THE HOCHSCHILD CLASS OF THE CHERN CHARACTER FOR SEMIFINITE SPECTRAL TRIPLES

Alan L. Carey
Mathematical Sciences Institute
Australian National University
Canberra, ACT. 0200, AUSTRALIA
e-mail: acarey@maths.anu.edu.au

John Phillips
Department of Mathematics and Statistics
University of Victoria
Victoria, B.C. V8W 3P4, CANADA[a]
e-mail: phillips@math.uvic.ca

Adam Rennie
School of Mathematical and Physical Sciences
University of Newcastle
Callaghan, NSW, 2308 AUSTRALIA
e-mail: adam.rennie@newcastle.edu.au

and

Fyodor A. Sukochev
School of Informatics and Engineering
Flinders University
Bedford Park S.A 5042 AUSTRALIA
e-mail: sukochev@maths.flinders.edu.au

RUNNING TITLE: THE HOCHSCHILD CLASS OF THE CHERN CHARACTER

All authors were supported by grants from ARC (Australia) and NSERC (Canada), in addition the third named author acknowledges a University of Newcastle early career researcher grant.

[a]Address for correspondence

Abstract

We provide a proof of Connes' formula for a representative of the Hochschild class of the Chern character for (p, ∞)-summable spectral triples. Our proof is valid for all semifinite von Neumann algebras, and all integral $p \geq 1$. We employ the minimum possible hypotheses on the spectral triples. [b]

[b]AMS Subject classification: Primary: 19K56, 46L80; secondary: 58B30, 46L87. Keywords and Phrases: von Neumann algebra, Fredholm module, cyclic cohomology, chern character.

1 Introduction

A key result in the quantised calculus of Alain Connes ([9, IV.2.γ]) is the formula for the Hochschild class of the Chern character of a (p, ∞)-summable spectral triple (these notions are explained below).

Our aim is to generalise this formula to encompass the situation in which one uses, instead of the bounded operators on Hilbert space and its various ideals of compact operators, a general semifinite von Neumann algebra and the analogous ideals as described for example in [18, 26]. Moreover we aim to prove the formula in the greatest possible generality with only the absolutely essential side conditions. This is a delicate matter as regards the amount of smoothness or regularity necessary. The result has been stated, [21, Theorem 10.32], with the hypothesis that the algebra be 'twice quantum differentiable' (see below), but the proof appearing in [21, pp 470-479] does not quite hold with this hypothesis. We employ the hypothesis of '$\max\{2, p-2\}$ quantum differentiability', and while this is sufficient, the necessity of this condition is unknown to us. Indeed, we only require this stronger hypothesis at one (crucial) point, Proposition 23, but we isolate the particular statement which uses this hypothesis in Lemma 2.

A rationale for the extension of Connes' spectral geometry to the case of general semifinite von Neumann algebras is presented in [1]. Examples where this notion arises naturally are non-smooth foliations [1, 25], the L^2-index theorem (see [24] and references therein) and L^2 spectral flow [3, 4].

In order to describe our results some preliminary machinery is needed (all of this is contained in [9] in the type I case). We deal with this in Section 2. We first describe spectral triples for semifinite von Neumann algebras, including definitions of smoothness and summability. We then briefly recall the Hochschild and cyclic cohomology theories, and explain what the Hochschild class of the Chern character is, and what kind of information it contains.

The last preliminary subsection describes results from [6], where a proof of the connection between the trace of the heat kernel and the Dixmier trace is presented. The idea has previously appeared, [9, p 563], but this is the first proof simultaneously valid for the case $p = 1$ and the general semifinite case. It is a key tool in our proof.

Section 3 begins with a statement of the main result and its main corollaries. The expert reader can skip straight to Subsections 3.1 and 3.2 for our result, its corollaries, and how it relates to the significant body of previous work on this general topic. We then set out the proof as clearly as possible, and in the greatest possible generality. The proof is considerably simplified by the assumption of invertibility of the 'Dirac' operator $\mathcal{D}$, but a standard construction in K-homology and computations contained in the Appendix show that the result remains true even when this is not the case.

2 Background Material and Preliminary Results

2.1 Spectral Triples

We begin with some semifinite versions of standard definitions and results.

Definition 1 *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that*

1) $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$

2) $(\lambda - \mathcal{D})^{-1} \in \mathcal{K}(\mathcal{N})$ for all $\lambda \notin \mathbf{R}$

Here $\mathcal{K}(\mathcal{N})$ is the ideal of τ-compact operators in $\mathcal{N}$ (this is explained in the next section). We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even if in addition there is a $\mathbf{Z}_2$-grading such that $\mathcal{A}$ is even and $\mathcal{D}$ is odd. That is an operator Γ such that $\Gamma = \Gamma^*$, $\Gamma^2 = 1$, $\Gamma a = a\Gamma$ for all $a \in \mathcal{A}$ and $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Otherwise we say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd.

Remark We will write Γ in all our formulae, with the understanding that if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is odd, $\Gamma = 1$ and of course, we drop the assumption that $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Alas, we will also employ the gamma function in this paper, but the meaning of the symbol 'Γ' should be clear from context. Henceforth we omit the term semifinite as it is implied by the use of a faithful normal semifinite trace τ on $\mathcal{N}$ in all of the subsequent text.

Definition 2 *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k for $k \geq 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k where $\delta(T) = [|\mathcal{D}|, T]$ is the (partially defined) derivation on $\mathcal{N}$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is smooth if it is QC^k for all $k \geq 1$.*

Remark The notation is meant to be analogous to the classical case, but we introduce the Q so that there is no confusion between quantum differentiability of $a \in \mathcal{A}$ and classical differentiability of functions. We may also speak about a QC^0 spectral triple, where only a and $[\mathcal{D}, a]$ are assumed bounded. We also note that if $T \in \mathcal{N}$, one can show that $[|\mathcal{D}|, T]$ is bounded if and only if $[(1 + \mathcal{D}^2)^{1/2}, T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}| - (1 + \mathcal{D}^2)^{1/2}$ is a bounded operator and lies in $\mathcal{N}$.

2.1.1 Summability

Recall from [18] that if $S \in \mathcal{N}$ the **t-th generalized singular value** of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{\| SE \| \ | \ E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1 - E) \leq t\}.$$

We write $T_1 \prec\prec T_2$ to mean that $\int_0^t \mu_s(T_1)ds \leq \int_0^t \mu_s(T_2)ds$ for all $t > 0$.

Definition 3 *If $\mathcal{I}$ is a $*$-ideal in $\mathcal{N}$ which is complete in a norm $\| \cdot \|_{\mathcal{I}}$ then we will call $\mathcal{I}$ a* **symmetric operator ideal** *if*

(1) $\| S \|_{\mathcal{I}} \geq \| S \|$ *for all* $S \in \mathcal{I}$*,*
(2) $\| S^* \|_{\mathcal{I}} = \| S \|_{\mathcal{I}}$ *for all* $S \in \mathcal{I}$*,*
(3) $\| ASB \|_{\mathcal{I}} \leq \| A \| \| S \|_{\mathcal{I}} \| B \|$ *for all* $S \in \mathcal{I}$*,* $A, B \in \mathcal{N}$*.*
Since $\mathcal{I}$ *is an ideal in a von Neumann algebra, it follows from I.1.6, Proposition 10 of [13] that if* $0 \leq S \leq T$ *and* $T \in \mathcal{I}$*, then* $S \in \mathcal{I}$ *and* $\| S \|_{\mathcal{I}} \leq \| T \|_{\mathcal{I}}$*.*

Such ideals are special cases of symmetric operator spaces (see [26] and references therein). The main examples of such ideals that we consider in this paper are the spaces

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \mid \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds < \infty \right\}.$$

and with $p > 1$,

$$\psi_p(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ t^{1-\frac{1}{p}} & \text{for } 1 \leq t \end{cases}$$

$$\mathcal{L}^{(p,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \mid \|T\|_{\mathcal{L}^{(p,\infty)}} := \sup_{t>0} \frac{1}{\psi_p(t)} \int_0^t \mu_s(T) ds < \infty \right\}.$$

For $p > 1$ there is also the equivalent definition (see for example [26, Section 5])

$$\mathcal{L}^{(p,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \mid \sup_{t>0} \frac{t}{\psi_p(t)} \mu_t(T) < \infty \right\}.$$

It is well-known (see e.g. [26]) that for $T_1 \in \mathcal{N}$, $T_2 \in \mathcal{L}^{(p,\infty)}(\mathcal{N})$, $p \in [1, \infty)$, the condition $T_1 \prec\prec T_2$ implies that $T_1 \in \mathcal{L}^{(p,\infty)}(\mathcal{N})$. We denote the norm on $\mathcal{L}^{(p,\infty)}$ by $\| \cdot \|_{(p,\infty)}$.
As we will not change $\mathcal{N}$ throughout the paper we will suppress the $(\mathcal{N})$ to lighten the notation. The reader should note that $\mathcal{L}^{(p,\infty)}$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of τ-measurable operators affiliated to $\mathcal{N}$. Our notation is however consistent with that of [9] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of τ-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that

$$\mu_\infty(T) := \lim_{t \to \infty} \mu_t(T) = 0.$$

Definition 4 *A spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is called* (p, ∞)*-summable if* $(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(p,\infty)}$*.*

We will also require the ideals $\mathcal{L}^p(\mathcal{N})$ and $\mathcal{L}^{(p,1)}(\mathcal{N})$, for $p \geq 1$. An operator $T \in \mathcal{N}$ is in $\mathcal{L}^p(\mathcal{N})$ if

$$\| T \|_p := \tau(|T|^p)^{1/p} < \infty.$$

In the Type I setting these are the usual Schatten ideals. Again we will simply write $\mathcal{L}^p$ for these ideals in order to simplify the notation, and denote the norm on $\mathcal{L}^p$ by $\| \cdot \|_p$. An operator $T \in \mathcal{N}$ is in $\mathcal{L}^{(p,1)}(\mathcal{N})$ if, [26],

$$\| T \|_{(p,1)} := (1/p \int_0^\infty (t^{1/p} \mu_t(T)) dt/t) < \infty.$$

For $p = 1$ the ideal $\mathcal{L}^{(p,1)}$ coincides with $\mathcal{L}^1$. We denote the norm on $\mathcal{L}^{(p,1)}$ by $\|\cdot\|_{(p,1)}$. If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the Köthe dual of $\mathcal{L}^{(p,1)}$ is $\mathcal{L}^{(q,\infty)}$, [16]. For $p = 1$ the Köthe dual of $\mathcal{L}^1$ is just $\mathcal{N}$.

We use the following results repeatedly. They tell us the summability of various operators associated to a (p,∞)-summable spectral triple. The results are established in [7, Propositions 1.1 and 1.2], namely that for any τ-measurable operators T_1 and T_2 we have

$$\mu(T_1T_2) \prec\prec \mu(T_1)\mu(T_2),$$

where $\mu(T)$ denotes the function $s \to \mu_s(T)$. Moreover, for any self-adjoint τ-measurable operators T and S with $T \geq 0$,

$$-T \leq S \leq T \ \ \textit{implies} \ \ S \prec\prec T.$$

Lemma 1 ([21]) *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a (p,∞)-summable QC^k spectral triple, $k \geq 0$, with $p \geq 1$ and $\mathcal{D}$ invertible. Then for all $a \in \mathcal{A}$*

$$[F,a],\ [F,\delta(a)],\ \ \cdots,\ [F,\delta^k(a)] \in \mathcal{L}^{(p,\infty)},$$

where $F = \mathcal{D}|\mathcal{D}|^{-1}$.

Proof We start with the formula

$$|\mathcal{D}|^{-1} = \frac{1}{\pi}\int_0^\infty (\lambda + \mathcal{D}^2)^{-1}\frac{d\lambda}{\sqrt{\lambda}}.$$

This is used to rewrite $[F,a]$ in the following way.

$$\begin{aligned}
[F,a] &= [\mathcal{D}|\mathcal{D}|^{-1}, a] = [\mathcal{D},a]|\mathcal{D}|^{-1} + \mathcal{D}[|\mathcal{D}|^{-1}, a] \\
&= \frac{1}{\pi}\int_0^\infty \Big([\mathcal{D},a](\lambda+\mathcal{D}^2)^{-1} + \mathcal{D}[(\lambda+\mathcal{D}^2)^{-1}, a]\Big)\frac{d\lambda}{\sqrt{\lambda}} \\
&= \frac{1}{\pi}\int_0^\infty \Big([\mathcal{D},a](\lambda+\mathcal{D}^2)^{-1} - \mathcal{D}^2(\lambda+\mathcal{D}^2)^{-1}[\mathcal{D},a](\lambda+\mathcal{D}^2)^{-1} \\
&\qquad\qquad -\mathcal{D}(\lambda+\mathcal{D}^2)^{-1}[\mathcal{D},a]\mathcal{D}(\lambda+\mathcal{D}^2)^{-1}\Big)\frac{d\lambda}{\sqrt{\lambda}} \\
&= \frac{1}{\pi}\int_0^\infty \Big(\lambda(\lambda+\mathcal{D}^2)^{-1}[\mathcal{D},a](\lambda+\mathcal{D}^2)^{-1} - \mathcal{D}(\lambda+\mathcal{D}^2)^{-1}[\mathcal{D},a]\mathcal{D}(\lambda+\mathcal{D}^2)^{-1}\Big)\frac{d\lambda}{\sqrt{\lambda}}.
\end{aligned}$$

The second last equality comes from [3, Lemma 2.3], whose proof requires only QC^0, as opposed to the usual resolvent calculation which requires QC^1. The final equality comes from $\mathcal{D}^2(\lambda+\mathcal{D}^2)^{-1} = 1 - \lambda(\lambda+\mathcal{D}^2)^{-1}$. We now suppose that $a^* = -a$ so that $[\mathcal{D},a]^* = [\mathcal{D},a]$ and similarly for $[F,a]$. Then we may employ the inequality

$$-\parallel [\mathcal{D},a] \parallel T^*T \leq T^*[\mathcal{D},a]T \leq \parallel [\mathcal{D},a] \parallel T^*T$$

for all $T \in \mathcal{N}$. Applying this inequality under the above integral yields

$$\begin{aligned}
[F,a] &\leq \frac{\parallel [\mathcal{D},a] \parallel}{\pi}\int_0^\infty (\lambda+\mathcal{D}^2)^{-1}\frac{d\lambda}{\sqrt{\lambda}} \\
&= \parallel [\mathcal{D},a] \parallel |\mathcal{D}|^{-1},
\end{aligned}$$

and similarly $[F,a] \geq - \parallel [\mathcal{D},a] \parallel |\mathcal{D}|^{-1}$. Thus $[F,a] \prec\prec \parallel [\mathcal{D},a] \parallel |\mathcal{D}|^{-1}$, in particular $[F,a] \in \mathcal{L}^{(p,\infty)}$, and by linearity this is true for all $a \in \mathcal{A}$. However, for not necessarily self-adjoint $a \in \mathcal{A}$, the precise inequality is

$$[F,a] \prec\prec (\parallel [\mathcal{D}, Re(a)] \parallel + \parallel [\mathcal{D}, Im(a)] \parallel)|\mathcal{D}|^{-1}. \tag{1}$$

From the comments in Definition 3, this shows that $[F,a] \in \mathcal{L}^{(p,\infty)}$. The remainder of the result is proved using the same argument by replacing a by $\delta^i(a)$, for $i = 1, ..., k$, and using the boundedness of $[\mathcal{D}, \delta^i(a)] = \delta^i([\mathcal{D},a])$. □

The following lemma is a consequence of the previous result. This is the point at which more smoothness than QC^2 is required. The analogous statement in [21, Lemma 10.27], is a little lax about the degree of smoothness necessary to perform the iterated commutators with $|\mathcal{D}|$ in the proof.

Lemma 2 *Let $p \geq 1$ and $k = \max\{1, p-2\}$. Suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^k (p,∞)-summable spectral triple with $\mathcal{D}$ invertible. For all $a_0, ..., a_{p-1} \in \mathcal{A}$, and $T \in \mathcal{N} \cap \mathrm{dom}(\delta)$, the operators*

$$|\mathcal{D}|^{p-2} a_0 [F,a_1] \cdots [F,a_{p-1}] F T |\mathcal{D}| \quad \text{and} \quad |\mathcal{D}| T a_0 [F,a_1] \cdots [F,a_{p-1}] F |\mathcal{D}|^{p-2}$$

are densely defined and bounded (or, more accurately, extend to bounded operators).

Proof The proof is essentially the same as that in [21]. First, the triple is at least QC^1, so $[F,a]|\mathcal{D}| = [\mathcal{D},a] - F\delta(a)$ is bounded for all $a \in \mathcal{A}$. This allows one to check the cases $p = 1, 2$. For $p > 2$ we have

$$\begin{aligned} & |\mathcal{D}|^{p-2} a_0 [F,a_1] \cdots [F,a_{p-1}] F T |\mathcal{D}| \\ = & \sum_{j=0}^{p-2} \begin{pmatrix} p-2 \\ j \end{pmatrix} \delta^j(a_0) |\mathcal{D}|^{p-2-j} [F,a_1] \cdots [F,a_{p-1}] F (|\mathcal{D}|T - \delta(T)), \end{aligned}$$

and similarly for the other operator. Now $[F,a_{p-1}](|\mathcal{D}|T - \delta(T))$ is bounded and in $\mathcal{N}$, since the triple is QC^1 and $T \in \mathrm{dom}(\delta)$. Similarly $j = p-1, p-2$, $|\mathcal{D}|^{p-2-j}[F,a_1]$ is bounded and in $\mathcal{N}$, and of course $\delta^j(a_0)$ is bounded for $0 \leq j \leq p-2$ since the triple is QC^{p-2}. Hence we may consider only those terms with $0 \leq j \leq p-3$.

One now continues to take commutators, observing that we obtain a sum of bounded operators in $\mathcal{N}$ plus the term

$$a_0 |\mathcal{D}| [F,a_1] |\mathcal{D}| [F,a_2] \cdots |\mathcal{D}| [F,a_{p-2}] [F,a_{p-1}] |\mathcal{D}| F T$$

which is bounded and in $\mathcal{N}$ since the triple is QC^1. Similar comments apply to the second operator. □

2.2 Hochschild and Cyclic Cohomology

For a locally convex unital algebra $\mathcal{A}$, we denote by $C^n(\mathcal{A})$ the linear space of continuous $n+1$-multilinear functionals on $\mathcal{A}^{n+1}$. The Hochschild coboundary of $\phi \in C^n(\mathcal{A})$ is the functional

$b\phi \in C^{n+1}(\mathcal{A})$ defined by

$$\begin{aligned}(b\phi)(a_0, ..., a_{n+1}) &= \phi(a_0 a_1, a_2, ..., a_{n+1}) \\ &+ \sum_{i=1}^{n}(-1)^i \phi(a_0, ..., a_i a_{i+1}, ..., a_{n+1}) \\ &+ (-1)^{n+1}\phi(a_{n+1}a_0, a_1, ..., a_n), \quad a_0, ..., a_{n+1} \in \mathcal{A}.\end{aligned}$$

One can easily check that $b^2 = 0$. The Hochschild cohomology, denoted $HH^*(\mathcal{A}, \mathcal{A}^*)$, is then the cohomology of the complex $(C^*(\mathcal{A}), b)$. The notation is explained in [23]. We denote the space of Hochschild k-cocycles by $Z^k(\mathcal{A})$.

Let $C^n_\lambda(\mathcal{A})$ be the subspace of $C^n(\mathcal{A})$ consisting of functionals ϕ such that

$$\phi(a_0, ..., a_n) = (-1)^n \phi(a_n, a_0, ..., a_{n-1}), \quad a_0, ..., a_n \in \mathcal{A}.$$

The Hochschild coboundary maps $C^n_\lambda(\mathcal{A})$ to $C^{n+1}_\lambda(\mathcal{A})$, so we can define the cyclic cohomology of $\mathcal{A}$, denoted $HC^*(\mathcal{A})$, to be the cohomology of the complex $(C^*_\lambda(\mathcal{A}), b)$. The space of cyclic cocycles is denoted $Z_\lambda(\mathcal{A})$.

Connes shows that there is a long exact sequence, [9, III.1.γ],

$$\cdots \xrightarrow{B} HC^p(\mathcal{A}) \xrightarrow{S} HC^{p+2}(\mathcal{A}) \xrightarrow{I} HH^{p+2}(\mathcal{A}, \mathcal{A}^*) \xrightarrow{B} HC^{p+1}(\mathcal{A}) \xrightarrow{S} \cdots$$

where I is the map induced on cohomology by the inclusion of complexes $C^n_\lambda(\mathcal{A}) \hookrightarrow C^n(\mathcal{A})$. The operator B will not concern us in this paper, see [23] and [9, III.1.γ], however the periodicity operator S is important for three reasons. The first reason is that cyclic cohomology groups are filtered by powers of S, so in general $HC^k(\mathcal{A})$ consists of (classes of) sums

$$\phi_k + S\phi_{k-2} + S^2\phi_{k-4} + ... + S^{[k/2]}\phi_{k-2[k/2]}$$

where ϕ_j is cyclic, $b\phi_j = 0$, $j = k - 2[k/2], ..., k$. As Image$S = \ker I$, we have

$$I(\phi_k + ... + S^{[k/2]}\phi_{k-2[k/2]}) = I(\phi_k).$$

Consequently, pairing a cyclic cocycle ϕ with a Hochschild cycle yields the same result as pairing the Hochschild class of the cyclic cocycle ϕ with a Hochschild cycle. Equivalently, any cocycle in the image of S has zero Hochschild class.

Secondly, because $S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A})$ for all n, we may define even periodic cyclic cohomology as the inductive limit $H^{ev}(\mathcal{A}) := \lim_\to(HC^{2n}(\mathcal{A}), S)$, and similarly the odd periodic cyclic cohomology as $H^{odd}(\mathcal{A}) := \lim_\to(HC^{2n+1}(\mathcal{A}), S)$.

Thirdly, we use constructions closely related to the periodicity operator in the Appendix to complete the proof of our main result for the case where our spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has $\mathcal{D}$ noninvertible.

The (dual) homology theories require more care regarding the completeness of $\mathcal{A}$ and the appropriate tensor product. We only need Hochschild cycles (not their classes), so we may ignore these difficulties in this paper. We only require the definition of the Hochschild boundary. If $c=\sum_{i=1}^n a_0^i\otimes a_1^i\otimes\cdots\otimes a_k^i$ then

$$\begin{aligned} bc=b(\sum_{i=1}^n a_0^i\otimes a_1^i\otimes\cdots\otimes a_k^i) &= \sum_{i=1}^n a_0^ia_1^i\otimes a_2^i\otimes\cdots\otimes a_k^i \\ &+ \sum_{i=1}^n\sum_{j=1}^{k-1}(-1)^j a_0^i\otimes\cdots\otimes a_j^ia_{j+1}^i\otimes\cdots\otimes a_k^i \\ &+ (-1)^k\sum_{i=1}^n a_k^ia_0^i\otimes\cdots\otimes a_{k-1}^i. \end{aligned}$$

We say that c is a Hochschild cycle if $bc=0$. When the Hochschild homology is well-defined we denote it by $HH_*(\mathcal{A})$.

An important point is that if ϕ is a $k-1$-multilinear functional, and c is a Hochschild k-cycle, $(b\phi)(c)=\phi(bc)=0$, so the pairing of a Hochschild coboundary with a Hochschild cycle vanishes. This follows immediately from the definitions.

Our only result in this section consists of a mild generalisation of a standard result for the behaviour of Hochschild (co)homology with respect to derivations. This result will simplify many later computations.

Lemma 3 *Let $\mathcal{N}$ be a semifinite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{A}\subset\mathcal{N}$ be a $*$-subalgebra, and $\mathcal{M}\subset\mathcal{N}$ an $\mathcal{A}$-bimodule. Suppose that $\delta_1,...,\delta_k:\mathcal{A}\to\mathcal{N}$ are derivations such that the products $\delta_1(a_1)\cdots\delta_k(a_k)T\in\mathcal{M}$ for all $a_1,...,a_k\in\mathcal{A}$, where $T\in\mathcal{N}$ is fixed. If $\phi:\mathcal{M}\to\mathbf{C}$ is a linear functional and we define $\tilde\phi\in C^k(\mathcal{A})$ via:*

$$\tilde\phi(a_0,...,a_k)=\phi(a_0\delta_1(a_1)\cdots\delta_k(a_k)T),$$

then the Hochschild coboundary of $\tilde\phi$ is

$$(b\tilde\phi)(a_0,...,a_{k+1})=(-1)^k\phi(a_0\delta_1(a_1)\cdots\delta_k(a_k)a_{k+1}T-a_{k+1}a_0\delta_1(a_1)\cdots\delta_k(a_k)T).$$

Proof We prove this by induction. For $k=1$ we have

$$\begin{aligned} (b\tilde\phi)(a_0,a_1,a_2) &= \phi(a_0a_1\delta(a_2)T-a_0\delta(a_1a_2)T+a_2a_0\delta(a_1)T) \\ &= -\phi(a_0\delta(a_1)a_2T-a_2a_0\delta(a_1)T). \end{aligned}$$

The derivation property shows that the first of these terms is still in $\mathcal{M}$, so the $k=1$ case is true. So we now suppose the result is true for all $n<k$. Then

$$\begin{aligned} (b\tilde\phi)(a_0,...,a_{k+1}) &= \phi(a_0a_1\delta_1(a_2)\cdots\delta_k(a_{k+1})T) \\ &\quad+\sum_{i=1}^k(-1)^i\phi(a_0\delta_1(a_1)\cdots\delta_i(a_ia_{i+1})\cdots\delta_k(a_{k+1})T) \\ &\quad+(-1)^{k+1}\phi(a_{k+1}a_0\delta_1(a_1)\cdots\delta_k(a_k)T) \\ &= (b\hat\phi)(a_0,...,a_k)-(-1)^k\phi(a_ka_0\delta_1(a_1)\cdots\delta_{k-1}(a_{k-1})\delta_k(a_{k+1})T) \\ &\quad+(-1)^k\phi(a_0\delta_1(a_1)\cdots\delta_{k-1}(a_{k-1})\delta_k(a_ka_{k+1})T \\ &\quad+(-1)^{k+1}\phi(a_{k+1}a_0\delta_1(a_1)\cdots\delta_k(a_k)T). \end{aligned}$$

Here

$$\hat{\phi}(a_0, ..., a_{k-1}) = \phi(a_0\delta_1(a_1)\cdots\delta_{k-1}(a_{k-1})(\delta_k(a_{k+1})T)).$$

Note that by hypothesis, the product of $\delta_1(a_1)\cdots\delta_{k-1}(a_{k-1})$ and $\delta_k(a_{k+1})T$ is in $\mathcal{M}$.

By induction we have

$$\begin{aligned}(b\hat{\phi})(a_0, ..., a_k) &= (-1)^{k-1}\phi(a_0\delta_1(a_1)\cdots\delta_{k-1}(a_{k-1})a_k\delta_k(a_{k+1})T) \\ &\quad -(-1)^{k-1}\phi(a_ka_0\delta_1(a_1)\cdots\delta_{k-1}(a_{k-1})\delta_k(a_{k+1})T).\end{aligned}$$

Thus we have

$$(b\tilde{\phi})(a_0, ..., a_{k+1}) = (-1)^k\phi(a_0\delta_1(a_1)\cdots\delta_k(a_k)a_{k+1}T - a_{k+1}a_0\delta_1(a_1)\cdots\delta_k(a_k)T).$$

□

Thus the derivations need not all be the same to obtain the usual result linking Hochschild homology and derivations, [23, p 84]. We will mostly be interested in the case where the bimodule $\mathcal{M}$ is the ideal $\mathcal{L}^{(1,\infty)}$, but we also use $\mathcal{M} = \mathcal{L}^1$.

Our next aim is to define the Chern character of a finitely summable Fredholm module. First we need a definition.

Definition 5 *A* **pre-Fredholm module** *for a unital Banach $*$-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, F)$ where $\mathcal{A}$ is (continuously) represented in $\mathcal{N}$ (a semifinite von Neumann algebra acting on $\mathcal{H}$) and F is a self-adjoint Breuer-Fredholm operator in $\mathcal{N}$ satisfying:*

1. $1 - F^2 \in \mathcal{K}_{\mathcal{N}}$, *and*

2. $[F, a] \in \mathcal{K}_{\mathcal{N}}$ *for* $a \in \mathcal{A}$.

If $1 - F^2 = 0$ *we drop the prefix "pre-".*

If, in addition, our module satisfies:

1.′ $1 - F^2 \in \mathcal{L}^{(p/2,\infty)}$

2.′ $[F, a] \in \mathcal{L}^{(p,\infty)}$ *for a dense set of $a \in \mathcal{A}$. we say $(\mathcal{H}, F)$ is (p,∞)-summable.*

Remark Here and throughout the rest of the paper, if $p < 2$ we interpret $T \in \mathcal{L}^{(p/2,\infty)}$ as indicating that $T^{p/2} \in \mathcal{L}^{(1,\infty)}$ which implies that $T \in \mathcal{L}^1$.

Let $(\mathcal{H}, F)$ be a $p+1$-summable Fredholm module for $\mathcal{A}$ with $F^2 = 1$, [9, IV.1.α], that is, we have $[F, a] \in \mathcal{L}^{p+1}(\mathcal{N})$ for all $a \in \mathcal{A}$. In particular if $(\mathcal{H}, F)$ is (p,∞)-summable then it is $(p+1)$-summable.

The Chern character of $(\mathcal{H}, F)$ is the class in periodic cyclic cohomology of the cocycles

$$\lambda_n\tau'(\Gamma a_0[F, a_1]\cdots[F, a_n]), \qquad a_0, ..., a_n \in \mathcal{A}, \quad n \geq p, \quad n - p \ \text{ even}.$$

Here λ_n are constants ensuring that this collection of cocycles yields a well-defined periodic class, and they are given by

$$\lambda_n = \begin{cases} (-1)^{n(n-1)/2}\Gamma(\frac{n}{2}+1) & n \quad \text{even} \\ \sqrt{2i}(-1)^{n(n-1)/2}\Gamma(\frac{n}{2}+1) & n \quad \text{odd} \end{cases}.$$

The 'conditional trace' (or, super-trace) τ' is defined by

$$\tau'(T) = \frac{1}{2}\tau(F(FT + TF)),$$

provided $FT + TF \in \mathcal{L}^1(\mathcal{N})$ (as it is in our case, see [9, p293]). Note that if $T \in \mathcal{L}^1(\mathcal{N})$ we have (using the trace property and $F^2 = 1$)

$$\tau'(T) = \tau(T). \tag{2}$$

This class is represented by the cyclic cocycle $Ch_F \in C^p_\lambda(\mathcal{A})$

$$Ch_F(a_0, ..., a_p) = \lambda_p \tau'(\Gamma a_0[F, a_1] \cdots [F, a_p]), \qquad a_0, ..., a_p \in \mathcal{A}.$$

If we only have a pre-Fredholm module $(\mathcal{H}, F)$, there is a canonical procedure described in [9, p 310] (and [1] in the general semifinite context) associating to $(\mathcal{H}, F)$ a Fredholm module $(\mathcal{H}', F')$. The Chern character of $(\mathcal{H}, F)$ is then defined to be the Chern character of $(\mathcal{H}', F')$. The Fredholm module $(\mathcal{H}', F')$ has the same summability as $(\mathcal{H}, F)$. We will not require the explicit form of this procedure, as we will now show that we have a more amenable procedure at our disposal.

Our next task is to show that if our spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is such that $\mathcal{D}$ is not invertible, we can replace it by a new spectral triple in the same K-homology class in which the unbounded operator is invertible. This is not a precise statement in the general semifinite case, as our spectral triples will not define K-homology classes in the usual sense. When we say that two spectral triples are in the same K-homology class, we shall take this to mean that the associated pre-Fredholm modules are operator homotopic up to the addition of degenerate Fredholm modules (see [22] for these notions, which make sense in our context).

Definition 6 *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. For any $m \in \mathbf{R} \setminus \{0\}$, define the 'double' of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to be the spectral triple $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_m)$ with $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$, and the action of $\mathcal{A}$ and $\mathcal{D}_m$ given by*

$$\mathcal{D}_m = \begin{pmatrix} \mathcal{D} & m \\ m & -\mathcal{D} \end{pmatrix}, \qquad a \to \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall a \in \mathcal{A}.$$

Remark Whether $\mathcal{D}$ is invertible or not, $\mathcal{D}_m$ always is invertible, and $F_m = \mathcal{D}_m|\mathcal{D}_m|^{-1}$ has square 1. This is the chief reason for introducing this construction. We need to ensure that by doing so we do not alter the (co)homological data.

Lemma 4 *The K-homology classes of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_m)$ are the same. A representative of this class is $(\mathcal{H}^2, F_m)$ with $F_m = \mathcal{D}_m|\mathcal{D}_m|^{-1}$.*

Proof The K-homology class of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is represented by the pre-Fredholm module $(\mathcal{H}, F_\mathcal{D})$ with $F_\mathcal{D} = D(1 + \mathcal{D}^2)^{-1/2}$ while $[(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_m)]$ is represented by the pre-Fredholm module $(\mathcal{H}^2, F_{\mathcal{D}_m})$ with $F_{\mathcal{D}_m} = \mathcal{D}_m(1 + \mathcal{D}_m^2)^{-1/2}$ (we describe (pre)-Fredholm modules in subsection 2.2). The one parameter family $(\mathcal{H}, F_{\mathcal{D}_m})_{0 \le m \le M}$ is a continuous operator homotopy, [22],[3], from $(\mathcal{H}^2, F_{\mathcal{D}_M})$ to the direct sum of two pre-Fredholm modules

$$(\mathcal{H}, F_{\mathcal{D}}) \oplus (\mathcal{H}, -F_{\mathcal{D}}),$$

and in the odd case, the second pre-Fredholm module is operator homotopic to $(\mathcal{H}, 1)$ by the straight line path, since $\mathcal{A}$ is represented by zero on this module. In the even case we find the second pre-Fredholm module is homotopic to

$$\left(\mathcal{H}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

the matrix decomposition being with respect to the $\mathbf{Z}_2$-grading of $\mathcal{H}$. Thus in both the even and odd cases the second module is degenerate, i.e. $F^2 = 1$, $F = F^*$ and $[F, a] = 0$ for all $a \in \mathcal{A}$, and so the K-homology class of $(\mathcal{H}^2, F_{\mathcal{D}_M})$, written $[(\mathcal{H}^2, F_{\mathcal{D}_M})]$, is the K-homology class of $(\mathcal{H}, F_{\mathcal{D}})$. In addition, the Fredholm module $(\mathcal{H}^2, F_m)$ with $F_m = \mathcal{D}_m|\mathcal{D}_m|^{-1}$ is operator homotopic to $(\mathcal{H}^2, F_{\mathcal{D}_m})$ via

$$t \to \mathcal{D}_m(t + \mathcal{D}_m^2)^{-1/2} \quad 0 \le t \le 1.$$

This provides the desired representative. □

The most basic consequence of Lemma 4, and the reason for proving it, comes from the following (see [9, IV.1.γ] and [12] for the proof).

Proposition 5 *The periodic cyclic cohomology class of the Chern character of a finitely summable Fredholm module depends only on its K-homology class.*

In the general semifinite case this should be interpreted as saying that two pre-Fredholm modules which are operator homotopic up to the addition of degenerate Fredholm modules have the same Chern character. In particular, therefore, the Chern characters of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and $(\mathcal{A}, \mathcal{H}^2, \mathcal{D}_m)$ have the same class in periodic cyclic cohomology, and this can be computed using the Fredholm module $(\mathcal{H}^2, F_m)$.

Using Connes' exact sequence, [9, III.1.γ],

$$\cdots \xrightarrow{B} HC^p(\mathcal{A}) \xrightarrow{S} HC^{p+2}(\mathcal{A}) \xrightarrow{I} HH^{p+2}(\mathcal{A}, \mathcal{A}^*) \xrightarrow{B} HC^{p+1}(\mathcal{A}) \xrightarrow{S} \cdots$$

we see that the Hochschild class of Ch_F is the image of the cyclic cohomology class of Ch_F under the map I induced by the inclusion of the cyclic complex in the Hochschild complex. This class is the noncommutative analogue of the integral representing the fundamental class. To see this, recall that for the Dirac operator $\mathcal{D}$ on a closed spin manifold X we have

$$Ch(\mathcal{D})(\cdot) = \text{const} \int_X \cdot \wedge \hat{A} = \text{const}(\int_X \cdot \wedge 1 + \int_X \cdot \wedge (-\frac{1}{24}p_1) + \cdots).$$

Here $\hat{A}$ is the A-roof genus, p_i are the Pontryagin classes, and regarding $Ch(\mathcal{D})$ as an element of de Rham homology, this formula tells us how to evaluate $Ch(\mathcal{D})$ on elements of the exterior algebra of the manifold. In particular, restricting to differential forms of top degree (volume forms) we have

$$Ch(\mathcal{D})(f_0 df_1 \wedge \cdots \wedge df_{\dim X}) = \mathrm{const} \int_X f_0 df_1 \wedge \cdots \wedge df_{\dim X}.$$

Hence the Hochschild class of the Chern character yields the usual integration of a $(\dim X)$-form. This gives not only justification for the identification and study of this Hochschild class, but also a heuristic for understanding the measurability described in Corollary 11 (see Subsection 3.1).

Before leaving Chern characters, we note that the hypothesis of (p,∞)-summability may be supplemented by Connes-Moscovici's discrete and finite dimension spectrum hypothesis, [11]. With this extra hypothesis one obtains a new representative of the Chern character expressed in terms of the operator $\mathcal{D}$. Using this representative, it is straightforward to identify the Hochschild class, and this agrees with the result stated in [9, IV.2.γ] and described here. However, the results concerning measurability (described later), arguably the most important consequence of Theorem 10, are rendered trivial, as the dimension spectrum hypothesis includes an assumption of measurability.

2.3 The Dixmier Trace and the Heat Kernel

Normally a Dixmier trace on the τ-compact operators means a positive linear functional which is constructed in the following way. One composes a positive element ω of the dual of $L^\infty(\mathbf{R}_+^*)$ with the map which takes compact operators to the Cesaro mean (described below) of their singular values (where the latter is thought of as an element of $L^\infty(\mathbf{R}_+^*)$. The positive functional ω is also required to agree with the ordinary limit on functions which have a limit at infinity.

The composition of any such ω from $L_\infty(\mathbf{R}_+^*)^*$ which is vanishing on $C_0(\mathbf{R}_+^*)$ with the Cesaro mean operator produces a functional which is (almost) dilation invariant and with which it is possible to define a non-normal trace (see [8]). We shall call such functionals Dixmier functionals and such non-normal traces Dixmier traces (see below).

A key technical lemma we will exploit uses the asymptotics of the trace of the heat operator for $\mathcal{D}$ to construct the singular or Dixmier trace that appears in Theorem 10 when we have a particular kind of Dixmier functional ω.

Definition 7 *The Cesaro mean on $L^\infty(\mathbf{R}_+^*)$, where $\mathbf{R}_+^*$ is the multiplicative group of the positive reals, is given by:*

$$M(g)(t) = \frac{1}{\log t}\int_1^t g(s)\frac{ds}{s} \ for\ g \in L^\infty(\mathbf{R}_+^*),\ t > 0.$$

Definition 8 *We define the following maps on $L^\infty(\mathbf{R}_+^*)$. Let D_a denote dilation by $a \in \mathbf{R}_+^*$ and let P^a denote exponentiation by $a \in \mathbf{R}_+^*$. That is,*

$$\begin{aligned} D_a(f)(x) &= f(ax)\ for\ f \in L^\infty(\mathbf{R}), and \\ P^a(f)(x) &= f(x^a)\ for\ f \in L^\infty(\mathbf{R}_+^*). \end{aligned}$$

G is the set $\mathbf{R}_+^* \times \mathbf{R}_+^*$ with multiplication:

$$(s,t)(x,y) = (sx^t, ty).$$

One of the main observations of [1] and [6] is that in addition to dilation and Cesaro invariance, invariance under the operators P^a ($a \in \mathbf{R}_+^*$) is critical in one key step of the proof of the zeta function representation of a Dixmier trace. We denote by $C_0(\mathbf{R}_+^*)$ the continuous functions on $\mathbf{R}_+^*$ vanishing at infinity. We will need the existence of a G-invariant, M invariant Dixmier functional on $L^\infty(\mathbf{R}_+^*)$.

Theorem 6 ([6]) *There exists a state Ω on $L^\infty(\mathbf{R}_+^*)$ satisfying the following conditions:*
(1) $\Omega(C_0(\mathbf{R}_+^*)) \equiv 0$.
(2) If f is real-valued in $L^\infty(\mathbf{R}_+^*)$ then

$$\textit{ess lim-inf}_{t\to\infty} f(t) \le \Omega(f) \le \textit{ess lim-sup}_{t\to\infty} f(t).$$

(3) If the essential support of f is compact then $\Omega(f) = 0$.
(4) For all $c \in \mathbf{R}_+^*$, $\Omega(D_c f) = \Omega(f)$ for all $f \in L^\infty(\mathbf{R}_+^*)$.
(5) For all $a \in \mathbf{R}_+^*$ and all $f \in L^\infty(\mathbf{R}_+^*)$ $\Omega(P^a f) = \Omega(f)$.
(6) For all $f \in L^\infty(\mathbf{R}_+^*)$, $\Omega(Mf) = \Omega(f)$.

The approach of [6] as described in Theorem 6 is to construct what might be more appropriately be termed a 'maximally invariant Dixmier functional'. This maximal invariance is what is required to establish the zeta function representation of a Dixmier trace (and hence the heat kernel formula for $\mathcal{L}^{(1,\infty)}$) in full generality. Weaker conditions suffice for the case of $L^{(p,\infty)}$, $p > 1$, essentially because the map $T \to T^p$ taking $L^{(p,\infty)}$ to $L^{(1,\infty)}$ is not surjective and in fact the image is a smaller ideal consisting of compact operators T whose singular values satisfy, for some $C > 0$, the inequality $\mu_s(T) \le C/s$ for s suficiently large; see [6] for further discussion.

A notation we will often use is to write, for a given function $f \in L^\infty(\mathbf{R}_+^*)$ and Dixmier functional ω, $\omega(f) = \omega\text{-}\lim_{\lambda\to\infty} f(\lambda)$. In particular we will be interested in applying such functionals to the function

$$\frac{1}{\log(1+t)} \int_0^t \mu_s(T) ds$$

where $T \in \mathcal{L}^{(1,\infty)}$ is positive. This is the Dixmier trace associated to the semifinite normal trace τ, denoted τ_ω, and we extend it to all of $\mathcal{L}^{(1,\infty)}$ by linearity. The Dixmier trace τ_ω is defined on the ideal $\mathcal{L}^{(1,\infty)}$, and vanishes on the ideal of trace class operators. This latter fact is used repeatedly throughout the paper without further comment.

Let $T \ge 0$ and define $e^{-T^{-2}}$ as the operator that is zero on $\ker T$ and on $\ker T^\perp$ is defined in the usual way by the functional calculus. We remark that if $T \ge 0$, $T \in \mathcal{L}^{(p,\infty)}$ for some $p \ge 1$ then $e^{-tT^{-2}}$ is trace class for all $t > 0$. Then we have

Theorem 7 ([6]) *If $A \in \mathcal{N}$, $T \ge 0$, $T \in \mathcal{L}^{(p,\infty)}$ then,*

$$\Omega\text{-}\lim_{\lambda\to\infty} \lambda^{-1} \tau(Ae^{-\lambda^{-2/p}T^{-2}}) = \Gamma(p/2+1)\tau_\Omega(AT^p)$$

for $\Omega \in L^\infty(\mathbf{R}_+^*)^*$ satisfying the conditions of Theorem 6.

Remark. The reason for the citation of [6] for this result is that we require the case $p = 1$, and this is the only place where this is established. For $p > 1$, however, see [9, p563] and [21].

To use this result in this paper we will apply it to the case where $T = (1+\mathcal{D}^2)^{-1/2}$ or $T = |\mathcal{D}|^{-1}$ if $\mathcal{D}$ has bounded inverse. Then a simple but useful corollary of this theorem is that for $p \geq 1$ and $|\mathcal{D}|^{-1} \in \mathcal{L}^{(p,\infty)}$ the function on $\mathbf{R}_+^*$ given by

$$t \to t^p \tau(A e^{-t^2\mathcal{D}^2})$$

is bounded. This follows from setting $\lambda^{-1} = t^p$ and $T = |\mathcal{D}|^{-1}$. Or in other words

Lemma 8 *If $p \geq 1$ and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a (p,∞)-summable spectral triple with $\mathcal{D}$ invertible, then there exists a constant $C_p > 0$ such that*

$$\tau(e^{-t^2\mathcal{D}^2}) \leq C_p t^{-p} \quad \textit{for} \quad t > 0.$$

A fact that we will frequently require is the following.

Proposition 9 ([6, 8, 19]) *The Dixmier trace τ_ω associated to a Dixmier functional ω defines a trace on the algebra of a QC^1 (p,∞)-summable spectral triple via*

$$a \mapsto \tau_\omega(a(1+\mathcal{D}^2)^{-p/2}).$$

3 The Hochschild Class of the Chern Character

3.1 Statement of the Main Result

Our main result is the general semifinite version of a Type I result in [9, IV.2.γ] which identifies the Hochschild class of the Chern character of a (p,∞)-summable spectral triple. With the preliminary definitions out of the way, we can now state our main result:

Theorem 10 *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^k (p,∞)-summable spectral triple with $p \geq 1$ integral and $k = \max\{2, p-2\}$. Then*

1) A Hochschild cocycle on $\mathcal{A}$ is defined by

$$\phi_\omega(a_0, ..., a_p) = \lambda_p \tau_\omega(\Gamma a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_p](1+\mathcal{D}^2)^{-p/2}),$$

2) For all Hochschild p-cycles $c \in C_p(\mathcal{A})$ (i.e., $bc = 0$),

$$\langle \phi_\omega, c \rangle = \langle Ch_{F_\mathcal{D}}, c \rangle,$$

where $Ch_{F_\mathcal{D}}$ is the Chern character in cyclic cohomology of the pre-Fredholm module over $\mathcal{A}$ with $F_\mathcal{D} = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$.

Remark Here τ_ω is the Dixmier trace associated to any Dixmier functional ω. The two most important corollaries of Theorem 10 are the following.

Corollary 11 *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be as in Theorem 10. If $c=\sum_i a_0^i\otimes a_1^i\otimes\cdots\otimes a_p^i$ is a Hochschild p-cycle, then*

$$\Gamma\sum_i a_0^i[\mathcal{D},a_1^i]\cdots[\mathcal{D},a_p^i](1+\mathcal{D}^2)^{-p/2}$$

is measurable.

Remark An operator $T\in\mathcal{L}^{(1,\infty)}$ is measurable (in the sense of Connes) if the ω-limit

$$\omega\text{-}\lim_{t\to\infty}\frac{1}{\log(1+t)}\int_0^t\mu_s(T)ds$$

is independent of the choice of ω. We will include a proof of this important result (Corollary 11) as part of the proof of Theorem 10.

Corollary 12 *With $(\mathcal{A},\mathcal{H},\mathcal{D})$ as in Theorem 10, and supposing that $Ch_{F_\mathcal{D}}$ pairs nontrivially with $HH_p(\mathcal{A})$, then*

$$\tau_\omega((1+\mathcal{D}^2)^{-p/2})\neq 0.$$

Remark The hypothesis of the Corollary is that there exists some Hochschild p-cycle such that $\langle ICh_{F_\mathcal{D}},c\rangle\neq 0$. Computing this pairing using Theorem 10 above, we see that $(1+\mathcal{D}^2)^{-p/2}$ can not have zero Dixmier trace for any choice of Dixmier functional ω. For if $(1+\mathcal{D}^2)^{-p/2}$ did have vanishing Dixmier trace, and $c=\sum_i a_0^i\otimes\cdots\otimes a_p^i$ is any Hochschild cycle

$$\begin{aligned}|\langle ICh_{F_\mathcal{D}},c\rangle| &= \left|\sum_i\tau_\omega\left(\Gamma a_0^i[\mathcal{D},a_1^i]\cdots[\mathcal{D},a_p^i](1+\mathcal{D}^2)^{-p/2}\right)\right|\\ &\leq \sum_i\parallel\Gamma a_0^i[\mathcal{D},a_1^i]\cdots[\mathcal{D},a_p^i]\parallel\tau_\omega\left((1+\mathcal{D}^2)^{-p/2}\right)=0.\end{aligned}$$

Hence if the pairing is nontrivial, the Dixmier trace can not vanish on $(1+\mathcal{D}^2)^{-p/2}$.

During the course of the proof *we will always suppose* that we have a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ with $\mathcal{D}$ invertible, by replacing $(\mathcal{A},\mathcal{H},\mathcal{D})$ by $(\mathcal{A},\mathcal{H}^2,\mathcal{D}_m)$ if necessary. Despite knowing that the cyclic classes of the Chern characters of these two triples coincide, by Lemma 4 and Proposition 5, and so their Hochschild classes also coincide, we do not know that this is true for the specific representative displayed in Theorem 10, and this is something we will need to determine. A proof that this is indeed the case can be found in the Appendix.

Before discussing the proof any further, we show that the functional ϕ_ω is indeed a Hochschild cocycle.

Lemma 13 *Let $p\geq 1$ and suppose that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a QC^1 (p,∞)-summable spectral triple. Then the multilinear functional*

$$\phi_\omega(a_0,...,a_p)=\lambda_p\tau_\omega(\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_p](1+\mathcal{D}^2)^{-p/2})$$

is a Hochschild cocycle.

Proof By Lemma 3 and the trace property of the Dixmier trace, we have

$$\begin{aligned}(b\phi_\omega)(a_0,...,a_p) &= (-1)^{p-1}\lambda_p\tau_\omega(\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_{p-1}]a_p(1+\mathcal{D}^2)^{-p/2}) \\ &\quad -(-1)^{p-1}\lambda_p\tau_\omega(\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_{p-1}](1+\mathcal{D}^2)^{-p/2}a_p).\end{aligned}$$

As $(\mathcal{A},\mathcal{H},\mathcal{D})$ is QC^1,

$$[(1+\mathcal{D}^2)^{-p/2},a_p] = -\sum_{k=0}^{p-1}(1+\mathcal{D}^2)^{-(p-k)/2}[(1+\mathcal{D}^2)^{1/2},a_p](1+\mathcal{D}^2)^{-(1+k)/2},$$

and this is trace class. So $a_p(1+\mathcal{D}^2)^{-p/2}=(1+\mathcal{D}^2)^{-p/2}a_p$ modulo trace class operators, and so the two terms above cancel. □

Thus to show that ϕ_ω is a Hochschild cocycle is relatively simple, and does not require the full smoothness assumptions of Theorem 10. Of course the important aspects of Theorem 10 are that ϕ_ω is a representative of the Hochschild class of the Chern character, and the measurability of 'p-forms'.

3.2 What was previously known

This theorem, for $\mathcal{N}=\mathcal{B}(\mathcal{H})$ and $1<p<\infty$ (p integral) was proved in lectures by Alain Connes at the Collège de France in 1990. A version of this argument appeared in [21]. The extension of this argument to general semi-finite von Neumann algebras, with the additional hypothesis that $\mathcal{D}$ have bounded inverse, is presented in the preprint of Benameur-Fack, [1] and we thank the authors for bringing it to our attention. It provided an impetus to our work. Some supplementary details in the proof were given to us by Thierry Fack, and we thank him for his notes, [19]. In addition, a simpler strategy using the pseudodifferential calculus of Connes-Moscovici, [11], was communicated to us by Nigel Higson. In conjunction with the results in [6], Higson's argument appears to generalise to the semifinite case as well as giving an alternate proof of Theorem 10, however we will not describe the details here.

The extension of these earlier results which our Theorem 10 implies are

1) for the first time we provide a proof for the case $p=1$ (the proof in this case overcomes some serious technical obstacles).

2) We dispense with the hypothesis in the type II_∞ case that $\mathcal{D}$ has bounded inverse. This is crucial due to the 'zero-in-the-spectrum' phenomenon for $\mathcal{D}$ (that is, for type II $\mathcal{N}$, zero is generically in the point and/or continuous spectrum, [20]) and is not just the simple problem posed by non-trivial $\ker\mathcal{D}$.

3) Importantly, our strategy of proof is the same for all $p\geq 1$, is independent of the type of the von Neumann algebra $\mathcal{N}$ and is simpler than previously published arguments.

We now come to the proof of Theorem 10. The general form of the technical estimates, and so the basic structure of the analytic parts of the proof, are based on a synthesis of our understanding of the arguments in [1] and [21]. These in turn have their origin in the original arguments of Connes. The latter parts of the argument where we need to construct various cohomologies in the Hochschild theory to arrive at the functional in the statement of the theorem, closely follow the argument in [21].

Our method of proof is in some ways more direct than these other approaches. In particular we do not need to prove our technical estimates for general functions of $\mathcal{D}$, only the particular functions that allow us to employ the heat kernel approach to the Dixmier trace. The chief novelty (and difficulty) of this direct approach is that we can deal with the case $p = 1$. For this approach the assumption of [21] that the functions of $\mathcal{D}$ involved are compactly supported is of no use and various technical estimates in [1, 9, 21] are not available.

3.3 Functional Calculus Preliminaries

In this subsection we establish some trace and commutator estimates for certain functions of $|\mathcal{D}|$. We will work exclusively with one function, however the definition of this function depends on the value of p. Moreover there are substantial differences between the even and odd cases, and for technical reasons we also require estimates involving square roots of functions.

For $p \geq 1$ an integer, and $x \geq 0$ define

$$erf_p(x) = \frac{p}{\Gamma(\frac{p}{2}+1)} \int_0^x r^{p-1} e^{-r^2} dr. \tag{3}$$

Using

$$\int_0^\infty r^{p-1} e^{-r^2} dr = \frac{\Gamma(p/2)}{2} = \frac{\Gamma(\frac{p}{2}+1)}{p},$$

we have $erf_p(\infty) = 1$ and $erf_p(0) = 0$. Now define

$$f_p(x) = \begin{cases} 1 - erf_p(x) & x \geq 0 \\ 1 - (-1)^p erf_p(-x) & x \leq 0 \end{cases} \tag{4}$$

Then we have $f_p(0) = 1$, $f_p(\infty) = 0$ and

$$f_p'(x) = \frac{-p}{\Gamma(\frac{p}{2}+1)} x^{p-1} e^{-x^2}. \tag{5}$$

For p even and all $x \in \mathbf{R}$ or p odd and $x \geq 0$ we can write

$$\begin{aligned} f_p(x) &= 1 - erf_p(|x|) \\ &= \frac{p}{\Gamma(p/2+1)} \int_1^\infty x^p s^{p-1} e^{-s^2x^2} ds \\ &= c(p) \int_1^\infty x^p s^{p-1} e^{-s^2x^2} ds. \end{aligned} \tag{6}$$

We will see shortly that f_p is Schwartz class for p even. However for p odd, while $f_p(x) \to 0$ rapidly as $x \to +\infty$, as $x \to -\infty$, $f_p(x) \to 2$. The reason we have defined the function in this way is to obtain smoothness at $x = 0$, and the important part of the definition is for $x \geq 0$ anyway. For instance, we have our first estimate.

Lemma 14 *Let $p \geq 1$ and suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^0 (p, ∞)-summable spectral triple with $\mathcal{D}$ invertible. If h is either of the functions f_p or $\sqrt{f_p}$ then for $t > 0$*

$$\| h(t|\mathcal{D}|) \|_1 \leq C_h t^{-p}.$$

Proof Let $d\phi_\lambda = d\tau(E_\lambda)$ be the scalar spectral measure for $|\mathcal{D}|$, and consider the function $\sqrt{f_p}$. We have by Lemma 8.2 of [4]

$$\begin{aligned}
\tau(\sqrt{f_p}(t|\mathcal{D}|)) &= \left(\frac{p}{\Gamma(\frac{p}{2}+1)}\right)^{1/2} \tau\left(\left(\int_1^\infty s^{p-1}t^p|\mathcal{D}|^p e^{-s^2t^2\mathcal{D}^2}ds\right)^{1/2}\right) \\
&= \left(\frac{t^p p}{\Gamma(\frac{p}{2}+1)}\right)^{1/2} \int_0^\infty \left(\int_1^\infty s^{p-1}\lambda^p e^{-s^2t^2\lambda^2}ds\right)^{1/2} d\phi_\lambda \\
&\leq \left(\frac{t^p p}{\Gamma(\frac{p}{2}+1)}\right)^{1/2} \int_0^\infty \lambda^{p/2}e^{-t^2\lambda^2/4}\left(\int_1^\infty s^{p-1}e^{-s^2t^2\lambda^2/2}ds\right)^{1/2} d\phi_\lambda \\
&\leq \left(\frac{t^p p}{\Gamma(\frac{p}{2}+1)}\right)^{1/2} \int_0^\infty \lambda^{p/2}e^{-t^2\lambda^2/4}\left(\int_0^\infty s^{p-1}e^{-s^2t^2\lambda^2/2}ds\right)^{1/2} d\phi_\lambda \\
&= \left(\frac{t^p p}{\Gamma(\frac{p}{2}+1)}\right)^{1/2} \int_0^\infty \lambda^{p/2}e^{-t^2\lambda^2/4}(\Gamma(p/2)\lambda^{-p}t^{-p}2^{\frac{p}{2}-1})^{1/2}d\phi_\lambda \\
&= 2^{\frac{p}{4}}\int_0^\infty e^{-t^2\lambda^2/4}d\phi_\lambda \\
&= 2^{\frac{p}{4}}\tau(e^{-(t/2)^2\mathcal{D}^2}) \\
&\leq Ct^{-p},
\end{aligned}$$

where the last line follows from the heat kernel estimate Lemma 8. The same method applies to yield the result for $h = f_p$ also. $\square$

The above Lemma required knowledge of f_p for positive arguments, so for p odd, we are free to alter the definition in any reasonable way for negative values.

So for $p \geq 1$ odd, and for some $k > 0$, define

$$f_p(x) = \begin{cases} 1 - erf_p(x) & x \geq 0 \\ 1 + erf_p(-x) & -k \leq x \leq 0 \\ g(x) & x \leq -k \end{cases}$$

Here we take $g(x) = Q(x)e^{-x^2}$, where Q is a polynomial. We may choose to make f_p a C^l function at $-k$, and this will require taking Q to be of order l.

Lemma 15 *For p even*

$$f_p(x) = \sum_{i=0}^{[p-2/2]} \frac{c(p-2i)}{2} x^{p-2-2i}e^{-x^2}$$

When p is odd and $p \geq 3$, and $x \geq 0$ we have

$$f_p(x) = \sum_{i=0}^{[p-2/2]} \frac{c(p-2i)}{2} x^{p-2-2i} e^{-x^2} + f_1(x).$$

Proof Integration by parts using the formulae in Equations (6), and the observation that for all integers $p \geq 3$

$$\frac{c(p)}{c(p-2)} \frac{(p-2)}{2} = 1.$$

□

Observe that the Lemma shows that for p even, the function f_p is Schwartz class. Indeed, $f_p^{1/2}$ is Schwartz class. This follows because $f_p(x) = P(x)e^{-x^2}$ where P is an even polynomial with a nonzero constant term. Thus

$$\frac{df_p^{1/2}}{dx} = \frac{(P'(x) - 2xP(x))e^{-x^2/2}}{P(x)^{1/2}}, \tag{7}$$

from which it is easy to see that $f_p^{1/2}$ has derivatives of all orders, and they are all of rapid decrease.

For p odd we will have a rapidly decaying function also, but only C^l at $-k$, where we may choose l as large as we like. To see this for large positive x we require the following result, [17].

Lemma 16 *There is an asymptotic expansion of*

$$f_1(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-s^2} ds$$

as $x \to +\infty$ of the form

$$f_1(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}x} \left[1 - \frac{2!}{(2x)^2} + \frac{4!}{2!(2x)^4} - \frac{6!}{3!(2x)^6} + \cdots \right].$$

Proof The function f_1 is precisely the complementary error function (for positive x), and there is a standard asymptotic expansion for x large and positive

$$erfc(x) = f_1(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}x} \left[1 - \frac{2!}{(2x)^2} + \frac{4!}{2!(2x)^4} - \frac{6!}{3!(2x)^6} + \cdots \right].$$

□

At this point we know enough to proceed when p is even, but for the estimates we wish to prove next, we require more information for p odd.

For p odd, our definition of f_p ensures that $x^m f_p^{1/2}$ is integrable for all $m \geq 0$, so the Fourier transform of $f_p^{1/2}$ is smooth, and of course lies in $C_0(\mathbf{R})$. If we define f_p so that it is C^l, then the first l derivatives of $f_p^{1/2}$ will also have smooth Fourier transform, contained in $C_0(\mathbf{R})$, using Lemma 15 and an argument similar to that in Equation 7. So for $l \geq i+2$, the Fourier transform of $\partial^i f_p^{1/2}$ is in $L^1(\mathbf{R})$. This follows because

$$(\widehat{\partial^{i+2} f_p^{1/2}})(\xi) \to 0 \quad \text{as} \quad |\xi| \to \infty,$$

so

$$|\xi^2 (\widehat{\partial^i f_p^{1/2}})(\xi)| \to 0 \quad \text{as} \quad |\xi| \to \infty,$$

which tells us that

$$|(\widehat{\partial^i f_p^{1/2}})(\xi)| = o(|\xi|^{-2}) \quad \text{as} \quad |\xi| \to \infty.$$

Choosing $l \geq 4$ then tells us that in both the even and odd cases

$$\int_{\mathbf{R}} |\xi^i \widehat{f_p^{1/2}}(\xi)| d\xi < \infty, \quad i = 1, 2. \tag{8}$$

We use this to formulate two commutator estimates.

Lemma 17 *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^1 (p, ∞)-summable spectral triple then*

$$\| [f_p(t|\mathcal{D}|), a] \|_1 \leq C_{a,f_p,p} t^{-p+1}.$$

Proof Writing $h = f_p^{1/2}$, we have the straightforward calculation

$$\begin{aligned} \| [f_p(t|\mathcal{D}|), a] \|_1 &= \| h(t|\mathcal{D}|)[h(t|\mathcal{D}|), a] + [h(t|\mathcal{D}|), a] h(t|\mathcal{D}|) \|_1 \\ &\leq 2 \| [h(t|\mathcal{D}|), a] \|_\infty \| h(t|\mathcal{D}|) \|_1 \\ &\leq 2C_p t \| [|\mathcal{D}|, a] \|_\infty \| \widehat{h}(\xi)\xi \|_1 \, t^{-p} \\ &= C_{a,f_p,p} t^{-p+1}. \end{aligned}$$

The last inequality comes from Lemma 14 and

$$\begin{aligned} [h(t|\mathcal{D}|), a] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{h}(s)[e^{its|\mathcal{D}|}, a] ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{h}(s) is \int_0^1 e^{itsr|\mathcal{D}|}[t|\mathcal{D}|, a] e^{i(1-r)st|\mathcal{D}|} dr ds. \end{aligned}$$

The finiteness of (8) for $i = 1$ completes the proof. $\square$

Lemma 18 *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^2 (p, ∞)-summable spectral triple then*

$$\| [f_p(t|\mathcal{D}|), a] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), t[|\mathcal{D}|, a]\} \|_1 \leq C_{a,p,f_p} t^{-p+2},$$

where $\{T, S\} = TS + ST$.

Proof Again we write $h = f_p^{1/2}$, and again this is just a computation.

$$\begin{aligned}
& \| \, [f_p(t|\mathcal{D}|), a] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), t[|\mathcal{D}|, a]\} \, \|_1 \\
= & \| \, h(t|\mathcal{D}|)[h(t|\mathcal{D}|), a] + [h(t|\mathcal{D}|), a]h(t|\mathcal{D}|) \\
& \quad -h(t|\mathcal{D}|)h'(t|\mathcal{D}|)t[|\mathcal{D}|, a] - t[|\mathcal{D}|, a]h(t|\mathcal{D}|)h'(t|\mathcal{D}|) \, \|_1 \\
\leq & \| \, [h(t|\mathcal{D}|), a] - th'(t|\mathcal{D}|)[|\mathcal{D}|, a] \, \|_\infty \| \, h(t|\mathcal{D}|) \, \|_1 \\
& \quad + \| \, [h(t|\mathcal{D}|), a] - t[|\mathcal{D}|, a]h'(t|\mathcal{D}|) \, \|_\infty \| \, h(t|\mathcal{D}|) \, \|_1 \\
\leq & t^{-p+2} C_p \int_{\mathbf{R}} |\widehat{h}(\xi)\xi^2| d\xi.
\end{aligned}$$

The final inequality follows from Lemma 14 and writing, [1]

$$A(t) := [h(t|\mathcal{D}|), a] - h'(t|\mathcal{D}|)[t|\mathcal{D}|, a],$$

we have

$$\begin{aligned}
A(t) &= \int_{\mathbf{R}} \hat{h}(u) \int_0^1 (e^{iuts|\mathcal{D}|}[iut|\mathcal{D}|, a]e^{iut(1-s)|\mathcal{D}|} - e^{iut|\mathcal{D}|}[iut|\mathcal{D}|, a])dsdu \\
&= \int_{\mathbf{R}} \hat{h}(u) \int_0^1 e^{iuts|\mathcal{D}|}[[iut|\mathcal{D}|, a], e^{iut(1-s)|\mathcal{D}|}]dsdu \\
&= -\int_{\mathbf{R}} \hat{h}(u) \int_0^1 e^{iuts|\mathcal{D}|} \int_0^1 e^{iut(1-s)r|\mathcal{D}|}[iut(1-s)|\mathcal{D}|, [iut|\mathcal{D}|, a]]e^{iut(1-s)(1-r)|\mathcal{D}|}dsdrdu.
\end{aligned}$$

A similar result holds for $B(t) = [h(t|\mathcal{D}|), a] - [t|\mathcal{D}|, a]h'(t|\mathcal{D}|)$. In both cases $\| \, A(t) \, \|_\infty$ and $\| \, B(t) \, \|_\infty$ are $O(t^2)$ as $t \to 0$. The finiteness of (8) for $i = 2$ completes the proof. □

Estimates like those presented in the last two Lemmas may be regarded as approximate extensions of familiar rules of calculus to the 'quantum' setting. Both the previous Lemmas extend to a large class of functions, but as we only require these very particular results, we do not pursue these matters here.

3.4 From the Chern Character to a Hochschild Cocycle

Now that we know something about f_p, we can begin the proof. The first step is to bring $|\mathcal{D}|$ into the picture, and to do so in a way that will allow us, eventually, to make use of its summability.

Lemma 19 ([1, 21]) *Let $p \geq 1$ be integral and suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^0 (p, ∞)-summable spectral triple with $\mathcal{D}$ invertible. Let $c = \sum_i a_0^i \otimes \cdots \otimes a_p^i$ be a Hochschild p-cycle. Then*

$$\langle ICh^*(F), c\rangle = -\lim_{t\to 0} \lambda_p \sum_i \tau(\Gamma a_0^i[F, a_1^i] \cdots [F, a_{p-1}^i]F[f_p(t|\mathcal{D}|), a_p^i]),$$

where $F_\mathcal{D} = D|\mathcal{D}|^{-1}$.

Proof Ignore i momentarily, and set $A = \Gamma a_0[F, a_1] \cdots [F, a_p]$. As $t \to 0$ we have $f_p(t|\mathcal{D}|) \to 1$ (strong operator topology), so

$$\langle ICh^*(F_\mathcal{D}), c\rangle = \lim_{t\to 0} \tau'(f_p(t|\mathcal{D}|)A).$$

Here we have used $F f_p(t|\mathcal{D}|) = f_p(t|\mathcal{D}|)F$ to see that the right hand side is equal to

$$\tau(f_p(t|\mathcal{D}|)F(FA + AF)).$$

As $F(FA + AF)$ is trace class, [14, I.6.1,p93] shows that the above equality holds.

For $t > 0$, the operator $f_p(t|\mathcal{D}|)$ is trace class by Lemma 14, so we may replace τ' by τ, using Equation 2. Making this change and expanding the last factor of A gives

$$\tau'(f_p(t|\mathcal{D}|)A) = \tau(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F a_p f_p(t|\mathcal{D}|)) - \tau(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]a_p F f_p(t|\mathcal{D}|)).$$

Using the fact that $a \to [F, a]$ is a derivation, we can use Lemma 3 and

$$\sum_i b(a_0^i \otimes a_1^i \otimes \cdots \otimes a_p^i) = 0,$$

to see that

$$\tau(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]a_p F f_p(t|\mathcal{D}|)) - \tau(\Gamma a_p a_0[F, a_1] \cdots [F, a_{p-1}]F f_p(t|\mathcal{D}|)) = 0,$$

as this is a Hochschild coboundary paired with a Hochschild cycle. This proves the Lemma. $\square$

Note this *only* works when we pair with a Hochschild cycle. For an arbitrary chain we can not swap a_p around to the front. Nevertheless, for any $a_0, ..., a_p \in \mathcal{A}$ we can define a one-parameter family of multilinear functionals

$$\psi_t(a_0, ..., a_p) := -\lambda_p \tau(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F[f_p(t|\mathcal{D}|), a_p]),$$

and we have already shown that the pairing of the Chern character with Hochschild cycles is given by pairing the Hochschild cycle with ψ_t and taking the limit as $t \to 0$. However, we *have not* yet defined a multilinear functional which represents ICh_F. *If* we knew that for all $a_0, ..., a_p \in \mathcal{A}$

$$\phi(a_0, ..., a_p) := \lim_{t\to 0} \psi_t(a_0, ..., a_p)$$

existed, and we could show that $b\phi = 0$, then we would have

$$[\phi] = [ICh_F] \in HH^*(\mathcal{A}, \mathcal{A}^*).$$

In general we can not assert the existence of the above limit, and this is why we do not yet have a representative of the Hochschild class of the Chern character.

The strategy is to show that $|\psi_t(a_0, ..., a_p)|$ is *bounded* as $t \to 0$ for any $a_0, ..., a_p \in \mathcal{A}$, so that we may define a functional by taking the ω-limit. We will then rewrite this result in terms of the associated Dixmier trace. Once achieved, we will have a well-defined multilinear functional on $\mathcal{A}$ which depends on the choice of ω. However, the pairing of this functional with Hochschild cycles will return the true limit, no matter what choice of ω is employed. This is the origin of Corollary 11, but the precise form must wait until we have identified ϕ_ω as a representative of the Hochschild class.

So to begin, let us obtain the estimate which will allow us to show that ψ_t is bounded.

Lemma 20 *Let $p \geq 1$ and suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^1 (p, ∞)-summable spectral triple with $\mathcal{D}$ invertible. Then*

$$\| [f_p(t|\mathcal{D}|), a] \|_{(p,1)} \quad \text{is bounded as } t \to 0.$$

Proof We have the estimate, Lemma 17, for all $p \geq 1$

$$\| [f_p(t|\mathcal{D}|), a] \|_1 \leq \tilde{C}_f t^{-p+1} \| [|\mathcal{D}|, a] \|_\infty .$$

So for $p = 1$ we are done. For $p > 1$ we have the interpolation inequality

$$\| T \|_{(p,1)} \leq C_p \| T \|_1^{1/p} \| T \|_\infty^{1-1/p}, \quad T \in \mathcal{L}^1.$$

In Lemma 17 we also estimated the norm, obtaining

$$\| [f_p(t|\mathcal{D}|), a] \|_\infty = O(t)$$

which allows us to finish the proof since

$$\| [f_p(t|\mathcal{D}|), a] \|_{(p,1)} \leq \mathcal{C}_{f_p,p} (t^{-p+1})^{1/p} t^{1-1/p} = \mathcal{C}_{f_p,p}.$$

□

Remark The use of the interpolation inequality in the previous proof is standard in the type I setting, [9, IV, Appendix B]. For the type II case we note that it is sufficient to obtain the result for the commutative von Neumann algebra $L^\infty(0, \infty)$ and apply the results of [15].

Lemma 21 *Let $p \geq 1$ and suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^1 (p, ∞)-summable spectral triple with $\mathcal{D}$ invertible. Then for all $a_0, ..., a_p \in \mathcal{A}$ the function*

$$t \to \psi_t(a_0, ..., a_p)$$

is bounded as $t \to 0$.

Proof By Lemma 1, $[F, a_i] \in \mathcal{L}^{(p,\infty)}$, $i = 1, ..., p$. So

$$\Gamma a_0 [F, a_1] \cdots [F, a_{p-1}] F \in \mathcal{L}^{(q,\infty)},$$

where $q = p/(p-1)$ (for $p = 1$ replace the (q, ∞) norm with the operator norm). The Köthe dual of $\mathcal{L}^{(p,1)}$ is $\mathcal{L}^{(q,\infty)}$, so as $t \to 0$

$$\begin{aligned} |\psi_t(a_0, ..., a_p)| &\leq \| \Gamma a_0 [F, a_1] \cdots [F, a_{p-1}] F \|_{(q,\infty)} \| [f_p(t|\mathcal{D}|), a_p] \|_{(p,1)} \\ &\leq \mathcal{C}_{f_p,p} \| \Gamma a_0 [F, a_1] \cdots [F, a_{p-1}] F \|_{(q,\infty)}, \end{aligned}$$

by Lemma 20. □

As ψ_t is bounded as $t \to 0$, we are justified in taking the ω-limit of the function $1/t \to \psi_t(a_0, ..., a_p)$, for t sufficiently small.

Definition 9 *For $p \geq 1$ and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ a QC^1 (p, ∞)-summable spectral triple, set*

$$\zeta_p(a_0, ..., a_p) = \Omega\text{-}\lim_{1/t \to \infty} \psi_t(a_0, ..., a_p),$$

for any (fixed) functional Ω satisfying the conditions of Theorem 6.

Remark From what we have shown already, ζ_p is a representative of the Hochschild class of the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and when ζ_p is evaluated on a Hochschild cycle, the Ω limit is a true limit. Thus the value of ζ_p on Hochschild cycles is independent of the choice of Dixmier functional ω, whether ω satisfies the extra invariance conditions of Theorem 6 or not.

We need a preliminary result before we can obtain our first formula for ζ_p in terms of the Dixmier trace associated to the functional Ω.

Lemma 22 *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^2 (p, ∞)-summable spectral triple, with $p \geq 1$ and $\mathcal{D}$ invertible. Then*

$$\| \, [f_p(t|\mathcal{D}|), a] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), [t|\mathcal{D}|, a]\} \, \|_{(p,1)} \to 0 \quad as \quad t \to 0,$$

where $\{T, S\} = TS + ST$.

Proof By Lemma 18, we can estimate the trace norm by

$$\| \, [f_p(t|\mathcal{D}|), a] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), [t|\mathcal{D}|, a]\} \, \|_1 \leq Ct^{-p+2}.$$

This completes the proof for $p = 1$, and for $p > 1$ we will employ interpolation as in Lemma 20. In Lemma 18 we also estimated the operator norm of this difference, obtaining

$$\| \, [f_p(t|\mathcal{D}|), a] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), [t|\mathcal{D}|, a]\} \, \|_\infty = O(t^2).$$

Applying the interpolation inequality

$$\| \, T \, \|_{(p,1)} \leq C \, \| \, T \, \|_1^{1/p} \| \, T \, \|_\infty^{1-1/p},$$

yields

$$\| \, [f_p(t|\mathcal{D}|), a] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), [t|\mathcal{D}|, a]\} \, \|_{(p,1)} \leq \mathcal{C}_{f,p}(t^{-p+2})^{1/p} t^{2-2/p} = \mathcal{C}_{f,p} t \to 0.$$

□

With these tools in hand, we can now obtain our first Dixmier trace formula for ζ_p. This result is where we use the invariance properties of the Dixmier functional Ω, as this is a necessary condition for Theorem 7 to hold, at least when $p = 1$.

Proposition 23 *If $p \geq 1$, $k = \max\{2, p-2\}$ and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^k (p, ∞)-summable spectral triple with $\mathcal{D}$ invertible, then for all $a_0, ..., a_p \in \mathcal{A}$*

$$\zeta_p(a_0, ..., a_p) = p\lambda_p \tau_\Omega(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]\mathcal{D}^{-1}[|\mathcal{D}|, a_p]).$$

Proof We begin by noting that we can write $\psi_t(a_0, ..., a_p)$ as

$$-\lambda_p \tau \left(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F\left(\frac{1}{2}\{f_p'(t|\mathcal{D}|), t\delta(a_p)\} + [f_p(t|\mathcal{D}|), a_p] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), t\delta(a_p)\}\right)\right).$$

This addition of zero inside the trace is justified as $f_p'(t|\mathcal{D}|)$ is trace class and $\delta(a_p)$ is bounded. Thus the Ω-limit $\Omega\text{-}\lim_{1/t\to\infty} \psi_t(a_0, ..., a_p)$, is given by the sum of two terms,

$$-\lambda_p \Omega\text{-}\lim_{1/t\to\infty} \tau\left(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F\frac{1}{2}\{f_p'(t|\mathcal{D}|), t\delta(a_p)\}\right) \tag{9}$$

$$-\lambda_p \Omega\text{-}\lim_{1/t\to\infty} \tau\left(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F\left([f_p(t|\mathcal{D}|), a_p] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), t\delta(a_p)\}\right)\right). \tag{10}$$

The second term, (10), is zero. To see this, we use the same estimates as in Lemma 21,

$$\begin{aligned} &\tau\left(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F\left([f_p(t|\mathcal{D}|), a_p] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), t\delta(a_p)\}\right)\right) \\ \leq\ & \| \Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F \|_{(q,\infty)} \| [f_p(t|\mathcal{D}|), a_p] - \frac{1}{2}\{f_p'(t|\mathcal{D}|), t\delta(a_p)\} \|_{(p,1)} \\ \leq\ & \mathcal{C}_{f_p,p} t \| \Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F \|_{(q,\infty)} \to 0 \ \text{ as } \ t \to 0, \end{aligned}$$

the last inequality following from Lemma 22. Here we again replace the (q, ∞) norm by the operator norm when $p = 1$. Hence the (ordinary) limit of the second term exists and is zero. This means that the first term, (9), is bounded as $t \to 0$, by Lemma 21, so we have

$$\begin{aligned} \zeta_p(a_0, ..., a_p) &= -\lambda_p \Omega\text{-}\lim_{1/t\to\infty} \tau(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F[f_p(t|\mathcal{D}|), a_p]) \\ &= -\lambda_p \Omega\text{-}\lim_{1/t\to\infty} \tau(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F\frac{1}{2}\{f_p'(t|\mathcal{D}|), [t|\mathcal{D}|, a_p]\}), \end{aligned}$$

the second line following from Lemma 22 and the above argument. Using

$$f_p'(t|\mathcal{D}|) = -\frac{p}{\Gamma(\frac{p}{2}+1)} t^{p-1}|\mathcal{D}|^{p-1} e^{-t^2\mathcal{D}^2},$$

we have

$$\begin{aligned} \zeta_p(a_0, ..., a_p) &= \frac{p\lambda_p}{2\Gamma(\frac{p}{2}+1)} \Omega\text{-}\lim_{1/t\to\infty} \tau(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F|\mathcal{D}|^{p-1} t^p e^{-t^2\mathcal{D}^2}[|\mathcal{D}|, a_p]) \\ &\quad + \frac{p\lambda_p}{2\Gamma(\frac{p}{2}+1)} \Omega\text{-}\lim_{1/t\to\infty} \tau(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]F[|\mathcal{D}|, a_p]|\mathcal{D}|^{p-1} t^p e^{-t^2\mathcal{D}^2}). \end{aligned}$$

For $p = 1, 2$, we use Lemma 2 and Theorem 7 to obtain

$$\begin{aligned}\zeta_p(a_0, ..., a_p) &= \frac{p\lambda_p}{2}\tau_\Omega(\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]F|\mathcal{D}|^{-1}[|\mathcal{D}|, a_p]) \\ &\quad + \frac{p\lambda_p}{2}\tau_\Omega(\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]F[|\mathcal{D}|, a_p]|\mathcal{D}|^{-1}).\end{aligned}$$

As

$$\delta(a_p)|\mathcal{D}|^{-1} = -|\mathcal{D}|^{-1}\delta^2(a_p)|\mathcal{D}|^{-1} + |\mathcal{D}|^{-1}\delta(a_p),$$

we may commute the $|\mathcal{D}|^{-1}$ past $\delta(a_p)$ in the second term, only picking up a term which vanishes under the Dixmier trace. Hence

$$\zeta_p(a_0, ..., a_p) = p\lambda_p\tau_\Omega(\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]\mathcal{D}^{-1}[|\mathcal{D}|, a_p]).$$

For $p > 2$, we use the fact that $|\mathcal{D}|^{p-2}e^{-t^2\mathcal{D}^2}$ and $|\mathcal{D}|e^{-t^2\mathcal{D}^2}$ are trace class, to rewrite $\zeta_p(a_0, ..., a_p)$ as

$$\begin{aligned}&\frac{p\lambda_p}{2\Gamma(\frac{p}{2}+1)}\Omega\text{-}\lim_{1/t\to\infty}\tau(t^p e^{-t^2\mathcal{D}^2}|\mathcal{D}|[|\mathcal{D}|, a_p]\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]|\mathcal{D}|^{p-2}F) \\ &+ \frac{p\lambda_p}{2\Gamma(\frac{p}{2}+1)}\Omega\text{-}\lim_{1/t\to\infty}\tau(\Gamma t^p e^{-t^2\mathcal{D}^2}|\mathcal{D}|^{p-2}a_0[F, a_1]\cdots[F, a_{p-1}]F[|\mathcal{D}|, a_p]|\mathcal{D}|).\end{aligned}$$

Now Lemma 2 (and the fact that $|\mathcal{D}|$ commutes with Γ) tells us that both of these terms are the trace of a bounded operator times $t^p e^{-t^2\mathcal{D}^2}$. So, by Theorem 7, see also [6], we have

$$\begin{aligned}\zeta_p(a_0, ..., a_p) &= \frac{1}{2}p\lambda_p\tau_\Omega(|\mathcal{D}|[|\mathcal{D}|, a_p]\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]F|\mathcal{D}|^{-2}) \\ &\quad + \frac{1}{2}p\lambda_p\tau_\Omega(|\mathcal{D}|^{-2}\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]F[|\mathcal{D}|, a_p]|\mathcal{D}|).\end{aligned}$$

Since $[F, a_i]$, $[F, \delta(a_i)]$ and $|\mathcal{D}|^{-1}$ are in $\mathcal{L}^{(p,\infty)}$, commuting $|\mathcal{D}|$ through these expressions gives, modulo terms of trace class which are killed by the Dixmier trace,

$$\begin{aligned}\zeta_p(a_0, ..., a_p) &= \frac{1}{2}p\lambda_p\tau_\Omega([|\mathcal{D}|, a_p]\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]F|\mathcal{D}|^{-1}) \\ &\quad + \frac{1}{2}p\lambda_p\tau_\Omega(|\mathcal{D}|^{-1}\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]F[|\mathcal{D}|, a_p]).\end{aligned}$$

In the first term we may cycle $\delta(a_p)$ around to the end using the trace property of the Dixmier trace (since $\delta(a_p)$ is bounded while the product of the remaining terms is in $\mathcal{L}^{(1,\infty)}$), while in the second we may commute the $|\mathcal{D}|^{-1}$ through the product, picking up trace class terms from each commutator and these vanish. So

$$\begin{aligned}\zeta_p(a_0, ..., a_p) &= \frac{1}{2}p\lambda_p\tau_\Omega(\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]\mathcal{D}^{-1}[|\mathcal{D}|, a_p]) \\ &\quad + \frac{1}{2}p\lambda_p\tau_\Omega(\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]\mathcal{D}^{-1}[|\mathcal{D}|, a_p]) \\ &= p\lambda_p\tau_\Omega(\Gamma a_0[F, a_1]\cdots[F, a_{p-1}]\mathcal{D}^{-1}[|\mathcal{D}|, a_p]). \qquad (11)\end{aligned}$$

□

Lemma 24 *Let $p \geq 1$, $k = \max\{2, p-2\}$ and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^k (p, ∞)-summable spectral triple with $\mathcal{D}$ invertible. If ω is any Dixmier functional then the multilinear functional $\tilde{\zeta}_p$ defined by*

$$\tilde{\zeta}_p(a_0, ..., a_p) = p\lambda_p \tau_\omega(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]\mathcal{D}^{-1}[|\mathcal{D}|, a_p])$$

represents the Hochschild class of the Chern character.

Proof Let Ω be a Dixmier functional satisfying the additional requirements of Theorem 6. Then by Proposition 23, the Hochschild class of the Chern character is represented by

$$\zeta_p(a_0, ..., a_p) := p\lambda_p \tau_\Omega(\Gamma a_0[F, a_1] \cdots [F, a_{p-1}]\mathcal{D}^{-1}[|\mathcal{D}|, a_p]).$$

Let $\sum_i a_0^i \otimes a_1^i \otimes \cdots \otimes a_p^i$ be a Hochschild cycle, and write the operator

$$p\lambda_p \sum_i \Gamma a_0[F, a_1] \cdots [F, a_{p-1}]\mathcal{D}^{-1}[|\mathcal{D}|, a_p]$$

as a sum $T_1 - T_2 + iT_3 - iT_4$ where $T_i \geq 0$, $i = 1, ..., 4$ and $T_i \in \mathcal{L}^{(1,\infty)}$. Then

$$\begin{aligned}
\sum_i \zeta_p(a_0^i, ..., a_p^i) &= \tau_\Omega(T_1) - \tau_\Omega(T_2) + i\tau_\Omega(T_3) - i\tau_\Omega(T_4) \\
&= \Omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_t(T_1)dt - \cdots - i\Omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_t(T_4)dt \\
&= \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_t(T_1)dt - \cdots - i \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_t(T_4)dt \\
&= \omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_t(T_1)dt - \cdots - i\omega\text{-}\lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_t(T_4)dt \\
&= \sum_i \tilde{\zeta}_p(a_0^i, ..., a_p^i).
\end{aligned}$$

The equality between the Ω-limit and the true limit follows from Lemma 19 and Proposition 23, since

$$\begin{aligned}
\sum_i \zeta_p(a_0^i, ..., a_p^i) &= \Omega\text{-}\lim_{1/t\to\infty} \sum_i \psi_t(a_0^i, ..., a_p^i) \\
&= \lim_{t\to 0} \sum_i \psi_t(a_0^i, ..., a_p^i).
\end{aligned}$$

Since this is a true limit, any Dixmier functional will also return the same value. Note that we are *not* asserting that Proposition 23 is true for an arbitrary Dixmier functional ω, nor are we asserting that ζ_p and $\tilde{\zeta}_p$ are equal as multilinear functionals. What we are asserting is that it makes sense to apply either of τ_ω or τ_Ω to any finite sum of operators of the form

$$p\lambda_p \sum_i \Gamma a_0^i[F, a_1^i] \cdots [F, a_{p-1}^i]\mathcal{D}^{-1}[|\mathcal{D}|, a_p^i], \quad a_j^i \in \mathcal{A},$$

and moreover, that if $c = \sum_i a_0^i \otimes a_1^i \otimes \cdots \otimes a_p^i$ is a Hochschild cycle, then τ_ω and τ_Ω yield the same result. The end result of this is that $\tau_\omega - \tau_\Omega$ vanishes on all Hochschild cycles. Hence τ_ω is cohomologous to τ_Ω, and so $\tau_\omega \in [ICh_F]$. □

Corollary 25 *Let $p \geq 1$, $k = \max\{2, p-2\}$ and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^k (p, ∞)-summable spectral triple with $\mathcal{D}$ invertible. For any Hochschild cycle $\sum_i a_0^i \otimes a_1^i \otimes \cdots \otimes a_p^i$, the operator*

$$p\lambda_p \sum_i \Gamma a_0^i [F, a_1^i] \cdots [F, a_{p-1}^i] \mathcal{D}^{-1} [|\mathcal{D}|, a_p^i]$$

is measurable.

3.5 Identification of $[\phi_\omega]$ as the Hochschild Class

We now come to the cohomological part of the argument where we relate ζ_p to the functional ϕ_ω appearing in the statement of Theorem 10. As mentioned, this part of the proof closely follows [21, pp477-478].

For any choice of Dixmier functional ω, define cochains ζ_k, $1 \leq k \leq p$, by

$$\zeta_k(a_0, ..., a_p) = p\lambda_p \tau_\omega(\Gamma a_0 [F, a_1] \cdots \mathcal{D}^{-1} [|\mathcal{D}|, a_k] \cdots [F, a_p]).$$

These are well-defined as the argument of the Dixmier trace in each case is an element of $\mathcal{L}^{(1,\infty)}$ as is readily checked using Lemma 1. Note that here we are replacing the definition of ζ_p given in Definition 9, where we required a Dixmier functional satisfying the conditions of Theorem 6, by the above definition using a general Dixmier functional. The two definitions yield cohomologous Hochschild cocycles by Lemma 24.

Lemma 26 *Let $p \geq 1$ be integral and suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^2 (p, ∞)-summable spectral triple with $\mathcal{D}$ invertible. The cochains $\zeta_1, ..., \zeta_p$ are Hochschild cocycles which are mutually cohomologous.*

Proof We first show that the ζ_k are Hochschild cocycles. First we need to rewrite ζ_k. We wish to rewrite $\mathcal{D}^{-1}\delta(a_k)$ as $\delta(a_k)\mathcal{D}^{-1}$+something in $\mathcal{L}^{(p/2,\infty)}$. First,

$$[|\mathcal{D}|^{-1}, T] = -|\mathcal{D}|^{-1} [|\mathcal{D}|, T] |\mathcal{D}|^{-1}, \tag{12}$$

where $T = [|\mathcal{D}|, a], a, [\mathcal{D}, a]$ or $[F, a]$. So

$$\mathcal{D}^{-1}\delta(a_k) = F\delta(a_k)|\mathcal{D}|^{-1} - F|\mathcal{D}|^{-1}\delta^2(a_k)|\mathcal{D}|^{-1},$$

and the latter term is in $\mathcal{L}^{(p/2,\infty)}$. Using Lemma 1, we see that

$$F\delta(a_k)|\mathcal{D}|^{-1} = \delta(a_k)\mathcal{D}^{-1} + [F, \delta(a_k)]|\mathcal{D}|^{-1}$$

is equal to $\delta(a_k)\mathcal{D}^{-1}$ modulo $\mathcal{L}^{(p/2,\infty)}$. Since each $[F, a_j] \in \mathcal{L}^{(p,\infty)}$, if $T \in \mathcal{L}^{(p/2,\infty)}$ then we have

$$[F, a_1] \cdots [F, a_{k-1}] T [F, a_{k+1}] \cdots [F, a_p] \in \mathcal{L}^1. \tag{13}$$

Hence

$$\zeta_k(a_0, ..., a_p) = p\lambda_p \tau_\omega(\Gamma a_0 [F, a_1] \cdots [|\mathcal{D}|, a_k] \mathcal{D}^{-1} [F, a_{k+1}] \cdots [F, a_p]).$$

To move $\mathcal{D}^{-1}$ all the way to the right, we note that because $F^2 = 1$, $F[F,T] = -[F,T]F$ for all $T \in \mathcal{N}$, we have

$$\zeta_k(a_0, ..., a_p) = (-1)^{p-k} p\lambda_p \tau_\omega(\Gamma a_0[F, a_1] \cdots [|\mathcal{D}|, a_k]|\mathcal{D}|^{-1}[F, a_{k+1}] \cdots [F, a_p]F).$$

Now

$$|\mathcal{D}|^{-1}[F,a] = [F,a]|\mathcal{D}|^{-1} + [|\mathcal{D}|^{-1}, [F,a]] = [F,a]|\mathcal{D}|^{-1} - |\mathcal{D}|^{-1}[F, \delta(a)]|\mathcal{D}|^{-1}$$

and so the operators $|\mathcal{D}|^{-1}[F,a]$ and $[F,a]|\mathcal{D}|^{-1}$ differ by an element of $\mathcal{L}^{(p/3,\infty)}$ (where for $p < 3$ we mean the trace class).

Thus we can move $\mathcal{D}^{-1}$ to the right to obtain

$$\zeta_k(a_0, ..., a_p) = (-1)^{p-k} p\lambda_p \tau_\omega(\Gamma a_0[F, a_1] \cdots [|\mathcal{D}|, a_k] \cdots [F, a_p]\mathcal{D}^{-1}).$$

Applying Lemma 3 and using the trace property of τ_ω, we find that the Hochschild coboundary of ζ_k is given by

$$(b\zeta_k)(a_0, ..., a_{p+1}) = (-1)^{k-1} p\lambda_p \tau_\omega(\Gamma a_0[F, a_1] \cdots [|\mathcal{D}|, a_k] \cdots [F, a_p][\mathcal{D}^{-1}, a_{p+1}]).$$

Repeating the argument of Equations 12 and 13 shows that this is zero.

The second statement requires that we produce p Hochschild $(p-1)$-cocycles η_k, $k = 1, ..., p$, such that

$$b\eta_k(a_0, ..., a_p) = \zeta_k - \zeta_{k-1}.$$

The difference on the right hand side is given by

$$(-1)^{p-k} p\lambda_p \tau_\omega(\Gamma a_0[F, a_1] \cdots [F, a_{k-2}]([F, a_{k-1}]\delta(a_k) + \delta(a_{k-1})[F, a_k])[F, a_{k+1}] \cdots [F, a_p]\mathcal{D}^{-1}).$$

Set $R_{k,k-1} = [F, a_{k-1}]\delta(a_k) + \delta(a_{k-1})[F, a_k]$. Then we have

$$[F, \delta(a_{k-1}a_k)] = R_{k,k-1} + a_{k-1}[F, \delta(a_k)] + [F, \delta(a_{k-1})]a_k. \tag{14}$$

So the linear map $a \to [F, \delta(a)]$ is 'almost' a derivation. Defining

$$\eta_k(a_0, ..., a_{p-1}) := (-1)^p p\lambda_p \tau_\omega(\Gamma a_0[F, a_1] \cdots [F, \delta(a_k)] \cdots [F, a_{p-1}]\mathcal{D}^{-1}),$$

it is straightforward to show that $b\eta_k = \zeta_k - \zeta_{k-1}$ using Equation 14 and Lemma 3. □

Proposition 27 *Let $p \geq 1$ be integral and suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^2 (p,∞)-summable spectral triple with $\mathcal{D}$ invertible. The cochain $\phi_\omega - \frac{1}{p}(\zeta_1 + \cdots + \zeta_p)$ is a Hochschild coboundary.*

Proof We first show that

$$\phi_\omega(a_0, ..., a_p) = p\lambda_p \tau_\omega(\Gamma a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_p]|\mathcal{D}|^{-p})$$

is equal to the cochain $\tilde{\phi}_\omega$ given by

$$\tilde{\phi}_\omega(a_0, ..., a_p) = p\lambda_p \tau_\omega(\Gamma a_0[\mathcal{D}, a_1]|\mathcal{D}|^{-1}[\mathcal{D}, a_2]|\mathcal{D}|^{-1} \cdots [\mathcal{D}, a_p]|\mathcal{D}|^{-1}).$$

To do this, we use the argument of Equation (12) in the last Lemma to write

$$\phi_\omega(a_0, ..., a_p) = p\lambda_p\tau_\omega(\Gamma a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_{p-1}](|\mathcal{D}|^{-1}[\mathcal{D}, a_p]|\mathcal{D}|^{-p+1} + |\mathcal{D}|^{-1}\delta([\mathcal{D}, a_p])|\mathcal{D}|^{-p})).$$

The second term is trace class, and so

$$\phi_\omega(a_0, ..., a_p) = p\lambda_p\tau_\omega(\Gamma a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_{p-1}]|\mathcal{D}|^{-1}[\mathcal{D}, a_p]|\mathcal{D}|^{-p+1}).$$

Repeating this process of moving one factor of $|\mathcal{D}|^{-1}$ to the left at a time (which only requires the triple to be QC^1) we see that $\phi_\omega = \tilde{\phi}_\omega$.

Next write

$$[\mathcal{D}, a_j]|\mathcal{D}|^{-1} = [F, a_j] + \delta(a_j)\mathcal{D}^{-1} + [F, \delta(a_j)]|\mathcal{D}|^{-1},$$

and observe that by Lemma 1 $[F, \delta(a_j)]|\mathcal{D}|^{-1} \in \mathcal{L}^{(p/2,\infty)}$. This allows us to replace

$$[\mathcal{D}, a_j]|\mathcal{D}|^{-1} \text{ by } [F, a_j] + \delta(a_j)\mathcal{D}^{-1}$$

in the formula for $\tilde{\phi}_\omega = \phi_\omega$, using an observation similar to that in Equation (13) in the previous Lemma. Making this substitution will produce 2^p functionals, and we will deal with them in order of how many terms of the form $\delta(a_j)\mathcal{D}^{-1}$ they contain. First, we deal with the single functional containing no $\delta(a_j)\mathcal{D}^{-1}$ terms, which is given by $\lambda_p\tau_\omega(\Gamma a_0[F, a_1]\cdots[F, a_p])$. Now $a_0 = F[F, a_0] + Fa_0F$, and $F[F, a_0][F, a_1]\cdots[F, a_p]$ is trace class. So

$$\begin{aligned} \lambda_p\tau_\omega(\Gamma a_0[F, a_1]\cdots[F, a_p]) &= (-1)^p\lambda_p\tau_\omega(\Gamma F a_0[F, a_1]\cdots[F, a_p]F) \\ &= (-1)^{p-1}(-1)^p\tau_\omega(F\Gamma a_0[F, a_1]\cdots[F, a_p]F) \\ &= -\lambda_p\tau_\omega(\Gamma a_0[F, a_1]\cdots[F, a_p]). \end{aligned}$$

Hence this functional is zero. The functionals containing precisely one $\delta(a_j)\mathcal{D}^{-1}$ term add up to $p^{-1}(\zeta_1 + \cdots + \zeta_p)$.

So now we come to the functionals containing two or more terms $\delta(a_j)\mathcal{D}^{-1}$. So in the following suppose that $\Delta(a) = [F, a]$ or $\delta(a)$, and consider a functional with a total of l terms of the form $\delta(a)\mathcal{D}^{-1}$, $2 \le l \le p$. We begin by considering functionals with two consecutive $\delta(a)\mathcal{D}^{-1}$ terms. So, modulo an overall sign arising from moving all powers of $\mathcal{D}^{-1}$ to the right, we need to show that

$$\psi_j(a_0, ..., a_p) = \lambda_p\tau_\omega(\Gamma a_0\Delta(a_1)\cdots\Delta(a_{j-1})\delta(a_j)\delta(a_{j+1})\Delta(a_{j+2})\cdots\Delta(a_p)\mathcal{D}^{-l})$$

is a coboundary. Now

$$\delta^2(a_ja_{j+1}) = 2\delta(a_j)\delta(a_{j+1}) + a_j\delta^2(a_{j+1}) + \delta^2(a_j)a_{j+1},$$

so δ^2 is almost a derivation, and is well-defined on $\mathcal{A}$ since we suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^2. Setting

$$\chi_j(a_0, ..., a_{p-1}) = \frac{1}{2}(-1)^j\lambda_p\tau_\omega(\Gamma a_0\Delta(a_1)\cdots\delta^2(a_j)\cdots\Delta(a_{p-1})\mathcal{D}^{-l}),$$

we have, by Lemma 3,

$$(b\chi_j)(a_0, ..., a_p) = \psi_j(a_0, ..., a_p).$$

So now we are left with functionals in which we do not have two consecutive $\delta(a_j)\mathcal{D}^{-1}$ terms. We will show that such functionals are cohomologous to functionals with consecutive $\delta(a_j)\mathcal{D}^{-1}$ terms, and so are coboundaries by the previous argument. Again suppose that we have a total of l, $2 \le l \le p$, $\delta(a_j)\mathcal{D}^{-1}$ terms. Consider first

$$\xi_{j,j+2}(a_0, ..., a_p) = -\lambda_p\tau_\omega(\Gamma a_0\Delta(a_1)\cdots\Delta(a_{j-1})\delta(a_j)\Delta(a_{j+1})\delta(a_{j+2})\cdots\Delta(a_p)\mathcal{D}^{-l}),$$

where again $\Delta(a) = [F, a]$ or $\delta(a)$. Let

$$\xi_j(a_0, ..., a_p) = \lambda_p\tau_\omega(\Gamma a_0\Delta(a_1)\cdots\Delta(a_{j-1})\delta(a_j)\delta(a_{j+1})\Delta(a_{j+2})\cdots\Delta(a_p)\mathcal{D}^{-l}),$$

which is the same as $\xi_{j,j+2}$ except we have swapped the derivations on the $j+1$ and $j+2$ terms, and introduced an overall minus sign. The difference $(\xi_{j,j+2} - \xi_j)(a_0, ..., a_p)$ is given by

$$\lambda_p\tau_\omega(\Gamma a_0\Delta(a_1)\cdots\Delta(a_{j+1})\delta(a_j)(\Delta(a_{j+1})\delta(a_{j+2}) + \delta(a_{j+1})\Delta(a_{j+2}))\cdots\Delta(a_p)\mathcal{D}^{-l}),$$

and this is a coboundary. This is because Δ and δ are commuting derivations so that

$$\Delta(\delta(a_{j+1}a_{j+2})) = a_{j+1}\Delta(\delta(a_{j+2})) + \Delta(\delta(a_{j+1}))a_{j+2} + \Delta(a_{j+1})\delta(a_{j+2}) + \delta(a_{j+1})\Delta(a_{j+2}).$$

Consequently setting

$$\chi(a_0, ..., a_{p-1}) = (-1)^j\lambda_p\tau_\omega(\Gamma a_0\Delta(a_1)\cdots\Delta(\delta(a_{j+1}))\cdots\Delta(a_{p-1})\mathcal{D}^{-l}),$$

Lemma 3 along with the argument following Equation 14 shows that

$$(b\chi)(a_0, ..., a_p) = \xi_{j,j+2} - \xi_j.$$

Thus any of the functionals containing two or more $\delta(a_j)\mathcal{D}^{-1}$ terms are cohomologous to zero. This completes the proof. □

This proves Theorem 10 for the case where $\mathcal{D}$ has bounded inverse. That this is the case is due to the fact that we can now express the pairing of the Chern character with Hochschild homology in terms of any of the functionals ζ_k, which is the same as employing $p^{-1}(\zeta_1+\cdots+\zeta_p)$, and the last Proposition says this is the same as employing ϕ_ω. Theorem 10 is also true for the case where $\mathcal{D}$ does not have bounded inverse; the remaining details are in Appendix 1.

We can also complete the proof of Corollary 11. Let ω and Ω be any two Dixmier functionals, and $\sum_i a_0^i \otimes a_1^i \otimes \cdots \otimes a_p^i$ a Hochschild cycle. Then

$$\begin{aligned}\sum_i \phi_\omega(a_0^i, ..., a_p^i) &= \sum_i \tau_\omega(\Gamma a_0^i[\mathcal{D}, a_1^i]\cdots[\mathcal{D}, a_p^i]|\mathcal{D}|^{-p}) \\ &= \sum_i \tau_\omega(\Gamma a_0^i[F, a_1^i]\cdots[F, a_{p-1}^i]\mathcal{D}^{-1}[|\mathcal{D}|, a_p]) \\ &= \sum_i \tau_\Omega(\Gamma a_0^i[F, a_1^i]\cdots[F, a_{p-1}^i]\mathcal{D}^{-1}[|\mathcal{D}|, a_p]) \\ &= \sum_i \tau_\Omega(\Gamma a_0^i[\mathcal{D}, a_1^i]\cdots[\mathcal{D}, a_p^i]|\mathcal{D}|^{-p}) \\ &= \sum_i \phi_\Omega(a_0^i, ..., a_p^i).\end{aligned}$$

The first equality is the definition of ϕ_ω, the second follows from Propositions 26 and 27, the third follows from the measurability obtained in Lemma 24 and Corollary 25, and the final two equalities follow from Propositions 26 and 27 and the definition. Hence the operator

$$\sum_i \Gamma a_0^i[\mathcal{D},a_1^i]\cdots[\mathcal{D},a_p^i]|\mathcal{D}|^{-p}$$

is measurable, and Corollary 11 is proved.

4 Appendix

Our chief remaining task is to determine the effects on our representative ϕ_ω of the Hochschild class of the Chern character of replacing $(\mathcal{A},\mathcal{H},\mathcal{D})$ by $(\mathcal{A},\mathcal{H}^2,\mathcal{D}_m)$.

In [9, III.1.β,Proposition 15], Connes shows that if $\phi\in Z^n_\lambda(\mathcal{A})$ is a cyclic cocycle, then $S\phi := \phi\#\sigma$ is a Hochschild coboundary. Here $\#$ is the cup product, [9, pp 191-193], and σ defined by $\sigma(1,1,1)=1$ is the cyclic cocycle generating the cyclic cohomology of $\mathbf{C}$. It is important to realise that σ is a Hochschild coboundary.

To define the periodicity operator on arbitrary cyclic cochains, one must introduce antisymmetrisation and some normalisation constants. This is not an appropriate procedure for Hochschild cochains, and it is in fact simply the cup product by the cyclic cocycle (Hochschild *coboundary*) σ which is important for us. Consequently, for any Hochschild cycle ϕ, we shall denote by $S\phi$ the Hochschild cocycle $\phi\#\sigma$. Note that this is not the usual definition of the periodicity operator S, but our definition coincides with the usual definition on cyclic cocycles. The important point is that if ϕ is a Hochschild cocycle, then $\phi\#\sigma$ is a Hochschild coboundary, [9, p 194].

Our strategy is to show that the representative of the Hochschild class of the Chern character we obtain in Theorem 10 when we use the operator

$$\begin{pmatrix} \mathcal{D} & m \\ m & -\mathcal{D} \end{pmatrix}$$

differs from our stated result by Hochschild coboundaries.

Definition 10 *Let $p\geq 1$ be integral and suppose that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a QC^2 (p,∞)-summable spectral triple. Let $(\mathcal{A},\mathcal{H}^2,\mathcal{D}_m)$ be the 'double' of $(\mathcal{A},\mathcal{H},\mathcal{D})$, and define*

$$\begin{aligned}\phi_\omega(a_0,...,a_p) &= \lambda_p\tau_\omega(\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_p](m^2+\mathcal{D}^2)^{-p/2})\\ &= \lambda_p\tau_\omega(\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_p](1+\mathcal{D}^2)^{-p/2})\end{aligned}$$

$$\phi_{\omega m}(a_0,...,a_p) = \lambda_p\tau_\omega(\Gamma a_0[\mathcal{D}_m,a_1]\cdots[\mathcal{D}_m,a_p]|\mathcal{D}_m|^{-p})$$

$$\begin{aligned}\tilde{\phi}^k_\omega(a_0,...,a_k) &= \lambda_p\tau_\omega(\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_k](m^2+\mathcal{D}^2)^{-p/2})\\ &= \lambda_p\tau_\omega(\Gamma a_0[\mathcal{D},a_1]\cdots[\mathcal{D},a_k](1+\mathcal{D}^2)^{-p/2}).\end{aligned}$$

The equalities in the definition follow from

$$(m^2+\mathcal{D}^2)^{-1}-(n^2+\mathcal{D}^2)^{-1}=(n^2-m^2)(m^2+\mathcal{D}^2)^{-1}(n^2+\mathcal{D}^2)^{-1},$$

which, by the BKS inequality [2], implies that

$$(m^2+\mathcal{D}^2)^{-p/2}-(n^2+\mathcal{D}^2)^{-p/2}\in\mathcal{L}^1.$$

Hence $\tau_\omega(A(m^2+\mathcal{D}^2)^{-p/2})=\tau_\omega(A(n^2+\mathcal{D}^2)^{-p/2})$ for all bounded $A\in\mathcal{N}$ and $n,m>0$.

It is straightforward to show using Lemma 3 and/or Lemma 13 that all of the functionals in Definition 10 are Hochschild cocycles. The explicit formula for $\phi\#\sigma$ where ϕ is any of the above (n-)cocycles, is [9, p 193],

$$\begin{aligned}(\phi\#\sigma)(a_0,...,a_{n+2}) &= \phi(a_0a_1a_2da_3\cdots da^{n+2}T)\\ &+\phi(a_0da_1(a_2a_3)da_4\cdots da_{n+2}T)+\cdots\\ &+\phi(a_0da_1\cdots da_{i-1}(a_ia_{i+1})da_{i+2}\cdots da_{n+2}T)+\cdots\\ &+\phi(a_0da_1\cdots da_n(a_{n+1}a_{n+2})T),\end{aligned}$$

where da denotes $[\mathcal{D},a]$ and we have written T generically for $(1+\mathcal{D}^2)^{-p/2}$ or $|\mathcal{D}_m|^{-p}$ etc.

We can now state the main result of the Appendix.

Proposition 28 *Let $p\geq 1$ be integral and suppose that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a QC^2 (p,∞)-summable spectral triple. Let $(\mathcal{A},\mathcal{H}^2,\mathcal{D}_m)$ be the 'double' of $(\mathcal{A},\mathcal{H},\mathcal{D})$. Then for all $a_0,...,a_p\in\mathcal{A}$*

$$\begin{aligned}\phi_{\omega m}(a_0,...,a_p) &= \phi_\omega(a_0,...,a_p)\\ &+\sum_{i=1}^{[p/2]}(-1)^i m^{2i}\frac{1}{i!}(S^i\tilde\phi_\omega^{p-2i})(a_0,...,a_p).\end{aligned}$$

Proof We begin by defining a collection of operators $\hat S^i$, $i\geq 1$, which we will use to work with elements of $\Omega^*_{\mathcal{D}}(\mathcal{A})$, the graded algebra generated by $\mathcal{A}$ and $[\mathcal{D},\mathcal{A}]$, rather than with the cocycles. We define $\hat S:\mathcal{A}^{\otimes n+1}\to\Omega^{n-2}_{\mathcal{D}}(\mathcal{A})$, for any n by

$$\hat S(a_0)=\hat S(a_0,a_1)=0,$$

$$\hat S(a_0,...,a_n)=\sum_{i=1}^{n-1}a_0d(a_1)\cdots d(a_{i-1})a_ia_{i+1}d(a_{i+2})\cdots d(a_n).$$

Here and below we write $d(a)=[\mathcal{D},a]$. To define 'powers' of $\hat S$, we employ the inductive definition

$$\begin{aligned}\hat S^k(a_0,...,a_n) &= \hat S^{k-1}(\sum_{i=1}^{n-1}a_0d(a_1)\cdots d(a_{i-1})a_ia_{i+1}d(a_{i+2})\cdots d(a_n))\\ &= \hat S^{k-2}(\sum_{i=1}^{n-1}\hat S(a_0,...,a_{i-1})a_ia_{i+1}d(a_{i+2})\cdots d(a_n))\end{aligned}$$

$$
\begin{aligned}
&+ \quad \hat{S}^{k-2}\Big(\sum_{i=1}^{n-1} a_0 d(a_1)\cdots d(a_{i-1})\hat{S}(a_i a_{i+1}, a_{k+2}, ..., a_n)\Big) \\
&= \quad \sum_{j=0}^{k-1} \begin{pmatrix} k-1 \\ j \end{pmatrix} \sum_{i=1}^{n-1} \hat{S}^j(a_0, ..., a_{i-1})\hat{S}^{k-j-1}(a_i a_{i+1}, ..., a_n).
\end{aligned}
$$

It is tedious but not difficult to check that

$$
(S^i\phi)(a_0, ..., a_n) = \phi(\hat{S}^i(a_0, ..., a_n)), \tag{15}
$$

for any of the Hochschild cocycles defined in Definition 10 (regarded as functionals on $\Omega^*_{\mathcal{D}}(\mathcal{A})$). We claim that for any $n \geq 0$ the product $a_0[\mathcal{D}_m, a_1]\cdots[\mathcal{D}_m, a_n]$ is given by

$$
\begin{pmatrix}
\begin{array}{l} a_0 d(a_1)\cdots d(a_n) \\ +\sum_{i=1}^{[n/2]} \frac{1}{i!}(-1)^i m^{2i}\hat{S}^i(a_0, ..., a_n) \end{array} & \begin{array}{l} m a_0 d(a_1)\cdots d(a_{n-1})a_n \\ +\sum_{i=1}^{[(n-1)/2]} m^{2i+1}(-1)^i \frac{1}{i!}\hat{S}^i(a_0, ..., a_{n-1})a_n \end{array} \\
0 & 0
\end{pmatrix}. \tag{16}
$$

Indeed, this is easy to verify for $n = 1, 2$. So if we suppose it to be true for all $k < n$ then using

$$
[\mathcal{D}_m, a_n] = \begin{pmatrix} d(a_n) & m a_n \\ -m a_n & 0 \end{pmatrix},
$$

we find that $a_0[\mathcal{D}_m, a_1]\cdots[\mathcal{D}_m, a_n]$ is given by (writing $c_i = \frac{1}{i!}(-1)^i m^{2i}$)

$$
\begin{aligned}
&\begin{pmatrix}
\begin{array}{l} a_0 d(a_1)\cdots d(a_{n-1}) \\ +\sum_{i=1}^{[(n-1)/2]} c_i\hat{S}^i(a_0, ..., a_{n-1}) \end{array} & \begin{array}{l} m a_0 d(a_1)\cdots d(a_{n-2})a_{n-1} \\ +m\sum_{i=1}^{[(n-2)/2]} c_i\hat{S}^i(a_0, ..., a_{n-2})a_{n-1} \end{array} \\
0 & 0
\end{pmatrix} [\mathcal{D}_m, a_n] \\
= &\begin{pmatrix}
\begin{array}{l} a_0 d(a_1)\cdots d(a_n) \\ +\sum_{i=1}^{[(n-1)/2]} c_i\hat{S}^i(a_0, ..., a_{n-1})d(a_n) \\ -m^2\sum_{i=0}^{[(n-2)/2]} c_i\hat{S}^i(a_0, ..., a_{n-2})a_{n-1}a_n \end{array} & \begin{array}{l} m a_0 d(a_1)\cdots d(a_{n-1})a_n \\ +\sum_{i=1}^{[(n-1)/2]} c_i\hat{S}^i(a_0, ..., a_{n-1})a_n \end{array} \\
0 & 0
\end{pmatrix}.
\end{aligned}
$$

In order to simplify this expression we note that

$$
\hat{S}(a_0, ..., a_{n-1})d(a_n) = \hat{S}(a_0, ..., a_n) - a_0 d(a_1)\cdots d(a_{n-2})a_{n-1}a_n, \tag{17}
$$

and for $i > 1$

$$
\hat{S}^i(a_0, ..., a_{n-1})d(a_n) = \hat{S}^i(a_0, ..., a_n) - i\hat{S}^{i-1}(a_0, ..., a_{n-2})a_{n-1}a_n.
$$

To see this, one first verifies the statement for $i = 2$ (which is straightforward using Equation 17 and a calculation similar to that below), and then we use induction. The computation is as follows.

$$\begin{aligned}
\hat{S}^{k+1}(a_0,...,a_n) &= \sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\sum_{i=1}^{n-1}\hat{S}^j(a_0,...,a_{i-1})\hat{S}^{k-j}(a_ia_{i+1},...,a_n) \\
&= \sum_{j=0}^{k-1}\begin{pmatrix} k \\ j \end{pmatrix}\sum_{i=1}^{n-1}\hat{S}^j(a_0,...,a_{i-1})\hat{S}^{k-j}(a_ia_{i+1},...,a_{n-1})d(a_n) \\
&+ \sum_{j=0}^{k-1}\begin{pmatrix} k \\ j \end{pmatrix}(k-j)\sum_{i=1}^{n-1}\hat{S}^j(a_0,...,a_{i-1})\hat{S}^{k-j-1}(a_ia_{i+1},...,a_{n-2})a_{n-1}a_n \\
&+ \sum_{i=1}^{n-2}\hat{S}^k(a_0,...,a_{i-1})a_ia_{i+1}d(a_{i+2})\cdots d(a_n) + \hat{S}^k(a_0,...,a_{n-2})a_{n-1}a_n \quad (18) \\
&= \sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\sum_{i=1}^{n-2}\hat{S}^j(a_0,...,a_{i-1})\hat{S}^{k-j}(a_ia_{i+1},...,a_{n-1})d(a_n) \\
&+ k\sum_{j=0}^{k-1}\begin{pmatrix} k-1 \\ j \end{pmatrix}\sum_{i=1}^{n-1}\hat{S}^j(a_0,...,a_{i-1})\hat{S}^{k-j-1}(a_ia_{i+1},...,a_{n-2})a_{n-1}a_n \\
&+ \hat{S}^k(a_0,...,a_{n-2})a_{n-1}a_n \quad (19) \\
&= \hat{S}^{k+1}(a_0,...,a_{n-1})d(a_n) + (k+1)\hat{S}^k(a_0,...,a_{n-2})a_{n-1}a_n. \quad (20)
\end{aligned}$$

The first line here follows from the definition. In 18 we apply the inductive hypothesis to the second term in each product, for $j \neq k$, and for $j = k$ we split the sum into the first $n-2$ terms, and the $(n-1)$-st. For $j \neq k$ we notice that the $(n-1)$-st term of the sum is zero, by the definition of $\hat{S}$, so in 19 we collect all these sums of $n-2$ terms. We also use the combinatorial identity

$$(k-j)\begin{pmatrix} k \\ j \end{pmatrix} = k\begin{pmatrix} k-1 \\ j \end{pmatrix}.$$

Finally, applying the definition of $\hat{S}$ we obtain the result 20.

Thus we have

$$\sum_{i=1}^{[(n-1)/2]} c_i\hat{S}^i(a_0,...,a_{n-1})d(a_n) - m^2\sum_{i=0}^{[(n-2)/2]} c_i\hat{S}^i(a_0,...,a_{n-2})a_{n-1}a_n$$

$$= \begin{cases} \sum_{i=1}^{[n/2]}(c_i\hat{S}^i(a_0,...,a_{n-1})d(a_n) + ic_i\hat{S}^{i-1}(a_0,...,a_{n-2})a_{n-1}a_n) & n \text{ odd} \\ \begin{array}{l}\sum_{i=1}^{[(n-1)/2]}(c_i\hat{S}^i(a_0,...,a_{n-1})d(a_n) + ic_i\hat{S}^{i-1}(a_0,...,a_{n-2})a_{n-1}a_n \\ +m^n(-1)^{n/2}\frac{1}{([n/2]-1)!}\hat{S}^{[n/2]-1}(a_0,...,a_{n-2})a_{n-1}a_n\end{array} & n \text{ even} \end{cases}$$

$$= \begin{cases} \sum_{i=1}^{[n/2]} c_i\hat{S}^i(a_0,...,a_n) & n \text{ odd} \\ \begin{array}{l}\sum_{i=1}^{[(n-1)/2]} c_i\hat{S}^i(a_0,...,a_n) \\ +m^n(-1)^{n/2}\frac{1}{([n/2]-1)!}\hat{S}^{[n/2]-1}(a_0,...,a_{n-2})a_{n-1}a_n\end{array} & n \text{ even} \end{cases}.$$

In the odd case we have used $[(n-1)/2] = [n/2]$, and we are left with the even case. For this we note that

$$k\hat{S}^{k-1}(a_0, ..., a_{2k-2})a_{2k-1}a_{2k} = \hat{S}^k(a_0, ..., a_{2k}),$$

since $\hat{S}^k(a_0, ..., a_{2k-1}) = 0$, so

$$m^n(-1)^{n/2}\frac{1}{([n/2]-1)!}\hat{S}^{[n/2]-1}(a_0, ..., a_{n-2})a_{n-1}a_n = m^{2[n/2]}(-1)^{[n/2]}\frac{1}{[n/2]!}\hat{S}^{[n/2]}(a_0, ..., a_n).$$

This completes the inductive step and proves the claim 16. Putting 16 together with 15 now completes the proof. □

Thus the Hochschild class of the Chern character can be represented by the cocycle

$$\phi_\omega(a_0, ..., a_p) = \lambda_p\tau_\omega(\Gamma a_0[\mathcal{D}, a_1]\cdots[\mathcal{D}, a_p](1+\mathcal{D}^2)^{-p/2}),$$

the other contributions appearing in Proposition 28 all being coboundaries with no effect on the Hochschild class.

References

[1] M-T. Benameur, T. Fack, On von Neumann Spectral Triples, arxiv posting: math.KT/0012233.

[2] M.S. Birman, L.S. Koplienko and M.Z. Solomyak, Estimates for the spectrum of the difference between fractional powers of two selfadjoint operators, Soviet Mathematics, **19**(3) (1975), 1-6.

[3] A. L. Carey, J. Phillips, Unbounded Fredholm Modules and Spectral Flow, Canadian J. Math., **50**(4)(1998), 673–718.

[4] A. L. Carey, J. Phillips, Theta Summable Fredholm Modules, Eta Invariants and the JLO Cocycle, to appear in K-Theory.

[5] A. L. Carey, J. Phillips, F. A. Sukochev, On Unbounded p-Summable Fredholm Modules, Advances in Math., **151** (2000) 140-163.

[6] A. L. Carey, J. Phillips, F. A. Sukochev, Spectral Flow and Dixmier Traces, Advances in Math., **173** (2003) 68-113.

[7] V.I. Chilin, F.A. Sukochev, Weak Convergence in Operator Spaces J. Operator Theory **31** (1991), pp 35-65

[8] F. Cipriani, D. Guido, S. Scarlatti, A Remark on Trace Properties of K-Cycles J. Oper. Theory **35** (1996) pp 179-189

[9] A. Connes, “Noncommutative Geometry” Academic Press, 1994

[10] A. Connes, Geometry From the Spectral Point of View Lett. Math. Phys. **34** (1995) pp 203-238

[11] A. Connes, H. Moscovici, The Local Index Formula in Noncommutative Geometry GAFA **5** (1995) pp 174-243

[12] A. Connes, Noncommutative Differential Geometry Publ. Math. IHES **39** (1985) pp 257-360

[13] J. Dixmier, Existence de Traces Non-normales C.R. Acad. Sci. Paris **A-B 262** (1966) A1107-A1108.

[14] J. Dixmier, "Von Neumann Algebras", North-Holland Mathematical Library, vol.27 (1981) (translation of : Les algebres d'opératurs dans l'espace hilbertien (algébres de Von Neumann), 1969).

[15] P.G. Dodds, T.K. Dodds, B. de Pagter, Fully Symmetric Operator Spaces, Integral Equations Operator Theory 15 (1992), no. 6, pp 942-972.

[16] P.G. Dodds, T.K. Dodds, B. de Pagter, Noncommutative Köthe Duality Trans. Amer. Math. Soc. **339** (2) (1993) pp 717-750

[17] H.B. Dwight, *Tables of Integrals and Other Mathematical Data, 4th Ed* Macmillan, 1961

[18] T. Fack, H. Kosaki, Generalised s-Numbers of τ-Measurable Operators, Pacific J. Math. **123** (1986), 269–300.

[19] T. Fack, private communication

[20] M. Farber, S. Weinberger, On the zero in the Spectrum Conjecture Annals of Math. **154** 2001, 139-154; M. Gromov, Large Riemannian Manifolds in 'Curvature and Toplogy of Riemannian Manifolds' ed K. Shiohama, T. Sakai, T. Sunada, Lecture Notes in Math., **1201** 1986, 108-121 and J. Lott, The Zero-in-the-Spectrum Question L'Enseignment Mathematique **42** (1996) 341-376.

[21] J.M. Gracia-Bondia, J.C. Varilly, H. Figueroa, "Elements of Noncommutative Geometry" Birkhauser, Boston, 2001

[22] G.G. Kasparov, The Operator K-Functor and Extensions of C^*-Algebras Izv. Akad. Nauk. SSSR Ser. Mat. **44** (1980) pp 571-636

[23] J-L Loday, "Cyclic Homology" 2nd Ed, Springer-Verlag, 1998

[24] V. Mathai, Spectral Flow, Eta Invariants and von Neumann Algebras Journal of Functional Analysis, **109** (1992), 442–456.

[25] R. Prinzis, Traces Residuelles et Asymptotique du Spectre d'Operateurs Pseudo-Differentiels Thèse, Université de Lyon, unpublished.

[26] F.A. Sukochev, Operator Estimates for Fredholm Modules Canad. J. Math. **52** (4) (2000), pp 849-896