The bulk-edge correspondence via Kasparov theory

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Abstract

We outline the Kasparov theory approach to proving the bulk-edge correspondence for topological insulators. As well as reviewing the KK background required, we indicate how the Kasparov approach relates to boundary maps in K-theory and the "pairing K-theory with K-homology" point of view.

Key words Kasparov theory, extensions of C^*-algebras, bulk-edge correspondence.

Key Objectives We aim to give sufficient detail about Kasparov theory to describe the KK-proof of the bulk-edge correspondence for topological insulators.

1 Introduction

For the involved history of the theoretical development of both the quantum Hall effect and topological phases of matter, we refer to the broader physics literature and the piece by Bernevig in this encyclopedia. The initial developments of the index theory approach to the quantum Hall effect is outlined in [51], and many of the topics we discuss can be reviewed in [58] in the complex case.

The use of K-theory to label topological phases of matter is by now well-established, see for instance [1, 4, 30, 48, 66]. Bellissard gave the Fredholm module picture of the quantum Hall effect, as summarised in [5], and the relation to cyclic cohomology is described by Connes in [23, IV.6]. An approach using unbounded Kasparov theory is more recent, [10, 11, 12, 14].

To describe the KK-approach to the bulk-edge correspondence, we will initially simplify Bellissard's picture by omitting disorder and focussing just on the quantum Hall effect. Later, we will incorporate disorder, along with the real and Real structures required to accommodate the various linear and anti-linear symmetries characterising different topological phases. Throughout we will only discuss the "tight-binding" models, mentioning continuum models briefly at the end.

On a 2-dimensional lattice $\mathbb{Z}^2$ without boundary, we take the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^2)$, and define the magnetic translations as unitaries U and V, and the Hamiltonian $H = U + U^* + V + V^*$. We choose the Landau gauge so that for $\xi \in \ell^2(\mathbb{Z}^2)$ we have

$$(U\xi)(m,n) = \xi(m-1,n), \qquad (V\xi)(m,n) = e^{-2\pi i\phi m}\xi(m,n-1), \quad m,n \in \mathbb{Z},$$

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where the real number ϕ has the interpretation as the magnetic flux through a unit cell. We note that $UV = e^{2\pi i\phi}VU$, so $C^*(U,V) \cong A_\phi$, the rotation algebra. Our choice of gauge also means that $C^*(U,V) \cong C^*(U) \rtimes_\alpha \mathbb{Z}$, where V is implementing the crossed-product structure via the automorphism $\alpha(U^m) = V^*U^mV$. See Kellendonk's discussion of observable algebras in this Encyclopedia.

The topological properties of the bulk conductance in the quantum Hall effect come from two pieces of information: the Fermi projection of the Hamiltonian and a spectral triple encoding the geometry of the (noncommutative) Brillouin zone. Provided the Fermi energy is in a gap of the spectrum of the Hamiltonian, the Fermi projection P_F defines a class in the complex K-theory group $[P_F] \in K_0(A_\phi)$.

The "spectral triple" ingredient was initially described in terms of a Fredholm module. Roughly the analogy is that a Fredholm module provides conformal geometry and its homotopy class in K-homology provides topological information. A spectral triple is a refinement of a Fredholm module incorporating metric and differential geometry: more information will be provided in Section 3.

Proposition 1.1 ([10])**.** *Let $\mathcal{A}_\phi$ be the dense $*$-subalgebra of A_ϕ generated by finite polynomials of U and V, and let X_1, X_2 be the position operators on $\ell^2(\mathbb{Z}^2)$ given by $(X_j\xi)(n_1,n_2) = n_j\xi(n_1,n_2)$. Then*

$$\left(\mathcal{A}_\phi,\ \ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^2 = \begin{pmatrix}\ell^2(\mathbb{Z}^2)\\ \ell^2(\mathbb{Z}^2)\end{pmatrix},\ \begin{pmatrix}0 & X_1 - iX_2\\ X_1 + iX_2 & 0\end{pmatrix},\ \gamma = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$$

is a complex spectral triple which defines a class in the K-homology group $K^0(A_\phi)$.

The topological invariance of the transverse conductivity in the quantum Hall effect can be expressed via the Fredholm index pairing [34] of K-theory and K-homology,

$$\begin{aligned} K_0(A_\phi)\times K^0(A_\phi) &\to K_0(\mathbb{C}) \cong \mathbb{Z}\\ ([P_F],[X]) &\mapsto \mathrm{Index}(P_F(X_1+iX_2)P_F), \end{aligned}$$

where $[X]$ is the K-homology class of the spectral triple from Proposition 1.1. Bellissard showed that the Kubo formula for the Hall conductivity gives an expression for this index pairing [5]. More precisely, the Hall conductivity is given by

$$\begin{aligned} \sigma_H &= \frac{e^2}{h}\,\mathrm{Index}(P_F(X_1+iX_2)P_F)\\ &= \frac{2\pi i e^2}{h}\mathcal{T}(P_F[X_1,P_F][X_2,P_F] - P_F[X_2,P_F][X_1,P_F]), \end{aligned} \tag{1.1}$$

where $\mathcal{T}$ is the trace per unit area. The second expression is the Kubo formula for the transverse conductivity, and in this context is also called the Chern number of the projection P_F. This tracial formula gives a computationally tractable expression for the index pairing, and arises (mathematically) from translating the index pairing into cyclic cohomology via the Chern character, [23, 22, 33].

While formulae in cyclic theory can be found for integer invariants, they can *not* naively be found for torsion invariants, e.g. the $\mathbb{Z}_2$-invariant associated to the time-reversal invariant systems [38, 32] (but see [43] for an interesting approach). Therefore cyclic formulae are not always available for general topological insulator systems and instead we must deal with the K-theoretic index pairing directly. Importantly our methods prove the bulk-edge correspondence even for torsion invariants.

To incorporate the boundary, Kellendonk, Schulz-Baldes and colleagues [44, 45, 46, 47] introduced the short exact sequence relating the edge algebra $C^*(U)$ and the bulk algebra A_ϕ via an intermediary "half-space" algebra T,

$$0 \to C^*(U) \otimes \mathcal{K}(\ell^2(\mathbb{N})) \to \mathcal{T} \to A_\phi \to 0. \tag{1.2}$$

In the two-dimensional case relevant to the quantum Hall effect, the half-space is a (discrete) half-plane $\mathbb{Z} \times \mathbb{N}$, with algebra $\mathcal{T}$ generated by a unitary shift U on $\mathbb{Z}$ and isometric shift V on $\mathbb{N}$. The edge algebra $C^*(U)$ is generated by a single unitary shift U, and the compacts $\mathcal{K}(\ell^2(\mathbb{N}))$ represent observables decaying away from the edge. The quotient map to the bulk algebra A_ϕ "pushes the boundary to infinity", or more algebraically quotients by the ideal generated by $V^*V - VV^*$ so that the bulk theory is described by two unitary shifts.

Associated to the exact sequence (1.2) are the six term exact sequences in K-theory and K-homology [7], given respectively by

$$\begin{tikzcd} K_0(C^*(U)) \arrow[r] & K_0(\mathcal{T}) \arrow[r] & K_0(A_\phi) \arrow[d, "\partial"] \\ K_1(A_\phi) \arrow[u, "\partial"] & K_1(\mathcal{T}) \arrow[l] & K_1(C^*(U)) \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} K^0(C^*(U)) \arrow[d, "\partial"'] & K^0(\mathcal{T}) \arrow[l] & K^0(A_\phi) \arrow[l] \\ K^1(A_\phi) \arrow[r] & K^1(\mathcal{T}) \arrow[r] & K^1(C^*(U)) \arrow[u, "\partial"'] \end{tikzcd} . \tag{1.3}$$

The class of the Fermi projector $[P_F]$ lies in $K_0(A_\phi)$. As noted above, the conductance is computed (up to an overall constant) by pairing with the spectral triple $(\mathcal{A}_\phi, \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2, X)$ with class $[X] \in K^0(A)$.

The analogue of the Fermi projector for the edge system is represented by some unitary $\tilde{U}$ with class $[\tilde{U}] \in K_1(C^*(U))$ (so $[\tilde{U}] = [U^m]$ for some $m \in \mathbb{Z}$ [34]). The edge current is computed as the pairing between $[\tilde{U}]$ and a spectral triple $(C^*(U), \ell^2(\mathbb{Z}), X_2)$ representing the Brillouin zone of the edge with class $[X_2] \in K^1(C^*(U))$. Such odd pairings are expressed using the relative index of a pair of projections [2]. Here the projections are $P := \chi_{[0,\infty)}(X_2)$ and UPU^*.

The bulk-edge correspondence relating the bulk conductance to the edge current follows from some general facts in KK-theory and a specific computation in KK-theory relating the geometry of the Brillouin zones for the bulk and edge systems.

Briefly, the general facts are that the index pairings and the boundary maps are compatible in specific ways [34]. In turn, this compatibility follows from the fact that the boundary maps in both K-theory and K-homology are given by Kasparov products with the KK-class $[ext]$ of the exact sequence (1.2), [40]. These general facts will be explained below.

The specific computation which enables us to use this information is that we can relate the bulk and edge geometries as

$$-[X] = \partial[X_2] = [ext] \otimes_{C^*(U)} [X_2].$$

Taken with the general facts above we learn that [58] in the language of pairings we have

$$\text{Bulk conductance} = \frac{e^2}{h}\langle [P_F] \mid [X] \rangle = \frac{e^2}{h}\langle [P_F] \mid \partial[X_2] \rangle = \frac{e^2}{h}\langle \partial[P_F] \mid [X_2] \rangle = \text{Edge conductance}$$

and the class $\partial[P_F] \in K_1(C^*(U))$ is identified with the class of the boundary translation operator. In the more sophisticated language of Kasparov theory and the Kasparov product we have

$$\begin{aligned}\text{Bulk conductance} = \frac{e^2}{h}[P_F] \otimes_A [X] &= \frac{e^2}{h}[P_F] \otimes_A \big([ext] \otimes_{C^*(U)} [X_2]\big) \\ &= \frac{e^2}{h}\big([P_F] \otimes_A [ext]\big) \otimes_{C^*(U)} [X_2] = \text{Edge conductance}\end{aligned}$$

and the bulk-edge correspondence is simply the associativity of the Kasparov product.

The aim of this note is to describe enough of the KK-framework to explain how it is used to prove the bulk-edge correspondence. In order to describe the general facts and the specific computation alluded to above, we spend the next few sections recalling some general facts with relevant examples.

2 Real algebras, gradings and disorder for general topological insulators

To address general topological insulators we need to accommodate real, Real and graded C^*-algebras, more general dimensions, and disorder.

2.1 Disorder and twisted crossed products

As the disorder is generally modelled by the "hull", see Kellendonk's article, we can consider the translational action α of $\mathbb{Z}^d$ (in d dimensions) on a probability space Ω. This gives us a crossed product $C(\Omega) \rtimes_\alpha \mathbb{Z}^d$. More generally, if there is a magnetic field present, we can consider a twisted crossed product $C(\Omega) \rtimes_{\alpha,\theta} \mathbb{Z}^d$, [54], which we now briefly describe.

Let B be a separable and unital C^*-algebra with an action α of $\mathbb{Z}^d$ and twisting cocycle $\theta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathcal{U}(B)$, with $\mathcal{U}(B)$ the unitaries of B. The twisted crossed product $A := B \rtimes_{\alpha,\theta} \mathbb{Z}^d$ is the universal C^*-completion of the algebraic crossed product $\mathcal{A} := C_c(\mathbb{Z}^d, B)$ given by finite sums $\sum_{n\in\mathbb{Z}^d} U^n b_n$ where $b_n \in B$, $n \in \mathbb{Z}^d$ is a multi-index and $U^n = U_1^{n_1} \cdots U_d^{n_d}$ is a product of powers of d abstract unitary elements U_i subject to the multiplication extending that of B by

$$U_i b = \alpha_i(b)U_i, \quad U_iU_j = \theta_{ij}U_jU_i, \quad U_i^* = U_i^{-1}, \quad i, j = 1, \ldots, d.$$

The map α_i is the automorphism corresponding to the action of $e_i \in \mathbb{Z}^d$ for e_i the standard generators of $\mathbb{Z}^d$. The elements θ_{ij} belong to B and can be obtained from the cocycle θ.

By modelling the neighbourhood of an edge by $\mathbb{Z}^{d-1} \times \mathbb{N}$, we can build an edge algebra using twisted crossed products by $\mathbb{Z}^{d-1}$. We will describe the process and the relation to the bulk algebra in Section 3.4.

2.2 $\mathbb{Z}_2$-graded, complex, real and Real algebras

In the following we need to consider real and complex C^*-algebras as well as Real and $\mathbb{Z}_2$-graded C^*-algebras, [8, 40, 64]. We recall the required definitions here.

Definition 2.1. A real C^*-algebra A is a real Banach $*$-algebra such that for all $a \in A$ $\|a^*a\| = \|a\|^2$ and $1 + a^*a$ is invertible.

A Real C^*-algebra A is a complex C^*-algebra together with a map $\sigma : A \to A$ such that for $a, b \in A$ and $\lambda \in \mathbb{C}$

$$\sigma(ab) = \sigma(a)\sigma(b), \quad \sigma(a^*) = \sigma(a)^*, \quad \sigma(\lambda a) = \overline{\lambda}\sigma(a). \tag{2.1}$$

One readily checks that the fixed point algebra $A^\sigma = \{a \in A : \sigma(a) = a\}$ is a real C^*-algebra.

Example 2.2. The Clifford algebra $Cl_{r,s}$ generated by the orthogonal/unitary elements γ^j, $1 \le j \le r+s$ subject to

$$\gamma^{j*} = \begin{cases} \gamma^j & 1 \le j \le r \\ -\gamma^j & r+1 \le j \le r+s \end{cases}, \qquad \gamma^j\gamma^k + \gamma^k\gamma^j = 0 \text{ for } j \neq k$$

is a real algebra. If we complexify then we obtain the complex Clifford algebras $\mathbb{C}\ell_{r+s} = Cl_{r,s} \otimes \mathbb{C}$, and the result depends only on $r+s$. If we wish to remember the values of r, s, we should consider the Real algebra $\mathbb{C}\ell_{r,s} = Cl_{r,s} \otimes \mathbb{C}$ together with $\sigma(\gamma^j) = \gamma^j$.

Definition 2.3. A $\mathbb{Z}_2$-graded C^*-algebra is a C^*-algebra A together with a homomorphism $\gamma : A \to A$ satisfying $\gamma^2 = 1$. We denote by A^0 the fixed point algebra and $A^1 = \{a \in A : \gamma(a) = -a\}$. Elements of A^0 are called even, while elements of A^1 are called odd. If $A = A^0$ we say that A is trivially graded.

Let A, B be $\mathbb{Z}_2$-graded C^*-algebras with gradings γ_A, γ_B. A $*$-homomorphism $\varphi : A \to B$ is called a $\mathbb{Z}_2$-graded $*$-homomorphism if $\varphi(\gamma_A(a)) = \gamma_B(\varphi(a))$.

Example 2.4. Most $\mathbb{Z}_2$-graded algebras encountered in this text are of the form $A \otimes \mathbb{C}\ell_{r,s}$ or $A \otimes Cl_{r,s}$ or $A \otimes \mathbb{C}\ell_n$, where A is trivially graded. The Clifford algebras are $\mathbb{Z}_2$-graded by

$$Cl_{r,s} = Cl^0_{r,s} \oplus Cl^1_{r,s}$$

where $Cl^0_{r,s}$ is the span of even products of the generators γ^j, and $Cl^1_{r,s}$ is the span of odd products.

We record an important fact about Clifford algebras.

Proposition 2.5. *On the graded vector space $\bigwedge^* \mathbb{R}^d$ of exterior powers (we denote the grading by $\gamma_{\bigwedge^* \mathbb{R}^d}$) there is a representation of $Cl_{d,0}$ and a representation of $Cl_{0,d}$. The generators γ^j of $Cl_{d,0}$ and the generators ρ^j of $Cl_{0,d}$ act by*

$$\gamma^j(w) = e_j \wedge w + \iota(e_j)w, \qquad \rho^j(w) = e_j \wedge w - \iota(e_j)w,$$

for $\{e_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$, $w \in \bigwedge^* \mathbb{R}^d$ and with $w = w_1 \wedge \cdots \wedge w_k$

$$\iota(v)w := \sum_{j=1}^{k} (-1)^{j-1} \langle v, w_j \rangle w_1 \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_k$$

is the contraction of w along v. The actions of $Cl_{d,0}$ and $Cl_{0,d}$ graded-commute, or equivalently $\gamma^j\rho^k + \rho^k\gamma^j = 0$ for all $j, k = 1, \ldots, d$.

In fact the $\mathbb{Z}_2$-graded tensor product (see [7]) $Cl_{d,0}\hat{\otimes}Cl_{0,d}$ is isomorphic to the $\mathbb{R}$-linear maps on $\bigwedge^* \mathbb{R}^d$.

There are many different pictures of Kasparov theory, depending on the choice of cycles used to develop the theory. We will use Kasparov modules and extensions (short exact sequences) of C^*-algebras. The basic tools for describing Kasparov modules are Hilbert modules and operators on them.

2.3 Hilbert modules for C^*-algebras

We need to make use of (right) Hilbert modules over C^*-algebras, and left actions of C^*-algebras on such modules as well. Standard references for Hilbert modules are [50, 60].

Definition 2.6. Let B be a real or complex C^*-algebra. A right Hilbert B-module is a real or complex Banach space X together with a right action $X \times B \to X$ of the C^*-algebra B and a B-valued inner product $(\cdot \mid \cdot)_B$. This inner product satisfies

$$(x + y \mid zb)_B = (x \mid z)_B b + (y \mid z)_B b, \quad (x \mid y)_B = (y \mid x)_B^*, \quad (x \mid x)_B \geq 0$$

for $x, y, z \in X$ and $b \in B$. The norm making X a Banach space is $\|x\|^2 := (x \mid x)_B$.

If B is a Real C^*-algebra with real structure σ_B, then X is a Real Hilbert module if there is an anti-linear involution $\sigma_X : X \to X$ such that $\sigma_X(xb) = \sigma_X(x)\sigma_B(b)$ for $b \in B$, $x \in X$.

Similarly, if B is $\mathbb{Z}_2$-graded (even trivially) by γ_B then X is $\mathbb{Z}_2$-graded if there is $\gamma_X : X \to X$ such that $\gamma_X(xb) = \gamma_X(x)\gamma_B(b)$.

Example 2.7. Every C^*-algebra B acts on itself by right multiplication, and we obtain a Hilbert module by defining $(a \mid b)_B = a^*b$ for $a, b \in B$.

Example 2.8. If $p = p^* = p^2 \in M_N(B)$ is a projection then pB^N is a Hilbert module.

Example 2.9. Given a C^*-algebra B, let $\ell^2(\mathbb{Z}^d, B)$ be the sequences $(b_n)_{n\in\mathbb{Z}^d}$, $b_n \in B$, such that $\sum_{n\in\mathbb{Z}^d} b_n^* b_n$ converges in B. Together with the inner product $((b_n) \mid (c_m))_B = \sum_{n\in\mathbb{Z}^d} b_n^* c_n$ we obtain a right Hilbert B-module, [40].

Definition 2.10. [50, 60] If X_B is a $\mathbb{Z}_2$-graded Hilbert module, a map $R : X_B \to X_B$ is adjointable if there exists $S : X \to X$ such that for all $x, y \in X_B$ we have $(Rx|y)_B = (x|Sy)_B$. If an adjoint exists it is unique and we denote it by $S = R^*$. Adjointable operators turn out to be B-linear, bounded and together form a C^*-algebra for the operator norm and composition as usual. We denote this algebra by $\mathrm{End}_B^*(X)$ (some authors use $\mathcal{L}(X)$).

Example 2.11. If B is non-trivially $\mathbb{Z}_2$-graded, the grading operator γ_X of X_B given by $\gamma_X(x) = x$ if $x \in X^0$ and $\gamma_X(x) = -x$ if $x \in X^1$ is NOT an adjointable endomorphism. Nevertheless γ_X defines a $\mathbb{Z}_2$-grading on $\mathrm{End}_B^*(X)$ by saying that an endomorphism $T : X \to X$ such that $T\gamma_X = -\gamma_X T$ is odd. Likewise a Real structure σ_X defines a Real structure on $\mathrm{End}_B^*(X)$ by $\sigma_{\mathrm{End}}(T) = \sigma_X \circ T \circ \sigma_X$.

Definition 2.12. [50, 60] Let X_B be a $\mathbb{Z}_2$-graded Hilbert module. Given $x, y \in X$ we can define a rank-one operator (a ket-bra) $\Theta_{x,y} : X \to X$ by $\Theta_{x,y}(z) = x(y|z)_B$. Then $\Theta_{x,y}$ is adjointable with adjoint $\Theta_{y,x}$, and the linear span of the rank one operators form a two-sided $*$-deal in $\mathrm{End}_B^*(X)$. The norm closure is denoted by $\mathrm{End}_B^0(X)$ and called the compact endomorphisms (some authors use $\mathcal{K}(X)$).

2.4 Correspondences

The notion of correspondences for C^*-algebras developed over many years from Rieffel and Paschke's use of the Mackey machine to define Morita equivalences, to the widespread modern use of correspondences to define Toeplitz- and Cuntz-Pimsner algebras, initiated by Pimsner [57].

Definition 2.13. Let X_B be a countably generated real or complex C^*-module over the real or complex C^*-algebra B, and let $\varphi : A \to \mathrm{End}^*_B(X)$ be a representation of the C^*-algebra A as adjointable endomorphisms. Then we call (A, X_B) a correspondence from A to B or an A-B-correspondence. When needed we write $(A, {}_\varphi X_B)$ to indicate the left action.

If A, B are Real C^*-algebras with real structures σ_A, σ_B, and $\sigma_X : X \to X$ is a Real structure then we require $\sigma_X(\varphi(a)xb) = \varphi(\sigma_A(a))\sigma_X(x)\sigma_B(b)$ for $a \in A$, $b \in B$, $x \in X$.

Similarly, if A, B are $\mathbb{Z}_2$-graded (even trivially) by γ_A, γ_B then X is $\mathbb{Z}_2$-graded if there is $\gamma_X : X \to X$ such that $\gamma_X(\varphi(a)xb) = \varphi(\gamma_A(a))\gamma_X(x)\gamma_B(b)$.

Example 2.14. A representation $\varphi : A \to \mathcal{B}(\mathcal{H})$ of a C^*-algebra on a complex Hilbert space is an A-$\mathbb{C}$ correspondence.

Example 2.15. Given a left action ρ of B on a right B-module M one obtains a left action π of $C_c(\mathbb{Z}, B) \subset B \rtimes_\alpha \mathbb{Z}$ on the module given by the algebraic tensor product $\ell^2(\mathbb{Z}) \odot M$. The action π is given on elementary tensors by

$$\pi(b)(e_j \otimes \xi) = e_j \otimes \rho(\alpha^{-j}(b))\xi, \qquad\qquad \pi(U)(e_j \otimes \xi) = e_{j+1} \otimes \xi \tag{2.2}$$

with $\{e_j\}_{j\in\mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$ and $\xi \in M$. We will show below that when M is a C^*-module the action extends to an action of the C^*-crossed product $B \rtimes_\alpha \mathbb{Z}$ on the Hilbert module completion of the algebraic tensor product $\ell^2(\mathbb{Z}) \otimes M$ given by $\ell^2(\mathbb{Z}, M)$.

Defining the left-action of $\mathcal{A}$ on the copy of $\mathcal{A} \subset \ell^2(\mathbb{Z}^d, B)$ via the twisted convolution multiplication in $C_c(\mathbb{Z}^d, B)$ yields a representation of $B \rtimes_{\alpha,\theta} \mathbb{Z}^d$ [54].

Proposition 2.16. *[12] Let $\mathcal{A} = C_c(\mathbb{Z}^d, B)$ be the finitely supported functions $\mathbb{Z}^d \to B$ with (twisted) convolution determined by α, θ as in Subsection 2.1. The left-action of $\mathcal{A}$ on $\ell^2(\mathbb{Z}^d, B)$ densely defined by twisted convolution extends to an adjointable representation of $A = B \rtimes_{\alpha,\theta} \mathbb{Z}^d$. So $(A, \ell^2(\mathbb{Z}^d, B)_B)$ is a correspondence, which is real, complex, Real and/or graded as B is.*

The essence of the proof is that the action of the compact dual group $\mathbb{T}^d$ provides an expectation $\Phi : B \rtimes_{\alpha,\theta} \mathbb{Z}^d \to B$ with which the completion to ℓ^2 is made. Since the inner product is positive and faithful, the bounded extension can be deduced as in [10, 12].

3 Kasparov theory

General references for Kasparov theory are scarce, but the original articles [40, 41] reward study, and more general introductory texts include [7, 34].

3.1 Kasparov modules bounded and unbounded

The following definition works for real, Real or complex $\mathbb{Z}_2$-graded C^*-algebras and C^*-modules. It is easiest to digest the definitions starting with trivially graded complex algebras.

Definition 3.1. Let A and B be $\mathbb{Z}_2$-graded C^*-algebras. A (bounded) Kasparov A-B-module $(A, {}_\varphi X_B, F)$ is a $\mathbb{Z}_2$-graded A-B-correspondence $(A, {}_\varphi X_B)$ together with an odd operator $F \in \mathrm{End}^*_B(X)$ such that for all $a \in A$ the operators

$$\varphi(a)(\mathrm{Id}_X - F^2), \quad \varphi(a)(F^* - F), \quad [F, \varphi(a)]_\pm \tag{3.1}$$

are compact. If A, B and X_B are trivially graded and $F \in \mathrm{End}^*_B(X)$ satisfies the conditions (3.1) then we say that $(A, {}_\varphi X_B, F)$ is an *odd* Kasparov module.

Example 3.2. Every A-B-correspondence $(A, {}_\varphi X_B)$ for which A acts compactly defines a Kasparov module $(A, {}_\varphi X_B, 0)$. This includes $*$-homomorphisms and imprimitivity bimodules. The Kasparov module of a $*$-homomorphism $\phi : A \to B$ is given by $(A, {}_\phi B_B, 0)$. If B is unital and $p \in M_N(B)$ is a projection then pB^N is called a finite projective module, and $(\mathbb{C}, pB^N_B, 0)$ is a Kasparov module.

Example 3.3. In the next subsection we will build an odd Kasparov module from an extension of C^*-algebras. To incorporate these odd modules into the theory we have the following construction (see [23, 27]).

Let (A, X_B, F) be an odd Kasparov module (so A and B are trivially graded). Let $e \in \mathbb{C}\ell_1$ be the nontrivial odd generator so $e = e^*$ and $e^2 = 1$. Then we define the Kasparov module

$$\left(A \otimes \mathbb{C}\ell_1, \begin{pmatrix} X \\ X \end{pmatrix}_B, \begin{pmatrix} 0 & -iF \\ iF & 0 \end{pmatrix} \right) \tag{3.2}$$

where $a \otimes e \in A \otimes \mathbb{C}\ell_1$ acts as $\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$.

The groups $KK(A, B)$ are built from homotopy classes of Kasparov modules: see [7, 40] for details. To obtain higher groups we define

$$\begin{aligned} KK^n(A, B) &= KK(A \hat{\otimes} \mathbb{C}\ell_n, B) && \text{complex case} \\ KKR^{r,s}(A, B) &= KKR(A \hat{\otimes} \mathbb{C}\ell_{r,s}, B) && \text{Real case} \\ KKO^{r,s}(A, B) &= KKO(A \hat{\otimes} Cl_{r,s}, B) && \text{real case} \end{aligned}$$

and in each case we use the $\mathbb{Z}_2$-graded tensor product of algebras, [40, Section 2]. In general we just write KK to indicate any of these particular cases.

Each $KK(A, B)$ is an abelian group under direct sum, the inverse of the class of the Kasparov module $(A, {}_\varphi X_B, F)$ is the class of $(A, {}_{\varphi^\circ} X^\circ, -F)$ where X° is X with the grading $\gamma_{X^\circ} = -\gamma_X$ and

$$\varphi^\circ(a^{even} + a^{odd}) = \varphi(a^{even}) - \varphi(a^{odd}).$$

We also have $KK(A \otimes \mathcal{K}, B) \cong KK(A, B \otimes \mathcal{K}) \cong KK(A, B)$ where $\mathcal{K}$ is the compact operators (or a full matrix algebra). For separable ungraded complex algebras, Kasparov theory recovers K-homology and K-theory via

$$\begin{aligned} KK(A, \mathbb{C}) \cong K^0(A),\ KK(A \otimes \mathbb{C}\ell_1, \mathbb{C}) \cong K^1(A), \\ KK(\mathbb{C}, A) \cong K_0(A),\ KK(\mathbb{C}, A \otimes \mathbb{C}\ell_1) \cong K_1(A). \end{aligned}$$

The analogous (more complicated) relations hold in the real and Real cases [12]. Many more properties can be found in [7, 40].

Definition 3.4. Let A and B be real or complex $\mathbb{Z}_2$-graded C^*-algebras, and let $\mathcal{A} \subset A$ be a dense $*$-subalgebra. An unbounded Kasparov A-B-module $(\mathcal{A}, {}_\varphi X_B, \mathcal{D})$ is a $\mathbb{Z}_2$-graded A-B-correspondence $(A, {}_\varphi X_B)$ together with an odd self-adjoint (unbounded) regular operator $\mathcal{D} : \operatorname{Dom} \mathcal{D} \to X$ such that for all $a \in \mathcal{A}$:
1) we have $\varphi(a) \operatorname{Dom} \mathcal{D} \subset \operatorname{Dom} \mathcal{D}$ and the densely-defined commutator

$$[\mathcal{D}, \varphi(a)] \tag{3.3}$$

is uniformly bounded on $\operatorname{Dom} \mathcal{D}$ and so extends to an adjointable operator on X;
2) the operator $\varphi(a)(1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism on X.

Remark 3.5. Condition 2) in Definition 3.4 is often called the local compactness of the resolvent in analogy with the classical case of elliptic operators [34]. If the algebra $\mathcal{A}$ is unital and acts unitally on X_B, then the operator $\mathcal{D}$ has compact resolvent.

Proposition 1.1 gives a special case of a Kasparov module.

Definition 3.6. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for the C^*-algebra A is an unbounded A-$\mathbb{C}$ Kasparov module.

Every class in $KK(A, B)$ has an unbounded representative [3]. The definition of the equivalence relation defining the groups can also be implemented directly in terms of unbounded cycles, [28, 35]. The reasons for introducing the unbounded version of the cycles is to assist in computing products (see below, and [3]), index pairings [19, 22], and to stay close to the defining geometric and physical origins of these cycles. For instance, the spectral triple for the quantum Hall effect in Proposition 1.1 is built from the position operators. Next we will build unbounded Kasparov modules for the bulk theory and the edge theory.

3.2 Bulk and edge classes

To construct Kasparov modules for the bulk and edge algebras, we start with the correspondences of Proposition 2.16. The last ingredient we need are Dirac-like operators on these correspondences, which we construct using the position operators, $X_j : C_c(\mathbb{Z}^d, B) \to \ell^2(\mathbb{Z}^d, B)$ for $j \in \{1, \dots, d\}$ defined by

$$X_j(e_m \otimes b) = m_j(e_m \otimes b), \quad m \in \mathbb{Z}^d.$$

We construct the Dirac-like operator by enlarging the module using a Clifford representation. On the tensor product space $\ell^2(\mathbb{Z}^d, B) \otimes \bigwedge^* \mathbb{R}^d$ we define

$$X := \sum_{j=1}^{d} X_j \otimes \gamma^j.$$

A simple check shows that X is odd, self-adjoint and regular on $\ell^2(\mathbb{Z}^d, B) \otimes \bigwedge^* \mathbb{R}^d$.

Proposition 3.7. *[12] Consider a possibly twisted $\mathbb{Z}^d$-action α, θ on a separable and unital C^*-algebra B. Let $A = B \rtimes_{\alpha,\theta} \mathbb{Z}^d$ be the associated crossed product with dense subalgebra $\mathcal{A} = C_c(\mathbb{Z}^d, B)$. The data*

$$\lambda^{(d)} = \left(\mathcal{A} \hat{\otimes} C\ell_{0,d},\, \ell^2(\mathbb{Z}^d, B)_B \otimes \bigwedge\nolimits^* \mathbb{R}^d,\, \sum_{j=1}^{d} X_j \otimes \gamma^j,\, \gamma_{\bigwedge^* \mathbb{R}^d} \right)$$

defines an unbounded $A \hat{\otimes} C\ell_{0,d}$-$B$ Kasparov module. The $C\ell_{0,d}$-action is generated by the operators ρ^j from Proposition 2.5. In the complex case we have $\mathbb{C}$ in place of $\mathbb{R}$ in the above formula and $\mathbb{C}\ell_d$ in place of $C\ell_{0,d}$.

For α, θ such that the action of $\mathbb{Z}^d$ restricts to an action of $\mathbb{Z}^{d-1}$ on B, we define the edge cycle as

$$\lambda^{(d-1)} = \left(C_c(\mathbb{Z}^{d-1}, B) \hat{\otimes} C\ell_{0,d-1},\, \ell^2(\mathbb{Z}^{d-1}, B)_B \otimes \bigwedge\nolimits^* \mathbb{R}^{d-1},\, \sum_{j=1}^{d-1} X_j \otimes \gamma^j,\, \gamma_{\bigwedge^* \mathbb{R}^{d-1}} \right)$$

We call $\lambda^{(d)}$ the fundamental K-cycle of the $\mathbb{Z}^d$-action because of its similarity to Kasparov's fundamental class [41] for oriented dimension d manifolds.

To relate the bulk and edge theories we will use extensions (aka short exact sequences) of observable algebras. Given an extension of the form

$$0 \to B \rtimes \mathbb{Z}^{d-1} \otimes \mathcal{K} \to T \to B \rtimes \mathbb{Z}^d \to 0 \tag{3.4}$$

we let $X_\partial = \sum_{j=1}^{d-1} X_j \otimes \gamma^j$ and $X = \sum_{j=1}^{d} X_j \otimes \gamma^j$ denote the operators associated to the "edge" and "bulk" respectively.

In the next section we show how an exact sequence of C^*-algebras gives rise to an odd Kasparov module. The class of the Kasparov module associated to the extension (3.4) is denoted $[ext]$. We will show that

$$[ext] \otimes_{B\rtimes\mathbb{Z}^{d-1}} [X_\partial] = (-1)^{d-1}[X]$$

and in fact the same relation holds on the level of cycles for an explicit construction of the product that we describe below.

3.3 Kasparov modules from short exact sequences

Given a short exact sequence of C^*-algebras

$$0 \to J \xrightarrow{\iota} A \xrightarrow{q} A/J \to 0 \tag{3.5}$$

which is split by a completely positive map $\rho : A/J \to A$ such that $q \circ \rho = \mathrm{Id}_{A/J}$ we can define a bounded Kasparov module and so a class in KK-theory. To construct the Kasparov module associated to the "semi-split extension" (3.5), we require Kasparov's version [39] of the Stinespring dilation theorem for Hilbert-modules.

Theorem 3.8 (Kasparov's Stinespring dilation theorem)**.** *(see [39, Theorem 3] and [50, Theorem 5.6]) Let A, B be C^*-algebras, let X_B be a right C^*-B-module and let $\rho : A \to \mathrm{End}^*_B(X)$ be a strict completely positive mapping.*

Then there is a Hilbert B-module Y_B, a $*$-homomorphism $\pi_\rho : A \to \mathrm{End}^*_B(Y)$ and an adjointable isometry $V : X \to Y$ such that $\pi_\rho(A)VX$ is dense in Y and for all $a \in A$ we have

$$\rho(a) = V^*\pi_\rho(a)V.$$

The data Y, π_ρ, V are unique in the sense that if we have $W : X \to Z$ and $\pi : A \to \mathrm{End}^*_B(Z)$ such that $\pi(A)WX$ is dense in Z and $\rho(a) = W^*\pi(a)W$ for all $a \in A$, then there is a unitary $U : Y \to Z$ such that $\pi(a) = U\pi_\rho(a)U^*$ for all $a \in A$.

Returning to our semi-split extension (3.5), we observe that since A acts by multipliers on the ideal $J \lhd A$, we can regard ρ as a map $\rho : A/J \to A \subset \mathrm{End}^*_J(J)$. Applying the Stinespring dilation theorem gives us a right C^*-J-module Y_J, an isometry $V : J \to Y$, and a $*$-homomorphism

$$\tilde{\rho} : A/J \to \mathrm{End}_J(Y) \quad \text{of the form} \quad V^*\tilde{\rho}(a)V = \rho(a).$$

Splitting the module Y with respect to the projection VV^*, we have

$$\tilde{\rho} = \begin{pmatrix} \rho & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}.$$

Example 3.9. Using the fact that $q \circ \rho = \mathrm{Id}_{A/J}$, it follows that σ is a homomorphism modulo J. Consequently one sees that $\rho_{12}([a])$ and $\rho_{21}([a])$ are compact endomorphisms of Y for all $[a] \in A/J$.

Letting $P = VV^* : Y_J \to Y_J$ be the projection, the definitions and Example 3.9 show that

$$(A/J, {}_{\tilde{\rho}}Y_J, 2P - 1)$$

is an odd Kasparov module. If instead we compress $\tilde{\rho}$ by the projection $\mathrm{Id}_Y - VV^*$ we see that ρ_{22} is also a completely positive map, and so we obtain another extension

$$0 \to J \to B \to A/J \to 0.$$

The class determined by ρ_{22} is the additive inverse of the class determined by ρ in $KK^1(A/J, J)$ [7, 40].

Definition 3.10. Given an extension of (graded) C^*-algebras

$$0 \to J \to A \to A/J \to 0$$

semi-split by the completely positive map $\sigma : A/J \to A$, the Kasparov class of the extension is the class $[(A/J, {}_{\tilde{\rho}}Y_J, 2P - 1)] \in KK^1(A/J, J)$. See [40, 65].

For an extension $0 \to J \to B \to C \to 0$ of a nuclear C^*-algebra C by a nuclear algebra J there is always a completely positive splitting, [40, 39]. In general, however, obtaining such a splitting can be difficult, and tantamount to finding the 'associated Kasparov module', see [14, 61] for hard examples. The next section describes the easy examples arising from the bulk edge correspondence of $\mathbb{Z}^d$ topological insulators.

The reason one might want the class of an extension is that the boundary maps in K-theory and K-homology are given by the Kasparov product with that class. Finding representatives of the class of an extension which enable more-or-less geometric realisations of the product on the level of cycles makes the physical applications simpler and more transparent.

3.4 The extension K-cycle

Relating the observables of bulk and edge algebras via an extension was pioneered in [44, 45, 46], and a good discussion appears in [47].

Under mild assumptions on the twist θ (see [47]), we can unwind the crossed product $A = B \rtimes_{\alpha,\theta} \mathbb{Z}^d$ such that, for $\alpha = (\alpha^{\parallel}, \alpha_d)$ and $\alpha^{\parallel}$ the restricted action of $\mathbb{Z}^{d-1}$,

$$A = \left(B \rtimes_{\alpha^{\parallel},\theta} \mathbb{Z}^{d-1}\right) \rtimes_{\alpha_d} \mathbb{Z} = C \rtimes_{\alpha_d} \mathbb{Z} \tag{3.6}$$

where $C = B \rtimes_{\alpha^{\parallel},\theta} \mathbb{Z}^{d-1}$. We link C and $C \rtimes_{\alpha_d} \mathbb{Z}$ by the Toeplitz extension, which we briefly recall.

Very similar to the construction of the crossed product $C \rtimes_{\alpha_d} \mathbb{Z}$, we can consider $C_{\alpha_d}\mathbb{N}$ the algebra given by finite sums $\sum_{k\in\mathbb{N}} \tilde{U}_d^k c_k + (\tilde{U}_d^*)^k c'_k$, where $c_k, c'_k \in C$ and $\tilde{U}_d$ is the operator such that

$$\tilde{U}_d b = \alpha_d(b)\tilde{U}_d, \qquad \tilde{U}_d^* b = \alpha_d^{-1}(b)\tilde{U}_d, \qquad \tilde{U}_d^*\tilde{U}_d = 1, \qquad \tilde{U}_d\tilde{U}_d^* = 1 - p$$

with $p = p^* = p^2$ a projection. Thus $\tilde{U}_d$ is no longer unitary but an isometry. There is a unique $*$-algebra morphism $q : C_{\alpha_d}\mathbb{N} \to C_{\alpha_d}\mathbb{Z}$ determined by $q(\tilde{U}_d) = U_d$ and is the identity on C. Its kernel is the ideal generated by p which can easily be seen to be isomorphic to $F \otimes C$ where F is the algebra of the finite rank operators. The exact sequence

$$0 \to F \otimes C \to C_{\alpha_d}\mathbb{N} \xrightarrow{q} C_{\alpha_d}\mathbb{Z} \to 0$$

is the algebraic version of the Toeplitz extension, the C^*-version is obtained by taking the universal C^*-closures. The C^*-closure of $C_{\alpha_d}\mathbb{N}$, denoted by $\mathcal{T}(\alpha_d)$, is the Toeplitz algebra of the $\mathbb{Z}$-action α_d and the closure of $F \otimes C$ is $\mathcal{K} \otimes C$, with $\mathcal{K}$ the algebra of complex operators on a separable (real or complex) Hilbert space. The short exact sequence

$$0 \to \mathcal{K} \otimes C \to \mathcal{T}(\alpha_d) \to C \rtimes_{\alpha_d} \mathbb{Z} \to 0 \tag{3.7}$$

gives rise to a class $[ext]$ in the group $KKO^{0,1}(A, C)$ (or $KKR^{0,1}(A, C)$ or $KK^1(A, C)$).

The extension class $[ext]$ serves to compute boundary maps in K-theory and K-homology, namely by taking Kasparov products with it. In order to make these maps computable in terms of the physical cycles, we construct an unbounded representative of $[ext]$.

Proposition 3.11. *[12] Let C be a separable and unital C^*-algebra and $A = C \rtimes_{\alpha_d} \mathbb{Z}$. On $C_c(\mathbb{Z}, C) \subset \ell^2(\mathbb{Z}, C)$ define $N(\sum_n e_n c_n) = \sum_n n e_n c_n$ to be the number operator. The extension class of the Toeplitz extension of Equation (3.7) is represented by the fundamental K-cycle of the $\mathbb{Z}$-action,*

$$\left(C_c(\mathbb{Z}, C)\hat{\otimes} C\ell_{0,1},\, \ell^2(\mathbb{Z}, C)_C \otimes {\textstyle\bigwedge}^* \mathbb{R},\, N \otimes \gamma^1\,, \gamma_{\bigwedge^* \mathbb{R}}\right). \tag{3.8}$$

There is an analogous result for complex algebras.

The method of proof is to use the Busby invariant of the exact sequence [7, 10] and seeing that the meaning of equivalence for odd Kasparov modules is equality of Busby invariants.

3.5 The Kasparov product

There are various instances of the product [40]. The one we require is a $\mathbb{Z}$-bilinear pairing

$$KK^i(A, B) \times KK^j(B, C) \to KK^{i+j}(A, C). \tag{3.9}$$

Theorem 3.12. *[40] There is a well-defined $\mathbb{Z}$-bilinear pairing as described in (3.9). Given classes $[F_1] \in KK^i(A, B)$ and $[F_2] \in KK^j(B, C)$, there exists a unique class $[F_1 \# F_2] \in KK^{i+j}(A, C)$ which depends covariantly on A and contravariantly on C.*

Given an extension $0 \to J \to A \to A/J \to 0$ with completely positive splitting, we have seen that there is a class $[ext] \in KK^1(A/J, J)$ representing the extension.

Theorem 3.13. *The product with the class $[ext]$ gives the boundary maps in K-theory and K-homology, so $\cdot \otimes_{A/J} [ext] : K_*(A/J) \to K_{*+1}(J)$ and $[ext] \otimes_J \cdot : K^*(J) \to K^{*+1}(A/J)$ are the boundary maps.*

The proof of the existence of a well-defined product given by [40] is not constructive. Nevertheless, there are various ways to compute the product $[F_1 \# F_2]$, and even in the noncommutative case there are guess and check methods [24, Appendix].

More recently methods of computing the product on the level of representatives, rather than on classes have been developed. The principal benefit of these methods, when they apply, is that the product can be computed directly in terms of geometric and/or physical data. These methods can provide deeper understanding of the noncommutative geometry which represents the topological data.

It is not the case that we can always compute representatives of the product of two classes in terms of the input cycles. There are numerous different sufficient conditions for: 1) recognising when a class represents the product [24, 49]; 2) for knowing ahead of time that the construction described below produces a representative of the product [37, 52, 53] given two unbounded cycles representing composable classes.

In the specific context of the models of topological insulators we have described, *all* of the methods of computing the Kasparov product are available. In particular the constructive procedure outlined next allows us to compute up to unitary equivalence the product of the cycle representing *ext* and the cycle representing the boundary.

The simplicity of the constructive Kasparov product for topological insulators is essentially because (ignoring disorder) we have the flat principal fibre bundle $\mathbb{T} \to \mathbb{T}^d \to \mathbb{T}^{d-1}$ arising from the $\mathbb{Z}^d$ action, [18, 29] This structure is visible in the extension class (3.8) for $C = B \rtimes \mathbb{Z}^{d-1}$. More difficult examples are presented in [14].

To compute a representative of the class of the product of unbounded Kasparov modules

$$(\mathcal{A}, X_B, S) \quad \text{and} \quad (\mathcal{B}, Y_C, T)$$

we need the additional ingredient of a (densely defined) connection $\nabla : \mathcal{X}_{\mathcal{B}} \to X \otimes \mathrm{End}^*_C(Y)$. This is a $\mathbb{C}$ (or $\mathbb{R}$) linear map satisfying $\nabla(xb) = \nabla(x)b + x \otimes [T, b]$ for $x \in \mathcal{X}_{\mathcal{B}}$ and $b \in \mathcal{B}$.

Given these ingredients, we try

$$(\mathcal{A}, X\hat{\otimes}_B Y, S\hat{\otimes}1 + 1\hat{\otimes}_\nabla T)$$

where we use the fact that for $x \in \mathcal{X}$ we have $\nabla(x) : Y \to X \otimes_B Y$ to define

$$1\hat{\otimes}_\nabla T(x\hat{\otimes}y) = \nabla(x)y + x \otimes Ty, \qquad x \otimes y \in \mathcal{X} \otimes \mathrm{Dom}(T).$$

One readily checks that $S\hat{\otimes}1 + 1\hat{\otimes}_\nabla T$ is well-defined and symmetric.

The three points of difficulty are: existence of a suitable connection; self-adjointness of $S\hat{\otimes}1 + 1\hat{\otimes}_\nabla T$; boundedness of the commutators. All can fail in various types of example, but are straightforward for the examples arising from topological insulators, including the aperiodic examples from [14]. All of these issues are discussed in [37, 52, 53].

Example 3.14. We start with the boundary cycle

$$\lambda^{(d-1)} = \left(C_c(\mathbb{Z}^{d-1}, B)\hat{\otimes}C\ell_{0,d-1},\, \ell^2(\mathbb{Z}^{d-1}, B)_B \otimes \bigwedge\nolimits^* \mathbb{R}^{d-1},\, \sum_{j=1}^{d-1} X_j \otimes \gamma^j,\, \gamma_{\bigwedge^* \mathbb{R}^{d-1}}\right)$$

and the extension cycle

$$ext = \left(C_c(\mathbb{Z}, B \rtimes \mathbb{Z}^{d-1})\hat{\otimes}C\ell_{0,1},\, \ell^2(\mathbb{Z}, B \rtimes \mathbb{Z}^{d-1})_{B\rtimes\mathbb{Z}^{d-1}} \otimes \bigwedge\nolimits^* \mathbb{R},\, N \otimes \gamma^1,\, \gamma_{\bigwedge^* \mathbb{R}}\right).$$

The internal product of the underlying Hilbert modules is described by an explicit unitary isomorphism

$$U : \left(\ell^2(\mathbb{Z}, B \rtimes \mathbb{Z}^{d-1})_{B\rtimes\mathbb{Z}^{d-1}} \otimes \bigwedge\nolimits^* \mathbb{R}\right)\hat{\otimes}\left(\ell^2(\mathbb{Z}^{d-1}, B)_B \otimes \bigwedge\nolimits^* \mathbb{R}^{d-1}\right) \xrightarrow{\cong} \ell^2(\mathbb{Z}^d, B) \otimes \bigwedge\nolimits^* \mathbb{R}^d$$

On $C_c(\mathbb{Z}, B \rtimes \mathbb{Z}^{d-1}) \subset \ell^2(\mathbb{Z}, B \rtimes \mathbb{Z}^{d-1})$ we can define

$$\nabla(\sum_n e_n c_n) = \sum_n e_n \otimes [X_\partial, c_n]$$

where X_∂ is the unbounded operator defining the boundary (or edge) cycle. Then for $\sum_n e_n c_n \in C_c(\mathbb{Z}, B \rtimes \mathbb{Z}^{d-1})$ and $\xi \in \ell^2(B \rtimes \mathbb{Z}^{d-1}) \otimes \wedge^* \mathbb{R}^{d-1}$ we have

$$1 \otimes_\nabla X_\partial(\sum_n e_n c_n \otimes \xi) = \sum_n e_n \otimes X_\partial(c_n \xi).$$

Then as proved in detail in [12, Theorem 3.4],

$$U(N \hat{\otimes} 1 + 1 \hat{\otimes}_\nabla X_\partial) U^* = (-1)^{d-1} X$$

where the sign needs to be determined by the change of orientation in the Clifford algebra arising from relabelling γ^1 from the extension module to γ^d in the bulk module.

4 The bulk-edge correspondence via Kasparov theory

The bulk-edge correspondence relies on the general properties of the Kasparov product, especially associativity, and the following summary of the calculation from Example 3.14.

Theorem 4.1. *[10, 12] Let B be a separable and unital real or complex C^*-algebra with fundamental K-cycles $\lambda^{(d)}$ and $\lambda^{(d-1)}$ for (possibly twisted) $\mathbb{Z}^d$ and $\mathbb{Z}^{d-1}$-actions. Then the unbounded Kasparov product of the extension Kasparov module from Proposition 3.11 with $\lambda^{(d-1)}$ gives, up to unitary equivalence and a cyclic permutation of the Clifford generators, the fundamental K-cycle $\lambda^{(d)}$. On the level of KK-classes this means*

$$[ext] \hat{\otimes}_C [\lambda^{(d-1)}] = (-1)^{d-1} [\lambda^{(d)}],$$

where $-[x]$ denotes the additive inverse of the KK-class.

Consequently, the associativity of the Kasparov product tells us that when we pair with the (bulk) Fermi projector, we obtain $[P_F] \otimes_A [\lambda^{(d)}] = (-1)^{d-1}([P_F] \otimes_A [ext]) \otimes_C [\lambda^{(d-1)}]$. The class $(-1)^{d-1}([P_F] \otimes_A [ext]) \in K_1(C)$ defines a unitary on the boundary. More generally, we have

Corollary 4.2. *The pairing of a K-theory class $[z] \in KO_j(B \rtimes_{\alpha,\theta} \mathbb{Z}^d)$ (or complex) with $\lambda^{(d)}$ is, up to a sign, the same as the pairing of $\partial[z] \in KO_{j-1}(B \rtimes_{\alpha^\parallel,\theta} \mathbb{Z}^{d-1})$ with $\lambda^{(d-1)}$.*

Proof. Using Theorem 4.1 and associativity of the Kasparov product,

$$\begin{aligned} [z] \hat{\otimes}_A [\lambda^{(d)}] &= (-1)^{d-1} [z] \hat{\otimes}_A \big([ext] \hat{\otimes}_C [\lambda^{(d-1)}]\big) \\ &= (-1)^{d-1} \big([z] \hat{\otimes}_A [ext]\big) \hat{\otimes}_C [\lambda^{(d-1)}] \\ &= (-1)^{d-1} \partial[z] \hat{\otimes}_C [\lambda^{(d-1)}] \end{aligned}$$

as the product with $[ext]$ implements the boundary map in K-theory. □

5 Continuum theory, disorder, van Daele K-theory and semifinite index theory

There are numerous aspects we have only touched on, or not addressed at all. We summarise some of them here along withs starting points in the literature for the interested reader.

Continuum models Throughout we have described the tight-binding model for topological insulators. Instead of modelling a d-dimensional sample by $\mathbb{Z}^d$ we could also use $\mathbb{R}^d$ [5, 67], where the Hamiltonian operator is a differential operator. Certainly the discussion becomes more technical, but the main results about the bulk-edge correspondence and Kasparov modules continue to hold as shown in [15].

Bulk-defect correspondence In [14] the methods described here were extended to incorporate aperiodicity and "edges" of different codimension. The bulk-edge correspondence, compatibility with pairings etc go through as expected. In [59] the notion of defect was used to describe, roughly speaking, a codimension 0 edge. These defects naturally give rise to a short exact sequence of groupoid algebras, and so determines a Kasparov class.

Edge vs bulk disorder One very strange feature of the class of models we have described, whether tight-binding or continuum, is that the same disorder space arises for the bulk and edge theories. This is certainly less than desirable. To the best of our knowledge, this problem is not yet understood.

Extension of pairing to strong disorder More positively, as Bellissard showed, the cocycles which compute the index pairings extend continuously to much larger algebras of "smooth-but-not-continuous" observables, [5]. Ultimately these continuous extensions are what allows us to describe the index pairings in the presence of disorder. This is not just mathematical artifice: the "non-smooth" algebras are characterised in terms of Sobolev-type norms defined in terms of "localisation length". As the localisation length diverges, mathematically the pairing loses meaning, and physically a new electron is promoted to the conduction band. These observations are consistent with the Kasparov approach to the bulk-edge correspondence [15], but go beyond the Kasparov framework.

van Daele K-theory The version of K-theory developed by van Daele [25, 26] is designed to be a K-theory capable of discussing all complex, real, Real and graded variants at once, just as Kasparov theory is. Consequently van Daele theory is a natural tool for topological insulators and their invariants [42, 43]. Unsurprisingly, van Daele classes can be related to the KK-language used to prove the bulk-edge correspondence [62, 63, 13]. The relationship can be made explicit at the level of cycles [13], so as to be compatible with the constructive Kasparov product discussed in previous sections.

Semifinite index theory In the genuinely disordered case, the bulk and edge fundamental classes are Kasparov modules over the disorder space $B = C(\Omega)$. Since, typically, the algebra $C(\Omega)$ is infinite dimensional, the bulk and edge Kasparov modules do not define spectral triples.

Instead, if we have a positive norm-lower semicontinuous trace $\tau : B \to \mathbb{C}$, we can define a scalar product on $C_c(\mathbb{Z}^d, B) \subset \ell^2(\mathbb{Z}^d, B)$ by

$$\langle e_n c_n, e_k d_k \rangle = \delta_{n,k} \tau(c_n^* d_n),$$

where $e_n \otimes c_n, e_k \otimes d_k \in C_c(\mathbb{Z}^d) \otimes B$ are simple tensors.

The passage from the Kasparov module to its completion

$$(\mathcal{A}, \ell^2(\mathbb{Z}^d, B)_B, \mathcal{D}) \mapsto (\mathcal{A}, L^2(\ell^2(\mathbb{Z}^d, B), \tau), \overline{\mathcal{D}})$$

yields what is called a semifinite spectral triple, [6, 20, 21], as was eventually recorded in [12, 31]. In the context of topological phases of matter, where $B = C(\Omega)$ the trace τ arises from a probability measure on Ω.

Semifinite spectral triples define index pairings with K-theory just like "vanilla" spectral triples, though these index pairings in general yield a real number [16, 17, 55, 56]. Nonetheless, the semifinite index pairings respect the Kasparov product [19], and the bulk-edge correspondence for the numerical index pairings follows from the Kasparov arguments above: see [12, 15].

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