

are easily seen to algebraically satisfy the relations for twisted b, B -cocycles with twisting coming from σ . This is basically the situation found in [CPR2] for the Cuntz algebras. However here the presence of the eta correction term complicates the analytic aspects of the cocycle condition and so we will defer this analysis to a future work. We will also defer discussion of the question of dependence of spectral flow on the homotopy class of a modular unitary to another place; the proof needs considerable additional information about modular unitaries.

8.2. The modular spectral triple and the index pairing. Our final aim is to construct a modular spectral triple for $A = SU_q(2)$, and compute the twisted index pairing. In the following, \mathcal{N} refers to the von Neumann algebra of Lemma 16 acting on the GNS Hilbert space for the Haar state of $SU_q(2)$. We let \tilde{h} be the faithful, normal, semifinite trace defined on \mathcal{N} in Lemma 16.

Definition 14. *Let $h_{\mathcal{D}}$ be the trace \tilde{h} twisted by the modular operator, that is*

$$(53) \quad h_{\mathcal{D}}(T) := \tilde{h}(HT)$$

for all operators T such that $HT \in \mathcal{N}_{\tilde{h}}$.

The functional $h_{\mathcal{D}}$ is no longer a trace on \mathcal{N} , but is a faithful, normal, semifinite weight. The restriction of $h_{\mathcal{D}}$ to $\mathcal{M} := \mathcal{N}^\sigma$ is a faithful, normal, semifinite trace.

Lemma 21. *The operators Φ_m are $h_{\mathcal{D}}$ -compact in the von-Neumann algebra \mathcal{M} , and*

$$\begin{aligned} \text{Res}_{r=1/2} h_{\mathcal{D}}(\tilde{T}_k \tilde{T}_k^* (1 + \mathcal{D}^2)^{-r}) &= \frac{1}{2} q^{2k} (1 - q^2) = \frac{1}{2} h(\tilde{T}_k \tilde{T}_k^*) \\ \text{Res}_{r=1/2} h_{\mathcal{D}}(T_k T_k^* (1 + \mathcal{D}^2)^{-r}) &= \frac{1}{2} q^{2k+2} = \frac{1}{2} h(T_k T_k^*). \end{aligned}$$

In particular, $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1, \infty)}(\mathcal{M}, h_{\mathcal{D}})$.

Remark. Both \tilde{h} and $h_{\mathcal{D}}$ give us $(1, \infty)$ summability, but only $h_{\mathcal{D}}$ recovers the Haar state using the Dixmier trace. In particular, the functional $a \rightarrow \text{Res}_{r=1/2} h_{\mathcal{D}}(a(1 + \mathcal{D}^2)^{-r})$ is faithful on A .

Proof. We will prove compactness by showing that the Φ_m are finite for $h_{\mathcal{D}}$ using the formulae of Lemma 8. We can apply the trace \tilde{h} to both sides of Equation (35) to obtain

$$\tilde{h}(\Phi_m) = \begin{cases} \tilde{h}(\Theta_{T_m, T_m}) + \tilde{h}(\Theta_{\tilde{T}_m, \tilde{T}_m}), & m \geq 1, \\ \tilde{h}(\Theta_{1, 1}), & m = 0, \\ \tilde{h}(\Theta_{T_m^*, T_m^*}) + \tilde{h}(\Theta_{\tilde{T}_m^*, \tilde{T}_m^*}), & m \leq -1. \end{cases}$$

Equation (51) then gives

$$(54) \quad \tilde{h}(\Phi_m) = \begin{cases} h(T_m^* T_m) + h(\tilde{T}_m^* \tilde{T}_m) & m \geq 1 \\ h(1) & m = 0 \\ h(T_m T_m^*) + h(\tilde{T}_m \tilde{T}_m^*) & m \leq -1 \end{cases}.$$

The relation (6) implies $T_m^* T_m = p_v$ and $\tilde{T}_m^* \tilde{T}_m = p_w$. On the other hand we have formulae for the trace on these and the other elements given above, in Equations (19) and (20). Substituting these values we obtain

$$(55) \quad \tilde{h}(\Phi_m) = q^{\max(0, -2m)}, \quad m \in \mathbb{Z}.$$

So by Equation (49) we have

$$h_{\mathcal{D}}(\Phi_m) = \tilde{h}(H\Phi_m) = q^{2m} \tilde{h}(\Phi_m) = q^{2m + \max(0, -2m)} = q^{\max(0, 2m)}, \quad m \in \mathbb{Z}.$$

Since $a\Theta_{x,y} = \Theta_{ax,y}$ for $a \in \mathcal{A}$, we can multiply both sides of Equation (35) by $T_k T_k^*$ to obtain

$$(56) \quad T_k T_k^* \Phi_m = \begin{cases} \Theta_{T_k T_k^* T_m, T_m} + \Theta_{T_k T_k^* \tilde{T}_m, \tilde{T}_m}, & m \geq 1, \\ \Theta_{T_k T_k^*, 1}, & m = 0, \\ \Theta_{T_k T_k^* T_m^*, T_m^*} + \Theta_{T_k T_k^* \tilde{T}_m^*, \tilde{T}_m^*}, & m \leq -1. \end{cases}$$

Taking the trace of both sides and again applying Equation (51), we obtain

$$\tilde{h}(T_k T_k^* \Phi_m) = \begin{cases} h(T_m^* T_k T_k^* T_m) + h(\tilde{T}_m^* T_k T_k^* \tilde{T}_m), & m \geq 1, \\ h(T_k T_k^*), & m = 0, \\ h(T_m T_k T_k^* T_m^*) + h(\tilde{T}_m T_k T_k^* \tilde{T}_m^*), & m \leq -1. \end{cases}$$

Now we apply Lemma 17 to find

$$\tilde{h}(T_k T_k^* \Phi_m) = \begin{cases} 1 & m \geq k+1, \\ q^{2(k-m+1)} & 0 \leq m \leq k, \\ q^{2(k+|m|+1)} & m \leq -1. \end{cases}$$

Since $T_k T_k^*$ commutes with H , we find

$$h_{\mathcal{D}}(T_k T_k^* \Phi_m) = \begin{cases} q^{2m} & m \geq k+1, \\ q^{2(k+1)} & 0 \leq m \leq k, \\ q^{2(k+1)} & m \leq -1. \end{cases} = \begin{cases} q^{2m} & m \geq k+1 \\ q^{2(k+1)} & m \leq k \end{cases}.$$

From this the summability of \mathcal{D} is computed as follows;

$$(57) \quad h_{\mathcal{D}}((1 + \mathcal{D}^2)^{-s/2}) = \sum_{m \in \mathbb{Z}} q^{\max(2m, 0)} (1 + m^2)^{-s/2} = \sum_{m=0}^{\infty} (1 + m^2)^{-s/2} + C(s),$$

where $C(s)$ is finite for all $s \geq 1$. Thus the whole sum is finite for $\Re(s) > 1$ and has a simple pole at $s = 1$. Similarly we have

$$(58) \quad h_{\mathcal{D}}(T_k T_k^* (1 + \mathcal{D}^2)^{-s/2}) = q^{2k+2} \sum_{m=k}^{\infty} (1 + m^2)^{-s/2} + C_k(s),$$

Putting $r = s/2$ and taking the residues of both sides, we obtain

$$(59) \quad \text{Res}_{r=1/2} h_{\mathcal{D}}((1 + \mathcal{D}^2)^{-r}) = 1/2, \quad \text{Res}_{r=1/2} h_{\mathcal{D}}(T_k T_k^* (1 + \mathcal{D}^2)^{-r}) = q^{2k+2}/2$$

A similar calculation yields

$$h_{\mathcal{D}}(\tilde{T}_k \tilde{T}_k^* \Phi_m) = \begin{cases} 0 & m \geq k+1 \\ q^{2k}(1 - q^2) & m \leq k \end{cases},$$

and so $\text{Res}_{r=1/2} h_{\mathcal{D}}(\tilde{T}_k \tilde{T}_k^* (1 + \mathcal{D}^2)^{-r}) = q^{2k}(1 - q^2)/2$. □

Theorem 3. *The triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ carries the additional structure of a modular spectral triple. The index pairings with the modular unitaries u_{T_k} and $u_{\tilde{T}_k}$ are given by*

$$\langle [u_{T_k}], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = kq^2(1 - q^{2k}), \quad \langle [u_{\tilde{T}_k}], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = k(1 - q^2)(1 - q^{2k}).$$

Proof. That $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a modular spectral triple is a consequence of our constructions. We are interested in computing the index pairing. Since for $f \in F$ we have $h_{\mathcal{D}}(fP_{\ker \mathcal{D}}) = \tilde{h}(fP_{\ker \mathcal{D}}) = h(f)$, for any modular unitary we have $h_{\mathcal{D}}((\sigma(u^*)u - 1)P_{\ker \mathcal{D}}) = h(\sigma(u^*)u - 1) = 0$. For the eta corrections we first compute, with $v = T_k, \tilde{T}_k$,

$$\sigma(u_v)u_v - 1 = \begin{pmatrix} \sigma(v)v^* - vv^* & 0 \\ 0 & \sigma(v^*)v - v^*v \end{pmatrix},$$

and so

$$h_{\mathcal{D}}((\sigma(u_v)u_v - 1)\Phi_m) = q^{2m}\tilde{h}(((q^{-2k} - 1)vv^* + (q^{2k} - 1)v^*v)\Phi_m).$$

Using our previous computations we find

$$h_{\mathcal{D}}((\sigma(u_{T_k})u_{T_k} - 1)\Phi_m) = \begin{cases} q^{2m}(q^{2k} + q^{-2k} - 2) & m \geq k + 1 \\ q^2(1 - q^{2k}) + q^{2m}(q^{2k} - 1) & 1 \leq m \leq k \\ 0 & m \leq 0 \end{cases}$$

and

$$h_{\mathcal{D}}((\sigma(u_{\tilde{T}_k})u_{\tilde{T}_k} - 1)\Phi_m) = \begin{cases} 0 & m > k \\ (1 - q^2)(1 - q^{2k}) & 1 \leq m \leq k \\ (1 - q^2)(1 + (1 - q^{2k})) & m = 0 \\ 0 & m \leq -1 \end{cases}.$$

Just as in the semifinite case we may replace the integral over $[1, \infty)$ by an integral over $[0, \infty)$ without affecting the residue, and interchange the sum and integral. Proceeding just as in that case we have modulo functions of r holomorphic at $r = 1/2$,

$$\int_1^\infty h_{\mathcal{D}}((\sigma(u_v)u_v - 1)\mathcal{D}(1 + s\mathcal{D}^2)^{-r})s^{-1/2}ds = C_r \begin{cases} kq^2(1 - q^{2k}) & v = T_k \\ k(1 - q^2)(1 - q^{2k}) & v = \tilde{T}_k \end{cases}.$$

So the contribution from the eta invariants is

$$\text{Res}_{r=1/2} \frac{1}{2}(\eta_r(u_v\mathcal{D}u_v) - \eta_r(\mathcal{D})) = \frac{1}{2} \begin{cases} kq^2(1 - q^{2k}) & v = T_k \\ k(1 - q^2)(1 - q^{2k}) & v = \tilde{T}_k \end{cases}.$$

The remaining piece of the computation is

$$\text{Res}_{r=1/2} h_{\mathcal{D}}(u_v[\mathcal{D}, u_v](1 + \mathcal{D}^2)^{-r}) = \frac{1}{2} \begin{cases} kq^2(1 - q^{2k}) & v = T_k \\ k(1 - q^2)(1 - q^{2k}) & v = \tilde{T}_k \end{cases}.$$

Combining these two pieces, we arrive at the final index as stated in the theorem. \square

9. CONCLUDING REMARKS

(i) The algebra $A = SU_q(2)$ contains as a subalgebra a copy of $C(S^1)$. The map in odd K -theory $K_1(C(S^1)) \rightarrow K_1(A)$ induced by the inclusion $C(S^1) \rightarrow A$ is an isomorphism. Therefore for any odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where \mathcal{A} is smooth in A , we can restrict to $\mathcal{B} := C(S^1) \cap \mathcal{A}$ to obtain a spectral triple $(\mathcal{B}, \mathcal{H}, \mathcal{D})$, with \mathcal{B} smooth in $C(S^1)$ to get the following isomorphisms

$$K_1(\mathcal{B}) \rightarrow K_1(C(S^1)) \rightarrow K_1(A) \leftarrow K_1(\mathcal{A}).$$

Therefore an odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, from the point of view of index pairings with unitaries, contains the same information as $(\mathcal{B}, \mathcal{H}, \mathcal{D})$ where \mathcal{B} is a smooth subalgebra of $C(S^1)$.

(ii) The semifinite index is known to be related to pairings in KK -theory, [CPR1, KNR], but the modular index introduced here is still mysterious. We will return to an investigation of this new index pairing in a later work.

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