Nonunital Spectral Triples Associated to Degenerate Metrics

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Abstract

We show that one can define (p, ∞)-summable spectral triples using degenerate metrics on smooth manifolds. Furthermore, these triples satisfy Connes-Moscovici's discrete and finite dimension spectrum hypothesis, allowing one to use the Local Index Theorem, [1], to compute the pairing with K-theory. We demonstrate this with a concrete example.

1 Introduction

Let X be a p-dimensional, geodesically complete, paracompact, σ-compact Riemannian spin manifold with metric g. Our aim is to show that if $\tilde{g}$ is another 'metric' which is allowed to be degenerate on a submanifold of measure zero, then a (p, ∞)-summable spectral triple can be constructed by employing the Dirac operator associated to this degenerate metric.

The next section makes some preliminary definitions and fixes notation. Some of these definitions are modifications of standard definitions necessary to be able to encompass the nonunital setting. More information will be found in [2, 3]. Section 3 describes the spectral triples and presents our main theorems. The final section provides a detailed example.

The original aim of the constructions in this paper was to find an explicit example of a spectral triple with nonsimple dimension spectrum. This would provide an example where the index pairing could be computed using Connes and Moscovici's Local Index Theorem, [1], and hopefully there would be contributions arising from the higher order poles. This would be of benefit in obtaining greater understanding of the various terms in the Local Index Theorem. This original aim failed, but several interesting results were obtained.

This paper shows that one must work quite hard to obtain such an example. The 'Dirac' operator of our main example seems to contain a double pole in its zeta function, however when one considers for a smooth function a

$$s \to \mathrm{Trace}(a(1 + \mathcal{D}^2)^{-s}), \quad Re(s) > 1,$$

either there is a simple pole at $s = 1$, or the operator $a(1 + \mathcal{D}^2)^{-s}$ is not compact for any $s \in \mathbf{C}$, so the trace fails to make sense. In some sense there is a trade off between the size of the (nonzero) point spectrum and the kernel of $\mathcal{D}$, so that the zeta function has a simple pole, or we do not even obtain a spectral triple.

2 Definitions

Here we review the relevant definitions, language and notation we will employ in the remainder of the paper.

Definition 1 *A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C^*-algebra A which is stable under the holomorphic functional calculus.*

Definition 2 *An algebra $\mathcal{A}$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}$, there exists $\phi \in \mathcal{A}$ such that for each i*

$$\phi a_i = a_i \phi = a_i.$$

Definition 3 *Let $\mathcal{A}$ be a Fréchet algebra and $\mathcal{A}_c \subset \mathcal{A}$ be a dense ideal with local units. Then we call $\mathcal{A}$ a local algebra (when $\mathcal{A}_c$ is understood.)*

Remark Localizable would be a more descriptive word, and local is over used, but it will do for now. Note that unital algebras are automatically local. Furthermore, the dense ideal $\mathcal{A}_c$ is saturated in the following sense. If $a \in \mathcal{A}$ and $\exists \phi \in \mathcal{A}_c$ such that $\phi a = a\phi = a$, then $a \in \mathcal{A}_c$. This follows because $\mathcal{A}_c$ is an ideal.

Example The basic example of a smooth local algebra is $C_0^\infty(X)$, where X is a noncompact manifold, and $C_0^\infty(X)$ denotes the smooth functions all of whose derivatives vanish at infinity. This is Fréchet, stable under the holomorphic functional calculus, and the dense ideal of compactly supported functions has local units.

Numerous properties of, and constructions with, local algebras are presented in [2, 3]. Next we present the definition of spectral triples appropriate to our situation, modelled on Connes' definitions, [9, Chap. VI].

Definition 4 *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by*

1) A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a local $*$-algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}$.

2) A self-adjoint (unbounded, densely defined) operator $\mathcal{D} : \mathrm{dom}\mathcal{D} \to \mathcal{H}$ such that $[\mathcal{D}, a]$ extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$ and $a(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ is compact for all $a \in \mathcal{A}$.

The triple is said to be even if there is an operator $\Gamma = \Gamma^*$ such that $\Gamma^2 = 1$, $[\Gamma, a] = 0$ for all $a \in \mathcal{A}$ and $\Gamma\mathcal{D} + \mathcal{D}\Gamma = 0$ (i.e. Γ is a $\mathbf{Z}_2$-grading such that $\mathcal{D}$ is odd and $\mathcal{A}$ is even.) Otherwise the triple is called odd.

Definition 5 *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, then we define $\Omega^*_{\mathcal{D}}(\mathcal{A})$ to be the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$.*

Definition 6 *A spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ is smooth if*

$$\mathcal{A} \text{ and } [\mathcal{D},\mathcal{A}] \subseteq \bigcap_{m\geq 0} \text{dom } \delta^m$$

where for $x \in \mathcal{B}(\mathcal{H})$, $\delta(x) = [|\mathcal{D}|, x]$.

Remark Note the difference between the definitions of smooth for topological algebras and spectral triples. In [4] such triples are called regular. In fact we have the following, [2].

Lemma 1 *If $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a smooth spectral triple, then $(\mathcal{A}_\delta,\mathcal{H},\mathcal{D})$ is also a smooth spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the locally convex topology determined by the seminorms*

$$q_n(a) = \| \delta^n(a) \|_{\mathcal{D}}, \text{ where } \quad \| a \|_{\mathcal{D}} = \| a \| + \| [\mathcal{D}, a] \| .$$

Moreover, $\mathcal{A}_\delta$ is a smooth algebra.

The following definition is, if not crucial, hugely simplifying for summability issues, [3].

Definition 7 *A local spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a spectral triple with $\mathcal{A}$ a local algebra (see Definition 3) such that $\Omega^*_{\mathcal{D}}(\mathcal{A}_c) \subseteq \Omega^*_{\mathcal{D}}(\mathcal{A})$ is a local algebra.*

We may assume without loss of generality that a local spectral triple has a local approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n$ and $\phi_{n+1}[\mathcal{D},\phi_n] = [\mathcal{D},\phi_n]$.

Definition 8 *A local spectral triple is (p,∞)-summable if $p \geq 1$ and for all λ in the resolvent set of $\mathcal{D}$*

$$a(\mathcal{D}-\lambda)^{-1} \in \mathcal{L}^{(p,\infty)}(\mathcal{H}) \quad \forall a \in \mathcal{A}_c.$$

We call it θ-summable if

$$\text{Trace}(ae^{-t(1+\mathcal{D}^2)}) < \infty$$

for all $a \in \mathcal{A}_c$ and $t > 0$.

Remark If $\mathcal{A}$ is unital, $\ker \mathcal{D}$ is finite dimensional. This case is fairly well described in the literature, see for instance [9, Chap VI] and [5]. Note that the summability requirements are only for $a \in \mathcal{A}_c$. We do not assume that elements of the algebra $\mathcal{A}$ are all 'integrable'. Note that Lemma 1 does not guarantee that elements of the completion of $\mathcal{A}$ for the seminorms arising from the derivation δ satisfy the above summability condition in the nonunital case. Of course, there is no difficulty in the unital case.

In [3], we show that if $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a (p,∞)-summable local spectral triple, then the operator $A = a(1+\mathcal{D}^2)^{-p/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$. As such, we may apply any Dixmier trace Tr_ω, [9, IV.2.β], to the operator A. An operator $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ is called *measurable* if the number $Tr_\omega(T)$ is independent of the choice of Dixmier trace Tr_ω.

The other main summability requirement is Connes-Moscovici's 'discrete and finite dimension spectrum' hypothesis, [1, 5].

Definition 9 *A smooth spectral triple* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *has discrete dimension spectrum* Sd *if the set* $Sd \subset \{z \in \mathbf{C} : Re(z) \le p\}$, $p \ge 1$, *is discrete and for any* $b \in \mathcal{B}(\mathcal{A}_c)$ *the function*

$$\zeta_b(z) := \mathrm{Trace}(b(1+\mathcal{D}^2)^{-\frac{z}{2}}), \tag{1}$$

is defined for all $z \in \mathbf{C}$ *with* $Re(z) > p$ *and extends holomorphically to* $\mathbf{C}\backslash Sd$. *Furthermore we require that*

$$\Gamma(z)\zeta_b(z)$$

is of rapid decay on vertical lines with $Re(z) > 0$. *We say that the discrete dimension spectrum* Sd *is of finite multiplicity* k *if for all* $b \in \mathcal{B}(\mathcal{A}_c)$, ζ_b *has a pole of order at most* k. *We say that* Sd *is simple if* $k = 1$. *Here* Γ *denotes the gamma function, and* $\mathcal{B}(\mathcal{A}_c)$ *is the algebra generated by* $\delta^k(a)$, $\delta^n([\mathcal{D},a])$ *for* $a \in \mathcal{A}_c$ *and* $k, n \ge 0$.

It is important to note that in the case of simple dimension spectrum, this definition implies the measurability of all the operators $b(1+\mathcal{D}^2)^{-p/2}$, $b \in \mathcal{B}(\mathcal{A}_c)$, by [3, Corollary 18].

The Local Index Theorem of Connes-Moscovici, [1, 3], computes the index pairing between the K-theory of the algebra $\mathcal{A}$ and a smooth local spectral triple with discrete and finite dimension spectrum.

In the following $k = (k_1, ..., k_n) \in \mathbf{N}^n$, $|k| = k_1 + \cdots k_n$, $da = [\mathcal{D}, a]$ for $a \in \mathcal{A}$, and $(da)^{(k)} = \nabla^k(da)$ where $\nabla(T) = [\mathcal{D}^2, T]$. Finally, if the function

$$z \longrightarrow \mathrm{Trace}(T(1+\mathcal{D}^2)^{-m-z}), \qquad T \in \mathcal{B}(\mathcal{H})$$

has a Laurent expansion around $z = 0$, let

$$\tau_q(T(1+\mathcal{D}^2)^{-m})$$

be the coefficient of z^{-q-1} in this expansion.

Theorem 2 (Local Index Theorem [1, 3]) *Let* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *be a smooth spectral triple, with discrete and finite dimension spectrum contained in the half plane* $\{z : Re(z) \le p\}$, *and suppose that* $\Omega^*_{\mathcal{D}}(\mathcal{A})$ *is local. Then if* $\mathcal{A}$ *is unital, the following formulae define the components of a cyclic cocycle in the* (b, B) *bicomplex of* $\mathcal{A}$ *whose class coincides with the class of the Chern character in* $HC^*(\mathcal{A})$. *If* $\mathcal{A}$ *is nonunital, then the following formulae define cyclic cocycles in the distributional sense, and their class coincides with that of the Chern character in the cyclic cohomology* $HC^*(\mathcal{A}_c)$.

a) *For* $(\mathcal{A},\mathcal{H},\mathcal{D})$ *even and summing over* $q \le |k| + \frac{n}{2}$ *and* $|k| + n \le p$,

$$\phi_n(a_0, ..., a_n)$$
$$= \sum_{k,q} \frac{(-1)^{|k|}}{k_1! \cdots k_n!} \alpha_{k,n} \sigma_q(|k| + \frac{n}{2}) \tau_q(\Gamma a_0 (da_1)^{(k_1)} \cdots (da_n)^{(k_n)} (1+\mathcal{D}^2)^{-\frac{(2|k|+n)}{2}})$$

for $n \ne 0$ *even, while*

$$\phi_0(a_0) = \tau_{-1}(\Gamma a_0)$$

where

$$\tau_{-1}(b) = \mathrm{res}_{z=0} z^{-1}\mathrm{Trace}(b(1+\mathcal{D}^2)^{-z}).$$

The σ_q *are the symmetric functions of the numbers* $1, 2, ..., |k| + \frac{n}{2}$,

$$\prod_{i=1}^{|k|+\frac{n}{2}} (s+i) = \sum_{j=0}^{|k|+\frac{n}{2}-1} \sigma_j(|k| + \frac{n}{2})s^j,$$

and

$$\alpha_{k,n}^{-1} = (k_1+1)(k_1+k_2+2)\cdots(k_1+k_2+\cdots+k_n+n).$$

b) *For* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *odd and summing over* $q \leq |k| + \frac{n-1}{2}$ *and* $|k| + n \leq p$,

$$\phi_n(a_0, ..., a_n)$$
$$= \sqrt{2\pi i}\sum_{k,q} \frac{(-1)^{|k|}}{k_1!\cdots k_n!}\alpha_{k,n}\sigma_{m-q}(m)\tau_q(a_0(da_1)^{(k_1)}\cdots(da_n)^{(k_n)}(1+\mathcal{D}^2)^{-\frac{(2|k|+n)}{2}})$$

where $m = |k| + \frac{n-1}{2}$ *and* σ_j *is defined by*

$$\prod_{l=0}^{m-1}\left(z + \frac{(2l+1)}{2}\right) = \sum z^j \sigma_{m-j}(m).$$

This statement is slightly different to that in [1], in that it has been extended to the nonunital case as described in [3]. More details can be found in these papers.

3 Construction of the Triples

Let X be a p-dimensional, geodesically complete, paracompact, σ-compact Riemannian spin manifold with metric g. Let $S_{\mathbf{C}} \to X$ be the complex spinor bundle canonically associated to the spin structure, [6, Appendix D], and $\mathcal{D} : \Gamma(S_{\mathbf{C}}) \to \Gamma(S_{\mathbf{C}})$ the Dirac operator of the spin structure. So in local coordinates $x_1, ..., x_p$ we have

$$dx_i \cdot dx_j + dx_j \cdot dx_i = -2g(dx_i, dx_j), \quad \mathcal{D} = \sum_{i=1}^{p} dx_j \cdot \nabla_j^{LC},$$

where $\cdot$ denotes Clifford multiplication and ∇^{LC} is the lift of the Levi-Civita connection on the cotangent bundle to the spinor bundle. Finally, let $\omega_{\mathbf{C}}$ be the complex volume form, [6], which in local coordinates is given by

$$\omega_{\mathbf{C}} = i^{[\frac{p+1}{2}]}dx_1\cdots dx_p.$$

If we define $C_0^\infty(X)$ to be the smooth complex-valued functions all of whose partial derivatives vanish at infinity, we have

Proposition 3 *The tuple* $(C_0^\infty(X), L^2(X, S_{\mathbf{C}}, g), \mathcal{D}, \omega_{\mathbf{C}})$ *is a spectral triple. It is* (p, ∞)*-summable, where* $p = \dim X$*, and has discrete and simple dimension spectrum. For all compactly supported* $a \in C_0^\infty(X)$*, the operator* $a(1+\mathcal{D}^2)^{-p/2}$ *is measurable, and so for any Dixmier trace*

$$Tr_\omega(a(1+\mathcal{D}^2)^{-p/2}) = c(p)\int_X a(x)dvol(x),$$

where $c(p)$ *is a constant depending only on* p *and* $dvol$ *is the Riemannian volume form.*

Proof In [2] it is shown that for a complete spin manifold, the topology (on smooth functions) of convergence in the seminorms $q_n(a) = \parallel \delta^n(a) \parallel_{\mathcal{D}}$, $a : X \to \mathbf{C}$, $\delta(a) = [|\mathcal{D}|, a]$, is the topology of uniform convergence of all derivatives. Thus by Lemma 1, it suffices to show that $(C_c^\infty(X), L^2(X, S_{\mathbf{C}}, g), \mathcal{D}, \omega_{\mathbf{C}})$ is a spectral triple, where $C_c^\infty(X)$ denotes the smooth compactly supported functions. The first step is to show that $\mathcal{D}$ is essentially self-adjoint, and so can be extended to a closed self-adjoint operator on $L^2(X, S_{\mathbf{C}}, g)$. An integration by parts shows that $\mathcal{D}$ is symmetric. The completeness of X and the finite propagation speed of the Dirac operator allows us to employ [7, Proposition 10.2.11], which shows that $\mathcal{D}$ is essentially self-adjoint.

The compactness and (p, ∞)-summability results are proven in [3]. In particular, for all compactly supported functions a on X, $a(1+\mathcal{D}^2)^{-p/2} \in \mathcal{L}^{(1,\infty)}(L^2(X, S_{\mathbf{C}}, g))$. The statements on the dimension spectrum are implied by Seeley's results, [8, Theorem 4 and section 2], namely that for any function a with support contained in a single coordinate chart (with compact closure), the function

$$s \longrightarrow \mathrm{Trace}(a(1+\mathcal{D}^2)^{-s/2}), \quad s > p,$$

extends to a meromorphic function with at most simple poles. The value of the residue at $s = p$ is given by the Wodzicki residue [4, 8, 9, 10],

$$WRes(a(1+\mathcal{D}^2)^{-p/2}) = \frac{2^{[p/2]}}{p(2\pi)^p}\int_{S^*X} a(x) \parallel \xi \parallel^{-p} dS(\xi)dvol = c(p)\int_X a(x)dvol(x).$$

By Connes' trace theorem, [1, Appendix A], the operator $a(1+\mathcal{D}^2)^{-p/2}$ is in the Dixmier ideal $\mathcal{L}^{(1,\infty)}(L^2(X, S_{\mathbf{C}}, g))$, and the Wodzicki residue coincides with the value of any Dixmier trace on $a(1+\mathcal{D}^2)^{-p/2}$. These results depend crucially on the self-adjointness and uniform ellipticity of the Dirac operator.

To conclude that the dimension spectrum is simple we need to check that the above statements are still true when we replace $a \in C_c^\infty(X)$ with $b = \delta^k(a)$ or $b = \delta^k([\mathcal{D}, a])$. In both cases, b is an order zero pseudodifferential operator with principal symbol a compactly supported function, [1, 2, 4]. The lower order terms do not contribute to the Wodzicki residue.

The grading conditions are well known, [6], and we have

$$\omega_{\mathbf{C}}\mathcal{D} + (-1)^p\mathcal{D}\omega_{\mathbf{C}} = 0$$

and $\omega_{\mathbf{C}}$ may be normalised to 1 when p is odd. Hence if $\dim X$ is even the triple is even, and if $\dim X$ is odd, the triple is odd. $\square$

Now let $\tilde{g}$ be a positive semidefinite metric. That is a smooth, bounded, symmetric section of $TX \otimes TX$, possibly degenerate. Let

$$F = \{x \in X : \exists 0 \neq \alpha \in \Gamma(T^*X) \text{ such that } \tilde{g}(\alpha, \alpha)(x) = 0\},$$

be the degeneracy set of $\tilde{g}$. We assume that F is closed, measure zero (with respect to the Riemannian volume form defined by the original complete metric g) and a smooth submanifold (possibly with boundary) so that $X \setminus F$ is a smooth manifold. The Clifford algebra determined by the semidefinite metric $\tilde{g}$ allows one to define a new Dirac operator, so that in local coordinates on X

$$dx_i \bullet dx_j + dx_j \bullet dx_i = -2\tilde{g}(dx_i, dx_j), \quad \tilde{\mathcal{D}} = \sum_{i=1}^{p} dx_i \bullet \nabla_i^{LC},$$

where $\bullet$ is the Clifford multiplication determined by the new metric $\tilde{g}$ and ∇^{LC} is the lift of the Levi-Civita connection (with respect to the old metric g) on TX to the spinor bundle (again with respect to the old metric g). Thus we are retaining the spinor bundle and connection of the complete metric g, and using $\tilde{g}$ to obtain a new Clifford action and hence a new Dirac operator.

The only remaining issue is to define the action of the new Clifford algebra on the old spinor bundle. An obviously sufficient condition for this to be possible is that there is an inclusion of the algebras of sections

$$\Gamma(Cliff(TX, \tilde{g})) \subseteq \Gamma(Cliff(TX, g)).$$

A sufficient condition for this to hold is as follows. Suppose that in any local coordinates we have

$$\tilde{g}_{ij} = f_{ij} g_{ij},$$

with each f_{ij} a smooth nonnegative function. Provided that for each i, j we have either $f_{ij} = \sqrt{f_{ii} f_{jj}}$ or $f_{ij} = 0$, we can define a representation of the new Clifford algebra on the old spinor bundle by setting

$$dx_i \bullet \xi = \sqrt{f_{ii}} dx_i \cdot \xi, \quad \xi \in \Gamma(X, S_{\mathbf{C}}).$$

One can now check that the Clifford relations for the new metric are satisfied. In the even case we also have (as operators on the spinor bundle or on Hilbert space) that $dx_i \bullet$ and $\omega_{\mathbf{C}}$ anticommute.

In the following we assume that the new Clifford algebra acts on the old spinor bundle, by restricting to the above case if necessary.

Theorem 4 *The tuple $(C_c^\infty(X \setminus F), L^2(X, S_{\mathbf{C}}, g), \tilde{\mathcal{D}}, \omega_{\mathbf{C}})$, with F and $\tilde{\mathcal{D}}$ as above, is a spectral triple. It is local and (p, ∞)-summable, where $p = \dim X$, and has discrete and simple dimension spectrum. For all functions $a \in C_c^\infty(X \setminus F)$, the operator $a(1 + \tilde{\mathcal{D}}^2)^{-p/2}$ is measurable and for any Dixmier trace*

$$Tr_\omega(a(1 + \tilde{\mathcal{D}}^2)^{-p/2}) = \frac{1}{p(2\pi)^p} \int_X \int_{S^*X} a(x) \mathrm{Trace}(\tilde{g}(\xi, \xi)^{-p/2}) dS(\xi) dvol(x),$$

where $dS(\xi)dvol(x)$ is the volume form of the cosphere bundle S^*X for the complete metric g.

Proof The first portion of the proof is exactly the same as the last proposition, with $\tilde{\mathcal{D}}$ self-adjoint by the completeness of X and the finite propagation speed of $\tilde{\mathcal{D}}$, [7, Proposition 10.2.11], the finite propagation speed following from the boundedness of the semidefinite metric $\tilde{g}$. To apply Seeley's results, as in Proposition 3, to $a(1+\tilde{\mathcal{D}}^2)^{-s/2}$, we need to be sure that $\tilde{\mathcal{D}}^2$ is uniformly elliptic over the support of a. However, the support of a is disjoint from the set of degeneracy F, so over the support of a, the size of the smallest eigenvalue of the principal symbol of $\tilde{\mathcal{D}}^2$ is bounded from below (and is greater than zero). Thus Seeley's techniques can be applied, and we deduce the simplicity of the dimension spectrum. The remainder of the proof now follows as in Proposition 3, the value of the Dixmier trace being given by the Wodzicki residue, which is given by the formula in the statement of the proposition. □

Remark The example in the next section employs a degenerate metric which is not bounded, and so $\tilde{\mathcal{D}}$ does not have finite propagation speed. Nevertheless, explicit calculations in the next section show that $\tilde{\mathcal{D}}$ is in fact self-adjoint.

Lemma 5 *With $(C_c^\infty(X \setminus F), L^2(X, S_{\mathbf{C}}, g), \tilde{\mathcal{D}}, \omega_{\mathbf{C}})$ as in the proposition, and σ the principal symbol of $\tilde{\mathcal{D}}^2$, the algebra*

$$\mathcal{A}_h = \{a \in C_0^\infty(X\backslash F) : x \to (\partial^\alpha a)(x)\mathrm{Trace}_{S_{\mathbf{C}}}(\sigma(x,\xi)^{-p/2}) \text{ is integrable for all multi}-\text{indices } \alpha\}$$

is a smooth algebra. Here integrability is over the cosphere bundle of X with respect to the volume form of the original metric g.

Proof The algebra $\mathcal{A}_h$ is dense in $C_0(X\backslash F)$, since it contains the smooth compactly supported functions.

Define seminorms on $\mathcal{A}_h$ by

$$q_n(a) = \sup_{|\alpha|\leq n} \sup_{x\in X} |\partial^\alpha a(x)|, \quad q_{n1}(a) = \sup_{|\alpha|\leq n} \int_{S^*X} \left|(\partial^\alpha a)(x)\mathrm{Trace}(\sigma(x,\xi)^{-p/2})\right| dS(\xi)dvol(x).$$

These seminorms determine a locally convex metrisable topology on $\mathcal{A}_h$, and a standard $\epsilon/3$ proof shows that $\mathcal{A}_h$ is complete, and so Fréchet.

To show that $\mathcal{A}_h$ is stable under the holomorphic functional calculus, suppose that $a \in \mathcal{A}_h$ and $1 + a$ is invertible in $C((X \setminus F)^+)$, with inverse $1 + b$. Then $b \in C_0(X \setminus F)$ and b is smooth. This is because the equation $a + b + ab = 0$ implies that $b = -a/(1 + a)$, which has derivatives of all orders by hypothesis, and these all vanish at infinity. The integrability condition follows similarly, since

$$b\sigma^{-p/2} = -a\sigma^{-p/2} - ab\sigma^{-p/2} = -(1 + b)a\sigma^{-p/2},$$

and $1 + b$ is bounded whilst a satisfies the integrability criteria by hypothesis. Differentiating $b = -(1 + b)a$, and applying the Leibniz rule completes the proof. Hence $\mathcal{A}_h$ is stable under the holomorphic functional calculus, and so smooth. □

Corollary 6 *The results of Proposition 4 remain true with the compactly supported functions replaced by $\mathcal{A}_h$.*

Proof (Sketch) For $a \in \mathcal{A}_h$ positive, we may use the monotone convergence theorem for the measure

$$\mu(E) = \int_{S^*E} \mathrm{Trace}_{S_{\mathbf{C}}}(\sigma(x,\xi)^{-p/2})dS(\xi)dvol(x), \quad E \subset X \setminus F,$$

to show that the measurability results hold for $a \geq 0$. Linearity allows us to conclude for general $a \in \mathcal{A}_h$. This measurability result, and its proof, is essentially the same as [4, Corollary 7.22]. The boundedness of $[\mathcal{D}, a]$, $a \in \mathcal{A}_h$, follows from the smoothness of the functions in $\mathcal{A}_h$. □

This is important for computing the pairing with K-theory. Along with results proved in [2, 3], this means that we can apply the Local Index Theorem to compute the pairing of the K-homology class of the spectral triple of a degenerate metric on X with the K-theory of $X \setminus F$. This follows because

$$K_*(\mathcal{A}_h) \cong K_*(C_0(X \setminus F),$$

so any class $[x]$ in the right hand group has a representative $x \in M_N(\mathcal{A}_h)$ for some sufficiently large N. This result of course applies to the case where $\tilde{g}$ is not degenerate; in particular it aplies to g.

4 A Detailed Example

In this section we present an example which shows how the construction of spectral triples from degenerate metrics can be used to do index theory on mildly singular spaces. We are quite explicit in what follows, so that it is clear what prevents us being able to work with some algebra of functions which is nonzero on the set of degeneracy of the metric.

The computations in this section determine precisely for which functions we obtain a spectral triple. Once this is done, we compute the K-theory of the singular space on which we work, and identify generators of the even K-theory. The index pairing between the spectral triple on this singular space and the K-theory generators is then determined using the Local Index Theorem of Connes-Moscovici, [1], which reduces to a Wodzicki residue computation.

We build this triple by making a deliberately naive attempt to work on a singular space. The extremely simple space we choose is the double cone,

$$C = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 = \kappa^2 z^2\},$$

where $\kappa = \tan(\frac{\alpha}{2})$, and $\alpha \in (0, \pi)$ is the cone angle.

At every point $z \neq 0$, we have a well-defined cotangent space, and choosing cylindrical coordinates (z, θ), this cotangent space is spanned by $dz, d\theta$. At each such point, we define a Clifford action of these covectors on $\mathbf{C}^2$ by

$$dz = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}, \quad d\theta = \begin{pmatrix} 0 & \frac{\kappa z}{i} \\ \frac{\kappa z}{i} & 0 \end{pmatrix}.$$

These satisfy the Clifford relations for the metric

$$g(z, \theta) = \begin{pmatrix} \kappa^2 z^2 & 0 \\ 0 & \kappa^2 \end{pmatrix}.$$

Of course this metric is degenerate at $z = 0$, but elsewhere reproduces the correct distances on the cone. The next step is to define the corresponding Dirac operator,

$$\mathcal{D} = dz\partial_z + d\theta\partial_\theta = \begin{pmatrix} 0 & \frac{\kappa z}{i}\partial_\theta - \kappa\partial_z \\ \frac{\kappa z}{i}\partial_\theta + \kappa\partial_z & 0 \end{pmatrix}.$$

We initially regard $\mathcal{D}$, and $\mathcal{D}^2$, as defined on the smooth sections of the spinor bundle over the cylinder, which are of rapid decrease. In the Hilbert space completions below, this will mean that $\mathcal{D}$ is not closed, but by [7, Lemma 10.2.1], it is closable.

The Hilbert space we employ is $\mathcal{H} = L^2(Cyl, \mathbf{C}^2, dzd\theta)$, where Cyl denotes the doubly infinite cylinder of unit radius, $L^2(Cyl, \mathbf{C}^2)$ is the L^2 sections of the trivial plane bundle (the spinor bundle) over the cylinder, and $dzd\theta$ denotes the usual Riemannian volume form on the cylinder (not the above Clifford action). By making this choice of Hilbert space we are regarding the cone as the cylinder imbued with a degenerate metric.

The operator $\Gamma = idzd\theta$ is a $\mathbf{Z}_2$-grading for $\mathcal{H}$ anticommuting with $\mathcal{D}$. In this expression $dz, d\theta$ act via the usual Clifford action on the cylinder,

$$dz = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad d\theta = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So there is a good deal of interplay between the two metrics we have imposed on the cylinder.

Finally, our algebra of functions must encode the topology of the cone, and act on $\mathcal{H}$. We must expect trouble from the singularity at $z = 0$, so we adopt the definition

$$\mathcal{A} = \{a : C \to \mathbf{C} : z^k\partial_\theta^m\partial_z^l a \text{ is smooth and vanishes at } z = 0, \pm\infty \text{ for all } k, l, m \geq 0\}.$$

The vanishing of a function a at $z = 0, \pm\infty$ is taken in the usual topological sense, so $a(z) \to 0$ as $z \to 0, \pm\infty$. The algebra $\mathcal{A}$ has a local structure, with the dense ideal of functions compactly supported away from $z = 0, \pm\infty$ providing $\mathcal{A}_c \subseteq \mathcal{A}$. We let $\mathcal{A}$ act by multiplication on $\mathcal{H}$.

Next we compute the spectrum of the operator $\mathcal{D}$. The sensible way to tackle the spectrum of a Dirac operator is to first consider the associated Laplace equation.

Lemma 7 *The operator $\mathcal{D}^2$ is essentially self-adjoint with spectrum the nonnegative reals. The kernel is infinite dimensional, the point spectrum consists of the values $2\kappa^2 N$, $N > 0$ integral, with multiplicity $4d(N)$, where $d(N)$ is the divisor function of N.*

Proof We first consider the equation

$$\mathcal{D}^2\xi = \lambda^2\xi,$$

$$\begin{pmatrix} -z^2\partial_\theta^2 - \partial_z^2 - \frac{1}{i}\partial_\theta & 0 \\ 0 & -z^2\partial_\theta^2 - \partial_z^2 + \frac{1}{i}\partial_\theta \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \frac{\lambda^2}{\kappa^2} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

To solve this equation, we employ separation of variables. If we can span the Hilbert space with such solutions there will be no need to try anything more esoteric.

For the first component we write $\xi_1(z, \theta) = f(\theta)g(z)$, and we consider three possibilities.

(1) f is constant. In this case the equation for the first component reduces to

$$g''(z) = -\frac{\lambda^2}{\kappa^2} g(z).$$

If $\lambda^2 > 0$, then the only solutions are oscillatory, and do not vanish at infinity. Provided that such a λ^2 is not an eigenvalue, this shows that it is in the continuous spectrum of $\mathcal{D}^2$. If $\lambda^2 = 0$, we will obtain a linear solution, again not vanishing at infinity, but we will see later that there are in fact many solutions in the kernel of $\mathcal{D}^2$. Finally, if $\lambda^2 < 0$, we have the solutions

$$g_\lambda(z) = e^{\pm\sqrt{-\frac{\lambda^2}{\kappa^2}}z},$$

and these fail to vanish at one of $\pm\infty$ or the other, and they do not belong to the Hilbert space.

(2) $f(\theta) = e^{im\theta}$, $m > 0$. This yields the equation

$$g''(z) = (z^2m^2 - m - \frac{\lambda^2}{\kappa^2})g(z).$$

The substitution $g(z) = \tilde{g}(z)e^{-\frac{m}{2}z^2}$ reduces this to

$$\tilde{g}''(z) - 2mz\tilde{g}'(z) + \frac{\lambda^2}{\kappa^2}\tilde{g} = 0.$$

For $m = 1$ this is the defining equation for the Hermite polynomials, and it is not difficult from there to see that

$$g(z) = H_n(\sqrt{m}z)e^{-\frac{m}{2}z^2}, \quad \lambda^2 = 2\kappa^2 nm, \quad m > 0, \ n \geq 0$$

is the unique square integrable solution, [12, 13].

(3) $f = e^{-im\theta}$, $m > 0$. With the same ansatz as the last case we find

$$\tilde{g}''(z) - 2mz\tilde{g}'(z) - (2m - \frac{\lambda^2}{\kappa^2})\tilde{g}(z) = 0,$$

and for $\lambda^2 = 2\kappa^2 m(n+1)$ this is the same as for the last case. Thus the unique solution is

$$g(z) = H_n(\sqrt{m}z)e^{-\frac{m}{2}z^2}, \quad \lambda^2 = 2\kappa^2 m(n+1), \quad m > 0, \ n \geq 0.$$

The equation for the second component behaves exactly as the first when f is constant, while the rôles of the two cases $f(\theta) = e^{im\theta}$ and $f(\theta) = e^{-im\theta}$ are reversed.

For $n \geq 0$ and $0 \neq m \in \mathbf{Z}$, define the functions $s_{nm} = e^{im\theta}e^{-|m|z^2/2}H_n(\sqrt{m}z)$. For $n < 0$ we set $s_{nm} = 0$, and for $m = 0$ we set $s_{n0} = e^{-z^2/2}H_n(z)$. Next define spinors

$$\xi_{nm1}(z,\theta) = \begin{pmatrix} s_{nm} \\ 0 \end{pmatrix}, \quad \xi_{nm2}(z,\theta) = \begin{pmatrix} 0 \\ s_{nm} \end{pmatrix}.$$

Using the orthogonality relations

$$\int_{-\infty}^{\infty} H_n(w)H_m(w)e^{-w^2}dw = \delta_{nm}2^n n!\sqrt{\pi}, \qquad \int_0^{2\pi} e^{il\theta}e^{-im\theta}d\theta = \delta_{lm}2\pi$$

and the completeness of the Hermite and trigonometric polynomials, one can show that these spinors provide a complete orthogonal basis of $L^2(Cyl, \mathbf{C}^2)$.

The operator $\mathcal{D}^2$ is defined on all finite linear combinations of these spinors, which is a dense subset of the smooth spinors of rapid decrease. Thus it suffices to show that $\mathcal{D}^2$ is essentially self-adjoint on this subspace, for then the unique self-adjoint extension, given by the closure, will coincide with the closure of $\mathcal{D}^2$ defined on all smooth spinors of rapid decrease. Moreover the projections on to the (closures of the) following three subspaces commute with $\mathcal{D}^2$, so we can write $\mathcal{D}^2$ as the direct sum of the restrictions of $\mathcal{D}^2$ to these subspaces. The subspaces are:

• The kernel of $\mathcal{D}^2$ is the L^2 closure of the span of the spinors ξ_{0m1}, $m < 0$, and ξ_{0m2}, $m > 0$. Thus the restriction of $\mathcal{D}^2$ to this subspace is a closed operator.

• The restriction of $\mathcal{D}^2$ to finite linear combinations of the spinors ξ_{n0i}, $i = 1, 2$ is essentially self-adjoint. This follows because these basis vectors are independent of θ and so $\mathcal{D}^2$ acts as $-\partial_z^2$, which is known to be essentially self-adjoint with continuous spectrum the positive reals.

• Finally, the action of $\mathcal{D}^2$ on the subspace of finite linear combinations of the eigenspinors for nonzero eigenvalues is essentially self-adjoint. This follows from the denseness of the range of $\mathcal{D}^2 \pm i$ on this subspace and [11, p 257]. The denseness of the range of $\mathcal{D}^2 \pm i$ follows from the explicit computations above.

Since $\mathcal{D}^2$ is the direct sum of these three restrictions, $\mathcal{D}^2$ is essentially self-adjoint and so has a unique self-adjoint extension, which we shall also refer to as $\mathcal{D}^2$.

Thus the spectrum of $\mathcal{D}^2$ is the nonnegative real axis, with the points $2\kappa^2 N$, $N \in \mathbf{N}$, being eigenvalues and everything else being continuous spectrum. The multiplicity of each $\lambda^2 = 2\kappa^2 N$, $N > 0$, is $4d(N)$, where $d(N)$ is the divisor function, the number of divisors of N including 1 and N, [14]. The origin of the divisor function is clear; the four arises by counting the eigenvectors for $\lambda^2 = 2\kappa^2 nm$, $m > 0, n > 0$, namely $\xi_{n(-m)1}$, ξ_{nm2}, and those for $\lambda^2 = 2\kappa^2 m(n+1)$, $m > 0$, $n \geq 0$, which are ξ_{nm1}, $\xi_{n(-m)2}$. □

The presence of the divisor function, whose asymptotics are extremely subtle, [14], indicates that the zeta function of $\mathcal{D}^2$ will have very interesting behaviour.

Lemma 8 *The operator $\mathcal{D}$ is essentially self-adjoint with spectrum the whole real line. The kernel is infinite dimensional, the point spectrum consists of the values $\pm\kappa\sqrt{2N}$, $N > 0$ integral, with multiplicity $2d(N)$.*

Proof As in Lemma 7, for $n \geq 0$ and $m > 0$, define functions $s_{nm} = e^{im\theta}e^{-mz^2/2}H_n(\sqrt{m}z)$. For $n < 0$ we set $s_{nm} = 0$, and for $m = 0$ we set $s_{n0} = e^{-z^2/2}H_n(z)$.

Using the orthogonality relations and the completeness of the Hermite and trigonometric polynomials as in Lemma 7, it is easy to check that the spinors

$$\chi_{nm\pm} = \begin{cases} \begin{pmatrix} \sqrt{2n} s_{n-1,m} \\ \mp s_{nm} \end{pmatrix} & m < 0 \\ \begin{pmatrix} \pm s_{nm} \\ \sqrt{2n} s_{n-1,m} \end{pmatrix} & m > 0 \\ \begin{pmatrix} s_{n0} \\ \pm i s_{n0} \end{pmatrix} & m = 0 \end{cases}$$

provide a complete orthogonal basis for $L^2(Cyl, \mathbf{C}^2)$. For $m \neq 0$, $\mathcal{D}\chi_{nm\pm} = \pm\kappa\sqrt{2nm}\chi_{nm\pm}$, and for $m = 0$, $\mathcal{D}\chi_{n0\pm} = \pm(\kappa/i)\partial_z\chi_{n0\pm}$. As in Lemma 7, $\mathcal{D}$ is closed on its kernel (spanned by $\chi_{0m\pm}$, $m \neq 0$), acts as $\pm(\kappa/i)\partial_z$ on two copies of $L^2(\mathbf{R})$ (spanned by $\chi_{n0\pm}$), and $\mathcal{D} \pm i$ has dense range when restricted to the finite linear combinations of the eigenvectors $\chi_{nm\pm}$, $m \neq 0$. As $\mathcal{D}$ is the direct sum of these restrictions, and each is essentially self-adjoint, $\mathcal{D}$ is essentially self-adjoint, [11, p 257], and has a unique self-adjoint extension which we also denote by $\mathcal{D}$. $\Box$

In the following we will be estimating traces for operators of the form $a(1 + \mathcal{D}^2)^{-s}$, $a \in \mathcal{A}$. The basis described in Lemma 7 is more suitable than that in Lemma 8. The normalisations to obtain an orthonormal basis are

$$\xi_{nmi} \longrightarrow \frac{|m|^{1/4}}{\sqrt{2\pi\sqrt{\pi}2^n n!}}\xi_{nmi} \quad i = 1, 2, \ m \neq 0, \tag{2}$$

$$\xi_{n0i} \longrightarrow \frac{1}{\sqrt{2\pi\sqrt{\pi}2^n n!}}\xi_{n0i}, \quad i = 1, 2.$$

The only remaining item to check in order to show that $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$ is a spectral triple is the compactness of $a(1 + \mathcal{D}^2)^{-1/2}$, for all $a \in \mathcal{A}$ compactly supported away from zero.

Lemma 9 *If $a \in \mathcal{A}$ has compact support disjoint from the set $\{(z, \theta) \in \mathbf{R} \times [0, 2\pi) : z = 0\}$, then the operator $a(1 + \mathcal{D}^2)^{-1/2}$ is compact. If a is a function defined on the cone which is nonzero at $z = 0$, $a(1 + \mathcal{D}^2)^{-1/2}$ is not compact.*

Proof Write $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_p \oplus \mathcal{H}_k$ for the decomposition of $\mathcal{H}$ into closed subspaces corresponding to the continuous subspace, the nonzero eigenspaces and the kernel of $\mathcal{D}^2$, respectively. Let P_c, P_p, P_k be the corresponding projections. Then from what we already know about the spectrum of $\mathcal{D}^2$, and employing the closure of the compacts under adjoints,

$$a(1 + \mathcal{D}^2)^{-\frac{1}{2}} = \begin{pmatrix} ? & K & ? \\ K & K & K \\ ? & K & ? \end{pmatrix} \begin{pmatrix} \mathcal{H}_c \\ \mathcal{H}_p \\ \mathcal{H}_k \end{pmatrix},$$

where K indicates that the entry is a compact operator between the appropriate subspaces. So, we begin with

$$P_c a(1 + \mathcal{D}^2)^{-\frac{1}{2}} P_c = \tilde{a} \begin{pmatrix} 1 - \kappa^2\partial_z^2 & 0 \\ 0 & 1 - \kappa^2\partial_z^2 \end{pmatrix}^{-\frac{1}{2}},$$

where $a(z,\theta) = \tilde{a}(z) + \sum_{m\neq 0} a_m(z)e^{im\theta}$. In this case, [3, Proposition 21] shows that

$$P_c a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_c \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_c),$$

and so compact. In fact it is measurable and

$$⨍ P_c a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_c = \kappa c(2) \int_{-\infty}^{\infty} \tilde{a}(z)dz.$$

Note that this piece of the computation did not require that a be nonzero at $z = 0$. Next we consider

$$P_k a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_c.$$

The projection P_k projects on to the subspace spanned by

$$e^{im\theta}e^{-\frac{m}{2}z^2}, \quad e^{-im\theta}e^{-\frac{m}{2}z^2}, \; m > 0$$

while P_c projects on to the space spanned by $H_n(z)e^{-\frac{z^2}{2}}$, $n \geq 0$. Thus we need to estimate

$$\frac{m^{1/4}}{2\pi\sqrt{2\pi 2^k k!}} \left| \int_C a(z,\theta)H_k(z)e^{-\frac{m+1}{2}z^2}e^{-im\theta}dzd\theta \right|.$$

Let a_{mk} is the coefficient of $H_k(z)$ in the expansion of the function $a_m(z)$ in the basis provided by the Hermite functions (with respect to the measure $e^{-z^2}dz$). These coefficents a_{mk} are $o((mk)^{-1/2})$ for large k, m. Thus

$$\frac{m^{1/4}}{2\pi\sqrt{2\pi 2^k k!}} \left| \int_C a(z,\theta)H_k(z)e^{-\frac{m+1}{2}z^2}e^{-im\theta}dzd\theta \right| \leq \frac{m^{1/4}}{\sqrt{2}} a_{mk} \longrightarrow 0.$$

So $P_k a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_c$ is compact. In fact we have shown that this term remains compact even if a is not zero at $z = 0$.

We now come to the final term. It is now that we need the compact support away from $z = 0$ for the functions a that we consider. So let $supp(a(z,\theta)) \subseteq ([-K,-\epsilon] \cup [\epsilon, K]) \times [0, 2\pi]$ for some $K >> 1$. Then

$$P_k a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_k = P_k a P_k$$

is compact, and to show this we need to estimate

$$\frac{\sqrt{m}}{2\sqrt{\pi}} \left| \int_{-\infty}^{\infty} \int_0^{2\pi} e^{-mz^2} a(z,\theta)dzd\theta \right| \leq \frac{\sqrt{m}}{2\sqrt{\pi}} e^{-m\epsilon^2} \parallel \tilde{a}(z) \parallel_1,$$

and we see that this is compact. If a is compactly supported but nonzero at $z = 0$, the sequence of integrals

$$\left| \int_{-\infty}^{\infty} \int_0^{2\pi} e^{-mz^2} a(z,\theta)dzd\theta \right|$$

is $O(m^{-1/2})$, and so the operator $P_k a(1+\mathcal{D}^2)^{-\frac{1}{2}} P_k$ is bounded but not compact. □

From Theorem 4 we know that the triple we have built over the cone has discrete and simple dimension spectrum, and is $(2,\infty)$-summable. So for a compactly supported away from zero,

$$\mathrm{Trace}(a(1+\mathcal{D}^2)^{-s})$$

is meromorphic, where initially we suppose that $s >> 1$. This trace is the sum of three pieces

$$\begin{aligned}\mathrm{Trace}(a(1+\mathcal{D}^2)^{-s}) &= \mathrm{Trace}(P_k a(1+\mathcal{D}^2)^{-s}P_k) \\ &+ \mathrm{Trace}(P_c a(1+\mathcal{D}^2)^{-s}P_c) + \mathrm{Trace}(P_p a(1+\mathcal{D}^2)^{-s}P_p).\end{aligned}$$

As already noted, $\mathrm{Trace}(P_c a(1+\mathcal{D}^2)^{-s}P_c)$ is holomorphic for all s with $Re(s) > \frac{1}{2}$. The pole at $s = \frac{1}{2}$ is simple and the residue is given by

$$\mathrm{res}_{s=\frac{1}{2}}\mathrm{Trace}(P_c a(1+\mathcal{D}^2)^{-s}P_c) = \frac{\kappa}{2\pi^2}\int_{-\infty}^{\infty}\tilde{a}(z)dz,$$

with $\tilde{a}$ the piece of a independent of θ. Seeley's results, [8, Theorem 4 and section 2], and the compact support of $\tilde{a}$, allow us to conclude that this piece of the trace analytically continues to $\mathbf{C}$ with the exceptions of the half-integers less than or equal to $\frac{1}{2}$, and all poles are simple.

We have already seen that the contribution of

$$\mathrm{Trace}(P_k a(1+\mathcal{D}^2)^{-s}P_k) = \mathrm{Trace}(P_k a P_k)$$

is independent of s and in fact finite (*provided* a is supported away from $z = 0$), from our earlier estimate. So we are left with the point spectrum.

It is shown in [14, Thm 289, p250] that

$$\sum_{n=1}^{\infty}\frac{d(n)}{n^s} = \zeta(s)^2,\ \ s > 1,$$

where ζ denotes the Riemann zeta function.

To put this information to use, we estimate

$$\begin{aligned}\left|\mathrm{Trace}(P_p a(1+\mathcal{D}^2)^{-s}P_p)\right| &= \sum_{k,m>0}\frac{4\sqrt{m}(1+2\kappa^2 mk)^{-s}}{2\pi\sqrt{\pi}2^k k!}\left|\int_C a(z,\theta)H_k^2(\sqrt{m}z)e^{-mz^2}dzd\theta\right| \\ &\le 4\parallel\tilde{a}\parallel_\infty\sum_{k,m>0}(1+2\kappa^2 km)^{-s} \\ &\sim 4\parallel\tilde{a}\parallel_\infty 2^{2-s}\kappa^{-2s}\sum_{n=1}^{\infty}\frac{d(n)}{n^s}.\end{aligned}$$

Here $\sim$ indicates that

$$\lim_{s\to 1^+}(2^{2-s}\kappa^{-2s}\sum_{n=1}^{\infty}\frac{d(n)}{n^s} - \sum_{k,m>0}(1+2\kappa^2 km)^{-s}) = \text{constant}.$$

Indeed

$$\lim_{s\to 1^+}\sum_{m,k>0}(2mk)^{-s}(1+\frac{1}{2mk})^{-s} = \lim_{s\to 1^+}2^{-s}\sum_{n=1}^{\infty}\frac{d(n)}{n^s} + \sum_{k=1}^{\infty}\frac{(-1)^k}{2^{k+1}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{k+1}}.$$

So summing over the nonzero eigenvalues of $(1+\mathcal{D}^2)^{-s}$ gives asymptotically

$$\begin{aligned}\left|\mathrm{Trace}(P_p(1+\mathcal{D}^2)^{-s}P_p)\right| &\sim 2^{2-s}\kappa^{-2s}\sum_{n=1}^{\infty} d(n)n^{-s} \\ &= 2^{2-s}\kappa^{-2s}\left(\frac{1}{(1-s)^2}+\frac{\gamma}{s-1}+\gamma^2+\text{holomorphic}\right).\end{aligned}$$

This shows that $\left|\mathrm{Trace}(a(1+\mathcal{D}^2)^{-s})\right|$ contains at worst a double pole. The precise behaviour will depend on a. Here γ is Euler's constant, the value of $\phi(s)$ at $s=1$ where $\zeta(s)=\frac{1}{s-1}+\phi(s)$ with ϕ holomorphic.

In fact we have already shown that if the function a has support disjoint from the set $\{z=0\}$, there can only be a simple pole. This follows from Theorem 4 and Lemma 9. Computing the actual values of the residue requires a concrete form for the function a, and of course we are mostly interested in the case where the function a is (a component of) a projection or unitary representing a K-theory class.

The Local Index Theorem [1, 3] gives us a formula for components of the Chern character of $(\mathcal{A},\mathcal{H},\mathcal{D},\Gamma)$. Substituting the various constant terms and using the simplicity of the dimension spectrum we obtain

$$\phi_2(a_0,a_1,a_2)=\frac{1}{2}\tau_0(\Gamma a_0da_1da_2(1+\mathcal{D}^2)^{-1})$$

$$\phi_0(a_0)=\mathrm{res}_{z=0}\frac{1}{z}\mathrm{Trace}(\Gamma a_0(1+\mathcal{D}^2)^{-z}).$$

The top component involves the coefficient of $\frac{1}{z}$ in the Laurent expansion at $z=0$ of $\mathrm{Trace}(a(1+\mathcal{D}^2)^{-1-z})$, while the zero-th component involves the coefficient of the constant term. A routine calculation shows that

$$\Gamma a_0[\mathcal{D},a_1][\mathcal{D},a_2]=\begin{pmatrix}-a_0g(da_1,da_2)-iz\kappa^2a_0da_1\wedge da_2 & 0\\ 0 & a_0g(da_1,da_2)-iz\kappa^2a_0da_1\wedge da_2\end{pmatrix},$$

so the trace $\mathrm{Trace}(\Gamma a_0da_1da_2(1+\mathcal{D}^2)^{-s})$ is given by

$$2i\kappa^2\mathrm{Trace}_{\mathcal{H}^+}(a_0((\partial_z a_1)(\partial_\theta a_2)-(\partial_\theta a_1)(\partial_z a_2))z(1+\mathcal{D}^2)^{-s}) \tag{3}$$

where $\mathcal{H}^+$ is the $+1$ eigenspace of Γ. The factor of κ^2 is precisely what one would expect for a critical point at $s=1$ since $(\mathcal{D}^2+1)^{-s}\sim\kappa^{-2s}$; κ is a geometric feature, and the residues we are employing compute purely topological quantities, and so should be insensitive to the precise value of κ.

To compute the pairing with K-theory using the residue, we require a concrete form for the generators of the even K-group of the cone. We first compute the K-theory for the cone.

Lemma 10 *The K-theory of the cone is given by*

$$K_0(C_0(cone))\cong\mathbf{Z}^2,\quad K_1(C_0(cone))\cong\mathbf{Z}^2$$

Proof The (C^*-closure of the) algebra of functions we are employing decomposes as

$$\overline{\mathcal{A}} \cong C_0(\mathbf{R}^2 \setminus \{0\}) \oplus C_0(\mathbf{R}^2 \setminus \{0\}),$$

so $K_*(\overline{\mathcal{A}}) \cong K_0(C_0(\mathbf{R}^2 \setminus \{0\})) \oplus K_0(C_0(\mathbf{R}^2 \setminus \{0\}))$. To compute $K_0(C_0(\mathbf{R}^2 \setminus \{0\}))$, and find explicit generators, consider the exact sequence, [7, 4],

$$0 \longrightarrow C_0(\mathbf{R}^2 \setminus \{0\}) \longrightarrow C(D^2) \longrightarrow C(S^1) \oplus \mathbf{C} \longrightarrow 0,$$

where $C_0(\mathbf{R}^2 \setminus \{0\})$ is included as the continuous functions on the closed unit disk D^2 vanishing at 0 and on the boundary circle. The corresponding K-theory exact sequence is

$$0 \longrightarrow K_1(C(S^1) \oplus \mathbf{C}) \stackrel{Ind}{\longrightarrow} K_0(C_0(\mathbf{R}^2 \setminus \{0\})) \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \stackrel{exp}{\longrightarrow} K_1(C_0(\mathbf{R}^2 \setminus \{0\})) \longrightarrow 0,$$

since $K_1(C(D^2)) = \{0\}$. The Index map on the left is necessarily injective, and it is also onto. To see this, observe that the map from $K_0(D^2) \cong \mathbf{Z}$ to $K_0(C(S^1) \oplus \mathbf{C}) \cong \mathbf{Z} \oplus \mathbf{Z}$ takes the trival bundle of rank k on the disk to the trivial bundle of rank k on the circle union the point zero. Hence it is the diagonal map, and is injective, whence the map from $K_0(C_0(\mathbf{R}^2 \setminus \{0\}))$ to $K_0(D^2)$ is zero. Furthermore, the exponential map on the right is onto, taking (n, m) onto $n - m$. $\square$

So to obtain the generator of $K_0(C_0(\mathbf{R}^2 \setminus \{0\})) \cong \mathbf{Z}$, it suffices to find a generator for the odd K-group of the circle union a point, and apply the boundary map. The obvious generator of $K_1(C(S^1) \oplus \mathbf{C})$ is the function which is the identity on the circle, and equal to 1 on the adjoined point. This is unitary. To apply the boundary map, we first need to 'double' this unitary to an element of the connected component of the identity, (in a larger matrix algebra) so

$$Id_{S^1} \oplus 1 \longrightarrow w := \begin{pmatrix} Id_{S^1} \oplus 1 & 0 \\ 0 & Id^*_{S^1} \oplus 1 \end{pmatrix}$$

where $Id^*_{S^1} : z \to \overline{z}$. Then we need to lift this unitary to a function in $C(D^2)$ which is equal to w modulo $C_0(\mathbf{R}^2 \setminus \{0\})$. So choose any continuous function f on the closed disk such that f is the identity on the boundary, 1 at the centre and has $|f|^2 \leq 1$ on the whole disk; for example

$$f(re^{i\theta}) = re^{i\theta} + (1 - r)$$

will do. Then the required lift is

$$\tilde{w} = \begin{pmatrix} f & \sqrt{1 - |f|^2} \\ -\sqrt{1 - |f|^2} & \overline{f} \end{pmatrix}.$$

Finally, we obtain a generator of $K_0(C_0(\mathbf{R}^2 \setminus \{0\}))$ defined by

$$\begin{aligned} Bott_0 \; &:= \; \tilde{w} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{w}^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} |f|^2 - 1 & -f\sqrt{1 - |f|^2} \\ -\overline{f}\sqrt{1 - |f|^2} & 1 - |f|^2 \end{pmatrix} = p_B - 1. \end{aligned}$$

This is the analogue of the Bott generator on the punctured disk. To convert this into a projection in $M_2(C_0(\mathbf{R}^2 \setminus \{0\}))$ (as opposed to M_2(punctured disk)), we need to compose f with a diffeomorphism $h : (0,\infty) \to (0,1)$. We leave this choice until later. Finally, to obtain generators on the cone, we take the Bott generator on each half and extend them by zero to the other half.

To compute the Chern character pairing, first recall that for a projection p , [4],

$$Ch_0(p) = \sum_i p_{ii},$$

$$Ch_2(p) = -2\sum_{i,j,k}(p - \frac{1}{2})_{ij} \otimes p_{jk} \otimes p_{ki},$$

which is the trace of $(p - \frac{1}{2})dpdp \in \Omega^*(M_2(\mathcal{A}))$, the universal differential algebra, [4, p320]. We actually require $Ch_*(p_B) - Ch_*(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})$, but $Ch_2(1) = 0$ and

$$Ch_0(p_B) - \text{trace}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \text{trace}\left(p_B - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right),$$

and this trace is zero. Hence

$$Res_{s=0}\frac{1}{s}\text{Trace}_{\mathcal{H}^2}(\Gamma_2(Ch_0(p_B) - Ch_0(1))(1 + \mathcal{D}_2^2)^{-s}) = 0,$$

and we need only worry about the order two pairing. Since this has only a simple pole, we may compute it using the Wodzicki residue, [9, 1, 10], via Connes' trace theorem.

Proposition 11 *The index pairing between the Bott generator supported on the positive half of the cone and the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$ described above is*

$$\langle [p_B] - 1, [(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)]\rangle = 1.$$

Proof To complete the computation of the pairing, we must choose an explicit diffeomorphism $h : (0,\infty) \to (0,1)$. We take

$$h(z) = 1 - e^{-z^2},$$

as this allows effective computations. Substituting in the components of p_B into the formula for the Chern character yields $-4i\kappa^2 Kz$, where K is a complicated expression in terms of z and θ. The integral in the θ direction of K yields

$$\int_0^{2\pi} K d\theta = -4\pi i z(e^{-2z^2} - e^{-z^2}).$$

Together with the definition of the Wodzicki residue of an operator of order -2 on the cylinder,

$$Wres(A) = \frac{1}{2(2\pi)^2}\int_{Cyl}\int_{\|\xi\|=1} \sigma^A_{-2}(x,\xi)dS(\xi)dvol(x),$$

we can compute the pairing. The computation is as follows.

$$
\begin{aligned}
&= Wres(\sum_{ijk}(p-1/2)_{ij}[\mathcal{D},p_{jk}][\mathcal{D},p_{ki}](1+\mathcal{D}^2)^{-1}) \\
&= \frac{-2i}{(2\pi)^2}\int_{-\infty}^{\infty}\int_0^{2\pi} K\left(\int_{\xi_\theta^2+\xi_z^2=1}(z^2\xi_\theta^2+\xi_z^2)^{-1}dS\right) zdzd\theta \qquad (4)\\
&= \frac{-i}{2\pi^2}\int_{-\infty}^{\infty}\int_0^{2\pi} K\left(\int_0^{2\pi}(z^2\sin^2 t+\cos^2 t)^{-1}dt\right) zdzd\theta \qquad (5)\\
&= \frac{-i}{2\pi^2}\int_{-\infty}^{\infty}\int_0^{2\pi} K\frac{2\pi}{|z|} zdzd\theta \qquad (6)\\
&= \frac{-i}{\pi}\int_0^{\infty}(-4\pi i)z(e^{-2z^2}-e^{-z^2})dz \qquad (7)\\
&= -4\left(\int_0^{\infty} ze^{-2z^2}dz-\int_0^{\infty} ze^{-z^2}dz\right)\\
&= -4(\frac{\Gamma(1)}{2^2}-\frac{\Gamma(1)}{2})\\
&= -1+2=1.
\end{aligned}
\qquad (8)
$$

Equation (4) is just the definition of the Wodzicki residue, and we have replaced the sum of products of differentials of components of p_B with $-4iz\kappa^2K$, as described above. The integral in equation (5) is a standard one, and the equality between (6) and (7) follows from integrating K in the θ direction. Finally we recall, the Bott projector we employ is supported only on a half-line, so the $|z|$ term becomes simply z. $\square$

It is clear from the above computation that if we begin with the punctured Bott projector on the other half of the cone (i.e. $z<0$) we obtain the result -1.

A final point to notice is that the trace over the continuous subspace for $\mathcal{D}$ and the trace over the kernel of $\mathcal{D}$ do not contribute, since both are finite as $s\to 1$. For the kernel this follows from our previous estimates and the independence of this trace on s, and for the continuous subspace it follows from our earlier computation that the trace only becomes singular as $s\to\frac{1}{2}$. Thus only the point spectrum of $\mathcal{D}$ contributes to the above pairing.

The industrious reader will find that the explicit expression for the trace of $Ch_2(p_B)(1+\mathcal{D}^2)^{-s}$ is given by the function

$$
\begin{aligned}
T(s) &= \sum_{N>1}\frac{-4\kappa^2}{(1+2\kappa^2N)^s}\sum_{m|N}\sum_{l=1,2}(-1)^l\left(\frac{N+m}{m^2}A_{N/m+1,m,l}+\frac{N}{m}A_{N/m-1,m,l}\right.\\
&\qquad\left.+\frac{N}{m^2}A_{N/m,m,l}+\frac{N-m}{m}A_{N/m-2,m,l}\right),
\end{aligned}
$$

or,

$$\sum_{n,m>1} \frac{-4\kappa^2}{(1+2\kappa^2 nm)^s} \sum_{l=1,2} (-1)^l \left(\frac{n+1}{m} A_{n+1,m,l} + n A_{n-1,m,l} + \frac{n}{m} A_{n,m,l} + (n-1) A_{n-2,m,l} \right),$$

the two obviously being equal. Here

$$A_{n,m,l} = \sqrt{\frac{m}{\pi}} \sum_{k,p=0}^{[n/2]} \frac{(-1)^{k+p} n! 2^{n-2k-2p} m^{n-k-p} \Gamma(n-k-p+\frac{1}{2})}{k! p! (n-2k)! (n-2p)! (m+l)^{n-k-p+\frac{1}{2}}}.$$

Our computations have shown that T is a meromorphic function whose residue at $s=1$ is precisely 1.

5 Conclusion

Despite obtaining (p,∞)-summable spectral triples from degenerate metrics, our original aim of obtaining a spectral triple with non-simple dimension spectrum failed. We feel that explicit examples of non-simple dimension spectrum are an important step towards understanding the full content of the Local Index theorem of Connes-Moscovici, [1].

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References

[1] A. Connes, H. Moscovici *The Local Index Formula in Noncommutative Geometry* Geometric and Functional Analysis **5** 174-243 (1995)

[2] A. Rennie *Smoothness and Locality for Nonunital Spectral Triples* to appear in K-Theory **519** pp 1-39 (2003)

[3] A. Rennie *Summability for Nonunital Spectral Triples* to appear

[4] J. M. Gracia-Bondía, J. C. Varilly, H. Figueroa *Elements of Noncommutative Geometry* Birkhauser, Boston, 2001

[5] A. Connes *Geometry from the Spectral Point of View* Lett. Math. Phys. **34** 203-238 (1995)

[6] H. Lawson, M. Michelsohn *Spin Geometry* Princeton University Press, 1989

[7] N. Higson and J. Roe *Analytic K-Homology* Oxford Mathematical Monographs, OUP, New York (2000)

[8] R. T. Seeley *Complex Powers of an Elliptic Operator* Proc. Symp. Pure Math. **10** 288-307 (1967)

[9] A. Connes *Noncommutative Geometry* Academic Press, 1994

[10] M. Wodzicki *Noncommutative Residue. Chapter I: Fundamentals* Lecture Notes in Mathematics, **1289**, Springer, Berlin, 1987 pp 320-399

[11] M. Reed and B. Simon *Functional Analysis* Academic Press, 1980

[12] R. Courant, D. Hilbert *Methods of Mathematical Physics, Volume I* Interscience, 1931, New York

[13] Earl D. Rainville *Special Functions* The MacMillan Company, New York, 1960

[14] G.H. Hardy, E.M. Wright *An Introduction to the Theory of Numbers* Clarendon Press, Oxford, 1960