Spectral Triples: Examples and Index Theory

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1 Introduction

The main objective of these notes is to give some intuition about spectral triples and the role they play in index theory. The notes are basically a road map, with much detail omitted. To give a complete account of all the topics covered would require at least a book, so we have opted for a sketch.

All examples of spectral triples require a lot of effort to set up, and so we have taken most of our examples from classical index theory on manifolds, where the necessary background is readily available, [Gi, LM]. However, we do give some examples arising from singular spaces, group algebras and so on. One more lengthy and detailed example is the graph C^*-algebras, included here to motivate the semifinite extension of spectral triples.

Spectral triples are special representatives of K-homology classes for which computing the index pairing with K-theory becomes tractable. The interest of spectral triples beyond index theory arises because the construction of spectral triples invariably uses some form of geometric and/or physical input over and above their topological content as representatives of K-homology classes.

As a consequence of our choice of examples, much else has been omitted. While we will not discuss the families index theorem here, part of our motivation for developing these notes arises from the use of the Atiyah-Singer index theorem in identifying the obstructions to quantisation in gauge field theory known as anomalies. These remain one of the primary applications of index theory outside of mathematics, along with the noncommutative interpretation of the standard model of particle physics, and the work by Bellissard using noncommutative geometry techniques in the study of the quantum Hall effect (references and further details may be found in [C1]). In recent applications of noncommutative geometry to number theory, index theorems also play a role, and these notes provide the basic ideas for that application as well.

These notes have always been accompanied by lectures. Sometimes the audience has been people with a geometry background, at other times the audience has been more operator theoretic. As a consequence of this, and our focus on presenting some key examples, the background assumed is rather mixed.

We assume a fair amount of differential geometry and especially pseudodifferential operator theory. We have tried to quote the main results we use, and hope our discussion can serve as a guide to those seeking to learn this subject for index purposes. On the other hand, we spend some time on Clifford algebras and the Hodge $*$-operator. We also assume some elementary theory of C^*-algebras. We have tried to write the notes so that lack of C^*-knowledge does not intrude too much.

There are a number of excellent textbook presentations of foundational material. Presentations of index theory on manifolds which adapt well to noncommutative geometry can be found in [BGV, HR, LM, GVF, G]. An introduction to noncommutative geometry and spectral triples can be found in [GVF, Lan, V]. More sophisticated descriptions and applications appear in [C0, C1] and [CMa]. Noncommutative algebraic topology, that is K-theory and K-homology, are expounded very clearly in [HR]. More introductory books on K-theory are [RLL] and [WO]. For K-theory of spaces see [AK]. We take the view that the noncommutative analogue of differential topology is cyclic homology and cohomology. The description in [C1] remains one of the best, and further information is available in [L]. A wonderful exposition of the intertwining of spectral triples and the local index formula in noncommutative geometry is [H].

Chapter 2 begins by introducing the Fredholm index and Clifford algebras. Then we outline, for the Hodge-de Rham operator on a compact manifold, the sequence of arguments that leads to a well-defined Fredholm index. We then sketch the result that the index in this case is the Euler characteristic.

Throughout this sketch, we focus on those features[1] which are essential for the arguments to hold, and which can be generalised. The definition of spectral triple is then more readily seen to be a generalisation of the situation we have just studied for the Hodge-de Rham operator. We show that given a spectral triple, there is a well-defined Fredholm index arising from the data.

We finish Chapter 2 with a brief look at how suitable spectral triples define a metric. This highlights the geometric content of spectral triples. In addition it allows us to use metric ideas to help us construct spectral triples, and we illustrate this with some basic examples.

Chapter 3 returns to operators on manifolds. We define the signature operator, spin^c manifolds, and Dirac operators. The process of twisting a (Dirac-type) operator by a vector bundle to obtain a new operator is also described. We indicate how the Fredholm indices these various operators are described by the Atiyah-Singer index theorem. The Chapter finishes with the noncommutative torus, a noncommutative example very close to the compact manifold setting we have focussed on so far.

Chapter 4 gives a short description of the cohomological picture of index theory, K-theory and K-homology. We show that spectral triples define K-homology classes. We then introduce the index pairing between K-theory and K-homology. This shows that a spectral triple over an algebra $\mathcal{A}$ defines a map from the K-theory of $\mathcal{A}$ to the integers.

For the examples coming from operators on manifolds, the algebra we are looking at is just $C(M)$ or $C^\infty(M)$, where M is our manifold. Amazingly, K-theory and K-homology continue to make sense for any C^*-algebra, commutative or not. It is this feature that allows us to extend index theory to the noncommutative world.

The computation of the index pairing, in any practical sense, requires special properties of representatives of K-homology and K-theory classes. We conclude this Chapter with Connes' famous formula for the Chern character of a finitely summable Fredholm module. This Chern character 'computes' the index pairing, establishes a connection with cyclic cohomology, and so gives many more tools with which index theory problems can be studied. However, for practical computations of the index pairing, the Chern character is usually not helpful.

Chapters 5 and 6 aim to present formulae for the index pairing which are more practically computable. First we require a spectral triple which is regular, a notion generalising both smoothness of functions and elliptic regularity, and summable in some sense. Summability is related to dimension and integration, and we present several different flavours: finite summability, Dixmier summability, and θ-summability.

Chapter 5 finishes with analytic formulae for the index pairing. Like the Chern character formula, these

[1] It is here that we are most ruthless with the pseudodifferential calculus. We are interested in various properties of operators which, on a manifold, one proves to be true using the pseudodifferential calculus. For our purposes it is the properties of operators, for example Fredholmness or compactness, that are important, not how we prove that the operator has these properties. On a manifold one should, of course, use the pseudodifferential calculus to prove these various properties of differential operators.

are usually not suitable for practical calculations. Starting from these formulae and employing perturbation techniques leads to more reasonable formulae. These are the local index formula of Connes and Moscovici, and the JLO formula. Both of these formulae have an interpretation in cyclic cohomology, and Chapter 6 gives a brief overview of the definitions necessary to give this interpretation.

Chapter 7 is both more detailed and more advanced. The first five sections summarise the construction of a semifinite spectral triple for graph C^*-algebras from [PR]. This provides an accesible example of semifinite noncommutative geometry as developed in [BeF, CP1, CP2].

The constructions in the first five sections also demonstrate the relationship between semifinite spectral triples and KK-theory, in analogy to the relationship between ordinary spectral triples and K-homology. Without going into Kasparov's KK-theory in too much detail, we give a general statement about this relationship, following [KNR].

The final section is a brief preview of current research, where traces are replaced by twisted traces. There are a number of examples where this is a natural extension of the tracial theory, but there are few general statements. This section provides some background for the paper [CMR] which uses graph algebras and noncommutative geometry to study Mumford curves.

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2 Preliminaries

2.1 Conventions and notations. Throughout we assume Hilbert spaces are separable and complex. The bounded linear operators on a Hilbert space $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$. The ideal of compact operators on $\mathcal{H}$ is denoted by $\mathcal{K}(\mathcal{H})$; it is the unique norm closed ideal in $\mathcal{B}(\mathcal{H})$. We use [Sim] for the theory of compact operators and Schatten ideals. The Calkin algebra is written $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. All C^*-algebras in these notes are separable and complex, and unital unless otherwise stated. We will make use of some results about unbounded densely defined self-adjoint operators on Hilbert spaces and for the reader's convenience we include a brief discussion of the theory in the appendix.

A pair (M, g) means an n-dimensional compact oriented manifold M (with no boundary) equipped with a Riemannian metric g. We use X for a compact Hausdorff space and $C(X)$ is the C^*-algebra of continuous functions on X. We let $\Lambda^*M := \Lambda^*T^*M = \oplus_{k=0}^n \Lambda^k T^*M$ denote the bundle of exterior differential forms on M, and $\Gamma(\Lambda^*M)$ the smooth sections of Λ^*M.

In the last Chapter we will discuss the more complicated theory of spectral triples associated to a semifinite von Neumann algebra. We do not have space to develop the requisite theory of von Neumann algebras and index theory in this context. General references to the background are Dixmier [Dix], Fack and Kosaki [FK] and Breuer [B1, B2]. A careful exposition of index theory in this framework is contained in [CPRS3].

2.2 The Fredholm index. As we explained in the introduction, we see the roots of noncommutative geometry arising in index theory. The central classical problem here is to compute an integer, called the Fredholm index, associated with certain special operators on sections of vector bundles over smooth manifolds, the elliptic pseudodifferential operators. The solution to this problem was provided by Atiyah and Singer in the 1960's, and we will discuss numerous examples and present their theorem later. In this Section, we will review some definitions and fundamental results. More details may be found in the discussion of the Fredholm index in [LM], which is suitably set in the context of the Atiyah-Singer index theorem.

Definition 2.1. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and $F : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded linear operator. We say that F is Fredholm if*
1) $\mathrm{range}(F)$ *is closed in* $\mathcal{H}_2$,
2) $\ker(F)$ *is finite dimensional, and*
3) $\mathrm{coker}(F) := \mathcal{H}_2/\mathrm{range}(F)$ *is finite dimensional.*
If F is Fredholm we define

$$\mathrm{Index}(F) = \dim\ker(F) - \dim\mathrm{coker}(F).$$

Example 1. The simplest example of a Fredholm operator with non-zero index is the unilateral shift operator $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$. If $\{e_i : i = 1, 2 \dots\}$ is the usual basis of $l^2(\mathbb{N})$ then the shift is defined by

$$S\sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^{\infty} a_i e_{i+1}, \quad a_i \in \mathbb{C}.$$

The range of S has codimension one, and is easily seen to be closed. The kernel of S is $\{0\}$, and so

$$\mathrm{Index}(S) = \dim\ker(S) - \dim\mathrm{coker}(S) = 0 - 1 = -1.$$

Example 2. If $F : \mathcal{H} \to \mathcal{H}$ is a Fredholm operator and F is self-adjoint, then $\mathrm{Index}(F) = 0$. This is because $\mathrm{coker}(F) = \ker(F^*)$.

We recall the following definition [Sim]:

Definition 2.2. *A bounded linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called compact if T maps any bounded sequence $\{\xi_k\}_{k\geq 0} \in \mathcal{H}_1$ to a sequence $\{T\xi_k\}_{k\geq 0} \in \mathcal{H}_2$ with a convergent subsequence.*

It is a basic theorem that T is compact if and only if it is the norm limit of a sequence of finite rank operators [Sim]. The next result is known as Atkinson's theorem.

Proposition 2.3. *Let $F : \mathcal{H}_1 \to \mathcal{H}_2$. Then F is Fredholm if and only if there is an operator $S : \mathcal{H}_2 \to \mathcal{H}_1$ such that $FS - \mathrm{Id}_{\mathcal{H}_2}$ and $SF - \mathrm{Id}_{\mathcal{H}_1}$ are compact operators (on $\mathcal{H}_2$ and $\mathcal{H}_1$ respectively). Then S is also a Fredholm operator.*

Thus the Fredholm operators $F : \mathcal{H} \to \mathcal{H}$ are precisely those whose image in the Calkin algebra $\mathcal{Q}(\mathcal{H})$ is invertible. Given F and S as in the Proposition, S is said to be a parametrix or approximate inverse for F.

Exercise. Show that if $F, S : \mathcal{H} \to \mathcal{H}$ are both Fredholm operators, then $FS : \mathcal{H} \to \mathcal{H}$ is also Fredholm.

We now summarise some of the important properties of the Fredholm operators.

Theorem 2.4. *Let $\mathcal{F}$ denote the set of Fredholm operators on a fixed Hilbert space $\mathcal{H}$, and let $\pi_0(\mathcal{F})$ denote the set of (norm) connected components of $\mathcal{F}$.*
(i) The index is locally constant on $\mathcal{F}$ and induces a bijection

$$\mathrm{Index} : \pi_0(\mathcal{F}) \to \mathbb{Z}. \tag{2.1}$$

(ii) The index satisfies

$$\mathrm{Index}(F^*) = -\,\mathrm{Index}(F), \qquad \mathrm{Index}(FS) = \mathrm{Index}(F) + \mathrm{Index}(S),$$

and so the induced map (2.1) is a group isomorphism.
(iii) If F is Fredholm and T is compact then $F+T$ is Fredholm and

$$\mathrm{Index}(F+T) = \mathrm{Index}(F).$$

In particular, any two operators with the same index lie in the same connected component of $\mathcal{F}$ and the index is constant on these components which are open in the norm topology. Thus the index is constant under compact perturbations and also sufficiently small norm perturbations. It follows that if $\{F_t\}_{t\in[0,1]}$ is a norm continuous path of Fredholm operators, then $\mathrm{Index}(F_t)$ is a constant independent of t.

We will see later that operators acting on sections of vector bundles on manifolds give rise to Fredholm operators on Hilbert spaces. The topological properties of the index will enable us to construct invariants of the underlying manifold from these operators. Surprisingly, we can frequently extend this same strategy to noncommutative spaces.

2.3 Clifford algebras. Clifford algebras play a central role in the construction and analysis of many important geometric operators on manifolds. It is worth introducing them early, as it will streamline much of what we will do. References for this material include [ABS, BGV, GVF, LM].

Let V be a finite dimensional real vector space, and $(\cdot|\cdot) : V \times V \to \mathbb{R}$ an inner product, so for u, v, $w \in V$ and $\lambda \in \mathbb{R}$

$$(v|w) = (w|v), \quad (\lambda v|w) = \lambda(v|w), \quad (v+u|w) = (v|w) + (u|w), \quad (v|v) \geq 0.$$

We suppose also that the inner product is nondegenerate, so that $(v|v) = 0 \Rightarrow v = 0$.

Definition 2.5. *The Clifford algebra* $\mathrm{Cliff}(V, (\cdot|\cdot))$ *(we write* $\mathrm{Cliff}(V)$ *when* $(\cdot|\cdot)$ *is understood) is the universal unital associative algebra over* $\mathbb{R}$ *generated by all* $v \in V$ *and* $\lambda \in \mathbb{R}$ *subject to*

$$v \cdot w + w \cdot v = -2(v|w)\,\mathrm{Id}_{\mathrm{Cliff}(V)}\,.$$

Clifford algebras arise in much greater generality (see [LM]), but this is enough for our purposes. Observe that if v, w are orthogonal, then in the Clifford algebra they anticommute. If we let $\Lambda^*V = \oplus_{j=0}^{\dim V}\Lambda^j V$ denote the exterior algebra of V, then we have:

Lemma 2.6. *The two algebras* Λ^*V *and* $\mathrm{Cliff}(V)$ *are linearly isomorphic (although not isomorphic as algebras).*

Proof. Fix an orthonormal basis $\{v_1, \dots, v_n\}$ of the vector space V, setting the dimension to be n. We define the map $m : \Lambda^*V \to \mathrm{Cliff}(V)$ by

$$m(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = v_1 \cdot v_2 \cdot \cdots \cdot v_k.$$

We leave it as an exercise to check this is an isomorphism. □

The Clifford algebra is a filtered algebra, while the exterior algebra is graded.

Exercise. Show that the exterior algebra is the associated graded algebra of the Clifford algebra.

Hence we can regard the Clifford algebra as the exterior algebra with a 'deformed' product. In [BGV] the map m is called a quantization map.

Exercise. Write down the inverse to the isomorphism m.

Most of the time, we work with the complexification of the Clifford algebra, $\mathbb{C}\mathrm{liff}(V) = \mathrm{Cliff}(V)\otimes\mathbb{C}$. This is because we will be using complex Hilbert spaces.

Exercise. Show that

$$\mathbb{C}\mathrm{liff}(\mathbb{R}) = \mathbb{C}\oplus\mathbb{C}, \quad \mathbb{C}\mathrm{liff}(\mathbb{R}^2) = M_2(\mathbb{C}).$$

More generally we have

$$\mathbb{C}\mathrm{liff}(\mathbb{R}^k) = \begin{cases} M_{2^{(k-1)/2}}(\mathbb{C})\oplus M_{2^{(k-1)/2}}(\mathbb{C}) & k \text{ odd} \\ M_{2^{k/2}}(\mathbb{C}) & k \text{ even} \end{cases}.$$

The complex Clifford algebra also satisfies a universal property.

Lemma 2.7. *If A is a complex unital associative algebra and $c : V \to A$ is a linear map satisfying*

$$c(v)c(w) + c(w)c(v) = -2(v|w)1_A,$$

for all $v, w \in V$, then there is a unique algebra homomorphism $\tilde{c} : \mathbb{C}\mathrm{liff}(V) \to A$ extending c.

There is a special element in the Clifford algebra which we refer to as the (complex) volume form. It is defined as follows. Suppose V is n-dimensional and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of V. Define

$$\omega_{\mathbb{C}} = i^{[(n+1)/2]}e_1\cdot e_2\cdot\cdots\cdot e_n.$$

Then $\omega_{\mathbb{C}}^2 = 1$ and for all $v \in V$ we have

$$v\cdot\omega_{\mathbb{C}} = (-1)^{n-1}\omega_{\mathbb{C}}\cdot v.$$

If we give the Clifford algebra a complex involution or adjoint operation by setting

$$(\lambda e_1\cdots e_k)^* = (-1)^k\bar{\lambda}e_k\cdots e_1$$

then the Clifford algebra becomes a (finite dimensional) C^*-algebra, and $\omega_{\mathbb{C}} = \omega_{\mathbb{C}}^*$.

Exercise. Check that $\omega_{\mathbb{C}}^* = \omega_{\mathbb{C}}$.

It is useful in what follows to represent the Clifford algebra by linear transformations on the exterior algebra. To do this, we need to recall the **interior product** on Λ^*V. For $v \in V$ and $v_1\wedge v_2\wedge\cdots\wedge v_k$ we define

$$v\llcorner(v_1\wedge v_2\wedge\cdots\wedge v_k) = \sum_{i=1}^k(-1)^{i+1}(v_i|v)v_1\wedge\cdots\wedge\hat{v}_i\wedge\cdots\wedge v_k,$$

where $\hat{}$ denotes omission. The interior product satisfies

$$v\llcorner(\varphi\wedge\psi) = (v\llcorner\varphi)\wedge\psi + (-1)^k\varphi\wedge(v\llcorner\psi), \quad \varphi\in\Lambda^kV,$$

and so $v\llcorner\circ v\llcorner = 0$. Also observe that for $v \in V$ and $z \in \mathbb{C}$, $(v\otimes\bar{z})\llcorner$ is the adjoint of $(v\otimes z)\wedge$ for the inner product on the complexification of Λ^*V (see below). Then under the isomorphism $\Lambda^*V \cong \mathrm{Cliff}(V)$, we have for $v \in V$ and $\varphi \in \mathrm{Cliff}(V)$

$$v\cdot\varphi = v\wedge\varphi - v\llcorner\varphi. \tag{2.2}$$

Since for all $\varphi \in \Lambda^*V$ we have

$$v\wedge(w\llcorner\varphi) + w\llcorner(v\wedge\varphi) = (v|w)\varphi$$

one can check that the action of V on $\Lambda^* V$ defined by the formula in Equation (2.2) satisfies

$$v \cdot w + w \cdot v = -2(v|w)\,\mathrm{Id}_V$$

and so extends to an action of Cliff(V) on $\Lambda^* V$. Similarly right multiplication by v gives

$$\varphi \cdot v = (-1)^k (v \wedge \varphi + v \llcorner \varphi), \quad \varphi \in \Lambda^k V.$$

Exercise. Check these relations between $\wedge$, $\llcorner$, and $\cdot$.

By associativity, the left and right actions of Cliff(V) commute with one another, so $\Lambda^* V$ carries two commuting actions of the Clifford algebra. The complexification of $\Lambda^* V$ also carries commuting representations of the complexified Clifford algebra.

Starting from the inner product on V we define a sesquilinear (bilinear if V is real) map

$$(\cdot|\cdot)^p : \Lambda^p V \times \Lambda^p V \to \mathbb{R}$$

by

$$(u_1 \wedge \cdots \wedge u_p | v_1 \wedge \cdots \wedge v_p)^p := \det \begin{pmatrix} (u_1|v_1) & \cdots & (u_1|v_p) \\ \vdots & \ddots & \vdots \\ (u_p|v_1) & \cdots & (u_p|v_p) \end{pmatrix}.$$

Choose an oriented orthonormal basis $e_1, \dots, e_n$ of V and let $\sigma = e_1 \wedge \cdots \wedge e_n$. For $\lambda \in \Lambda^k V$ or $\lambda \in \Lambda^k(V \otimes \mathbb{C})$, the map

$$\lambda \wedge \cdot : \Lambda^{n-k} V \to \Lambda^n V,$$

is linear, and as $\Lambda^n V$ is one-dimensional, there exists a unique $f_\lambda \in Hom(\Lambda^{n-k} V, \mathbb{R})$ such that

$$\lambda \wedge \mu = f_\lambda(\mu)\sigma, \quad \text{for all} \quad \mu \in \Lambda^{n-k} V.$$

As $\Lambda^{n-k} V$ is an inner product space, every such linear form is given by the inner product with a fixed element of $\Lambda^{n-k} V$, which in this case depends on λ. Denote this element by $*\lambda$. So

$$f_\lambda(\mu) = (\mu| *\lambda)^{n-k},$$

and

$$\lambda \wedge \mu = (\mu| *\lambda)^{n-k}\sigma, \quad \text{for all} \quad \mu \in \Lambda^{n-k} V.$$

The map

$$* : \Lambda^k V \to \Lambda^{n-k} V, \qquad \lambda \longrightarrow *\lambda$$

is called the **Hodge Star Operator**.

Lemma 2.8. *If V has a positive definite inner product, and $\lambda, \mu \in \Lambda^k V$, then*

$$*(*\lambda) = (-1)^{k(n-k)}\lambda,$$
$$\lambda \wedge *\mu = \mu \wedge *\lambda = (\lambda|\mu)^k \sigma.$$

This discussion of actions of the Clifford algebra on $\Lambda^* V$ and the Hodge star operator carry over to real Clifford algebras. Now, in general, $\omega_{\mathbb{C}}$ is not an element of the real Clifford algebra (supposing V to be the complexification of a real vector space). Nevertheless, when it is in the real Clifford algebra we will see that $\omega_{\mathbb{C}}$ and the Hodge star operator are closely related.

All of these constructions extend to the exterior algebra of differential forms on a manifold M. Here we consider the vector bundle $\Lambda^* M$ and the sections $\Gamma(\Lambda^* M)$ with all the above operations defined pointwise. Similarly, we let $\mathbb{C}$liff(M) denote the sections of the bundle of algebras $\mathbb{C}$liff(T^*M, g), where g is a Riemannian inner product on T^*M. The complex volume form $\omega_{\mathbb{C}}$ is defined using a partition of unity and local orthonormal bases of T^*M. One can then check that $\omega_{\mathbb{C}}$ is a globally defined section of $\mathbb{C}$liff(M).

2.4 The Hodge-de Rham operator. We now have sufficient information to build our first geometric example. References for all the material in this Section are [BGV, LM]. As usual (M, g) is a closed oriented Riemannian manifold and $L^2(\Lambda^* M, g)$ is the Hilbert space completion of the smooth sections of the exterior bundle $\Lambda^* T^*_{\mathbb{C}} M$ with respect to the inner product

$$\langle \omega, \rho \rangle_g = \int_M \omega \wedge *\overline{\rho}.$$

Here $*$ is the Hodge $*$-operator, described in the previous Section. In this inner product, forms of different degrees are orthogonal, and the inner product is positive definite, as

$$\omega \wedge *\bar{\omega} = (\omega|\bar{\omega}) dvol.$$

The exterior derivative d extends to a closed unbounded operator on $L^2(\Lambda^* M, g)$, [HR, Lemma 10.2.1]. We let d^* be the adjoint of the exterior derivative with respect to this inner product. We let $\mathcal{D} = d + d^*$, and call this the **Hodge-de Rham operator**. Since M is closed and so has no boundary, this operator is formally self-adjoint (and so symmetric) on the domain given by smooth forms, and hence by [HR, Corollary 10.2.6], it extends uniquely to a self-adjoint operator on $L^2(\Lambda^* M, g)$.

2.4.1 The symbol and ellipticity. Before analysing this example any further, we need to recall the notion of principal symbol of a differential operator $D : \Gamma(E) \to \Gamma(F)$ between sections of vector bundles E, F over M. Letting $\pi : T^* M \to M$ be the projection, the principal symbol σ_D associates to each $x \in M$ and $\xi \in T^*_x M$ a linear map $\sigma_D(x, \xi) : \pi^*(E_x) \to \pi^*(F_x)$ defined as follows. If D is order m and in local coordinates we have

$$D = \sum_{|\alpha| \leq m} M_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \qquad \xi = \sum \xi_k dx^k \in T^*_x M$$

then

$$\sigma_D(x, \xi) = i^m \sum_{|\alpha| = m} M_\alpha(x) \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

This local coordinate description can be pasted together to give a globally defined map

$$\sigma_D : T^* M \to Hom(\pi^*(E), \pi^*(F)).$$

Lemma 2.9. *[HR, Chapter 10] Let D be a first order differential operator on a smooth compact manifold M. Then for $f \in C^\infty(M)$*

$$[D, f] = \sigma_D(df).$$

Proof. This is just a computation. □

Let's apply this result to the Hodge-de Rham operator. First observe that

$$\begin{aligned} (d(f\omega)|\rho) &= (df \wedge \omega|\rho) + (f d\omega|\rho) \\ &= (\omega|d\bar{f} \llcorner \rho) + (d\omega|\bar{f}\rho) \quad \text{since} \quad (v\wedge)^* = \bar{v}\llcorner \\ &= (\omega|d\bar{f} \llcorner \rho) + (\omega|d^*(\bar{f}\rho)) \end{aligned}$$

So

$$(\omega|d^*(\bar{f}\rho)) = (\omega|\bar{f} d^* \rho) - (\omega|d\bar{f} \llcorner \rho).$$

Since this is true for all forms ω, ρ and all smooth functions f, we deduce that for all forms φ and functions f

$$d^*(f\varphi) = f d^* \varphi - df \llcorner \varphi.$$

Now for $\varphi \in \Gamma(\Lambda^* M)$ we can compute

$$[d + d^*, f]\varphi = df \wedge \varphi + f d\varphi - df \llcorner \varphi + f d^* \varphi - f d\varphi - f d^* \varphi = df \wedge \varphi - df \llcorner \varphi.$$

Hence the principal symbol of $d+d^*$ is given by the left Clifford action on $\Lambda^* M$. In particular, for all $f \in C^\infty(M)$, the commutator $[d + d^*, f]$ extends to a bounded operator on $L^2(\Lambda^* M, g)$.

Much of the following relies on extending ideas based on differential operators to pseudodifferential operators. These form a wider class of operators that includes integral operators. The reason for expanding the class of operators is so that (approximate) inverses of differential operators can be treated by the same methods as differential operators. We will not cover the theory of pseudodifferential operators on manifolds in these notes, just quoting some results. The reader who wants to learn more can find many discussions, such as [Gi, LM, Sh].

The reason we are not emphasising the pseudodifferential calculus is that the main results of this theory are proved using local or pseudolocal constructions. This means that many key results rely on carrying out estimates in local coordinates, and pasting these together over the manifold. It is precisely these local estimates which do not persist in the noncommutative examples. What does tend to generalise to the noncommutative world are the properties which have global descriptions and proofs, [H].

One result about pseudodifferential operators we need is the following.

Lemma 2.10. *Suppose that Q, $P : \Gamma(E) \to \Gamma(E)$ are two (pseudo)differential operators on the same vector bundle $E \to M$ of orders q, $p \geq 0$ respectively. If their principal symbols commute, then*

$$\mathrm{order}([Q, P]) \leq q + p - 1.$$

Moreover if both Q, P are differential operators, so is the commutator.

Since the operator of multiplication of differential forms by a function f has principal symbol f Id, which commutes with any endomorphism, we find that for a first order differential operator such as $d + d^*$, the commutator $[d + d^*, f \text{ Id}]$ is order zero, namely an endomorphism.

Definition 2.11. *Let E, F be complex vector bundles over the compact manifold M. Let $P : \Gamma(E) \to \Gamma(F)$ be a differential operator with principal symbol $\sigma_P : T^*M \to Hom(\pi^*(E), \pi^*(F))$. If for all $x \in M$ and $0 \neq \xi \in T^*_x M$ we have that $\sigma_P(x, \xi)$ is an isomorphism, we call P an* **elliptic** *operator.*

Since $\xi \cdot \xi \cdot = -\|\xi\|^2$, where the norm is the one coming from the inner product in $T^*_x M$, we see that $(d + d^*)^2$ has invertible principal symbol for all $\xi \neq 0$, and so is elliptic. It then follows easily that $d + d^*$ is also elliptic.

Elliptic operators have approximate inverses, as we now describe. Let $P : \Gamma(E) \to \Gamma(E)$, where $E \to M$ is a vector bundle, be an elliptic pseudodifferential operator of order $m > 0$, say. Then there exists an elliptic operator $Q : \Gamma(E) \to \Gamma(E)$ of order $-m$ such that $PQ - \text{Id}$ and $QP - \text{Id}$ are both 'smoothing operators'. The smoothing operators act as compact operators on $L^2(E)$, and so we have something analogous to invertibility of P modulo compacts. However, there are subtleties, as operators of positive order such as P have unbounded realisations on Hilbert space.

2.4.2 Ellipticity and Fredholm properties. We now discuss the definition of unbounded Fredholm operators such as $d+d^*$. The first problem to handle is the fact that $d+d^*$ is not a bounded operator on $L^2(\Lambda^* M, g)$. The first difficulty unboundedness presents is that while an easy integration by parts shows that $d + d^*$ is symmetric on smooth forms, this is not the same as self-adjointness. However, every symmetric differential operator on a closed manifold is essentially self-adjoint, [HR, Corollary 10.2.6] and so $d + d^*$ has a unique self-adjoint extension given by its closure. We denote this self-adjoint extension by the same symbol.

Having dealt with self-adjointness, we can use the functional calculus to introduce Sobolev spaces. These allow us to view $d + d^*$ as a bounded operator between appropriate spaces. There are other definitions of Sobolev spaces based on local constructions; however, we employ a definition employing the spectral theorem since this is what we will be forced to use in the noncommutative case.

Definition 2.12. *Let M be a compact oriented n-dimensional Riemannian manifold. For $s \geq 0$, define*

$$L^2_s(\Lambda^* M, g) = \{\xi \in L^2(\Lambda^* M, g) : (1+\Delta)^{s/2}\xi \in L^2(\Lambda^* M, g)\},$$

where $\Delta = (d + d^*)^2$ is the Hodge Laplacian. Then $L^2_s(\Lambda^* M, g)$ is a Hilbert space for the inner product

$$\langle \xi, \eta\rangle_s := \langle \xi, \eta\rangle + \langle (1+\Delta)^{s/2}\xi, (1+\Delta)^{s/2}\eta\rangle$$

and we call this the s-th Sobolev space.

Remark. This construction may be generalised to any vector bundle E by choosing a connection ∇ on E, and defining the connection Laplacian $\Delta_E := \nabla^*\nabla$ which is positive.

The point of Sobolev spaces for us is the following proposition.

Proposition 2.13. *A differential operator $D : \Gamma(\Lambda^* M) \to \Gamma(\Lambda^* M)$ of order $m \geq 0$ extends to a bounded operator $D : L^2_s(\Lambda^* M) \to L^2_{s-m}(\Lambda^* M)$ for all $s \geq m$.*

Exercise. Prove Proposition 2.13 (see [LM, Proposition 2.13] for more information).

Even though the operator $d + d^*$ is Fredholm in a suitable sense, it is self-adjoint, so the index of $d + d^*$ will be zero. There is an algebraic aspect of $d + d^*$ we have neglected. If we define $\gamma : \Gamma(\Lambda^* M) \to \Gamma(\Lambda^* M)$ by

$$\gamma(\omega) = (-1)^k \omega, \quad \omega \in \Lambda^k M$$

then since both d and d^* change the degree of a form by one we have

$$\gamma(d + d^*) = -(d + d^*)\gamma.$$

Since $\gamma^2 = 1$, γ has ± 1 eigenvalues (on $\Lambda^{even} M$ and $\Lambda^{odd} M$) and we can write

$$\Lambda^* M = \Lambda^{even} M \oplus \Lambda^{odd} M, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad d + d^* = \begin{pmatrix} 0 & (d+d^*)^- \\ (d+d^*)^+ & 0 \end{pmatrix}.$$

The differential operator $(d+d^*)^+ : \Gamma(\Lambda^{even} M) \to \Gamma(\Lambda^{odd} M)$ has adjoint $(d+d^*)^- : \Gamma(\Lambda^{odd} M) \to \Gamma(\Lambda^{even} M)$. We will see that in a suitable sense the operator $(d + d^*)^+ : L^2(\Lambda^{even} M) \to L^2(\Lambda^{odd} M)$ is Fredholm. Our objective then is to compute the index of $(d + d^*)^+$.

What we would like to do is define the index of $(d + d^*)^+$ to be the index of

$$(d + d^*)^+ : L^2_s(\Lambda^{even}) \to L^2_{s-1}(\Lambda^{odd})$$

for $s \geq 1$. However, for this to be well defined we need to know that the index is independent of s. The following is the key result linking ellipticity of pseudodifferential operators with Fredholm theory.

Theorem 2.14. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic pseudodifferential operator of order $m \geq 0$ on M. Then*
1) For any open set $U \subset M$ and any $\xi \in L^2_s(E)$,

$$P\xi|_U \in C^\infty \Rightarrow \xi|_U \in C^\infty \tag{2.3}$$

2) For each $s \geq m$, P extends to a bounded Fredholm operator $P : L^2_s(E) \to L^2_{s-m}(F)$ whose index is independent of s.
3) For each $s \geq m$ there is a constant C_s such that

$$\|\xi\|_s \leq C_s(\|\xi\|_{s-m} + \|P\xi\|_{s-m}) \qquad \text{elliptic estimate} \tag{2.4}$$

for all $\xi \in L^2_s(E)$. Hence the norms $\|\cdot\|_s$ and $\|\cdot\|_{s-m} + \|P\cdot\|_{s-m}$ on $L^2_s(E)$ are equivalent.

The key to proving this theorem is the existence of a parametrix. Once this is proved, the rest can be deduced reasonably simply; see [LM] for a clear proof. The elliptic estimate also says that we can always restrict attention to operators of fixed order. Mostly we consider operators of order zero and one, for if P is of order $m > 0$ we may consider the order zero operator $P(1+\Delta)^{-m/2}$, and so on.

So now we see that the index can be defined in a sensible way. However, it is natural to ask whether it may be related to the index of a bounded linear operator on $L^2(\Lambda^*M, g)$ without using Sobolev spaces.

Proposition 2.15. *Let $\mathcal{D}$ be a self-adjoint elliptic pseudodifferential operator on a bundle $E \to M$. Then the densely defined operator $(1+\mathcal{D}^2)^{-1/2} : L^2(E) \to L^2(E)$ is bounded and extends to a compact operator on $L^2(\Lambda^*M, g)$. Hence the operator $\mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ is a self-adjoint Fredholm operator.*

Proof. The functional calculus for self-adjoint operators tells us that $\mathcal{D}^2$ is positive and so $1+\mathcal{D}^2$ is (boundedly) invertible. The operator $(1+\mathcal{D}^2)^{-1/2}$ maps $L^2 = L^2_0$ onto L^2_1, and is bounded in norm by at most 1, again by the functional calculus. The inclusion of L^2_1 into L^2_0 is a compact linear operator by the Rellich Lemma (this uses the compactness of M), and so $(1+\mathcal{D}^2)^{-1/2} : L^2 \to L^2$ is a compact operator.

The second statement follows because

$$\Big(\mathcal{D}(1+\mathcal{D}^2)^{-1/2}\Big)^2 = \mathcal{D}^2(1+\mathcal{D}^2)^{-1} = 1 - (1+\mathcal{D}^2)^{-1}$$

and so $\mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ has a parametrix (itself), and so is Fredholm. □

Similarly, given a self-adjoint elliptic first order (pseudo)differential operator $\mathcal{D}$ which may be written $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$ in a fashion similar to that for the operator $d + d^*$, the index of $\mathcal{D}^+(1+\mathcal{D}^2)^{-1/2}$ is defined and equals that of all the closed extensions of $\mathcal{D}$ on Sobolev spaces by Proposition 2.15. Before discussing the index of $(d+d^*)^+$, we quote one further result about elliptic differential operators.

Theorem 2.16. *[LM, Thm 5.5] Let $P : \Gamma(E) \to \Gamma(E)$ be an elliptic self-adjoint differential operator over a compact Riemannian manifold. Then there is an L^2-orthogonal direct sum decomposition*

$$\Gamma(E) = \ker P \oplus \text{Image } P.$$

2.4.3 The index of the Hodge-de Rham operator. To determine the index of $(d+d^*)^+$, we are going to need a little more machinery. Let $\Delta = (d+d^*)^2$ be the Laplacian on forms, and observe that $\Delta = dd^* + d^*d$. Then $\text{Image}(\Delta) = \text{Image}(d) + \text{Image}(d^*)$ and so by Theorem 2.16 we have:

Proposition 2.17 (The Hodge Decomposition Theorem). *Let M be a compact oriented Riemannian manifold, and let $\mathbf{H}^p$ denote the kernel of $\Delta = (d+d^*)^2$ on p-forms. Then there is an L^2-orthogonal direct sum decomposition*

$$\Gamma(\Lambda^p M) = \mathbf{H}^p \oplus \text{Image}(d) \oplus \text{Image}(d^*), \quad p = 0, \ldots, n. \tag{2.5}$$

In particular, there is an isomorphism

$$\mathbf{H}^p \cong H^p_{dR}(M;\mathbb{R}), \quad p = 0, \ldots, n,$$

where $H^p_{dR}(M;\mathbb{R})$ denotes the p-th de Rham cohomology group.

Proof. (Sketch) The first statement follows directly from Theorem 2.16 and the fact that $d^2 = d^{*2} = 0$. For the second, we observe that Equation (2.5) says

$$\mathbf{H}^p \oplus \mathrm{Image}(d) = \mathrm{coker}(d^*) = \ker(d).$$

Hence

$$H^p_{dR}(M;\mathbb{R}) = \frac{\ker(d)}{\mathrm{Image}(d)} = \mathbf{H}^p.$$

□

From this result it is now not too difficult to prove the following:

Theorem 2.18. *The index of $(d + d^*)^+$ is*

$$\begin{aligned} \mathrm{Index}(d + d^*)^+ &= \sum_{k=0}^{n} (-1)^k \dim H^k_{dR}(M;\mathbb{R}) \\ &= \chi(M) = \text{the Euler characteristic of } M \\ &= \text{a homotopy invariant of } M \\ &= \text{independent of the metric } g. \end{aligned} \tag{2.6}$$

These index calculations depend on the analysis of pseudodifferential operators, which we have omitted. In particular, it is the pseudodifferential machinery which allows us to see that for P elliptic, $Pu \in C^\infty \Rightarrow u \in C^\infty$. This has the corollary that all elements of the kernel and cokernel of $(d + d^*)^+$ are smooth. In some sense it is this fact that tells us that the index is independent of s. For an introduction to this pseudodifferential theory, see [LM] or [G].

2.5 The definition of a spectral triple. The example of the previous Section motivates the following notions.

Definition 2.19. *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a $*$-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$, that is,*

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}), \quad \pi \text{ is a } *-\text{homomorphism}$$

along with a densely defined self-adjoint (typically unbounded) operator

$$\mathcal{D} : \mathrm{dom}\ \mathcal{D} \subset \mathcal{H} \to \mathcal{H}$$

satisfying the following conditions.
1) For all $a \in \mathcal{A}$, $\pi(a)\mathrm{dom}\ \mathcal{D} \subset \mathrm{dom}\ \mathcal{D}$ and the densely defined operator $[\mathcal{D}, \pi(a)] := \mathcal{D}\pi(a) - \pi(a)\mathcal{D}$ is bounded (and so extends to a bounded operator on all of $\mathcal{H}$ by continuity),
2) For all $a \in \mathcal{A}$ the operator $\pi(a)(1 + \mathcal{D}^2)^{-1/2}$ is a compact operator.

If in addition there is an operator $\gamma \in \mathcal{B}(\mathcal{H})$ with $\gamma = \gamma^*$, $\gamma^2 = 1$, $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$, and for all $a \in \mathcal{A}$ $\gamma\pi(a) = \pi(a)\gamma$, we call the spectral triple **even** *or* **graded***. Otherwise it is* **odd** *or* **ungraded***.*

Remarks. This appears to be an unwieldy definition. There are numerous ingredients and using unbounded operators makes things technically more difficult. However, as we have seen, this is actually the structure one encounters naturally when doing index theory, and has computational advantages, as we shall see. We will nearly always dispense with the representation π, treating $\mathcal{A}$ as a subalgebra of $\mathcal{B}(\mathcal{H})$.

In the sequel we will make a general simplifying assumption unless explicitly mentioned otherwise. Namely, in all spectral triples the algebra $\mathcal{A}$ is unital; i.e. there is $1 \in \mathcal{A}$ such that $1a = a1 = a$, $1^* = 1$. If $1 \in \mathcal{A}$ acts as the identity of the Hilbert space, then this implies in particular that $(1+\mathcal{D}^2)^{-1/2}$ is a compact operator. We will assume in the following that even if $1 \in \mathcal{A}$ does not act as the identity, we always have $(1+\mathcal{D}^2)^{-1/2}$ compact.

Example 3. The 'Hodge-de Rham' triple $(C^\infty(M), L^2(\Lambda^*M, g), d+d^*)$ of an oriented compact manifold M is an example of a spectral triple (we have in fact already verified this). It is always even, being graded by the degree of forms modulo 2.

Example 4. Let $\mathcal{H} = L^2(S^1)$ and $\mathcal{A} = C^\infty(S^1)$ act by multiplication operators on $\mathcal{H}$. The one-dimensional Dirac operator (see next Chapter) is

$$\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$$

where we are using local coordinates to define $\mathcal{D}$. This is an odd spectral triple, as a little Fourier theory will reveal. This is a useful example to understand in detail.

2.5.1 The Fredholm index in a spectral triple. We want to study index theory in a spectral triple by analogy with the example of the Hodge-de Rham operator. First we need to show that the unbounded operator $\mathcal{D}$ appearing in the spectral triple has a well-defined index and for this we need the notion of an unbounded Fredholm operator. To this end we form the Hilbert space $\mathcal{H}_1 = \{\xi \in \mathcal{H} : \mathcal{D}\xi \in \mathcal{H}\}$ with the inner product

$$\langle \xi, \eta \rangle_1 := \langle \xi, \eta \rangle + \langle \mathcal{D}\xi, \mathcal{D}\eta \rangle$$

Then $\mathcal{D}$ is a bounded operator from $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}$ (**Exercise**). We will say that $\mathcal{D}$ is an unbounded Fredholm operator if $\mathcal{D} : \mathcal{H}_1 \to \mathcal{H}$ is a Fredholm operator in the sense of Definition 2.1.

Lemma 2.20. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. Then $\mathcal{D}$ is unbounded Fredholm.*

Proof. To see this, we produce an inverse up to compacts. Such an approximate inverse (parametrix) is given by

$$\mathcal{D}(1+\mathcal{D}^2)^{-1} : \mathcal{H} \to \mathcal{H}_1$$

since

$$\mathcal{D} \cdot \mathcal{D}(1+\mathcal{D}^2)^{-1} = 1 - (1+\mathcal{D}^2)^{-1}.$$

□

Exercise: Fill in the details of this proof.

Since $\mathcal{D}$ is self-adjoint, it has zero index, but when $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is even, or graded, we define

$$\mathcal{D}^+ = \frac{1-\gamma}{2}\mathcal{D}\frac{1+\gamma}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}^+ : \mathcal{H}_1^+ \to \mathcal{H}^-.$$

For an even spectral triple, this is the operator of interest, and it too is Fredholm since $\mathcal{D}^+\mathcal{D}^-(1+\mathcal{D}^2)^{-1}$ is 'almost' the identity on $\mathcal{H}_-$. Since $\mathcal{D}$ is unbounded, we need to be careful about what we mean by the index. This is analogous to using Sobolev spaces for elliptic operators on manifolds.

Definition 2.21. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a spectral triple. For $s\geq 0$ define $\mathcal{H}_s=\{\xi\in\mathcal{H}:(1+\mathcal{D}^2)^{s/2}\xi\in\mathcal{H}\}$. With the inner product*

$$\langle\xi,\eta\rangle_s:=\langle\xi,\eta\rangle+\langle(1+\mathcal{D}^2)^{s/2}\xi,(1+\mathcal{D}^2)^{s/2}\eta\rangle,$$

$\mathcal{H}_s$ is a Hilbert space. Finally let

$$\mathcal{H}_\infty:=\bigcap_{s\geq 0}\mathcal{H}_s=\bigcap_{s\geq 0}\mathrm{dom}\,(1+\mathcal{D}^2)^{s/2}.$$

Corollary 2.22. *Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an even spectral triple with grading γ. Write $\mathcal{D}=\begin{pmatrix}0&\mathcal{D}^-\\ \mathcal{D}^+&0\end{pmatrix}$, and let $\mathcal{D}^+_s$ be the restriction $\mathcal{D}^+:\mathcal{H}_s\to\mathcal{H}_{s-1}$ where $\mathcal{H}_0=\mathcal{H}$, $\mathcal{H}_1=\mathrm{dom}\,(\mathcal{D})$. For all $s\geq 1$ we have*

$$\mathrm{Index}(\mathcal{D}^+_s)=\mathrm{Index}(\mathcal{D}^+)=\mathrm{Index}(\mathcal{D}^+(1+\mathcal{D}^2)^{-1/2}),$$

where the middle index is of $\mathcal{D}^+:\mathcal{H}^+_1\to\mathcal{H}^-$ and the last is the index of the bounded operator $\mathcal{D}^+(1+\mathcal{D}^2)^{-1/2}:\mathcal{H}^+\to\mathcal{H}^-$.

Proof. Suppose that $\mathcal{D}\xi=\lambda\xi$, so that ξ is an eigenvector. Then, since $\xi\in\mathrm{dom}\,\mathcal{D}$, we see that $\xi\in\mathcal{H}_\infty$. So all the eigenvectors of $\mathcal{D}$ lie in $\mathcal{H}_\infty$, and in particular, if $\mathcal{D}\xi=0$, $\xi\in\mathcal{H}_\infty$. Consequently, if $\mathcal{D}^+\xi=0$, $\xi\in\mathcal{H}^+_\infty$, and similarly for $\mathcal{D}^-$. Hence the kernel and cokernel of $\mathcal{D}^+$ consist of elements of $\mathcal{H}_\infty$, and the index is independent of which 'Sobolev space' we use. The equality with $\mathrm{Index}(\mathcal{D}^+(1+\mathcal{D}^2)^{-1/2})$ now follows from the invertibility of $(1+\mathcal{D}^2)^{-1/2}$. □

Example 5. In finite dimensions, i.e. $\dim\mathcal{H}<\infty$, we can take $\mathcal{A}$ to be finite dimensional, and so we are dealing with sums of matrix algebras. There is then really no constraint in the definition of spectral triple. If we have an even triple $(\mathcal{A},\mathcal{H},\mathcal{D},\gamma)$ which is finite dimensional in this sense, then

$$\mathrm{Index}(\mathcal{D}_+:\mathcal{H}_+\to\mathcal{H}_-)=\dim\mathcal{H}_+-\dim\mathcal{H}_-$$

by the rank nullity theorem.

2.5.2 Connes' metric for spectral triples. Here we mention one important geometric feature of spectral triples, the metric on the state space of the algebra. This is of some importance for the construction of spectral triples for particular algebras. The heuristic idea that the operator '$\mathcal{D}$' of a spectral triple is some sort of differentiation allows us to use metric ideas to construct '$\mathcal{D}$' so it is compatible with the notion of difference quotients coming from a given metric. We give examples below.

A state on a unital C^*-algebra is a linear functional $\phi:A\to\mathbb{C}$ with $\phi(1)=1=\|\phi\|$ and $\phi(a^*a)\geq 0$ for all $a\in A$. This is a convex space, and the extreme points (those states that are not convex combinations of other states) are called pure states. We denote the state space by $\mathcal{S}(A)$ and the pure states by $\mathcal{P}(A)$. The pure states of a commutative C^*-algebra, $C(X)$, correspond to point evaluations. So, for $x\in X$, defining $\phi_x(f)=f(x)$ for all $f\in C(X)$ gives a pure state, and they are all of this form. Indeed, the weak* topology on $\mathcal{P}(C(X))$ is the original topology on X, and $\mathcal{P}(C(X))\simeq X$ (Gel´fand Theory).

We now show that spectral triples are 'noncommutative metric spaces'. We begin with the definition of the metric.

Lemma 2.23. *Suppose that $(\mathcal{A},\mathcal{H},\mathcal{D})$ is a spectral triple such that*

$$\{a\in\mathcal{A}\setminus\{\mathbb{C}\cdot 1\}:\ \|[\mathcal{D},a]\|\leq 1\}\tag{2.7}$$

is a norm bounded set in the Banach space $\overline{\mathcal{A}}/\mathbb{C}\cdot 1$. Then

$$d_{\mathcal{D}}(\phi,\psi)=\sup_{a\in\mathcal{A}}\{|\phi(a)-\psi(a)|:\ \|[\mathcal{D},a]\|\leq 1\}$$

defines a metric on $\mathcal{P}(\overline{\mathcal{A}})$, the pure state space of $\overline{\mathcal{A}}$.

Proof. The triangle inequality is a direct consequence of the definition. To see that $d(\phi,\psi)=0$ implies $\phi=\psi$, suppose $\phi\neq\psi$. Then there is some $a\in\overline{\mathcal{A}}$ with $\phi(a)\neq\psi(a)$, and we can use the density of $\mathcal{A}$ in $\overline{\mathcal{A}}$ to find an element of $b\in\mathcal{A}$ such that $\phi(b)\neq\psi(b)$, and so $d(\phi,\psi)\neq 0$. The norm boundedness of the set in (2.7) gives the finiteness of the distance between any two pure states. □

In the future, when we mention the metric associated to a spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$, we implicitly assume that the condition (2.7) is met. In particular, it means that no element of $\mathcal{A}$ except scalars commutes with $\mathcal{D}$.

The metric is actually defined on the whole state space $\mathcal{S}(A)$, but the metric on $\mathcal{S}(A)$ need not be determined by the restriction of the metric to the pure states, even for commutative examples, [Rie3]. Much more information about 'compact quantum metric spaces' is contained in [Rie1, Rie2, Rie3] and references therein. In particular, Rieffel proves that if the set in Equation 2.7 is in fact pre-compact in $\overline{\mathcal{A}}/\mathbb{C}\cdot 1$, then the metric induces the weak* topology on the state space.

Note that when $\mathcal{A}$ is commutative, so that $\mathcal{A}$ is an algebra of (at least continuous for the weak* topology) functions on $X=\mathcal{P}(\overline{\mathcal{A}})$, the metric topology on $\mathcal{P}(\overline{\mathcal{A}})$ is automatically finer than the weak* topology. In the case of a smooth Riemannian spin manifold, whose algebra of smooth functions is finitely generated by the (local) coordinate functions, not only do the topologies on the pure state space agree, so do the metrics, [C1, C2].

Lemma 2.24. *If $(C^\infty(M),L^2(E),\mathcal{D})$ is the spectral triple of any 'Dirac type operator' of the Clifford module E on a compact Riemannian spin manifold (M,g), then*

$$d_{\mathcal{D}}(\phi,\psi)=d_\gamma(\phi,\psi),\quad \textit{for all}\ \ \phi,\,\psi\in\mathcal{P}(C(M)),$$

where d_γ is the geodesic distance on X.

For a discussion of general Dirac type operators, see [BGV, LM]. It is enough for us that $d+d^*$ and the Dirac operator of a spin or spin^c structure (discussed in the next Chapter) are of Dirac type.

More generally, whenever $\mathcal{A}$ is commutative, we can take $\mathcal{A}\subseteq \mathrm{Lip}_d(\mathcal{P}(\mathcal{A}))$, the Lipschitz functions with respect to the metric topology, since

$$|a(\phi)-a(\psi)|:=|\phi(a)-\psi(a)|\leq \|[\mathcal{D},a]\|d(\phi,\psi)$$

for all $a\in\mathcal{A}$, $\phi,\psi\in\mathcal{P}(\mathcal{A})$.

Example 6. Here is a simple way to use metric ideas to build a spectral triple. Let $\mathcal{A}=\mathbb{C}\oplus\mathbb{C}$ be the continuous functions on two points. Let $\mathcal{A}$ act on the Hilbert space $\mathcal{H}=\mathbb{C}^2$ by multiplication. Let $0\neq m\in\mathbb{R}$ and set $\mathcal{D}=\begin{pmatrix}0 & m\\ m & 0\end{pmatrix}$ with the grading $\gamma=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. The index here is zero, but the distance between the two points is $\frac{1}{m}$. Check this as an **Exercise**. If we want a nonzero index as well, let $(a,b)\in\mathcal{A}$ act on $\mathbb{C}^3$ by

$$\pi(a,b)\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_2\end{pmatrix}=\begin{pmatrix}a\xi_1\\ b\xi_2\\ b\xi_3\end{pmatrix}$$

and define $\mathcal{D}=\begin{pmatrix}0_2 & \bar{m}\\ (\bar{m})^T & 0\end{pmatrix}$ where now $\bar{m}=\begin{pmatrix}m\\ 0\end{pmatrix}$ and $\gamma=1\oplus -1_2$. As a further **Exercise**, calculate the distance and the index for this example.

These simple cases lead quickly to more complicated examples of similar type with different indices and different numbers of points and so on. However, as the number of points goes up, the expression for the distance (given a generic operator $\mathcal{D}$) becomes more and more complicated. It has been shown that for a particular class of examples of this form, polynomials of degree 5 and more arise, so the distance is generically not computable using arithmetic and the extraction of roots, [IKM].

Another level of complexity is added when we consider matrix algebras instead of copies of $\mathbb{C}$. This is because we can have much more complicated commutation relations. We refer to [IKM] for a fuller discussion of these examples, but recommend that the interested reader first gain some experience by computing special cases. Note that these are *not* just toy models. Taking the product of the Dirac spectral triple of a manifold with certain spectral triples for sums of matrix algebras yields spectral triples with close relationships with particle physics. The reconstruction of the (classical) Lagrangian of the standard model of particle physics from such a procedure is a fundamental example. See [GV] for an introduction and a guide to some of the extensive literature on this subject. For more information on these finite spectral triples, see [IKM, K, PS].

There are many other examples of spectral triples built with the intention of recovering or studying metric data (also dimension type data). Some interesting examples are contained in the recent papers of Erik Christensen and Cristina Ivan, as well as their coauthors. See [CIL, CI1, CI2, AC]. We present one more example here which we will look at again when we come to dimensions.

Example 7. This example illustrates how to use metric ideas to construct a spectral triple. Many thanks to Nigel Higson for relating it to us. Consider the Cantor 'middle thirds' set K. So start with the unit interval, and remove the (open) middle third. Then remove the open middle third of the two remaining subintervals, etc. Observe that end points of removed intervals come in pairs, e_-, e_+ where e_- is the left endpoint of a gap and e_+ is the right endpoint of a gap. Every end point except $0, 1$ is one (and only one) of these two types, and we take $0, 1$ as a pair.

Let $\mathcal{H} = l^2(\text{end points})$ and $\mathcal{A}$ be the locally constant functions on K. Recall that a function $f : K \to \mathbb{C}$ is locally constant if for all $x \in K$ there is a neighbourhood U of x such that f is constant on U. Define $\mathcal{D} : \mathcal{A} \cap \mathcal{H} \to \mathcal{H}$ by

$$(\mathcal{D}f)(e_+) = \frac{-if(e_-)}{e_+ - e_-}, \quad (\mathcal{D}f)(e_-) = \frac{if(e_+)}{e_+ - e_-}.$$

The closure of this densely defined operator is self-adjoint (**Exercise**). Also

$$[\mathcal{D}, f]g(e_+) = i\frac{f(e_+) - f(e_-)}{e_+ - e_-}g(e_-), \quad [\mathcal{D}, f]g(e_-) = i\frac{f(e_+) - f(e_-)}{e_+ - e_-}g(e_+).$$

For f locally constant the commutator $[\mathcal{D}, f]$ defines a bounded operator.

Let δ_{e_+} be the function which is one on e_+ and zero elsewhere, and similarly for δ_{e_-}. Now these are not locally constant functions, but are in the domain of (the closure of) $\mathcal{D}$. We observe that $\text{span}\{\delta_{e_+}, \delta_{e_-}\}$ is invariant under $\mathcal{D}$ since

$$\mathcal{D}\delta_{e_+} = \frac{i\delta_{e_-}}{e_+ - e_-}, \quad \mathcal{D}\delta_{e_-} = -\frac{i\delta_{e_+}}{e_+ - e_-}.$$

Indeed, in the basis given by $\delta_{e_+}, \delta_{e_-}$, $\mathcal{D} = \frac{i}{e_+ - e_-}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence $\mathcal{D}$ has eigenvalues $\pm 1/(e_+ - e_-) = \pm 3^n$ if the points appear in the n-th stage of the construction, and their multiplicity is 2^{n-1}. Thus $(1 + \mathcal{D}^2)^{-1}$ is certainly compact and invertible.

It is now an **Exercise** to show that Connes' metric is precisely the usual metric on the Cantor set. We will return to this example when we discuss summability.

3 More spectral triples from manifolds

Our aim in this Chapter is to produce more examples of spectral triples from the theory of elliptic differential operators on manifolds. To achieve this we will need a little more differential geometry, as well as the Clifford algebra formalism we have introduced previously.

3.1 The signature operator. In dimensions $n = 4k$ there is another grading on the space $\Gamma(\Lambda^*T^*M)$ that allows us to define a new spectral triple. In these dimensions, the complex volume form $\omega_{\mathbb{C}}$ is given by $(-1)^k\omega = (-1)^k e_1 \cdots e_{4k}$ in terms of a local orthonormal basis. Consequently, the Clifford action by $\omega_{\mathbb{C}}$ maps the space of real sections of Λ^*M into itself. Moreover, for $\varphi \in \Lambda^pM$ we have

$$\omega_{\mathbb{C}} \cdot \varphi = (-1)^{k+p(p-1)/2} * \varphi,$$

where $*$ is the Hodge star operator. Since (in even dimensions) $d^* = - * d*$, we see that $d + d^*$ anticommutes with the action of $\omega_{\mathbb{C}}$, and we get a new grading. We already know that $d + d^*$ has compact resolvent, so when $\dim M = n = 4k$,

$$(C^\infty(M), L^2(\Lambda^*M, g), d + d^*, \omega_{\mathbb{C}})$$

is an even spectral triple. Now we ask: what is the index? The answer takes several steps.

Claim: The identification up to sign of $\omega_{\mathbb{C}}\cdot$ and $*$ gives us isomorphisms

$$\omega_{\mathbb{C}} : \mathbf{H}^p \to \mathbf{H}^{4k-p}$$

for each $p = 0, 1, \ldots, 2k$.

Proof. We know from the Hodge decomposition theorem that $\varphi \in \mathbf{H}^p$ if and only if $d\varphi = d^*\varphi = 0$. So let $\varphi \in \mathbf{H}^p$. Then $d\omega_{\mathbb{C}}\varphi = \pm\omega_{\mathbb{C}}d^*\varphi = 0$ and similarly, $d^*\omega_{\mathbb{C}}\varphi = \pm\omega_{\mathbb{C}}d\varphi = 0$. Since $\omega_{\mathbb{C}}^2 = 1$, we are done. □

So for $p = 0, \ldots, 2k-1$ the space $\mathbf{H}(p) = \mathbf{H}^p \oplus \mathbf{H}^{4k-p}$ has a decomposition

$$\mathbf{H}(p) = \mathbf{H}^+(p) \oplus \mathbf{H}^-(p) = \frac{(1+\omega_{\mathbb{C}})}{2}\mathbf{H}(p) \oplus \frac{(1-\omega_{\mathbb{C}})}{2}\mathbf{H}(p).$$

Claim: The subspaces $\frac{1}{2}(1 \pm \omega_{\mathbb{C}})\mathbf{H}(p)$ have the same dimension.

Proof. If $\{\varphi_1, \ldots, \varphi_m\}$ is a basis of $\mathbf{H}^p$, the subspace $\mathbf{H}^+(p)$ has basis $\{\varphi_1 + \omega_{\mathbb{C}}\varphi_1, \ldots, \varphi_m + \omega_{\mathbb{C}}\varphi_m\}$. Likewise $\mathbf{H}^-(p)$ has basis $\{\varphi_1 - \omega_{\mathbb{C}}\varphi_1, \ldots, \varphi_m - \omega_{\mathbb{C}}\varphi_m\}$. □

This allows us to compute

$$\ker \mathcal{D}^\pm = \mathbf{H}^\pm := \mathbf{H}^\pm(0) \oplus \mathbf{H}^\pm(1) \oplus \cdots \oplus \mathbf{H}^\pm(2k-1) \oplus (\mathbf{H}^{2k})^\pm,$$

where $(\mathbf{H}^{2k})^\pm = (1 \pm \omega_{\mathbb{C}})\mathbf{H}^{2k}$. Since the index is $\dim\ker\mathcal{D}^+ - \dim\ker\mathcal{D}^-$, we have

$$\mathrm{Index}(d + d^*) = \dim(\mathbf{H}^{2k})^+ - \dim(\mathbf{H}^{2k})^-.$$

Observe that on $\Lambda^{2k}M$, $\omega_{\mathbb{C}}$ and the Hodge star operator coincide. We define a new inner product on $\mathbf{H}^{2k}$ by

$$\langle \phi, \psi \rangle^{new} := \int_M \phi \wedge \psi.$$

Then, using $* = \omega_{\mathbb{C}}$ and $\int_M \phi \wedge *\phi = \|\phi\|^2$, we see that the signature of this inner product (the number of positive eigenvalues minus the number of negative eigenvalues) is precisely the difference in dimensions of the ± 1 eigenspaces of the $*$ operator. So, letting signature(M) denote the signature of this inner product we have our answer:

$$\mathrm{Index}((d+d^*)^+, \omega_{\mathbb{C}}) = \mathrm{signature}(M).$$

Remarkably, this is a homotopy invariant of the manifold.

3.2 Connections and twistings. To go further than the preceding Section it is very useful to make much of our discussion more streamlined by exploiting Clifford algebras. Recall that if W is a left module over a $*$-algebra $\mathcal{A}$, a Hermitian form is a map $(\cdot|\cdot) : W \times W \to \mathcal{A}$ such that for all u, v, $w \in W$ and $a \in \mathcal{A}$

$$(v+w|u) = (v|u) + (w|u), \quad (av|u) = a(v|u), \quad (v|u) = (u|v)^*.$$

If $\mathcal{A}$ is a pre-C^*-algebra, we can also ask for $(v|v) \geq 0$ in the sense of the C^*-closure of $\mathcal{A}$, and that $(v|v) = 0 \Rightarrow v = 0$. We assume that all our Hermitian forms satisfy these properties.

For a vector bundle $E \to M$, we know that $\Gamma(E)$ is a module over $C^\infty(M)$ via $(f\sigma)(x) = f(x)\sigma(x)$ for all $f \in C^\infty(M)$, $\sigma \in \Gamma(E)$ and $x \in M$. A Hermitian form is then a collection of positive definite inner products $(\cdot|\cdot)_x$ on E_x such that for all smooth sections $\sigma, \rho \in \Gamma(E)$, the function $x \to (\sigma(x)|\rho(x))_x$ is smooth. All complex vector bundles have such a smooth inner product.

Now let $E \to M$ be a smooth vector bundle, and let $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ be a connection. So ∇ is $\mathbb{C}$-linear and for all $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma.$$

We can extend ∇ to a map $\nabla : \Gamma(\Lambda^*M \otimes E) \to \Gamma(\Lambda^*M \otimes E)$ by defining for $\omega \in \Gamma(\Lambda^k M)$ and $\sigma \in \Gamma(E)$

$$\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^k \omega \otimes \nabla\sigma.$$

Claim: The operator ∇^2 is linear over $C^\infty(M)$.

Proof. Let $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$. Then

$$\nabla^2(f\sigma) = \nabla(df \otimes \sigma + f\nabla\sigma) = d^2 f \otimes \sigma - df \otimes \nabla\sigma + df \otimes \nabla\sigma + f\nabla^2\sigma = f\nabla^2\sigma.$$

□

Thus ∇^2 is a two-form with values in the endomorphisms of E (locally a matrix of two-forms). It is called the curvature of E.

A connection ∇ on a vector bundle $E \to M$ with Hermitian form $(\cdot|\cdot)$ is said to be compatible with $(\cdot|\cdot)$ if for all $\sigma, \rho \in \Gamma(E)$

$$d(\sigma|\rho) = (\nabla\sigma|\rho) + (\sigma|\nabla\rho),$$

where to interpret the right hand side we write in local coordinates $\nabla\sigma = \sum_i dx^i \otimes \sigma_i$ and $\nabla\rho = \sum_j dx^j \otimes \rho_j$ and then

$$(\nabla\sigma|\rho) + (\sigma|\nabla\rho) := \sum_i dx^i(\sigma_i|\rho) - \sum_j dx^j(\sigma|\rho_j).$$

Example 8. The Levi-Civita connection on TM or T^*M is compatible with the Riemannian metric. The curvature of the Levi-Civita connection is, by definition, the curvature of the manifold.

Lemma 3.1 (see [LM]). *Let M be a compact oriented manifold, and let c denote the usual left action of* $\mathbb{C}\mathrm{liff}(M)$ *on Λ^*M. Let ∇ denote the Levi-Civita connection on T^*M. Then*

$$d + d^* = c \circ \nabla.$$

Thus the Hodge-de Rham and Signature operators are both given by composing the Levi-Civita connection with the Clifford action. This is a very general recipe, and allows us to construct versions of these operators that are 'twisted' by a vector bundle.

If $E, F \to M$ are both vector bundles, with connections ∇^E, ∇^F respectively, we can define a connection $\nabla^{E,F}$ on the tensor product $E \otimes F$ by defining for all $\sigma \in \Gamma(E)$ and $\rho \in \Gamma(F)$

$$\nabla^{E,F}(\sigma \otimes \rho) = (\nabla^E \sigma) \otimes \rho + \sigma \otimes (\nabla^F \rho).$$

If E is a $\mathbb{C}\mathrm{liff}(M)$ module, then so is $E \otimes F$ by letting $\mathbb{C}\mathrm{liff}(M)$ act only on E. Provided that for all $\sigma \in \Gamma(E)$ and $\varphi \in \Gamma(T^*M)$ we have

$$\nabla^E(c(\varphi)\,\sigma) = c(\nabla^{T^*M}\varphi)\,\sigma + c(\varphi)\nabla^E\sigma,$$

we can form the operator

$$c \circ \nabla^{E,F} : \Gamma(E \otimes F) \to \Gamma(E \otimes F),$$

where c denotes the Clifford action. Choose a Hermitian structure on E so that for any one-form φ

$$(c(\varphi)\rho|\sigma) = -(\rho|c(\bar{\varphi})\sigma), \quad \rho, \sigma \in E,$$

where c denotes the Clifford action. Such an inner product always exists. The choice of inner product and connection ensures that $c \circ \nabla^{E,F}$ is (formally) self-adjoint.

Applying this recipe to $d + d^*$ allows us to 'twist' $d + d^*$ by any vector bundle

$$(d + d^*) \otimes_\nabla \mathrm{Id}_E := c \circ \nabla^{T^*M,E}.$$

By choosing a connection compatible with a product Hermitian structure, and using the integral to define a scalar inner product, we can construct a new spectral triple

$$(C^\infty(M), L^2(\Lambda^*M \otimes E, g \otimes (\cdot|\cdot)), (d + d^*) \otimes_\nabla \mathrm{Id}_E).$$

Remark. We have made use of the commutativity of the algebra at a few points in the above discussion. For example, we have identified the right and left actions of functions on sections by multiplication. This allows us to use $\Gamma(E \otimes F) = \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$.

Many of these arguments are unavailable in the noncommutative case. If we have a spectral triple $(\mathcal{A}\otimes\mathcal{A}^{op}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}^{op}$ is the opposite algebra, then we can twist by finite projective modules $p\mathcal{A}^N$ or equivalently $p\mathcal{A}^{op,N}$, and be left with a spectral triple for one copy of $\mathcal{A}$. This point of view underlies Poincaré duality in K-theory.

3.3 The spin^c condition and the Dirac operator. Building differential operators from connections and Clifford actions yields operators which depend on the Riemannian metric and the Clifford module. The index of such an operator is an invariant of the manifold and the Clifford module, [BGV, Thm 3.51]. Since for both the Hodge-de Rham and signature operator the Clifford module depends only on the manifold, we see that the Euler characteristic and the signature are invariants of the manifold. Thus it is natural to ask what other operators can one build in this way. For spin and spin^c manifolds, there is an interesting answer. The spin^c condition was originally formulated in differential geometry language, involving double covers of principal SO bundles on a manifold. This brings in the spin groups and their representations. This will take us too far afield, so we will take a different approach more suitable for noncommutative geometry. The definition of spin^c has been shown by Plymen, [P], to be equivalent to the following straightforward characterisation in terms of Clifford algebras.

Definition 3.2 ([P]). *Let (M,g) be an oriented Riemannian manifold. Then we say that (M,g) is spin^c if there exists a complex vector bundle $S \to M$ such that for all $x \in M$ the vector space S_x is an irreducible representation space for $\mathbb{C}\mathrm{liff}_x(M,g)$.*

A spin^c structure on a Riemannian manifold (M,g) is then the choice of an orientation and irreducible representation bundle S of $\mathbb{C}\mathrm{liff}(M,g)$. The bundle S is called a (complex) spinor bundle.

When M has at least one spin^c structure S, we can build a new operator called the Dirac operator. Choose a Hermitian form $(\cdot|\cdot)$, and let $\nabla : \Gamma(S) \to \Gamma(T^*M \otimes S)$ be any connection compatible with $(\cdot|\cdot)$, which satisfies

$$\nabla^E(c(\varphi)\,\sigma) = c(\nabla^{T^*M}\varphi)\,\sigma + c(\varphi)\nabla^E\sigma, \tag{3.1}$$

for all $\sigma \in \Gamma(E)$ and $\varphi \in \Gamma(T^*M)$. Then we compose the connection with the Clifford action

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{c} \Gamma(S).$$

The resulting operator $\mathcal{D} = c \circ \nabla : \Gamma(S) \to \Gamma(S)$ is the Dirac operator on the (complex) spinor bundle S.

Using the more geometric definitions in terms of principal bundles, one can obtain a more canonical Dirac operator by taking ∇ to be a 'lift' of the Levi-Civita connection to S. This guarantees the existence of a connection satisfying Equation (3.1).

By considering connections on a $\mathbb{C}\mathrm{liff}(M)$ module, we see that we have a general construction of a Dirac operator on any such module. The following Lemma describes these modules.

Lemma 3.3 (see [BGV]). *If M is a spin^c manifold, then every $\mathbb{C}\mathrm{liff}(M)$ module is of the form $S \otimes W$ where S is an irreducible $\mathbb{C}\mathrm{liff}(M)$ module and W is a complex vector bundle.*

Thus, on a spin^c manifold we can describe every 'Dirac type' operator as a twisted version of the Dirac operator of an irreducible Clifford module. In fact, using Poincaré duality in K-theory, one can show that up to homotopy and change of the order of the operator, every elliptic operator on a compact spin^c manifold is a twisted Dirac operator. See [HR, R1].

One very important difference between the Dirac operator of a spin^c structure and the Hodge-de Rham operator must be mentioned. Recall the complex volume form defined in terms of a local orthonormal basis of the cotangent bundle by $\omega_{\mathbb{C}} = i^{[(n+1)/2]}e_1 \cdot e_2 \cdot \cdots \cdot e_n$. This element of the Clifford algebra is globally parallel, meaning $\nabla\omega_{\mathbb{C}} = 0$, where ∇ is the Levi-Civita connection. Since for any differential one-form φ we have

$$\varphi\omega_{\mathbb{C}} = (-1)^{n-1}\omega_{\mathbb{C}}\varphi$$

we have the following two situations.

When n is odd, $\omega_{\mathbb{C}}$ is central with eigenvalues ± 1. Since the Clifford algebra is (pointwise) $M_{2^{(n-1)/2}}(\mathbb{C}) \oplus M_{2^{(n-1)/2}}(\mathbb{C})$, we can take $\omega_{\mathbb{C}} = 1 \oplus -1$. An irreducible representation of $\mathbb{C}\mathrm{liff}(M)$ then corresponds (pointwise) to a representation of one of the two matrix subalgebras. Without loss of generality we choose the representation with $\omega_{\mathbb{C}} = 1$. In this case the spectral triple $(C^\infty(M), L^2(S), \mathcal{D})$ is ungraded, or odd.

When n is even, the ± 1 eigenspaces of $\omega_{\mathbb{C}}$ provide a global splitting of $S = S^+ \oplus S^-$. The Clifford action of a one-form maps S^+ to S^- and vice versa. Hence

$$\omega_{\mathbb{C}}\mathcal{D} = -\mathcal{D}\omega_{\mathbb{C}}.$$

As the Clifford algebra is (pointwise) a single matrix algebra, we get a representation of the whole Clifford algebra. The resulting spectral triple $(C^\infty(M), L^2(S), \mathcal{D})$ is graded by the action of $\omega_{\mathbb{C}}$, and we get an even spectral triple.

Thus the Dirac operator of a spin^c structure gives an even spectral triple if and only if the dimension of M is even. The same remains true if we twist the Dirac operator by any vector bundle.

One can define a spin structure in terms of representations of real Clifford algebras. This is not quite a straightforward generalisation, but does go through: see [LM, GVF, P]. Thus one can also talk about the Dirac operator of a spin structure. In many ways the spin case is easier, and certainly is so from the differential geometry point of view; see [LM, Appendix D].

3.4 Relationship to the Atiyah-Singer index theorem. The Hodge-de Rham operator has index equal to the Euler characteristic of the manifold. In dimension 2, the Gauss-Bonnet Theorem asserts that

$$\chi(M) = \frac{1}{2\pi}\int_M r\, d\mathrm{vol},$$

where r is the scalar curvature of M. This is a remarkable formula, because it allows us to compute a topological quantity, $\chi(M)$, using geometric quantities. More blatantly, it says that by doing explicitly computable calculus operations, we can compute this topological invariant. The answer does not depend on which coordinates or metric we choose to compute with.

The Atiyah-Singer index theorem generalises[2] this theorem to any elliptic operator on a compact oriented n-dimensional manifold. Specifically, it says:

- take a first order elliptic operator $D : \Gamma(E) \to \Gamma(F)$ between sections of (complex) vector bundles $E,\ F \to M$;
- then there is a sum of even differential forms $\omega(D) = \omega_0(D) + \cdots + \omega_{2[n/2]}(D)$ so that

$$\mathrm{Index}(D) = \int_M \omega_{2[n/2]}(D).$$

In particular, if n is odd the index is zero.

However, more is true. If we take a vector bundle W and twist everything to get $D \otimes_\nabla \mathrm{Id}_W$ then

$$\mathrm{Index}(D \otimes_\nabla \mathrm{Id}_W) = \int_M \omega(D) \wedge \mathrm{Ch}(W).$$

Here $\mathrm{Ch}(W)$ is the Chern character of W defined by $\mathrm{Ch}(W) = \mathrm{Trace}(e^{-\nabla^2})$ where ∇ is any connection on W and Trace is a pointwise matrix trace. We will give another description of the Chern character of vector bundles later.

Thus the Atiyah-Singer Index Theorem allows us to not only compute the index of D, but all twisted versions of D also. This means that D is a machine for turning vector bundles into integers via

$$W \mapsto \mathrm{Index}(D \otimes_\nabla \mathrm{Id}_W) \in \mathbb{Z}.$$

Noncommutative geometry and the local index formula are about extending this kind of picture to the noncommutative world. Exchanging vector bundles over a space for projections 'over' an algebra enables pairings of spectral triples (differential operators) with K-theory. However there is also odd K-theory, described in terms of unitaries 'over' an algebra, and one may ask what the Atiyah-Singer theorem says in this case.

[2] There were several distinct motivations for Atiyah and Singer's work, from the integrality of the $\hat{A}$-genus to the Riemann-Roch theorem. The main point is that a range of different theorems became special instances of the Atiyah-Singer index theorem. See [Y] for more information.

Actually, it is 'the same' in odd dimensions, the difference being that the Chern character of a unitary u (see subsection 6.1) has only odd degree differential form components. Thus $\omega(D) \wedge \mathrm{Ch}(u)$ is an odd form, and so there can be forms of degree $\dim M$ to integrate over M. Also, it is not so straightforward to write down a differential operator whose index we are computing (it can be done) and we should think of the odd case in terms of generalised Toeplitz operators. That is, the odd index theorem should be thought of as computing $\mathrm{Index}(PuP)$, where $P = \chi_{[0,\infty)}(\mathcal{D})$ is the non-negative spectral projection of $\mathcal{D}$.

Example 9. Hodge-de Rham. Here the sum of differential forms ω is given in [Gi, p175], and the term in top degree is given by $(2\pi)^{-n/2}\mathrm{Pf}(-R)$, where Pf is short for the Pfaffian of an antisymmetric matrix, and R is the curvature of the Levi-Civita connection. The Pfaffian satisfies $\mathrm{Pf}(A)^2 = \det(A)$, and changes sign if the orientation changes.

In the case where $\dim M = 2$, $\mathrm{Pf}(-R) = r$, the scalar curvature, and so

$$\mathrm{Index}((d + d^*)^+) = \chi(M) = \frac{1}{2\pi}\int_M r\, d\mathrm{vol},$$

and we recover the classical Gauss-Bonnet theorem.

Example 10. For the signature operator,

$$\mathrm{Index}((d + d^* \otimes_\nabla \mathrm{Id}_E)^+) = (\pi i)^{-n/2}\int_M L(M) \wedge \mathrm{Ch}(E),$$

where the L-genus is $L = 1 + \frac{1}{24}\mathrm{Tr}(R^2) + \cdots$.

Example 11. For the spin Dirac operator,

$$\mathrm{Index}((\mathcal{D} \otimes_\nabla \mathrm{Id}_E)^+) = (2\pi i)^{-n/2}\int_M \hat{A}(M) \wedge \mathrm{Ch}(E),$$

where $\hat{A}$ is called the 'A-roof' or 'A-hat' class. So

$$\mathrm{Index}(\mathcal{D}^+) = (2\pi i)^{-n/2}\int_M \hat{A}(M).$$

Since $\hat{A}(M)$ is defined in terms of the curvature, the index is independent of the spin structure. The right hand side is always a rational number, and if it is not an integer, then the manifold has no spin structure.

These examples, and a discussion of the Atiyah-Singer index theorem, are presented in detail in [BGV, G, LM].

A natural question at this point is whether the schematic

$$\text{Index} \equiv \text{integral of differential forms}$$

has any sensible generalisation for noncommutative algebras. In a very real sense, cyclic homology is the generalisation of de Rham cohomology, and obtaining formulae for the index in terms of cyclic homology and cohomology is analogous to integrating differential forms to compute the index. We will take this up later.

3.5 The noncommutative torus and isospectral deformations. One of the nicest and most thoroughly studied spectral triples is defined for the irrational rotation algebra (and its higher dimensional relatives). The spectral triple defined below satisfies every proposed condition intended to characterise what we mean by a noncommutative manifold, [C3].

The noncommutative torus is the universal unital C^*-algebra A_θ generated by two unitaries U, V subject to the commutation relations

$$UV = e^{-2\pi i\theta}VU, \quad \theta \in [0, 1).$$

For $\theta = 0$ this is clearly the algebra of continuous functions on the torus. For θ rational, we obtain an algebra Morita equivalent to the functions on the torus. We will be interested in the case where θ is irrational. In this case, A_θ is simple.

There are two other descriptions of A_θ, which we mention for those interested in pursuing the noncommutative geometry of foliations and crossed products. The first is (up to stable isomorphism) as the C^*-algebra associated to the Kronecker foliation of the (ordinary) 2-torus given by the differential equation

$$dy = \theta dx.$$

Spectral triples can be constructed for more general foliations also; see [Ko] and references therein.

The other description of A_θ is as a crossed product algebra (see [W] for a thorough introduction to this topic). For this description we have

$$A_\theta = C(S^1) \times_{R_\theta} \mathbb{Z}.$$

Here R_θ is the automorphism of $C(S^1)$ induced by a rotation of the circle by $2\pi\theta$. So for $g \in C(S^1)$ and $z \in S^1$ we have $(R_\theta(g))(z) = g(e^{2\pi i\theta}z)$. The connection to our generators U and V is as follows. The unitary $U \in C(S^1)$ is the generator of functions on the circle, the function $z \mapsto z$. The unitary $V \in A_\theta$ is the image of $1 \in \mathbb{Z}$ in the crossed product algebra. The unitary V implements the automorphism R_θ of $C(S^1)$. So, just considering the action of R_θ on U, we have

$$R_\theta(U) = VUV^* = e^{2\pi i\theta}U.$$

There is to our knowledge no general construction of spectral triples for crossed products $A \rtimes \Gamma$ given a discrete group Γ and spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Some special cases can be found in the literature, and we point out the recent paper [BMR] which analyses spectral triples for crossed products by $\mathbb{Z}$ from a metric point of view, in the sense of Chapter 2.

In order to build a spectral triple encoding geometry on the noncommutative torus, we need a smooth algebra, an unbounded operator and a Hilbert space. We begin with the algebra. Let

$$\mathcal{A}_\theta = \left\{ \sum_{n,m\in\mathbb{Z}} c_{nm}U^nV^m : |c_{nm}|(|n| + |m|)^q \text{ is a bounded double sequence for all } q \in \mathbb{N} \right\}.$$

Fourier theory on the ordinary torus suggests viewing this algebra as the smooth functions on the noncommutative torus. It is not much work to see that $\mathcal{A}_\theta$ is indeed a Fréchet pre-C^*-algebra (see Section 5.1).

Next we require a Hilbert space. Recall that for θ irrational, A_θ has a unique faithful normalised trace ϕ given (on polynomials in the generators) by

$$\phi(a) = \phi\left(\sum c_{ij}U^iV^j\right) = c_{00}.$$

If we set $\langle a, b\rangle = \phi(b^*a)$, then $\langle \cdot, \cdot \rangle$ is an inner product and this makes A_θ a pre-Hilbert space. Completing with respect to the topology given by the inner product gives us a Hilbert space which we write as $L^2(A_\theta, \phi)$. The algebra A_θ acts on $L^2(A_\theta, \phi)$ in the obvious way as multiplication operators. We set

$$\mathcal{H} = L^2(A_\theta, \phi) \oplus L^2(A_\theta, \phi).$$

This doubling up of the Hilbert space is motivated by the dimension of spinor bundles on the ordinary torus, or equivalently, the dimension of irreducible representations of $\mathbb{C}\text{liff}(\mathbb{C}^2)$. Thus, loosely speaking, we are aiming to build a Dirac triple rather than a Hodge-de Rham triple.

In order to specify a Dirac operator for our triple, we need to look at how geometric data are encoded for classical tori. The problem is that any quadrilateral with opposite sides identified gives rise to a *geometric* object which is homeomorphic to the torus. To specify the extra geometric content given by our original quadrilateral, we embed it in the first quadrant of the complex plane with one vertex at the origin and another at $1 \in \mathbb{R}$ (we could scale the geometry up by putting one corner at $r \in \mathbb{R}$, but this is more or less irrelevant). The resulting geometry is then specified by the ratio of the edge lengths as complex numbers, or, with our description, a single complex number τ which is the coordinate of the other independent vertex. In particular, $\text{Im}(\tau) > 0$. The usual 'square' torus corresponds to the choice $\tau = i$.

With this in mind we define two derivations on A_θ by

$$\delta_1(U) = 2\pi i U \qquad \delta_1(V) = 0$$

$$\delta_2(U) = 0 \qquad \delta_2(V) = 2\pi i V.$$

We then find (using $UU^* = 1$ etc and the Leibniz rule) that $\delta_1(U^*) = -2\pi i U^*$, $\delta_2(V^*) = -2\pi i V^*$, $\delta(U^n) = n2\pi i U^n$ and so on. These rules correspond to the derivatives of exponentials generating the functions on a torus. Using these derivations and a choice of τ with $\text{Im}(\tau) > 0$ we define

$$\mathcal{D} = \begin{pmatrix} 0 & \delta_1 + \tau\delta_2 \\ -\delta_1 - \bar{\tau}\delta_2 & 0 \end{pmatrix}.$$

Lastly, we set

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that we have the following heuristic. Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \tau \\ -\bar{\tau} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \tau \\ -\bar{\tau} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2\text{Re}(\tau) & 0 \\ 0 & -2\text{Re}(\tau) \end{pmatrix},$$

it appears we are working with the Clifford algebra of the inner product

$$g = \begin{pmatrix} 1 & \text{Re}(\tau) \\ \text{Re}(\tau) & |\tau|^2 \end{pmatrix}.$$

Again, if $\tau = i$, we are reduced to the usual Euclidean inner product.

Hard question Can we encode a nonconstant metric using this heuristic?

We claim $(\mathcal{A}_\theta, \mathcal{H}, \mathcal{D})$ defines an even spectral triple with grading γ for each such choice of τ. First we must show that for all $a \in \mathcal{A}_\theta$ we have $[\mathcal{D}, a]$ bounded. So let $a = \sum c_{nm} U^n V^m \in \mathcal{A}_\theta$. Then

$$\begin{aligned} \mathcal{D}a - a\mathcal{D} &= \begin{pmatrix} 0 & \delta_1 + \tau\delta_2 \\ -\delta_1 - \bar{\tau}\delta_2 & 0 \end{pmatrix}\begin{pmatrix} \sum c_{nm}U^nV^m & 0 \\ 0 & \sum c_{nm}U^nV^m \end{pmatrix} \\ &- \begin{pmatrix} \sum c_{nm}U^nV^m & 0 \\ 0 & \sum c_{nm}U^nV^m \end{pmatrix}\begin{pmatrix} 0 & \delta_1 + \tau\delta_2 \\ -\delta_1 - \bar{\tau}\delta_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\pi i \sum c_{nm}U^nV^m(n + m\tau) \\ -2\pi i \sum c_{nm}U^nV^m(n + m\bar{\tau}) & 0 \end{pmatrix}, \end{aligned}$$

and as $|c_{nm}|$ is 'Schwartz class', this converges in norm and so is bounded.

Next we must show that $\mathcal{D}$ has compact resolvent. As this is equivalent to $\mathcal{D}$ having only eigenvalues of finite multiplicity (which must go to infinity so that $\mathcal{D}$ is unbounded) we will prove this instead. We begin by looking at $\mathcal{D}^2$,

$$\mathcal{D}^2 = \begin{pmatrix} -\delta_1^2 - |\tau|^2\delta_2^2 - \bar\tau\delta_1\delta_2 - \tau\delta_2\delta_1 & 0 \\ 0 & -\delta_1^2 - |\tau|^2\delta_2^2 - \tau\delta_1\delta_2 - \bar\tau\delta_2\delta_1 \end{pmatrix}.$$

Applying this to the monomial $U^nV^m \oplus U^nV^m$ gives

$$\begin{aligned} D^2 \begin{pmatrix} U^nV^m \\ U^nV^m \end{pmatrix} &= (2\pi)^2(n^2 + |\tau|^2 m^2 + nm(\tau+\bar\tau)) \begin{pmatrix} U^nV^m \\ U^nV^m \end{pmatrix} \\ &= (2\pi)^2 |n+\tau m|^2 \begin{pmatrix} U^nV^m \\ U^nV^m \end{pmatrix}. \end{aligned}$$

This shows that all of these monomials are eigenvectors of $\mathcal{D}^2$. Note that

$$\begin{aligned} \phi(V^{-l}U^{-k}U^nV^m) &= \phi(V^{-l}U^{n-k}V^m) \\ &= \phi(e^{-2\pi i l\theta(n-k)}U^{n-k}V^{m-l}) \\ &= \delta_{n,k}\delta_{m,l}, \end{aligned}$$

so that the monomials U^nV^m form an orthonormal basis of $L^2(A_\theta, \phi)$ (they clearly span). As $\mathcal{D}^2$ preserves the splitting of $\mathcal{H}$, we see that these are all the eigenvalues of $\mathcal{D}^2$ and that they give the whole spectrum of $\mathcal{D}^2$. Also note in passing that

$$\ker \mathcal{D}^2 = \mathrm{span}_{\mathbb{C}}\{1\} \oplus \mathrm{span}_{\mathbb{C}}\{1\} = \mathbb{C} \oplus \mathbb{C}.$$

Our results so far are actually enough to conclude, but let us make the eigenvalues and eigenvectors of $\mathcal{D}$ explicit.

The eigenvalues of $\mathcal{D}$ are given by the square roots of the eigenvalues of $\mathcal{D}^2$, and so are

$$\pm 2\pi |n + \tau m| \quad n, m \in \mathbb{Z}.$$

The corresponding eigenvectors are

$$\text{positive sign} \quad \begin{pmatrix} \frac{i(n+\tau m)}{|n+\tau m|} U^nV^m \\ U^nV^m \end{pmatrix}, \quad \text{negative sign} \quad \begin{pmatrix} \frac{i(n+\tau m)}{|n+\tau m|} U^nV^m \\ -U^nV^m \end{pmatrix}.$$

The multiplicity of these eigenvalues depends on the value of τ, but is always finite. Thus $\mathcal{D}$ has compact resolvent, and for any choice of τ with $\mathrm{Im}(\tau) > 0$ we have an even spectral triple.

Using the noncommutative torus we can construct other 'noncommutative manifolds', which are isospectral deformations of classical manifolds. The word 'isospectral' refers to the fact that while the new spectral triple is in a suitable sense a deformation of the original, the spectrum of the Dirac operator remains unchanged.

Part of the interest here is that for a classical geometric space (i.e. a manifold or domain in $\mathbb{R}^n$) the spectrum of the Laplace operator comes very close to characterising the space plus metric up to isometry. An important point is that there are non-isometric spaces whose Laplacians have the same spectrum, but nevertheless the spectrum of the Laplacian remains a strong invariant.

This goes back to the famous paper 'Can one hear the shape of a drum?', [Kac], and lives on in the subject known as spectral geometry. A good overview can be found in [Gi], while an interesting modern contribution is [ABE]. See also references in `http://en.wikipedia.org/wiki/Hearing_the_shape_of_a_drum`.

Thus, making an isospectral deformation, but making the algebra become noncommutative, should result in an 'almost' complete characterisation of the resulting noncommutative space up to isometry.

Let (M, g) be a compact Riemannian spin manifold with an isometric action of $\mathbb{T}^2$ (we can do this with higher dimensional noncommutative tori also). Then we define $C^\infty(M_\theta)$ to be the fixed point for the diagonal action of $\mathbb{T}^2$ on $C^\infty(M) \otimes A_\theta$. This is similar to gluing in a noncommutative torus in place of each torus orbit in M. The algebra $C^\infty(M_\theta)$ is noncommutative.

The same kind of procedure (though somewhat more subtle) applied to the Hilbert space of spinors allows one to take $(C^\infty(M), L^2(S), \mathcal{D})$ and produce $(C^\infty(M_\theta), L^2(S_\theta), \mathcal{D})$, where $\mathcal{D}$ is essentially the same (Dirac) operator in both triples, and certainly has the same spectrum. The interested reader can look at the papers by Alain Connes and Michel Dubois-Violette, [CD].

4 K-theory, K-homology and the index pairing

4.1 K-theory. This section is the briefest of overviews of K-theory for C^*-algebras. If the discussion here is unfamiliar, try [WO, RLL, HR]. We will follow [HR]. The reasons we spend much more time on K-homology than on K-theory are that there are many more texts on K-theory, and because spectral triples represent K-homology classes.

Definition 4.1. *Given a unital C^*-algebra A, we denote by $K_0(A)$ the abelian group with one generator $[p]$ for each projection p in each matrix algebra $M_n(A)$, $n = 1, 2, \ldots$ and the following relations:*

a) if p, $q \in M_n(A)$ and p, q are joined by a norm continuous path of projections in $M_n(A)$ then $[p] = [q]$;

b) $[0] = 0$ for any square matrix of zeroes;

c) $[p] + [q] = [p \oplus q]$ for any $[p]$, $[q]$.

In a), we say that p and q are homotopic. If $p \in M_n(A)$ we say that p is a projection over A. Projections in matrix algebras $M_n(A)$ and $M_m(A)$, $m \neq n$ can be compared by using the fact that these matrix algebras form an increasing union in the obvious way (that is, $M_n(A) \subset M_{n+1}(A)$).

The group structure arises using the Grothendieck construction: every element of $K_0(A)$ can be written as a formal difference $[p] - [q]$, and two elements $[p] - [q]$ and $[p'] - [q']$ are equal if and only if there is a projection r such that
$$p \oplus q' \oplus r \ \text{ is homotopic to } \ p' \oplus q \oplus r.$$

The assignment $A \to K_0(A)$ is a covariant functor from C^*-algebras to abelian groups. If $\phi : A \to B$ is a $*$-homomorphism, then applying ϕ element by element to the matrix $p \in M_n(A)$ gives a projection $\phi(p) \in M_n(B)$. This yields a group homomorphism $\phi_* : K_0(A) \to K_0(B)$.

Exercise. Show that two projections in $M_n(\mathbb{C})$ are homotopic if and only if they have the same rank, and that $[p] \to \text{Rank}(p)$ is an isomorphism from $K_0(\mathbb{C})$ to $\mathbb{Z}$.

Example 12. (**Important**) If $A = C(X)$, where X is a compact Hausdorff space, we find that $K_0(A) = K^0(X)$, where $K^0(X)$ is the topological K-theory defined by vector bundles.

This is because if $E \to X$ is a complex vector bundle, there is a projection $p \in M_N(C(X))$ for some N such that $\Gamma(X, E) \cong pC(X)^N$ as a $C(X)$-module. Similarly, any $C(X)$-module of the form $pC(X)^N$ is the space of sections of a vector bundle. This is the Serre-Swan Theorem, [S], and we recommend the book [AK] as a wonderful place to discover this and other aspects of topological K-theory.

This exchange between projections and vector bundles is one of the many important instances of exchanging topological information for algebraic information, with the Gel´fand Theory (exchanging abelian C^*-algebras and Hausdorff spaces) being one of the main motivations of noncommutative geometry.

Definition 4.2. *Given a unital C^*-algebra A, we denote by $K_1(A)$ the abelian group with one generator $[u]$ for each unitary u in each matrix algebra $M_n(A)$, $n = 1, 2, \ldots$ and the following relations:*

a) if u, $v \in M_n(A)$ and u, v are joined by a norm continuous path of unitaries in $M_n(A)$ then $[u] = [v]$;

b) $[1] = 0$ for any square identity matrix;

c) $[u] + [v] = [u \oplus v]$ for any $[u]$, $[v]$.

Note that unitaries in matrix algebras of different dimensions can be compared by using the trick of sending u to

$$\begin{pmatrix} u & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Exercise. Let $\sim$ denote the relation of path-connectedness through unitaries. Let u, $v \in A$ be unitary. Prove that in $M_2(A)$ we have

$$u \oplus 1 \sim 1 \oplus u, \quad u \oplus v \sim uv \oplus 1 \sim vu \oplus 1, \quad u \oplus u^* \sim 1 \oplus 1.$$

Hint: consider the rotation matrix

$$R_t = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

From the exercise we see that $[u] + [u^*] = [u \oplus u^*] = [1] = 0$, and so $-[u] = [u^*]$.

Exercise. Show that $K_1(\mathbb{C}) = 0$.

This section on K-theory could be made arbitrarily long, but it is not the main focus of these notes, and so we leave K-theory for now with the warning that here we have seen the definitions and nothing more.

4.2 Fredholm modules and K-homology. One of the central reasons that the techniques employed by Atiyah and Singer to compute the index of elliptic differential operators on manifolds continues to work for noncommutative spaces is the way K-theory enters the proof. Essentially, K-theory, and the dual theory K-homology, make perfectly good sense for C^*-algebras, commutative or not. The best reference for K-homology is the book [HR]. The origins of the ideas can be seen in [A], and the modern form is developed in [Ka1]. While we were very brief with K-theory, we will spend a little longer on K-homology as it is much closer to spectral triples. Indeed, spectral triples are 'just' nice representatives of classes in K-homology. For example, cohomology of manifolds is studied using tools of calculus by introducing differential forms. Spectral triples play an almost exactly analogous role.

Definition 4.3. *Let A be a separable C^*-algebra. A Fredholm module over A is given by a Hilbert space $\mathcal{H}$, a $*$-representation $\rho : A \to \mathcal{B}(\mathcal{H})$ and an operator $F : \mathcal{H} \to \mathcal{H}$ such that for all $a \in A$*

$$(F^2 - 1)\rho(a), \quad (F - F^*)\rho(a), \quad [F, \rho(a)] := F\rho(a) - \rho(a)F$$

are all compact operators. We say that $(\rho, \mathcal{H}, F)$ is even (or graded) if there is an operator $\gamma : \mathcal{H} \to \mathcal{H}$ such that $\gamma^2 = 1$, $\gamma = \gamma^*$, $\gamma F + F\gamma = 0$ and for all $a \in A$, $\gamma\rho(a) = \rho(a)\gamma$. Otherwise we call $(\rho, \mathcal{H}, F)$ odd.

We will usually consider algebras A which are unital and for which $\rho(1_A) = \mathrm{Id}_{\mathcal{H}}$, and this simplifies the first two conditions on F: $F^2 - 1$ and $F - F^*$ are compact. In this case we have the following descriptions.

An odd Fredholm module is given by a (unital) representation ρ on $\mathcal{H}$ and an operator $F = 2P - 1 + K$ where K is compact and P is a projection commuting with $\rho(A)$ modulo compact operators

An even Fredholm module is given by a pair of representations ρ_+, ρ_- on Hilbert spaces $\mathcal{H}_+, \mathcal{H}_-$ respectively, and

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \rho = \begin{pmatrix} \rho_+ & 0 \\ 0 & \rho_- \end{pmatrix}, \ \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix} \tag{4.1}$$

with $F_- = (F_+)^* + K$ with K compact. The conditions defining the Fredholm module tell us that $F_+ : \mathcal{H}_+ \to \mathcal{H}_-$ is a Fredholm operator.

Example 13. Let $\rho : \mathbb{C} \to \mathcal{B}(\mathcal{H})$ be the unique unital representation. Then an ungraded Fredholm module is given by an essentially self-adjoint Fredholm operator F. Likewise, a graded Fredholm module is given by an essentially self-adjoint Fredholm operator of the form (4.1).

For an even Fredholm module, we denote by $\mathrm{Index}(\rho, \mathcal{H}, F)$ the index of $F_+ : \mathcal{H}_1 \to \mathcal{H}_2$.

Example 14. Let $\mathcal{H} = L^2(S^1)$ and represent $C(S^1)$ on $\mathcal{H}$ as multiplication operators. So for $f \in C(S^1)$ and $\xi \in L^2(S^1)$, $(f\xi)(x) = f(x)\xi(x)$ for $x \in S^1$.

Let $P \in \mathcal{B}(\mathcal{H})$ be the projection onto $\overline{\mathrm{span}}\{z^k : k \geq 0\}$. Since P is a projection, the operator $F = 2P - 1$ is self-adjoint and has square one. Thus, to check that $(\mathcal{H}, F)$ (along with the multiplication representation) is an odd Fredholm module for $C(S^1)$, we just need to check that $[F, f]$ is compact for all $f \in C(S^1)$.

First, every $f \in C(S^1)$ is a norm (uniform) limit of trigonometric polynomials (Stone-Weierstrass) and so the norm limit of finite sums of z^k, $k \in \mathbb{Z}$, where $z : S^1 \to \mathbb{C}$ is the identity function.

Hence it suffices to show that $[F, z^k]$ is compact for each k, and so it is enough to show that $[P, z^k]$ is compact. Let $\xi \in \mathcal{H}$ so $\xi = \sum_{n\in\mathbb{Z}} c_n z^n$ (this sum converges in the Hilbert space norm). Then

$$Pz^k\xi = P\sum_{n\in\mathbb{Z}} c_n z^{n+k} = \sum_{n\geq -k} c_n z^{n+k}$$

while

$$z^k P\xi = \sum_{n\geq 0} c_n z^{n+k}.$$

The difference is

$$[P, z^k]\xi = \begin{cases} \sum_{n=-k}^{0} c_n z^{n+k} & k \geq 0 \\ \sum_{n=0}^{-k} c_n z^{n+k} & k \leq 0 \end{cases}$$

Hence $[P, z^k]$ is a rank k operator, and so compact.

The operators $T_f := PfP : P\mathcal{H} \to P\mathcal{H}$, $f \in C(S^1)$, are called Toeplitz operators. One can show that

$$T_f^* = T_{\bar{f}} \quad \text{and} \quad T_f T_g = T_{fg} + K$$

where K is a compact operator. Composing with the quotient map $\pi : \mathcal{B}(P\mathcal{H}) \to \mathcal{Q}(P\mathcal{H})$ we get a $*$-homomorphism $C(S^1) \to \mathcal{Q}(P\mathcal{H})$ which is **faithful**!; see [HR] for instance. Hence we get an extension (short exact sequence)

$$0 \to \mathcal{K}(P\mathcal{H}) \to T \to C(S^1) \to 0$$

where T is the algebra generated by the T_f, $f \in C(S^1)$. This is called the Toeplitz extension.

Exercise. What is the relationship between the Fredholm module for $C(S^1)$ in Example 14 and the spectral triple in Example 4? *Hint*: Look at the next example.

Example 15. Let $\mathcal{H} = L^2(\Lambda^* M, g)$ and let $C^\infty(M)$ act as multiplication operators. If $\mathcal{D} = d + d^*$, then we know that $(1+\mathcal{D}^2)^{-1/2}$ is compact, and $F_\mathcal{D} = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ is bounded (by the functional calculus). Now we compute

$$F_\mathcal{D}^2 = \mathcal{D}^2(1+\mathcal{D}^2)^{-1} = 1 - (1+\mathcal{D}^2)^{-1}$$

which is the identity modulo compacts. Since $F_\mathcal{D}$ is self-adjoint and anticommutes with the grading γ of differential forms by degree, we need only check that $[F_\mathcal{D}, f]$ is compact for all $f \in C^\infty(M)$. So

$$[F_\mathcal{D}, f] = [\mathcal{D}, f](1+\mathcal{D}^2)^{-1/2} - \mathcal{D}(1+\mathcal{D}^2)^{-1/2}[(1+\mathcal{D}^2)^{1/2}, f](1+\mathcal{D}^2)^{-1/2}.$$

Since $[\mathcal{D}, f]$ is Clifford multiplication by df, the first term is compact. Likewise, the second term will be compact if we can see that $[(1+\mathcal{D}^2)^{1/2}, f]$ is bounded. But the symbol of f is $f\,\mathrm{Id}$, so the order of the commutator is $1+0-1=0$, and so we have a bounded pseudodifferential operator. Hence we get an even Fredholm module for the algebra $C^\infty(M)$.

Exercise. Does the Fredholm module of Example 15 extend to a Fredholm module for $C(M)$, the C^*-algebra of continuous functions? Why?

As the example suggests, every spectral triple defines a Fredholm module. For simplicity, we give a proof that relies on an extra assumption, though this can be removed (see [CP1]).

Proposition 4.4. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple such that $[|\mathcal{D}|, a]$ is bounded for all $a \in \mathcal{A}$, where $|\mathcal{D}| = \sqrt{\mathcal{D}^2}$ is the absolute value. Define $F_\mathcal{D} = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then $(\mathcal{H}, F_\mathcal{D})$ is a Fredholm module for the C^*-algebra $A := \overline{\mathcal{A}}$.*

Proof. For $a \in \mathcal{A}$ we have

$$\begin{aligned} [F_\mathcal{D}, a] &= [\mathcal{D}, a](1+\mathcal{D}^2)^{-1/2} + \mathcal{D}[(1+\mathcal{D}^2)^{-1/2}, a] \\ &= [\mathcal{D}, a](1+\mathcal{D}^2)^{-1/2} - \mathcal{D}(1+\mathcal{D}^2)^{-1/2}[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}. \end{aligned}$$

This is a compact operator. If $\{a_k\}_{k\geq 0} \subset \mathcal{A}$ is a sequence converging in (operator) norm then

$$\|[F_\mathcal{D}, a_k - a_m]\| \leq 2\|F_\mathcal{D}\|\,\|a_k - a_m\| \leq 2\|a_k - a_m\| \to 0.$$

Hence if $a = \lim_k a_k$ with $a_k \in \mathcal{A}$ and convergence in norm,

$$[F_\mathcal{D}, a] = \lim_k [F_\mathcal{D}, a_k]$$

and this is a limit of compact operators, and so compact. □

Definition 4.5. *Let $(\rho, \mathcal{H}, F)$ be a Fredholm module, and suppose that $U : \mathcal{H}' \to \mathcal{H}$ is a unitary. Then $(U^*\rho U, \mathcal{H}', U^*FU)$ is also a Fredholm module (with grading $U^*\gamma U$ if γ is a grading of $(\rho, \mathcal{H}, F)$) and we say that it is unitarily equivalent to $(\rho, \mathcal{H}, F)$.*

Definition 4.6. *Let $(\rho, \mathcal{H}, F_t)$ be a family of Fredholm modules parameterised by $t \in [0,1]$ with $\rho, \mathcal{H}$ constant. If the function $t \to F_t$ is norm continuous, we call this family an operator homotopy between $(\rho, \mathcal{H}, F_0)$ and $(\rho, \mathcal{H}, F_1)$, and say that these two Fredholm modules are operator homotopic.*

If $(\rho_1, \mathcal{H}_1, F_1)$ and $(\rho_2, \mathcal{H}_2, F_2)$ are Fredholm modules over the same algebra A, then $(\rho_1 \oplus \rho_2, \mathcal{H}_1 \oplus \mathcal{H}_2, F_1 \oplus F_2)$ is a Fredholm module over A, called the direct sum.

Definition 4.7. *Let $p = 0, 1$. The K-homology group $K^p(A)$ is the abelian group with one generator $[x]$ for each unitary equivalence class of Fredholm modules (even or graded if $p = 0$, and odd or ungraded for $p = 1$) with the following relations:*

1) If x_0 and x_1 are operator homotopic Fredholm modules (both even or both odd) then $[x_0] = [x_1]$ in $K^p(A)$, and

2) If x_0 and x_1 are two Fredholm modules (both even or both odd) then $[x_0 \oplus x_1] = [x_0] + [x_1]$ in $K^p(A)$.

The zero element is the class of the zero module, which is the zero Hilbert space, zero representation and naturally a zero operator. There are also other representatives of this class, which we require in order to be able to display inverses.

Definition 4.8. *A Fredholm module $(\rho, \mathcal{H}, F)$ is called degenerate if $F = F^*$, $F^2 = 1$ and $[F, \rho(a)] = 0$ for all $a \in A$.*

Exercise. The class of a degenerate module is zero in K-homology. *Hint*: Consider $\oplus^\infty(\rho, \mathcal{H}, F)$ and $(\rho, \mathcal{H}, F) \oplus \oplus^\infty(\rho, \mathcal{H}, F)$.

Lemma 4.9. *[HR] If $x = (\rho, \mathcal{H}, F)$ is an odd Fredholm module, then the class of $-[x]$ is represented by the Fredholm module $(\rho, \mathcal{H}, -F)$. For an even Fredholm module $x = (\rho, \mathcal{H}, F, \gamma)$ the inverse is represented by $(\rho, \mathcal{H}, -F, -\gamma)$.*

Proof. We do the even case, by showing that

$$\left(\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \mathcal{H} \oplus \mathcal{H}, \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}, \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}\right)$$

is operator homotopic to the degenerate module

$$\left(\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \mathcal{H} \oplus \mathcal{H}, \begin{pmatrix} 0 & \mathrm{Id}_{\mathcal{H}} \\ \mathrm{Id}_{\mathcal{H}} & 0 \end{pmatrix}, \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}\right).$$

We do this by displaying the homotopy

$$F_t = \begin{pmatrix} \cos(\pi t/2)F & \sin(\pi t/2)\,\mathrm{Id}_{\mathcal{H}} \\ \sin(\pi t/2)\,\mathrm{Id}_{\mathcal{H}} & -\cos(\pi t/2)F \end{pmatrix}.$$

We leave the details as an **Exercise**. □

Let $\psi : A \to B$ be a $*$-homomorphism, and $(\rho, \mathcal{H}, F)$ a Fredholm module over B. Then $(\rho \circ \psi, \mathcal{H}, F)$ is a Fredholm module over A. This allows us to define

$$\psi^* : K^*(B) \to K^*(A) \quad \text{by} \quad \psi^*[(\rho, \mathcal{H}, F)] = [(\rho \circ \psi, \mathcal{H}, F)]$$

and so K-homology is a contravariant functor from (separable) C^*-algebras to abelian groups. We write $K^*(A) = K^0(A) \oplus K^1(A)$.

Being able to work modulo compact operators gives us plenty of freedom, and allows us to build a good (co)homology theory. Sometimes however, it is better to have 'nice' representatives of K-homology classes.

Lemma 4.10. *[HR] Every K-homology class in $K^*(A)$ can be represented by a Fredholm module $(\rho, \mathcal{H}, F)$ with $F = F^*$ and $F^2 = 1$.* **Alternatively**, *we may suppose that $(\rho, \mathcal{H}, F)$ is nondegenerate in the sense that $\rho(A)\mathcal{H}$ is dense in $\mathcal{H}$. In general we cannot do both these things at the same time.*

We will call any Fredholm module with $F = F^*$ and $F^2 = 1$ a **normalised Fredholm module**. In [HR], this is called an involutive Fredholm module. Usually, we will omit the representation, and refer to a Fredholm module $(\mathcal{H}, F)$ for a C^*-algebra A.

Later we will give an explicit way of obtaining a normalised Fredholm module from a spectral triple. Here are simple methods for achieving the same end when we begin with a Fredholm module.

Exercise Given a Fredholm module $(\rho, \mathcal{H}, F)$ for a C^*-algebra A, show that $t \mapsto F_t = (1-t)F + \frac{t}{2}(F + F^*)$ is an operator homotopy, and F_1 is self-adjoint.

Exercise Let $(\rho, \mathcal{H}, F)$ be a Fredholm module for a C^*-algebra A with $F = F^*$. Show that $(\rho, \mathcal{H}, F)$ is equivalent (in K-homology) to

$$\left(\rho \oplus 0, \mathcal{H} \oplus \mathcal{H}, \tilde{F} = \begin{pmatrix} F & (1-F^2)^{1/2} \\ (1-F^2)^{1/2} & -F \end{pmatrix}\right),$$

and $\tilde{F}^2 = 1$.

4.3 The index pairing. The pairing between K-theory and K-homology is given in terms of the Fredholm index. As we need to handle matrix algebras over $\mathcal{A}$ we observe that if $(\mathcal{H}, F)$ is a Fredholm module for an algebra $\mathcal{A}$, then $(\mathcal{H}^k, F \otimes \mathrm{Id}_k)$ is a Fredholm module for $M_k(\mathcal{A})$. If $(\mathcal{H}, F)$ is normalised so is $(\mathcal{H}^k, F \otimes \mathrm{Id}_k)$. We leave this as an **Exercise**.

Let $(\mathcal{H}, F, \gamma)$ be an even Fredholm module for an algebra $\mathcal{A}$ and $p \in M_k(\mathcal{A})$ a projection. Then the pairing between $[p] \in K_0(\mathcal{A})$ and $[(\mathcal{H}, F, \gamma)] \in K^0(\mathcal{A})$ is given by

$$\langle [p], [(\mathcal{H}, F, \gamma)] \rangle := \mathrm{Index}(p(F^+ \otimes \mathrm{Id}_k)p : p\mathcal{H}^k \to p\mathcal{H}^k).$$

When $(\mathcal{H}, F)$ is an odd Fredholm module over $\mathcal{A}$, and $u \in M_k(\mathcal{A})$ is a unitary, the pairing between $[u] \in K_1(\mathcal{A})$ and $[(\mathcal{H}, F)] \in K^1(\mathcal{A})$ is given by

$$\langle [u], [(\mathcal{H}, F)] \rangle := \mathrm{Index}(P_k u P_k - (1 - P_k) : \mathcal{H}^k \to \mathcal{H}^k)$$

where $P_k = \frac{1}{2}(1 + F) \otimes \mathrm{Id}_k$. If P_k is actually a projection, then this formula simplifies to

$$\langle [u], [(\mathcal{H}, F)] \rangle := \mathrm{Index}(P_k u P_k : P_k\mathcal{H}^k \to P_k\mathcal{H}^k).$$

The reason is that $1 - P_k$ maps $(1 - P_k)\mathcal{H}^k$ to itself, and is self-adjoint and so has zero index. When P_k is not a projection, we cannot just consider $P_k u P_k$, as it does not necessarily preserve any subspace. Adding $1 - P_k$ preserves the Fredholm property on the whole Hilbert space and does not change the index when P_k is a projection.

Since we can always find a normalised Fredholm module in any K-homology class, it is more useful to always think of $\mathrm{Index}(P_k u P_k)$.

Exercise. With p, u, P_k, F as above, show that $p(F^+ \otimes \mathrm{Id}_k)p$ and $P_k u P_k - (1 - P_k)$ are Fredholm operators.

Exercise. Show that the index pairing between a projection and an even Fredholm module depends only on the K classes. Similarly for the odd case.

When $P = (1 + F)/2$ is a projection (say when $F = F^*$ and $F^2 = 1$), operators such as $P_k u P_k$ are called generalised Toeplitz operators. The index of $P_k u P_k$ is also equal to spectral flow. This **spectral flow** has a rigorous analytic definition due to Phillips, [Ph1, Ph2], which we won't pursue here. The original topological definition is due to Lusztig and Atiyah-Patodi-Singer, [APS1].

Example 16. Let $\mathcal{D}^+ : \Gamma(E) \to \Gamma(F)$ be a first order elliptic differential operator on the manifold M. Let $(C^\infty(M), L^2(E) \oplus L^2(F), \mathcal{D})$ be the even spectral triple with $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}^- \\ \mathcal{D}^+ & 0 \end{pmatrix}$, where $\mathcal{D}^- = (\mathcal{D}^+)^*$.

If W is another vector bundle, we can associate to it a projection $p \in M_N(C^\infty(M))$ so that

$$L^2(E \otimes W) = pL^2(E)^N,$$

and similarly for $F \otimes W$. Then $\mathcal{D} \otimes_\nabla \mathrm{Id}_W = p(\mathcal{D} \otimes 1_N)p+$ order zero terms, and we see that

$$\mathrm{Index}((\mathcal{D} \otimes_\nabla \mathrm{Id}_W)^+) = \mathrm{Index}(p(\mathcal{D}^+ \otimes 1_N)p).$$

Exercise. Prove the equalities in Example 16. *Hint*: If $\Gamma(E) = pC^\infty(M)^N$ then the composition

$$pC^\infty(M)^N \xrightarrow{i} C^\infty(M)^N \xrightarrow{d} \Gamma(T^*M)^N \xrightarrow{p} \Gamma(E \otimes T^*M)$$

is a connection.

From now on for notational convenience we will assume that any projection or unitary we want to pair with a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ lives in the algebra $\mathcal{A}$ rather than $M_k(\mathcal{A})$.

4.4 The index pairing for finitely summable Fredholm modules. In this Section we briefly describe how we can compute the index pairing for (the class of) certain special Fredholm modules. Before stating the definition, we need a comment. There is a problem with C^*-algebras as can be seen in the commutative case: if we want to start computing the index of $p\mathcal{D}p$ where $\mathcal{D}$ is a first order elliptic differential operator, we have to require that p is at least C^1. That means we cannot use just any representative of a K-theory class. Next there needs to be some additional constraint on the Fredholm module in order to obtain explicit formulae. For finite dimensional situations one uses 'finite summability' of commutators $[F, a]$, which we now proceed to define. Note that as usual our reference for operator ideals is [Sim].

Definition 4.11. *For any $p \geq 1$, define the p-th Schatten ideal of the Hilbert space $\mathcal{H}$ to be*

$$\mathcal{L}^p(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \mathrm{Trace}(|T|^p) < \infty\}, \quad |T| = \sqrt{T^*T}. \tag{4.2}$$

Remarks. 1) These ideals are all two-sided, but are not norm closed. As the compact operators are the only norm closed ideal of the bounded operators on Hilbert space (when the Hilbert space is separable), all the ideals $\mathcal{L}^p(\mathcal{H})$ have norm closure equal to the compact operators.

2) The ideal $\mathcal{L}^2(\mathcal{H})$ is called the Hilbert-Schmidt class, and is a Hilbert space for the inner product

$$\langle T, S\rangle = \mathrm{Trace}(S^*T) \quad \text{for all } T,\, S \in \mathcal{L}^2(\mathcal{H}). \tag{4.3}$$

In particular, it is complete for the norm $\|T\|_2 = \mathrm{Trace}(T^*T)^{1/2}$.

3) More generally, $\mathcal{L}^p(\mathcal{H})$ is complete for the norm

$$\|T\|_p := \mathrm{Trace}(|T|^p)^{1/p},\ T \in \mathcal{L}^p(\mathcal{H}). \tag{4.4}$$

Moreover, if $B, C \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{L}^p(\mathcal{H})$, then $\|BTC\|_p \leq \|B\|\,\|T\|_p\,\|C\|$, where $\|\cdot\|$ is the usual operator norm.

4) Two useful (and immediate) facts are
a) If $T \in \mathcal{L}^p(\mathcal{H})$ then $T^p \in \mathcal{L}^1(\mathcal{H})$.

b) If $T_i \in \mathcal{L}^{p_i}(\mathcal{H})$ then $T_1 \cdots T_k \in \mathcal{L}^p(\mathcal{H})$ where

$$\frac{1}{p} = \sum_{j=1}^{k} \frac{1}{p_j}. \tag{4.5}$$

This last can be shown using the Hölder inequality. Note the analogy with the ℓ^p spaces of classical analysis. After this very brief summary of these operator ideals, we make the following key definition.

Definition 4.12 (Connes). *Let $\mathcal{A}$ be a unital $*$-algebra. A (normalised) Fredholm module $(\mathcal{H}, F)$ for $\mathcal{A}$ is $(p+1)$-summable, $p \in \mathbb{N}$, if for all $a \in \mathcal{A}$ we have*

$$[F, a] \in \mathcal{L}^{p+1}(\mathcal{H}).$$

Example 17. When we construct the Hodge-de Rham Fredholm module, instead of starting with $d + d^*$, we can start with

$$\mathcal{D}_m = \begin{pmatrix} d + d^* & m \\ m & -(d + d^*) \end{pmatrix}, \quad m > 0,$$

acting on $\mathcal{H}_2 = L^2(\Lambda^* M, g) \oplus L^2(\Lambda^* M, g)$ with the grading $\gamma \oplus -\gamma$. The representation of $C^\infty(M)$ on $\mathcal{H}_2$ is as multiplication operators in the first copy, and by zero in the second. Since $\mathcal{D}_m$ is invertible, we are free to define $F_{\mathcal{D}_m} = \mathcal{D}_m |\mathcal{D}_m|^{-1}$. Again we obtain a Fredholm module, but this time $F^2_{\mathcal{D}_m} = \mathrm{Id}_{\mathcal{H}_2}$, and so we have a normalised Fredholm module. From Weyl's Theorem, see Theorem 5.11, $|\mathcal{D}_m|^{-1}$ has eigenvalues $\mu_1 \geq \mu_2 \geq \cdots$ satisfying

$$\mu_n = C\, n^{-1/\dim M} + o(n^{-1/\dim M}).$$

From this and the standard commutator tricks, it is easy to see that $(\mathcal{H}_2, F_{\mathcal{D}_m})$ is $(\dim(M) + 1)$-summable.

For finitely summable normalised Fredholm modules we can define cyclic cocycles whose class in periodic cyclic cohomology is called the **Chern character**. We defer a detailed discussion of this cohomology theory to subsection 6.1. We now give the definition of the Chern character and its main application.

Definition 4.13 (Connes). *Let $(\mathcal{H}, F)$ be a $(p+1)$-summable normalised Fredholm module for the $*$-algebra $\mathcal{A}$. For any $n \geq p$ of the same parity as the Fredholm module we define cyclic cocycles by*

$$\mathrm{Ch}_n(\mathcal{H}, F)(a_0, a_1 \ldots, a_n) = \frac{\lambda_n}{2} \mathrm{Trace}(\gamma\, [F, a_0]\, [F, a_1] \cdots [F, a_n]),$$

where γ is 1 if the module is odd, and the normalisation constants are

$$\lambda_n = \begin{cases} (-1)^{n(n-1)/2} \Gamma(n/2 + 1) & (\textit{even}) \\ \sqrt{2i}(-1)^{n(n-1)/2} \Gamma(n/2 + 1) & (\textit{odd}) \end{cases}$$

The Chern character $\mathrm{Ch}_*(\mathcal{H}, F)$ is the class of these cocycles in periodic cyclic cohomology.

There is a notational device for making the odd and even cases similar. For $T \in \mathcal{B}(\mathcal{H})$ such that $FT + TF \in \mathcal{L}^1(\mathcal{H})$, define the 'conditional trace'

$$\mathrm{Trace}'(T) = \frac{1}{2} \mathrm{Trace}(F(FT + TF)).$$

Note that if $T \in \mathcal{L}^1(\mathcal{H})$ then $\mathrm{Trace}'(T) = \mathrm{Trace}(T)$. Then define

$$\mathrm{Trace}_s(T) = \mathrm{Trace}'(\gamma T).$$

Here $\gamma = \mathrm{Id}_{\mathcal{H}}$ if n is odd. Then we can write

$$\mathrm{Ch}_n(\mathcal{H}, F, \gamma)(a_0, a_1, \ldots, a_n) = \lambda_n\, \mathrm{Trace}_s(a_0 [F, a_1] \cdots [F, a_n]).$$

Remark. This is the definition as given by Alain Connes, and the next theorem we take from [C1] also, except for a minus sign in the odd case. This minus sign is a persistent nuisance in the literature, and we have addressed it by retaining the definition of the Chern character and introducing an additional minus sign in the pairing.

Theorem 4.14 (Connes). *Let $(\mathcal{H}, F)$ be a finitely summable normalised Fredholm module over $\mathcal{A}$. Then for any $[e] \in K_0(\mathcal{A})$*

$$\langle [e], [(\mathcal{H}, F)] \rangle = \mathrm{Ch}_*(\mathcal{H}, F)(e) := \frac{1}{(n/2)!}\mathrm{Ch}_n(\mathcal{H}, F)(e, e, \ldots, e)$$

for n large enough and even. For $[u] \in K_1(\mathcal{A})$

$$\langle [u], [(\mathcal{H}, F)] \rangle = -\mathrm{Ch}_*(\mathcal{H}, F)(u) := -\frac{1}{\sqrt{2i}2^n\Gamma(n/2+1)}\mathrm{Ch}_n(\mathcal{H}, F)(u^*, u, \ldots, u)$$

for n large enough and odd.

Proof. For the even case the proof, and the result, is just as in [C0], and the strategy in the odd case is also the same. However, we present the proof in the odd case in order to clarify the sign convention mentioned above. A similar proof can be given starting from [CP2, Corollary 3.3].

Using a simple modification of [GVF, Proposition 4.2] we know that

$$\mathrm{Index}(PuP) = \mathrm{Trace}((P - Pu^*PuP)^n) - \mathrm{Trace}((P - PuPu^*P)^n)$$

where $n > (p+1)/2$ is an integer. First we observe that $P - Pu^*PuP = -P[u^*, P]uP$, and by replacing P by $(1+F)/2$ we have

$$P[u^*, P]uP = \frac{1}{4}[F, u^*]\,[F, u]\,\frac{1+F}{2}.$$

Since $F[F, a] = -[F, a]F$ for all $a \in \mathcal{A}$, cycling a single $[F, u^*]$ around using the trace property yields

$$\begin{aligned}
\mathrm{Index}(PuP) &= \mathrm{Trace}((P - Pu^*PuP)^n) - \mathrm{Trace}((P - PuPu^*P)^n)\\
&= \mathrm{Trace}\left(\left(-\frac{1}{4}[F, u^*]\,[F, u]\,\frac{1+F}{2}\right)^n\right) - \mathrm{Trace}\left(\left(-\frac{1}{4}[F, u]\,[F, u^*]\,\frac{1+F}{2}\right)^n\right)\\
&= (-1)^n\frac{1}{4^n}\,\mathrm{Trace}\left(\frac{1+F}{2}([F, u^*][F, u])^n - \right.\\
&\qquad\qquad \left. [F, u^*][F, u][F, u^*]\frac{1+F}{2}[F, u][F, u^*]\cdots\frac{1+F}{2}[F, u]\frac{1-F}{2}\right)\\
&= (-1)^n\frac{1}{4^n}\,\mathrm{Trace}\left(\left(\frac{1+F}{2} - \frac{1-F}{2}\right)([F, u^*][F, u])^n)\right)\\
&= (-1)^n\frac{1}{4^n}\,\mathrm{Trace}(F([F, u^*][F, u])^n)\\
&= (-1)^n\frac{1}{2^{2n-1}}\,\mathrm{Trace}'(u^*[F, u]\cdots[F, u^*][F, u]),
\end{aligned}$$

where in the last line there are $2n-1$ commutators. Comparing the normalisation for the index pairing in [C0] and the formula above we find

$$\mathrm{Index}(PuP) = -\frac{1}{\sqrt{2i}2^n\Gamma(n/2+1)}\mathrm{Ch}_n(\mathcal{H}, F)(u).$$

An independent check can be made on the circle, using the unitary u given by multiplication by $e^{i\theta}$ on $L^2(S^1)$ and the Dirac operator $\frac{1}{i}\frac{d}{d\theta}$. In this case $\mathrm{Index}(PuP) = -1$. □

For the reader's benefit, we note that in [CPRS2] the signs used are all correct, however in [CPRS4] we introduced an additional minus sign (in error) in the formula for the odd case. This disguised the fact that we were not taking a homotopy to the Chern character (as defined above) but rather to minus the Chern character.

The pairing only depends on the K-theory class of e or u and the K-homology class of $(\mathcal{H}, F)$. Whilst the Chern character in this form is very useful for proving basic facts about the index pairing, and relating it to cyclic cohomology, it is not the most computable form for examples.

Imagine trying to compute the pairing of the Hodge-de Rham Fredholm module with K-theory this way. First take the Hilbert space of L^2 differential forms, then construct an invertible version as in Example 17

$$\mathcal{D}_m = \begin{pmatrix} d+d^* & m \\ m & -(d+d^*) \end{pmatrix},$$

and then take the phase $F = \mathcal{D}_m|\mathcal{D}_m|^{-1}$. Now take commutators with a projection, multiply the commutators together and take a trace. This would appear to be (and is!) much harder to compute with than simply differentiating functions. The issue of finding a computable approach to the index pairing is the most fundamental reason for being interested in spectral triples. They capture the index pairings that arise naturally from the study of unbounded (differential) operators.

5 Spectral triples and computation of the index pairing

5.1 Regularity of spectral triples and algebras. In order to express the pairing of the K-homology class $[(\mathcal{A}, \mathcal{H}, \mathcal{D})]$ with K-theory directly in terms of the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we need more constraints. In particular, to compare our computations with the Chern character computations, we will need to know that $(\mathcal{H}, \mathcal{D}(1+\mathcal{D}^2)^{-1/2})$ is a finitely summable Fredholm module. On the other hand regularity is about having a noncommutative analogue of differentiability for elements of our algebra. To ensure that a spectral triple represents a K-homology class with a finitely summable representative, we need a summability assumption on the spectral triple, and some regularity as well. The interplay between regularity(=differentiability) and summability(=dimension) is more complicated than in the commutative case.

Definition 5.1. *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k for $k \geq 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k, where $\delta(T) = [|\mathcal{D}|, T]$ is the partial derivation on $\mathcal{B}(\mathcal{H})$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \geq 1$.*

Remark. The notation is meant to be analogous to the classical case, but we introduce the Q so that there is no confusion between quantum differentiability of $a \in \mathcal{A}$ and classical differentiability of functions.

Remarks concerning derivations and commutators. By partial derivation we mean that δ is defined on some subalgebra of $\mathcal{B}(\mathcal{H})$ which need not be (weakly) dense in $\mathcal{B}(\mathcal{H})$. More precisely, $\mathrm{dom}\,\delta = \{T \in \mathcal{B}(\mathcal{H}) : \delta(T) \text{ is bounded}\}$. We also note that if $T \in \mathcal{B}(\mathcal{H})$, one can show that $[|\mathcal{D}|, T]$ is bounded if and only if $[(1+\mathcal{D}^2)^{1/2}, T]$ is bounded, by using the functional calculus to show that $|\mathcal{D}| - (1+\mathcal{D}^2)^{1/2}$ extends to a bounded operator in $\mathcal{B}(\mathcal{H})$. In fact, writing $|\mathcal{D}|_1 = (1+\mathcal{D}^2)^{1/2}$ and $\delta_1(T) = [|\mathcal{D}|_1, T]$ we have

$$\mathrm{dom}\,\delta^n = \mathrm{dom}\,\delta_1^n \quad \text{for all } n.$$

Thus the condition defining QC^∞ can be replaced by

$$a, [\mathcal{D}, a] \in \bigcap_{n \geq 0} \mathrm{dom}\,\delta_1^n \quad \text{for all} \quad a \in \mathcal{A}.$$

This is important in situations where we cannot assume $|\mathcal{D}|$ is invertible.

We saw in Proposition 4.4 that spectral triples define Fredholm modules. In order that a spectral triple defines a finitely summable Fredholm module, and so a Chern character, we need finite summability of the spectral triple, which we take up in the next section. We finish this Section with some definitions and results for special algebras that arise in the context of spectral triples.

Definition 5.2. *A Fréchet algebra is a locally convex, metrizable and complete topological vector space with jointly continuous multiplication.*

We will always suppose that we can define the Fréchet topology of $\mathcal{A}$ using a countable collection of submultiplicative seminorms which includes the C^*-norm of $\overline{\mathcal{A}} = A$, and note that the multiplication is jointly continuous. By replacing any seminorm q by $\frac{1}{2}(q(a) + q(a^*))$, we may suppose that $q(a) = q(a^*)$ for all $a \in \mathcal{A}$.

Definition 5.3. *A subalgebra $\mathcal{A}$ of a C^*-algebra A is a* **pre-C^*-algebra** *or* **stable under the holomorphic functional calculus** *if, whenever $a \in \mathcal{A}$ is invertible in A, it is invertible in $\mathcal{A}$. Equivalently, $\mathcal{A}$ is a pre-C^*-algebra if, whenever $f : \mathbb{C} \to \mathbb{C}$ is a function holomorphic in a neighbourhood of the spectrum of $a \in \mathcal{A}$, then the element $f(a) \in A$ defined by the continuous functional calculus is in fact in $\mathcal{A}$, i.e. $f(a) \in \mathcal{A}$.*

Definition 5.4. *A $*$-algebra $\mathcal{A}$ is smooth if it is Fréchet and $*$-isomorphic to a proper dense subalgebra $i(\mathcal{A})$ of a C^*-algebra A which is stable under the holomorphic functional calculus.*

Thus, saying that $\mathcal{A}$ is smooth means that $\mathcal{A}$ is Fréchet and a pre-C^*-algebra. Asking for $i(\mathcal{A})$ to be a *proper* dense subalgebra of A immediately implies that the Fréchet topology of $\mathcal{A}$ is finer than the C^*-topology of A (since Fréchet means locally convex, metrizable and complete.)

Remark. It has been shown that if $\mathcal{A}$ is smooth in A then $M_n(\mathcal{A})$ is smooth in $M_n(A)$, [GVF, Sc]. This ensures that the K-theories of the two algebras are isomorphic, the isomorphism being induced by the inclusion map i. This definition ensures that a smooth algebra is a 'good' algebra, [GVF], so these algebras have a sensible spectral theory which agrees with that defined using the C^*-closure, and that the group of invertibles is open.

Lemma 5.5. *[[GVF, R1]] If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple, then $(\mathcal{A}_\delta, \mathcal{H}, \mathcal{D})$ is also a QC^∞ spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the locally convex topology determined by the seminorms*

$$q_{ni}(a) = \|\delta^n d^i(a)\|, \quad n \geq 0, \; i = 0, 1,$$

where $d(a) = [\mathcal{D}, a]$. Moreover, $\mathcal{A}_\delta$ is a smooth algebra.

Thus, whenever we have a QC^∞ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we may suppose without loss of generality that the algebra $\mathcal{A}$ is a Fréchet pre-C^*-algebra. Thus $\mathcal{A}$ suffices to capture all the K-theory of A. This is necessary if we are to use spectral triples to compute the index pairing. A QC^∞ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for which $\mathcal{A}$ is complete has not only a holomorphic functional calculus for $\mathcal{A}$, but also a C^∞ *functional calculus* for self-adjoint elements: we quote [R1, Prop. 22].

Proposition 5.6 (C^∞ Functional Calculus). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ spectral triple, and suppose $\mathcal{A}$ is complete in the δ-topology. Let $f : \mathbb{R} \to \mathbb{C}$ be a C^∞ function in a neighbourhood of the spectrum of $a = a^* \in \mathcal{A}$. If we define $f(a) \in A$ using the continuous functional calculus, then in fact $f(a)$ lies in $\mathcal{A}$.* □

Remark. For each $a = a^* \in \mathcal{A}$, the C^∞-functional calculus defines a continuous homomorphism $\Psi : C^\infty(U) \to \mathcal{A}$ where $U \subset \mathbb{R}$ is any open set containing the spectrum of a, and the topology on $C^\infty(U)$ is that of uniform convergence of all derivatives on compact subsets.

5.2 Summability for spectral triples.

5.2.1 Finite summability. Just as for Fredholm modules, we require a notion of summability for spectral triples as this is needed to write down explicit formulae for index pairings and representatives of the Chern character.

Definition 5.7. *A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is called finitely summable if there is some $s_0 > 0$ such that*

$$\mathrm{Trace}((1 + \mathcal{D}^2)^{-s_0/2}) < \infty.$$

This is then true for all $s > s_0$ and we call

$$p = \inf\{s \in \mathbb{R}_+ : \mathrm{Trace}((1 + \mathcal{D}^2)^{-s/2}) < \infty\}$$

the spectral dimension.

Remark. What finitely summable means for a spectral triple with $\mathcal{A}$ non-unital and $(1+\mathcal{D}^2)^{-1/2}$ not a compact operator, but of course with $a(1+\mathcal{D}^2)^{-1/2}$ compact for all $a \in \mathcal{A}$, is still an open question; see [GGISV, R1, R2].

Not all algebras have finitely summable spectral triples, even when they have finitely summable Fredholm modules (more on this later). We quote the following necessary condition.

Theorem 5.8 (Connes, [C2]). *Let A be a unital C^*-algebra and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ a finitely summable QC^1 spectral triple, with $\mathcal{A} \subset A$ dense. Then there exists a positive trace τ on A with $\tau(1) = 1$.*

So algebras with no normalised trace, such as the Cuntz algebras, do not have finitely summable spectral triples associated to them.

Proposition 5.9. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a finitely summable QC^1 spectral triple with spectral dimension $p \geq 0$, then $(\mathcal{H}, F_{\mathcal{D}})$ is a $([p] + 1)$-summable Fredholm module for $\mathcal{A}$, where $[p]$ is the largest integer less than or equal to p.*

Proof. Let $a \in \mathcal{A}$ and recall

$$[F_{\mathcal{D}}, a] = [\mathcal{D}, a](1 + \mathcal{D}^2)^{-1/2} - F_{\mathcal{D}}[(1 + \mathcal{D}^2)^{1/2}, a](1 + \mathcal{D}^2)^{-1/2} =: T(1 + \mathcal{D}^2)^{-1/2}.$$

Now observe that T is bounded, and we want to show

$$T(1 + \mathcal{D}^2)^{-1/2}T(1 + \mathcal{D}^2)^{-1/2} \cdots T(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^1(\mathcal{H})$$

where we have a product of $[p]+1$ terms. For each $\epsilon > 0$ we have $T(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{p+\epsilon}(\mathcal{H})$. As $[p] \leq p < [p]+1$, we can choose $p + \epsilon$ between p and $[p] + 1$, and so $T(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{[p]+1}(\mathcal{H})$, the product is in $\mathcal{L}^1(\mathcal{H})$ and we are done. □

Remark. As in Proposition 4.4, using [CP1] we can replace QC^1 by QC^0.

The finitely summable Fredholm module above is not normalised. To obtain a normalised finitely summable Fredholm module, we follow the same recipe that we applied to the Hodge-de Rham example.

Lemma 5.10. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. For any $m > 0$ we define the double of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ to be the spectral triple $(\mathcal{A}, \mathcal{H}_2, \mathcal{D}_m)$ with*

$$\mathcal{H}_2 = \mathcal{H} \oplus \mathcal{H}, \quad \mathcal{D}_m = \begin{pmatrix} \mathcal{D} & m \\ m & -\mathcal{D} \end{pmatrix}, a \to \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is graded by γ, the double is graded by $\gamma \oplus -\gamma$. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k, $k = 0, 1, \ldots, \infty$, so is the double. If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is finitely summable with spectral dimension p, the double is finitely summable with spectral dimension p. Moreover, the K-homology classes of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and its double coincide for any $m > 0$. This class can be represented by the normalised Fredholm module $(\mathcal{H}_2, \mathcal{D}_m|\mathcal{D}_m|^{-1})$.

Remark. The explicit identification of the K-homology classes and the normalised representative can be found in [CPRS1].

5.2.2 (n, ∞)**-summability and the Dixmier trace.** We recall the classical result of Weyl.

Theorem 5.11 (Weyl's theorem). *Let P be an order d elliptic differential operator on a compact oriented manifold M of dimension n. Let $\{\lambda_k\}$ denote the eigenvalues of P ordered so that $|\lambda_1| \le |\lambda_2| \le \cdots$ and repeated according to multiplicity. Then*

$$|\lambda_k| \sim Ck^{d/n} + O(k^{d/n-1}).$$

The constant C can also be computed, but we will leave that for a little while. First we will introduce some analytic machinery. If $T \in \mathcal{K}(\mathcal{H})$, let $\mu_n(T)$ denote the n-th singular number of T; that is, $\mu_n(T)$ is the n-th eigenvalue of $\sqrt{T^*T}$ when they are listed in nonincreasing order and repeated according to multiplicity. Let

$$\sigma_N(T) = \sum_{k=1}^{N} \mu_k(T)$$

be the N-th partial sum of the singular values. For $p > 1$ let

$$\mathcal{L}^{(p,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sigma_N(T) = O(N^{1-1/p})\}$$

and for $p = 1$

$$\mathcal{L}^{(1,\infty)}(\mathcal{H}) = \{T \in \mathcal{K}(\mathcal{H}) : \sigma_N(T) = O(\log N)\}.$$

We will be interested in $\mathcal{L}^{(1,\infty)}(\mathcal{H})$; however the following is useful: If $T_1, \ldots, T_m$ are in $\mathcal{L}^{(p_1,\infty)}, \ldots, \mathcal{L}^{(p_m,\infty)}$ respectively, and $1/p_1 + \cdots + 1/p_m = 1$ then $T_1T_2 \cdots T_m \in \mathcal{L}^{(1,\infty)}$. While the Schatten classes play a similar role to the L^p spaces of classical analysis, the $\mathcal{L}^{(p,\infty)}$ spaces play a role similar to weak L^p spaces.

What we would like to do is construct a functional on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ by defining for $T \ge 0$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \mu_k(T).$$

However, this formula need not define a linear functional, and it may not converge. The trick is to consider the sequence

$$\left(\frac{\sigma_2(T)}{\log 2}, \frac{\sigma_3(T)}{\log 3}, \frac{\sigma_4(T)}{\log 4}, \ldots\right)$$

and observe that this sequence is bounded. If it always converged, the limit would provide a linear functional which is a trace. As it does not always converge, we must consider certain generalised limits, whose explicit definition can be found in [CPS2]. We will denote by $\lim_\omega$ any such generalised limit, and observe that there are uncountably many such, as is explained in [CPS2, C1]. The resulting functionals on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ are called Dixmier traces.

Proposition 5.12. *For $T \ge 0$, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ and suitable $\omega \in \ell^\infty(\mathbb{N})^*$ we define the associated Dixmier trace*

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_{k=1}^{N} \mu_k(T).$$

Then

1) $\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2)$, *so we can extend it by linearity to all of* $\mathcal{L}^{(1,\infty)}(\mathcal{H})$

2) If $T \ge 0$ *then* $\mathrm{Tr}_\omega(T) \ge 0$

3) If $S \in \mathcal{B}(\mathcal{H})$ *and* $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ *then* $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$

Moreover, for any trace class operator T we have $\mathrm{Tr}_\omega(T) = 0$

While the above result holds for a range of functionals ω, in practise the value of a Dixmier trace on operators that arise in examples is independent of the choice of ω: we call such operators measurable.

Here is the key criterion for measurability. First, for $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, $T \geq 0$, define for $\mathrm{Re}(s) > 1$

$$\zeta_T(s) = \mathrm{Trace}(T^s) = \sum_{k=1}^{\infty} \mu_k(T)^s.$$

Then

Proposition 5.13. *With $T \geq 0$ as above the following are equivalent:*

1) $(s-1)\zeta_T(s) \to L$ as $s \searrow 1$;

2) $\frac{1}{\log N} \sum_{k=1}^{N} \mu_k(T) \to L$ as $N \to \infty$.

In this case, the residue at $s = 1$ of $\zeta_T(s)$ is precisely $\mathrm{Tr}_\omega(T)$ and so the value of Tr_ω on T is independent of ω. In fact, non-measurable operators are not natural; see [GVF] for an example.

Proposition 5.14 (Connes' Trace Theorem). *Let M be an n-dimensional compact manifold and let T be a classical pseudodifferential operator of order $-n$ (think of $T = (1+\mathcal{D}^2)^{-n/2}$ where $\mathcal{D}$ is of order 1) acting on sections of a complex vector bundle $E \to M$. Then*

1) The corresponding operator T on $L^2(M,E)$ belongs to the ideal $\mathcal{L}^{(1,\infty)}$

2) The Dixmier trace $\mathrm{Tr}_\omega(T)$ is independent of ω and is equal to the Wodzicki residue:

$$\mathrm{WRes}(T) = \frac{1}{n(2\pi)^n} \int_{S^*M} \mathrm{trace}_E(\sigma_T(x,\xi))d\mathrm{vol}.$$

Here S^*M is the cosphere bundle, $\{\xi \in T^*M : \|\xi\|^2 = g^{\mu\nu}\xi_\mu\xi_\nu = 1\}$.

The surprising fact about the Wodzicki residue is that it extends to a trace (the unique such trace) on the whole algebra of pseudodifferential operators of any order. This extension is simply to take the $-n$-th part of the symbol and integrate it over the co-sphere bundle. Thus the residue of the zeta function can be computed geometrically.

In the following we restrict attention to operators 'of Dirac type', by which we mean that the principal symbol of $\mathcal{D}$ is given by Clifford multiplication. This means that the symbol of $\mathcal{D}^2$ is given by $\sigma_{\mathcal{D}^2}(x,\xi) = \|\xi\|^2$.

Corollary 5.15. *Let $f \in C^\infty(M)$ and $\mathcal{D}$ be a first order self-adjoint elliptic operator 'of Dirac type' on the vector bundle E. Then the operator $f(1+\mathcal{D}^2)^{-n/2}$ acting on $L^2(E)$ is measurable and*

$$\mathrm{Tr}_\omega(f(1+\mathcal{D}^2)^{-n/2}) = \frac{\mathrm{rank}(E)\mathrm{Vol}(S^{n-1})}{n(2\pi)^n} \int_M f\, d\mathrm{vol}.$$

Hence the representation of functions as multiplication operators, along with the spectrum of $\mathcal{D}$, is enough to recover the integral on a manifold using the Dixmier trace on smooth functions. For a clear discussion of what happens for measurable functions, see [CS, LPS]. This has stimulated interest in other spectral triples satisfying the following summability hypothesis.

Definition 5.16. *A spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ is (n,∞)-summable if*

$$(1+\mathcal{D}^2)^{-n/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}).$$

This definition is definitely in the context of unital algebras $\mathcal{A}$. For an approach to this definition when $\mathcal{A}$ is non-unital see [GGISV, R1, R2]. Observe that if a spectral triple is (n,∞)-summable, then the associated Fredholm module is $(n+1)$-summable. Also, the spectral dimension of such a triple is n.

Example 18. Examining the eigenvalues of the 'Dirac' operator for the noncommutative torus in Section 3.5 (for simplicity set $\tau = i$), we see that the eigenvalues obey Weyl's Theorem. This is not surprising since $\mathcal{D}$ and $\mathcal{H}$ are actually the same as in the commutative case. Hence the spectral triple for the noncommutative torus is $(2,\infty)$-summable with spectral dimension $p = 2$.

Exercise. Compute the Dixmier trace of $(1+\mathcal{D}^2)^{-1}$ for the noncommutative torus. *Hint* See [GVF].

Example 19. The Cantor set spectral triple introduced in Example 7 is also illuminating. If the gap between e_- and e_+ appears at the n-th stage of our construction (counting the interval $[0,1]$ as the 0-th stage), then $e_+ - e_- = 3^{-n}$. How many gaps are there with this length? The answer is 2^{n-1}, except for $n = 0$. Including the extra 2 for the 2×2 matrix structure of $|\mathcal{D}|$, we find that the trace of $|\mathcal{D}|^{-s}$ for $s \gg 1$ is

$$\zeta(s) = 2 + \sum_{n=1}^{\infty} 2^n 3^{-ns} = 1 + \frac{1}{1-2/3^s}.$$

This is finite for $s > \frac{\log 2}{\log 3}$ and this formula provides a meromorphic continuation of $\zeta(s)$ whose only singularities are simple poles at $s = (\log 2 + 2k\pi i)/\log 3$. The number $\log 2/\log 3$ is the Hausdorff dimension of the Cantor set.

Exercise. Find the residue at $s = \log 2/\log 3$.

The relationship between the Dixmier trace and the zeta function, as well as the heat kernel asymptotics, is well described in [CPS2], and the definitive results are in [CRSS] and [LPS].

5.2.3 θ-summability.

Definition 5.17. *A spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ is θ-summable if for all $t > 0$ we have*

$$\mathrm{Trace}(e^{-t\mathcal{D}^2}) < \infty.$$

Example 20. There are many interesting spectral triples that are not finitely summable. This has the consequence that our main tool (which we describe later), the local index formula, is not available. The examples arising from supersymmetric quantum field theory are not finitely summable, but rather than take a detour into physics, we will look at examples coming from group C^*-algebras. All of this material, plus the construction of the metric on the state space first appeared in the beautiful paper [C2].

Let Γ be a finitely generated group, and let $\mathbb{C}\Gamma$ denote the group ring of Γ. The group ring $\mathbb{C}\Gamma$ acts by bounded operators on the Hilbert space $l^2(\Gamma)$. The action is defined on the dense linear subspace $\mathbb{C}\Gamma \subset l^2(\Gamma)$ by left multiplication. One shows that each $a \in \mathbb{C}\Gamma$ extends to a bounded operator on $l^2(\Gamma)$ and that we obtain a $*$-homomorphism $\mathbb{C}\Gamma \to \mathcal{B}(l^2(\Gamma))$. This is called the left regular representation.

Then the C^*-algebra $C^*_{red}(\Gamma)$, called the reduced C^*-algebra of Γ, is the norm closure of the image of $\mathbb{C}\Gamma$ in $\mathcal{B}(l^2(\Gamma))$. For $\psi \in l^2(\Gamma)$ and $g \in \Gamma$, the left regular representation is given by

$$(\lambda(g)\psi)(k) = \psi(g^{-1}k).$$

Define a length function on Γ to be a function $L : \Gamma \to \mathbb{R}^+$ such that

1) $L(gh) \le L(g) + L(h)$ for all $g, h \in \Gamma$

2) $L(g^{-1}) = L(g)$ for all $g \in \Gamma$

3) $L(1) = 0$

The prototypical example is the word length function. Let $G \subset \Gamma$ be a generating set. Then for all $g \in \Gamma$, $g = g_1 \cdots g_n$ for some n where $g_i \in G$ for all $i = 1, ..., n$. This expression is not unique, and we define

$$L(g) = \min\{n : g = g_1 \cdots g_n,\ g_i \in G,\ i = 1, \ldots, n\}.$$

Using length functions we can construct spectral triples.

Lemma 5.18. *Let Γ be a finitely generated discrete group and L a length function on Γ. Let $\mathcal{D}$ be the operator of multiplication by L on $\mathcal{H} = l^2(\Gamma)$. If L is unbounded on Γ then*

1) $(\mathbb{C}(\Gamma), \mathcal{H}, \mathcal{D})$ *is a spectral triple.*

2) $\|[\mathcal{D}, \lambda(g)]\| = L(g)$ *for all $g \in \Gamma$.*

Proof. To show that for a dense subalgebra $\mathcal{A} \subset C^*(\Gamma)$ the commutators $[\mathcal{D}, a]$ are bounded, it suffices to show that for all $g \in \Gamma$, the commutator $[\mathcal{D}, \lambda(g)]$ is bounded (the group ring $\mathbb{C}\Gamma$ is dense). We compute

$$\begin{aligned}(\mathcal{D}\lambda(g)\psi)(k) - (\lambda(g)\mathcal{D}\psi)(k) &= \mathcal{D}\psi(g^{-1}k) - \lambda(g)L(k)\psi(k) \\ &= L(g^{-1}k)\psi(g^{-1}k) - L(k)\psi(g^{-1}k).\end{aligned}$$

However

$$|L(g^{-1}k) - L(k)| \leq |L(g^{-1}) + L(k) - L(k)| = L(g),$$

so this is bounded.

Now for any real number $x \in \mathbb{R}$, let $K_x \subset \Gamma$ be those group elements with $L(g) = x$. Let ψ_x be the function in $l^2(\Gamma)$ with $\psi_x \equiv 1$ on K_x and zero elsewhere. Then

$$\mathcal{D}\psi_x = x\psi_x.$$

As Γ is discrete, L takes on only a discrete set of values. Thus there are a countable number of ψ_xs and corresponding eigenvalues x. With the assumption that $L(g) \to \infty$ as $g \to \infty$, we see that $\mathcal{D}$ is unbounded, has countably many eigenvalues of finite multiplicity, and this is enough to conclude that $\mathcal{D}$ has compact resolvent. This shows that we have a spectral triple. We do not know if it is even or odd.

Lastly, let ψ_1 be the function which is 1 on $1 \in \Gamma$ and zero elsewhere. Then

$$\begin{aligned}([\mathcal{D}, \lambda(g)]\psi_1)(k) &= (\mathcal{D}\lambda(g)\psi_1)(k) - (\lambda(g)\mathcal{D}\psi_1)(k) \\ &= (\mathcal{D}\psi_1)(g^{-1}k) - (\lambda(g)L(k)\psi_1)(k) \\ &= (L(g^{-1}k) - L(k))\psi_1(g^{-1}k) \\ &= \begin{cases} 0 & k \neq g^{-1} \\ -L(k) & k = g^{-1} \end{cases} \\ &= -L(g)\delta_{k,g^{-1}}.\end{aligned}$$

So as we showed that $\|[\mathcal{D}, \lambda(g)]\| \leq L(g)$, the above calculation shows that equality always holds, proving the second assertion of the lemma. $\square$

So we have a spectral triple, a priori odd (ungraded). We are interested in seeing whether it is finitely summable.

Theorem 5.19 (Connes, [C2]). *Let Γ be a discrete group containing the free group on two generators. Let $\mathcal{H}$ be any representation of $C^*(\Gamma)$, absolutely continuous with respect to the canonical trace on $C^*(\Gamma)$. Then there does not exist a self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that $(\mathcal{H}, \mathcal{D})$ is a finitely summable spectral triple for $C^*_{red}(\Gamma)$.*

Remark. There may be finitely summable Fredholm modules for such a group algebra. In particular, one is known for the free group on 2 generators. The culprit here is the lack of hyperfiniteness of the group von Neumann algebra.

Theorem 5.20 (Connes, [C2]). *If Γ is an infinite discrete group with property T, then there exists no finitely summable spectral triple for $C^*_{red}(\Gamma)$.*

Again, there are interesting finitely summable Fredholm modules for such groups but the fundamental problem is that some group C^*-algebras are 'infinite dimensional noncommutative spaces'.

Theorem 5.21 (Connes, [C2]). *Let Γ be a finitely generated discrete group, and L the word length function, relative to some generating subset. Let $\mathcal{H} = l^2(\Gamma)$, with $C^*_{red}(\Gamma)$ acting by multiplication and let $\mathcal{D}$ be multiplication by the word length function L. Then $(\mathcal{H}, \mathcal{D})$ is a θ-summable spectral triple for $C^*_{red}(\Gamma)$.*

5.3 Analytic formulae for the index. There are analytic formulae for the pairing between a spectral triple and K-theory with no reference to cyclic cohomology. Nevertheless, it is via these formulae that the link to cyclic cohomology is made in [CPRS2, CPRS3]. These formulae are truly purely analytic, and apart from a few special cases do not provide in themselves a computational tool. Rather, they allow us to derive a variety of different formulae from which we can connect to, say, topological formulae for the index in the case of manifolds.

Theorem 5.22 (McKean-Singer Formula). *Let $\mathcal{D}$ be an unbounded self-adjoint operator with compact resolvent. Let γ be a self-adjoint unitary which anticommutes with $\mathcal{D}$. Finally, let f be a continuous even function on $\mathbf{R}$ with $f(0) \neq 0$ and $f(\mathcal{D})$ trace-class. Let $\mathcal{D}^+ = P^\perp \mathcal{D} P$ where $P = (1+\gamma)/2$ and $P^\perp = 1 - P$. Then $D^+ : P\mathcal{H} \to P^\perp\mathcal{H}$ is Fredholm and*

$$\mathrm{Index}(D^+) = \frac{1}{f(0)} \mathrm{Trace}\left(\gamma f(D)\right). \tag{5.1}$$

This version (actually a stronger version valid in semfinite von Neumann algebras) can be found in [CPRS3]. The traditional function used in this context is $f(x) = e^{-tx^2}$, $t > 0$, so the formula becomes:

$$\mathrm{Index}(\mathcal{D}^+) = \mathrm{Trace}\left(\gamma e^{-t\mathcal{D}^2}\right).$$

The operator $e^{-t\mathcal{D}^2}$ is called the **heat operator**, being the solution of the 'heat equation' $\partial_t A(t) + \mathcal{D}^2 A(t) = 0$. However for a finitely summable spectral triple, functions such as $f(x) = (1+x^2)^{-s/2}$ for s large provide a natural alternative. As we remarked previously, in the case of odd spectral triples the pairing between K-theory and K-homology calculates spectral flow.

Theorem 5.23 (Spectral flow formula,[CP1, CP2]). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a finitely summable spectral triple with spectral dimension $p \geq 1$. Let $u \in \mathcal{A}$ be unitary and let P be the spectral projection of $\mathcal{D}$ corresponding to the interval $[0, \infty)$. Then for any $s > p$ the spectral flow along the line segment joining $\mathcal{D}$ to $u\mathcal{D}u^*$ is given by*

$$\mathrm{Index}(PuP) = \frac{1}{C_{s/2}} \int_0^1 \mathrm{Trace}(u[\mathcal{D}, u^*](1 + (\mathcal{D} + tu[\mathcal{D}, u^*])^2)^{-s/2}) dt, \tag{5.2}$$

with $C_{s/2} = \int_{-\infty}^{\infty} (1+x^2)^{-s/2} dx$.

Both of the analytic formulae are scale invariant. By this we mean that if we replace $\mathcal{D}$ by $\epsilon\mathcal{D}$, for $\epsilon > 0$, in the right hand side of (5.2) or (5.1), then the left hand side is unchanged, since in both cases the index is invariant with respect to change of scale. Rewriting the 'constant' $C_{s/2}$ as

$$C_{s/2} = \frac{\Gamma(s-1/2)\Gamma(1/2)}{\Gamma(s)}$$

we see that in fact the integral formula in (5.2) can be given a meromorphic continuation (as a function of s) by setting

$$\mathrm{Index}(PuP)C_{s/2} = \int_{\mathbf{0}}^{\mathbf{1}} \mathrm{Trace}(\mathbf{u}[\mathcal{D},\mathbf{u}^*](\mathbf{1}+(\mathcal{D}+\mathbf{tu}[\mathcal{D},\mathbf{u}^*])^{\mathbf{2}})^{-\mathbf{s/2}})\mathbf{dt}.$$

Here we have written the right hand side in bold face to indicate that we are thinking of the meromorphically continued function. Since the residue of $C_{s/2}$ at $s = 1/2$ is 1, we also have

$$\mathrm{Index}(PuP) = \mathrm{res}_{s=1/2}\int_{\mathbf{0}}^{\mathbf{1}} \mathrm{Trace}(\mathbf{u}[\mathcal{D},\mathbf{u}^*](\mathbf{1}+(\mathcal{D}+\mathbf{tu}[\mathcal{D},\mathbf{u}^*])^{\mathbf{2}})^{-\mathbf{s/2}})\mathbf{dt}.$$

This observation is the starting point for the proof of the local index formula in [CPRS2]. A suitable choice of functions allows a similar analysis in the even case; we refer to [CPRS3].

6 The Chern character of spectral triples

6.1 Cyclic homology and cohomology. A central feature of [C1] is the expression of the K-theory pairing in terms of cyclic theory in order to obtain index theorems. In this Section we will summarise the relevant notions of cyclic theory.

The aim is to associate to a suitable representative of a K-theory class, respectively a K-homology class, a class in periodic cyclic homology, respectively a class in periodic cyclic cohomology, called a Chern character in both cases. The principal result is then

$$\langle [x], [(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle = -\frac{1}{\sqrt{2\pi i}}\langle [\mathrm{Ch}_*(x)], [\mathrm{Ch}^*(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle, \tag{6.1}$$

where $[x] \in K_*(\mathcal{A})$ is a K-theory class with representative x and $[(\mathcal{A},\mathcal{H},\mathcal{D})]$ is the K-homology class of the spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$.

On the right hand side, $\mathrm{Ch}_*(x)$ is the Chern character of x, and $[\mathrm{Ch}_*(x)]$ its cyclic homology class. Similarly $[\mathrm{Ch}^*(\mathcal{A},\mathcal{H},\mathcal{D})]$ is the cyclic cohomology class of a representative of the Chern character of $(\mathcal{A},\mathcal{H},\mathcal{D})$.

We will use the normalised (b, B)-bicomplex (see [C1, L]). The reason for this is that one can easily realise the Chern character of a finitely summable Fredholm module, a cyclic cocycle, in the (b, B) picture, but going the other way requires substantial work, [CPRS4].

We introduce the following linear spaces. Let $C_m = \mathcal{A}\otimes\bar{\mathcal{A}}^{\otimes m}$ where $\bar{\mathcal{A}}$ is the quotient $\mathcal{A}/\mathbb{C}I$ with I being the identity element of $\mathcal{A}$ and (assuming with no loss of generality that $\mathcal{A}$ is complete in the δ-topology) we employ the projective tensor product. Let $C^m = \mathrm{Hom}(C_m,\mathbb{C})$ be the linear space of continuous multilinear functionals on C_m. We may define the (b, B)-bicomplex using these spaces (as opposed to $C_m = \mathcal{A}^{\otimes m+1}$ etc) and the resulting cohomology will be the same. This follows because the bicomplex defined using $\mathcal{A}\otimes\bar{\mathcal{A}}^{\otimes m}$ is quasi-isomorphic to that defined using $\mathcal{A}\otimes\mathcal{A}^{\otimes m}$.

A normalised $(\mathbf{b},\mathbf{B})$**-cochain**, ϕ, is a finite collection of continuous multilinear functionals on $\mathcal{A}$,

$$\phi = (\phi_m)_{m=1,2,\dots,M} \text{ with } \phi_m \in C^m.$$

It is a (normalised) $(\mathbf{b},\mathbf{B})$**-cocycle** if, for all m, $b\phi_m + B\phi_{m+2} = 0$ where $b : C^m \to C^{m+1}$, $B : C^m \to C^{m-1}$ are the coboundary operators given by

$$(B\phi_m)(a_0, a_1, \ldots, a_{m-1}) = \sum_{j=0}^{m-1} (-1)^{(m-1)j} \phi_m(1, a_j, a_{j+1}, \ldots, a_{m-1}, a_0, \ldots, a_{j-1})$$

$$(b\phi_{m-2})(a_0, a_1, \ldots, a_{m-1}) =$$

$$\sum_{j=0}^{m-2} (-1)^j \phi_{m-2}(a_0, a_1, \ldots, a_j a_{j+1}, \ldots, a_{m-1}) + (-1)^{m-1} \phi_{m-2}(a_{m-1}a_0, a_1, \ldots, a_{m-2})$$

We write $(b + B)\phi = 0$ for brevity. Thought of as functionals on $\mathcal{A}^{\otimes m+1}$ a normalised cocycle will satisfy $\phi(a_0, a_1, \ldots, a_n) = 0$ whenever any $a_j = 1$ for $j \geq 1$. An **odd (even)** cochain has $\phi_m = 0$ for m even (odd).

Similarly, a $(\mathbf{b^T},\mathbf{B^T})$**-chain**, c is a (possibly infinite) collection $c = (c_m)_{m=1,2,\ldots}$ with $c_m \in C_m$. The (b, B)-chain (c_m) is a $(\mathbf{b^T},\mathbf{B^T})$**-cycle** if $b^T c_{m+2} + B^T c_m = 0$ for all m. More briefly, we write $(b^T + B^T)c = 0$. Here b^T, B^T are the boundary operators of cyclic homology, and are the transpose of the coboundary operators b, B in the following sense.

The pairing between a (b, B)-cochain $\phi = (\phi_m)_{m=1}^M$ and a (b^T, B^T)-chain $c = (c_m)$ is given by ($M \in \mathbb{N}$ or $M = \infty$)

$$\langle \phi, c \rangle = \sum_{m=1}^{M} \phi_m(c_m).$$

This pairing satisfies

$$\langle (b + B)\phi, c \rangle = \langle \phi, (b^T + B^T)c \rangle.$$

We use this fact in the following way. We call $c = (c_m)_{m \text{ odd}}$ an odd normalised $(\mathbf{b^T},\mathbf{B^T})$**-boundary** if there is some even chain $e = (e_m)_{m \text{ even}}$ with $c_m = b^T e_{m+1} + B^T e_{m-1}$ for all m. If we pair a normalised (b, B)-cocycle ϕ with a normalised (b^T, B^T)-boundary c we find

$$\langle \phi, c \rangle = \langle \phi, (b^T + B^T)e \rangle = \langle (b + B)\phi, e \rangle = 0.$$

There is an analogous definition in the case of even chains $c = (c_m)_{m \text{ even}}$. All of the cocycles we consider in these notes are in fact defined as functionals on $\oplus_m \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes m}$. Henceforth we will drop the superscript on b^T, B^T and just write b, B for both boundary and coboundary operators as the meaning will be clear from the context.

We recall that the Chern character $\mathrm{Ch}_*(u)$ of a unitary $u \in \mathcal{A}$ is the following (infinite) collection of odd chains $\mathrm{Ch}_{2j+1}(u)$ satisfying $b\mathrm{Ch}_{2j+3}(u) + B\mathrm{Ch}_{2j+1}(u) = 0$,

$$\mathrm{Ch}_{2j+1}(u) = (-1)^j j! u^* \otimes u \otimes u^* \otimes \cdots \otimes u \quad (2j + 2 \ \text{ entries}).$$

Exercise. Check that $\mathrm{Ch}_*(u)$ is a (b, B)-cycle, and that $\mathrm{Ch}_*(u) + \mathrm{Ch}_*(u^*)$ is a coboundary.

Similarly, the (b, B) Chern character of a projection p in an algebra $\mathcal{A}$ is an even (b, B) cycle with $2m$-th term, $m \geq 1$, given by

$$\mathrm{Ch}_{2m}(p) = (-1)^m \frac{(2m)!}{2(m!)} (2p - 1) \otimes p^{\otimes 2m}.$$

For $m = 0$ the definition is $\mathrm{Ch}_0(p) = p$.

Exercise. Check that $\mathrm{Ch}_*(p)$ is a (b, B)-cycle.

Since the (b, B) Chern character of a projection or unitary has infinitely many terms which grow rapidly, we need some constraint on the cochains we pair them with. If we allow only finitely supported cochains, then we obtain the usual cyclic cohomology groups $HC^*(\mathcal{A})$. The Chern character of a finitely summable spectral triple is finitely supported.

If we allow infinitely supported cochains which satisfy some decay condition α, then we get something we shall call $HC^*_\alpha(\mathcal{A})$. The most commonly used condition is to look at entire cochains, and the reason for this is that the JLO cocycle is entire; see [C1]. Very often one finds that for any reasonable decay condition α we have $HC^*_\alpha(\mathcal{A}) \cong HC^*(\mathcal{A})$, but general statements are hard to find.

A final warning: cyclic (co)homology of a C^*-algebra is trivial. It is necessary to work with a smooth subalgebra, or employ a cyclic theory developed for C^*-algebras called local cyclic (co)homology, due to Puschnigg, [Pu]. Alternatively, one could use Kasparov's KK-theory with smooth algebras. This approach is developed by Cuntz, [Cu]. In general, however, the tension between continuous and smooth theories appears naturally.

6.2 The local index formula for finitely summable smooth spectral triples. A natural question is whether there is an analogue of the Atiyah-Singer index theorem in noncommutative geometry. This question was answered, in a sense, by Connes and Moscovici in the paper [CM]. What the local index formula of Connes and Moscovici provides is a schematic: a general formula which must still be massaged further to get interesting information for particular examples. One example of this process is the proof of the Atiyah-Singer index theorem by Ponge, [Po], starting from the Connes-Moscovici residue cocycle. Similarly the Atiyah-Singer index formula has been derived from the JLO cocycle by Block and Fox, [BF].

Generalisations and new proofs of Connes and Moscovici's local index formula have been given in [H, CPRS2, CPRS3]. The main point to appreciate is that given a finitely summable spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ there are various representatives of the Chern character. We have seen one already in the form of Connes' definition of the Chern character of the Fredholm module associated to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. In this Chapter we present several more.

The (finite) summability conditions give a half-plane where the function

$$z \mapsto \mathrm{Trace}((1 + \mathcal{D}^2)^{-z}) \tag{6.2}$$

is well-defined and holomorphic. In [C4, CM], a stronger condition was imposed in order to prove the local index formula. This condition not only specifies a half-plane where the function in (6.2) is holomorphic, but also that this function analytically continues to $\mathbb{C}$ minus some discrete set. We clarify this in the following definitions.

Definition 6.1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞ spectral triple. The algebra $\mathcal{B}(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H})$ is the algebra of polynomials generated by $\delta^n(a)$ and $\delta^n([\mathcal{D}, a])$ for $a \in \mathcal{A}$ and $n \geq 0$. A QC^∞ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has* **discrete dimension spectrum** *$Sd \subseteq \mathbb{C}$ if Sd is a discrete set and for all $b \in \mathcal{B}(\mathcal{A})$ the function $\mathrm{Trace}(b(1 + \mathcal{D}^2)^{-z})$ is defined and holomorphic for $\mathrm{Re}(z)$ large, and analytically continues to $\mathbb{C} \setminus Sd$. We say the dimension spectrum is* **simple** *if this zeta function has poles of order at most one for all $b \in \mathcal{B}(\mathcal{A})$,* **finite** *if there is a $k \in \mathbb{N}$ such that the function has poles of order at most k for all $b \in \mathcal{B}(\mathcal{A})$ and* **infinite***, if it is not finite.*

Connes and Moscovici impose the discrete dimension spectrum assumption to prove their original version of the local index formula. The dimension spectrum idea is quite attractive in a number of respects. The dimension spectrum of a direct sum of spectral triples is the union of the dimension spectra of the summands. The dimension spectrum of a product consists of sums of elements in the dimension spectra of the 'prodands'.

New proofs of the local index formula were presented by Nigel Higson, and by Carey, Phillips, Rennie, Sukochev. These were much simpler, more widely applicable and in the case of [CPRS1, CPRS2], required much less restriction on the zeta functions, and in particular did not require the discrete dimension spectrum hypothesis. We will introduce some notation and definitions and then state the local index formula using [CPRS1, CPRS2].

Introduce multi-indices $(k_1, \ldots, k_m)$, $k_i = 0, 1, 2, \ldots$, whose length m will always be clear from the context and let $|k| = k_1 + \cdots + k_m$. Define

$$\alpha(k) = \frac{1}{k_1!k_2!\cdots k_m!\,(k_1+1)(k_1+k_2+2)\cdots(|k|+m)}$$

and the numbers $\tilde{\sigma}_{n,j}$ and $\sigma_{n,j}$ are defined by the equalities

$$\prod_{j=0}^{n-1}(z+j+1/2) = \sum_{j=0}^{n} z^j \tilde{\sigma}_{n,j}, \qquad \text{and} \qquad \prod_{j=0}^{n-1}(z+j) = \sum_{j=1}^{n} \sigma_{n,j}.$$

If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ spectral triple and $T \in \mathcal{N}$ then $T^{(n)}$ is the n^{th} iterated commutator with $\mathcal{D}^2$, that is, $[\mathcal{D}^2, [\mathcal{D}^2, [\cdots, [\mathcal{D}^2, T]\cdots]]]$.

Definition 6.2. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞ finitely summable spectral triple, we call*

$$q = \inf\{k \in \mathbb{R} : \mathrm{Trace}((1+\mathcal{D}^2)^{-k/2}) < \infty\}$$

the spectral dimension of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has isolated spectral dimension if for all b of the form

$$b = a_0[\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}, \quad a_j \in \mathcal{A},$$

the zeta functions

$$\zeta_b(z-(1-q)/2) = \mathrm{Trace}(b(1+\mathcal{D}^2)^{-z+(1-q)/2})$$

have analytic continuations to a deleted neighbourhood of $z = (1-q)/2$.

Remark. Observe that we allow the possibility that the analytic continuations of these zeta functions may have an essential singularity at $z = (1-q)/2$. All that is necessary for us is that the residues at this point exist. Note that discrete dimension spectrum implies isolated spectral dimension.

Now let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple with isolated spectral dimension. For operators $b \in \mathcal{B}(\mathcal{H})$ of the form

$$b = a_0[\mathcal{D}, a_1]^{(k_1)} \cdots [\mathcal{D}, a_m]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|}$$

we can define the functionals

$$\tau_j(b) := \mathrm{res}_{z=(1-q)/2}(z-(1-q)/2)^j \zeta_b(z-(1-q)/2).$$

The hypothesis of isolated spectral dimension is clearly necessary here in order to define the residues. Let P be the spectral projection of $\mathcal{D}$ corresponding to the interval $[0, \infty)$.

In [CPRS2] we proved the following result:

Theorem 6.3 (Odd local index formula). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd finitely summable QC^∞ spectral triple with spectral dimension $q \geq 1$. Let $N = [q/2] + 1$ where $[\cdot]$ denotes the integer part, and let $u \in \mathcal{A}$ be unitary. Then*

1) $$\mathrm{Index}(PuP) = \frac{1}{\sqrt{2\pi i}} \mathrm{res}_{r=(1-q)/2} \left(\sum_{m=1,\text{odd}}^{2N-1} \phi_m^r(\mathrm{Ch}_m(u)) \right)$$

where for $a_0, \ldots, a_m \in \mathcal{A}$, $l = \{a + iv : v \in \mathbb{R}\}$, $0 < a < 1/2$, $R_s(\lambda) = (\lambda - (1+s^2+\mathcal{D}^2))^{-1}$ and $r > 0$ we define $\phi_m^r(a_0, a_1, \ldots, a_m)$ to be

$$\frac{-2\sqrt{2\pi i}}{\Gamma((m+1)/2)} \int_0^\infty s^m\, \mathrm{Trace}\left(\frac{1}{2\pi i} \int_l \lambda^{-q/2-r} a_0 R_s(\lambda)[\mathcal{D}, a_1] R_s(\lambda) \cdots [\mathcal{D}, a_m] R_s(\lambda) d\lambda \right) ds.$$

In particular the sum on the right hand side of 1) *analytically continues to a deleted neighbourhood of* $r = (1-q)/2$ *with* at worst *a simple pole at* $r = (1-q)/2$. *Moreover, the complex function-valued cochain* $(\phi_m^r)_{m=1,\text{odd}}^{2N-1}$ *is a* (b, B) *cocycle for* $\mathcal{A}$ *modulo functions holomorphic in a half-plane containing* $r = (1-q)/2$.

2) The index is also the residue of a sum of zeta functions:

$$\frac{1}{\sqrt{2\pi i}}\text{res}_{r=(1-q)/2}\Bigg(\sum_{m=1,\text{odd}}^{2N-1}\sum_{|k|=0}^{2N-1-m}\sum_{j=0}^{|k|+(m-1)/2}(-1)^{|k|+m}\alpha(k)\Gamma((m+1)/2)\tilde{\sigma}_{|k|+(m-1)/2,j}$$
$$(r-(1-q)/2)^j\,\text{Trace}\left(u^*[\mathcal{D},u]^{(k_1)}[\mathcal{D},u^*]^{(k_2)}\cdots[\mathcal{D},u]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|-r+(1-q)/2}\right)\Bigg).$$

In particular the sum of zeta functions on the right hand side analytically continues to a deleted neighbourhood of $r = (1-q)/2$ *and has* at worst *a simple pole at* $r = (1-q)/2$.

3) If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *also has isolated spectral dimension then*

$$\text{Index}(PuP) = \frac{1}{\sqrt{2\pi i}}\sum_m \phi_m(\text{Ch}_m(u))$$

where for $a_0, \ldots, a_m \in \mathcal{A}$

$$\begin{aligned}\phi_m(a_0,\ldots,a_m) &= \text{res}_{r=(1-q)/2}\phi_m^r(a_0,\ldots,a_m) = \sqrt{2\pi i}\sum_{|k|=0}^{2N-1-m}(-1)^{|k|}\alpha(k)\times\\ &\times\sum_{j=0}^{|k|+(m-1)/2}\tilde{\sigma}_{(|k|+(m-1)/2),j}\tau_j\left(a_0[\mathcal{D},a_1]^{(k_1)}\cdots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-|k|-m/2}\right),\end{aligned}$$

and $(\phi_m)_{m=1,\text{odd}}^{2N-1}$ *is a* (b, B)*-cocycle for* $\mathcal{A}$. *When* $[q] = 2n$ *is even, the term with* $m = 2N - 1$ *is zero, and for* $m = 1, 3, ..., 2N - 3$, *all the top terms with* $|k| = 2N - 1 - m$ *are zero.*

Corollary 6.4. *For* $1 \le p < 2$, *the statements in* 3) *of Theorem 6.3 are true without the assumption of isolated dimension spectrum.*

For even spectral triples we have

Theorem 6.5 (Even local index formula). *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *be an even* QC^∞ *spectral triple with spectral dimension* $q \ge 1$. *Let* $N = [\frac{q+1}{2}]$, *where* $[\cdot]$ *denotes the integer part, and let* $p \in \mathcal{A}$ *be a self-adjoint projection. Then*

1) $$\text{Index}(p\mathcal{D}^+p) = \text{res}_{r=(1-q)/2}\left(\textstyle\sum_{m=0,\text{even}}^{2N}\phi_m^r(\text{Ch}_m(p))\right)$$
where for $a_0, \ldots, a_m \in \mathcal{A}$, $l = \{a + iv : v \in \mathbb{R}\}$, $0 < a < 1/2$, $R_s(\lambda) = (\lambda - (1 + s^2 + \mathcal{D}^2))^{-1}$ *and* $r > 1/2$ *we define* $\phi_m^r(a_0, a_1, \ldots, a_m)$ *to be*

$$\frac{(m/2)!}{m!}\int_0^\infty 2^{m+1}s^m\,\text{Trace}\left(\gamma\frac{1}{2\pi i}\int_l \lambda^{-q/2-r}a_0R_s(\lambda)[\mathcal{D},a_1]R_s(\lambda)\cdots[\mathcal{D},a_m]R_s(\lambda)d\lambda\right)ds.$$

In particular the sum on the right hand side of 1) *analytically continues to a deleted neighbourhood of* $r = (1-q)/2$ *with* at worst *a simple pole at* $r = (1-q)/2$. *Moreover, the complex function-valued cochain* $(\phi_m^r)_{m=0,\text{even}}^{2N}$ *is a* (b, B)*-cocycle for* $\mathcal{A}$ *modulo functions holomorphic in a half-plane containing* $r = (1-q)/2$.

2) The index, $\mathrm{Index}(p\mathcal{D}^+p)$ *is also the residue of a sum of zeta functions:*

$$\mathrm{res}_{r=(1-q)/2}\Bigg(\sum_{m=0,\mathrm{even}}^{2N}\sum_{|k|=0}^{2N-m}\sum_{j=1}^{|k|+m/2}(-1)^{|k|+m/2}\alpha(k)\frac{(m/2)!}{2\,m!}\sigma_{|k|+m/2,j}\times$$
$$\times(r-(1-q)/2)^j\,\mathrm{Trace}\Big(\gamma(2p-1)[\mathcal{D},p]^{(k_1)}[\mathcal{D},p]^{(k_2)}\cdots[\mathcal{D},p]^{(k_m)}(1+\mathcal{D}^2)^{-m/2-|k|-r+(1-q)/2}\Big)\Bigg),$$

(for $m=0$ we replace $(2p-1)$ by $2p$). In particular the sum of zeta functions on the right hand side analytically continues to a deleted neighbourhood of $r=(1-q)/2$ and has at worst *a simple pole at $r=(1-q)/2$.*

3) If $(\mathcal{A},\mathcal{H},\mathcal{D})$ also has isolated spectral dimension then

$$\mathrm{Index}(p\mathcal{D}^+p)=\sum_{m=0,\mathrm{even}}^{2N}\phi_m(\mathrm{Ch}_m(p))$$

where for $a_0,\dots,a_m\in\mathcal{A}$ we have $\phi_0(a_0)=\mathrm{res}_{r=(1-q)/2}\phi_0^r(a_0)=\tau_{-1}(\gamma a_0)$ and for $m\geq 2$

$$\begin{aligned}\phi_m(a_0,\dots,a_m) &= \mathrm{res}_{r=(1-q)/2}\phi_m^r(a_0,\dots,a_m)=\sum_{|k|=0}^{2N-m}(-1)^{|k|}\alpha(k)\times\\ &\times\sum_{j=1}^{|k|+m/2}\sigma_{(|k|+m/2),j}\tau_{j-1}\Big(\gamma a_0[\mathcal{D},a_1]^{(k_1)}\cdots[\mathcal{D},a_m]^{(k_m)}(1+\mathcal{D}^2)^{-|k|-m/2}\Big),\end{aligned}$$

and $(\phi_m)_{m=0,\mathrm{even}}^{2N}$ is a (b,B)-cocycle for $\mathcal{A}$. When $[q]=2n+1$ is odd, the term with $m=2N$ is zero, and for $m=0,2,\dots,2N-2$, all the top terms with $|k|=2N-m$ are zero.

Corollary 6.6. *For $1\leq q<2$, the statements in* 3) *of Theorem 6.5 are true without the assumption of isolated dimension spectrum.*

Proposition 6.7 ([CPRS4]). *For a QC^∞ finitely summable spectral triple with isolated spectral dimension, the collection of functionals $(\phi_m)_{m=P,P+2,\dots,2N-P}$, $P=0$ or 1, defined in part 3 of Theorems 6.3 and 6.5 are (b,B)-cocycles. They represent the class of the Chern character, and we call this (b,B)-cocycle the* **residue cocycle***.*

We remark that the proof of the (b,B)-cocycle property is straightforward once one verifies that the functionals $(\phi_m^r)_{m=P,P+2,\dots,2N-1}$ of part 1 of the Theorems 6.3 and 6.5 are function-valued (b,B)-cocycles (they depend on the complex variable r) modulo functions holomorphic at the critical point $r=(1-q)/2$. This 'almost' cocycle is called the **resolvent cocycle**, and serves as a replacement for the JLO coycle (see below) when we have a finitely summable spectral triple.

Computing the cocycle given by the local index formula is often much easier than computing the Fredholm module version. Understanding *all* the terms and interpreting what they tell us about the 'geometry' of a spectral triple is a major undertaking, involving the construction of many new examples.

6.3 The JLO **cocycle.** The JLO-cocycle is another representative of the Chern character for finitely summable spectral triples [C1]. It may be derived in the even case, from the McKean-Singer formula, while in the odd case one uses the θ-summable spectral flow formula of [G, CP2] which we now describe. Let $\mathcal{D}_t=\mathcal{D}+tu[\mathcal{D},u^*]$ for $u\in\mathcal{A}$ unitary, and then the spectral flow along $\{\mathcal{D}_t\}$ is given by

$$\mathrm{Index}(PuP)=\frac{1}{\sqrt{\pi}}\int_0^1\mathrm{Trace}(u[\mathcal{D},u^*]e^{-\mathcal{D}_t^2})dt.$$

The derivation of the JLO-cocycle from this analytic formula in the odd case is in [G, CP2].

However, the principal application of the JLO cocycle is in the study of theta summable spectral triples, where the local index formula is not available. These are essential for infinite dimensional situations, for example in supersymmetric quantum field theory. The JLO cocycle is also the starting point for Connes and Moscovici's original proof of the local index formula [CM].

Historically θ-summable Fredholm modules and spectral triples were introduced by Connes in association with the study of entire cyclic cohomology, see [C1]. The JLO cocycle was discovered later by Jaffe, Lesniewski and Osterwalder [JLO]. Connes then proved that it is a representative for the Chern character of a theta summable spectral triple. We now describe this representative explicitly. It is given on even spectral triples by an infinite sequence of cochains $(\mathrm{JLO}_{2k})_{k\geq 0}$ defined by

$$\mathrm{JLO}_{2k}(a_0,a_1,...,a_{2k}) = \int_\Delta \mathrm{Trace}(\gamma a_0 e^{-t_0\mathcal{D}^2}[\mathcal{D},a_1]e^{-t_1\mathcal{D}^2}\cdots e^{-t_{2k-1}\mathcal{D}^2}[\mathcal{D},a_{2k}]e^{-t_{2k}\mathcal{D}^2})dt_0dt_1\cdots dt_{2k}.$$

Here $\Delta = \{(t_0,t_1,\dots,t_{2k})\in\mathbb{R}^{2k+1} : t_j\geq 0,\ t_0+t_1+\cdots+t_{2k}=1\}$ is the standard simplex.

In the odd case we have $(\mathrm{JLO}_{2k+1})_{k\geq 0}$ defined by

$$\mathrm{JLO}_{2k+1}(a_0,a_1,...,a_{2k+1}) = \sqrt{2\pi i}\int_\Delta \mathrm{Trace}(a_0 e^{-t_0\mathcal{D}^2}[\mathcal{D},a_1]e^{-t_1\mathcal{D}^2}\cdots e^{-t_{2k}\mathcal{D}^2}[\mathcal{D},a_{2k}]e^{-t_{2k+1}\mathcal{D}^2})dt_0dt_1\cdots dt_{2k+1}.$$

In the context of entire cyclic cohomology, the JLO cocycle represents the Chern character. That is, if $[p]\in K_0(\mathcal{A})$ and $(\mathcal{A},\mathcal{H},\mathcal{D})$ is θ-summable, then

$$\langle [p],[(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle = \langle [\mathrm{Ch}(p)],[\mathrm{JLO}(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle = \sum_{k=0}^\infty \mathrm{JLO}_{2k}(\mathrm{Ch}_{2k}(p)).$$

A similar statement holds in the odd case. The problem with the JLO cocycle is that it is difficult to compute with. For finitely summable spectral triples the local index formula is superior in this regard.

Example 21. Block and Fox showed, [BF], starting with the JLO cocycle and using Getzler scaling, that the Chern character for the Dirac operator on a compact spin manifold M can be represented by

$$\mathrm{Ch}_k(f_0,f_1,...,f_k) = c_k\int_M \hat{A} f_0\, df_1\wedge\cdots df_k,\quad f_j\in C^\infty(M),$$

where $\hat{A}$ is the A-roof class of the manifold M. This is the Atiyah-Singer index theorem for Dirac operators, from which the general statement can be deduced. Similarly, Ponge used Getzler scaling and Greiner's analysis of heat kernel asymptotics, [Gr], to obtain the same formula starting from the residue cocycle for $\mathcal{D}$, [Po].

Example 22. We explain the local index cocycle for the noncommutative torus. From known computations of the cyclic cohomology of the noncommutative torus, [C0], the cocycle arising from the local index formula must be a linear combination of the 0-cocycle τ_0 and the 2-cocycle τ_2 given by

$$\tau_0(a_0)=\tau(a_0),\qquad \tau_2(a_0,a_1,a_2)=\tau\left(a_0\left(\delta_1(a_1)\delta_2(a_2)-\delta_2(a_1)\delta_1(a_2)\right)\right).$$

Exercise. What is the linear combination? *Hint*: The index pairing with any projection is an integer. Consider $1\in A_\theta$ and the Powers-Rieffel projector p_θ; see for example [D]. What integers do we expect?

The reason it is worth expressing the index pairing in cyclic theory is highlighted by the last exercise. Cyclic cohomology can often be computed using homological algebra and the relation to Hochschild cohomology. Moreover, there are other operators on these homology theories, and relations between them reminiscent of, for example, Hodge decompositions and the relation between Lie derivatives and the exterior derivative, [C1]. All these tools make computations in cyclic theory more practical. However, this is a big topic and beyond the scope of these notes.

7 Semifinite spectral triples and beyond

7.1 Semifinite spectral triples. Applications of noncommutative geometry to number theory and physics require a generalisation of the notion of spectral triple. In this Section we discuss one such generalisation. We have already seen that the discussion of the Chern character relies on the theory of trace ideals in $\mathcal{B}(\mathcal{H})$. There are however situations where more general traces and their associated ideals arise [CP1, BeF]. This is the so-called semifinite theory. Thus we begin with some semifinite versions of the standard $\mathcal{B}(\mathcal{H})$ definitions and results.

Let τ be a fixed faithful, normal, semifinite trace on a von Neumann algebra $\mathcal{N}$. Let $\mathcal{K}_\mathcal{N}$ be the τ-compact operators in $\mathcal{N}$ (that is, the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\tau(E) < \infty$).

Definition 7.1. *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that*

1) $[\mathcal{D}, a]$ is densely defined and extends to a bounded operator for all $a \in \mathcal{A}$

2) $a(\lambda - \mathcal{D})^{-1} \in \mathcal{K}_\mathcal{N}$ for all $\lambda \notin \mathbb{R}$ and all $a \in \mathcal{A}$.

3) The triple is said to be even if there is $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$ and $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Otherwise it is odd.

Along with the notion of τ-compact, we naturally get a notion of τ-Fredholm: $T \in \mathcal{N}$ is τ-Fredholm if and only if T is invertible modulo $\mathcal{K}_\mathcal{N}$. The index of such operators is in general real-valued, but we can often constrain the possible values. Index pairings with K-theory still make sense, and we are still interested in computing such pairings.

Observe that while we can define a semifinite Fredholm module in a similar way, it is not at all clear at this point what the relation to K-homology is, if any.

Definition 7.2. *A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^k for $k \geq 1$ (Q for quantum) if for all $a \in \mathcal{A}$ the operators a and $[\mathcal{D}, a]$ are in the domain of δ^k, where $\delta(T) = [|\mathcal{D}|, T]$ is the partial derivation on $\mathcal{N}$ defined by $|\mathcal{D}|$. We say that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is QC^∞ if it is QC^k for all $k \geq 1$.*

Remark The comments made about derivations and domains after Definition 5.1 are just as relevant here. One additional feature is that if $T \in \mathcal{N}$ and $[\mathcal{D}, T]$ is bounded, then $[\mathcal{D}, T] \in \mathcal{N}$. Similar comments apply to $[|\mathcal{D}|, T]$, $[(1 + \mathcal{D}^2)^{1/2}, T]$. The proofs can be found in [CPRS2].

7.1.1 Non-unitality. The examples coming from graph algebras, described soon, are often non-unital. Here is a brief summary of what we require in this case. See [R1, R2] and [GGISV] for more information. Whilst smoothness does not depend on whether $\mathcal{A}$ is unital or not, many analytical problems arise because of the lack of a unit. As in [GGISV, R1, R2], we make two definitions to address these issues.

Definition 7.3. *An algebra $\mathcal{A}$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}$, there exists $\phi \in \mathcal{A}$ such that for each i*

$$\phi a_i = a_i \phi = a_i.$$

Definition 7.4. *Let $\mathcal{A}$ be a Fréchet algebra and $\mathcal{A}_c \subseteq \mathcal{A}$ be a dense subalgebra with local units. Then we call $\mathcal{A}$ a quasi-local algebra (when $\mathcal{A}_c$ is understood). If $\mathcal{A}_c$ is a dense ideal with local units, we call $\mathcal{A}_c \subset \mathcal{A}$ local.*

Separable quasi-local algebras have an approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that for all n, $\phi_{n+1}\phi_n = \phi_n$, [R1]; we call this a local approximate unit. We also require that when we have a spectral triple the operator $\mathcal{D}$ is compatible with the quasi-local structure of the algebra, in the following sense.

Definition 7.5. *If $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple, then we define $\Omega^*_{\mathcal{D}}(\mathcal{A})$ to be the algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$.*

Definition 7.6. *A local spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with $\mathcal{A}$ quasi-local such that there exists an approximate unit $\{\phi_n\} \subset \mathcal{A}_c$ for $\mathcal{A}$ satisfying*

$$\Omega^*_{\mathcal{D}}(\mathcal{A}_c) = \bigcup_n \Omega^*_{\mathcal{D}}(\mathcal{A})_n, \quad \text{where}$$

$$\Omega^*_{\mathcal{D}}(\mathcal{A})_n = \{\omega \in \Omega^*_{\mathcal{D}}(\mathcal{A}) : \phi_n\omega = \omega\phi_n = \omega\}.$$

Remark. A local spectral triple has a local approximate unit $\{\phi_n\}_{n\geq 1} \subset \mathcal{A}_c$ such that $\phi_{n+1}\phi_n = \phi_n\phi_{n+1} = \phi_n$ and $\phi_{n+1}[\mathcal{D}, \phi_n] = [\mathcal{D}, \phi_n]\phi_{n+1} = [\mathcal{D}, \phi_n]$, see [R1, R2]. We require this property to prove the summability results we require.

7.1.2 Semifinite summability. In the following, let $\mathcal{N}$ be a semifinite von Neumann algebra with faithful normal trace τ. Recall from [FK] that if $S \in \mathcal{N}$, the *t-th generalized singular value* of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{||SE|| \ : \ E \text{ is a projection in } \mathcal{N} \text{ with } \tau(1-E) \leq t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N})$ consists of those operators $T \in \mathcal{N}$ such that $\|T\|_1 := \tau(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will simply write $\mathcal{L}^1$ for this ideal in order to simplify the notation, and denote the norm on $\mathcal{L}^1$ by $\|\cdot\|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1$ if $\|T\|_1 := \int_0^\infty \mu_t(T)dt < \infty$.

Note that in the case where $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1$ need not be complete in this norm but it is complete in the norm $\|\cdot\|_1 + \|\cdot\|_\infty$. (where $\|\cdot\|_\infty$ is the uniform norm). Another important ideal for us is the domain of the Dixmier traces:

$$\mathcal{L}^{(1,\infty)}(\mathcal{N}) = \left\{ T \in \mathcal{N} \ : \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds < \infty \right\}.$$

We will suppress the $(\mathcal{N})$ in our notation for these ideals, as $\mathcal{N}$ will always be clear from the context. The reader should note that $\mathcal{L}^{(1,\infty)}$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of τ-measurable operators affiliated to $\mathcal{N}$, [FK]. Our notation is however consistent with that of [C1] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of τ-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that

$$\mu_\infty(T) := \lim_{t\to\infty} \mu_t(T) = 0.$$

Definition 7.7. *A semifinite local spectral triple is*

- *finitely summable if there is some $s_0 \in [0, \infty)$ such that for all $s > s_0$ we have*

$$\tau(a(1+\mathcal{D}^2)^{-s/2}) < \infty \quad \textit{for all } a \in \mathcal{A}_c;$$

- *(p, ∞)-summable if*

$$a(\mathcal{D}-\lambda)^{-1} \in \mathcal{L}^{(p,\infty)} \quad \textit{for all } a \in \mathcal{A}_c, \quad \lambda \in \mathbb{C} \setminus \mathbb{R};$$

- *θ-summable if for all $t > 0$ we have*

$$\tau(ae^{-t\mathcal{D}^2}) < \infty \quad \textit{for all } a \in \mathcal{A}_c.$$

Remark. If $\mathcal{A}$ is unital, and $(1+\mathcal{D}^2)^{-1}$ is τ-compact, $\ker\mathcal{D}$ is τ-finite dimensional. Note that the summability requirements are only for $a \in \mathcal{A}_c$. We do not assume that elements of the algebra $\mathcal{A}$ are all integrable in the non-unital case.

We need to briefly discuss Dixmier traces in the von Neumann setting, but fortunately we will usually be applying it in reasonably simple situations. For more information on semifinite Dixmier traces, see [CPS2]. For $T \in \mathcal{L}^{(1,\infty)}$, $T \geq 0$, the function

$$F_T : t \to \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds$$

is bounded. For certain elements [CPS2], $\omega \in L^\infty(\mathbb{R}_*^+)^*$, we obtain a positive functional on $\mathcal{L}^{(1,\infty)}$ by setting

$$\tau_\omega(T) = \omega(F_T).$$

This is a Dixmier trace associated to the semifinite normal trace τ, denoted τ_ω, and we extend it to all of $\mathcal{L}^{(1,\infty)}$ by linearity, where of course it is a trace. The Dixmier trace τ_ω is defined on the ideal $\mathcal{L}^{(1,\infty)}$, and vanishes on the ideal of trace class operators. Whenever the function F_T has a limit at infinity, all Dixmier traces return the value of the limit. We denote the common value of all Dixmier traces on measurable operators by $⨍$. So if $T \in \mathcal{L}^{(1,\infty)}$ is measurable, for any allowed functional $\omega \in L^\infty(\mathbb{R}_*^+)^*$, we have

$$\tau_\omega(T) = \omega(F_T) = ⨍ T.$$

Example. Let $\mathcal{D} = \frac{1}{i}\frac{d}{d\theta}$ act on $L^2(S^1)$. Then the spectrum of $\mathcal{D}$ consists of eigenvalues $\{n \in \mathbb{Z}\}$, each with multiplicity one. So, using the standard operator trace, the function $F_{(1+\mathcal{D}^2)^{-1/2}}$ is

$$N \to \frac{1}{\log 2N+1} \sum_{n=-N}^{N} (1+n^2)^{-1/2}$$

which is bounded. So $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}$ and for any Dixmier trace Trace_ω

$$\mathrm{Trace}_\omega((1+\mathcal{D}^2)^{-1/2}) = ⨍(1+\mathcal{D}^2)^{-1/2} = 2.$$

In [R1, R2] we proved numerous properties of local algebras. The introduction of quasi-local algebras in [GGISV] led us to review the validity of many of these results for quasi-local algebras. Most of the summability results of [R1] are valid in the quasi-local setting. In addition, the summability results of [R2] are also valid for general semifinite spectral triples since they rely only on properties of the ideals $\mathcal{L}^{(p,\infty)}$, $p \geq 1$, [C1, CPS2], and the trace property. We quote the version of the summability results from [R2] that we require below, stated just for $p = 1$. This is a non-unital analogue of a result from [CPS2].

Proposition 7.8 ([R2]). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a QC^∞, local $(1,\infty)$-summable semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Let $T \in \mathcal{N}$ satisfy $T\phi = \phi T = T$ for some $\phi \in \mathcal{A}_c$. Then*

$$T(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}.$$

For $\mathrm{Re}(s) > 1$, $T(1+\mathcal{D}^2)^{-s/2}$ *is trace class. If the limit*

$$\lim_{s\to 1/2^+} (s-1/2)\tau(T(1+\mathcal{D}^2)^{-s}) \tag{7.1}$$

exists, then it is equal to

$$\frac{1}{2} ⨍ T(1+\mathcal{D}^2)^{-1/2}.$$

In addition, for any Dixmier trace τ_ω, the function

$$a \mapsto \tau_\omega(a(1+\mathcal{D}^2)^{-1/2})$$

defines a trace on $\mathcal{A}_c \subset \mathcal{A}$.

The various analytic formulae for computing the index, the local index formula, JLO cocycle, Chern characters and the results connecting them all continue to hold in the semifinite case, [CP1, CP2, CPS2, CPRS1, CPRS2, CPRS3, CPRS4]. Just replace the operator trace in the statement by a general semifinite trace. In addition, the local index formula holds for local spectral triples, semifinite or not, [R2].

7.2 Graph algebras. Our aim is to give an example of the application of spectral triples in the noncommutative setting that is quite distinct from any of the classical examples discussed so far. This Section also illustrates the fact that the semifinite theory is needed for many cases. Graph algebras and their higher dimensional analogues (k-graphs) are quite a diverse zoo. One can find algebras in these classes (and their relatives such as topological graph algebras, Cuntz-Krieger algebras, etc) with almost any required property. This makes them a great laboratory. They also arise in applications to number theory [CMR]. The following account of semifinite spectral triples for graph algebras comes from [PR].

For a more detailed introduction to graph C^*-algebras we refer the reader to [BPRS, KPR] and the references therein. A directed graph $E = (E^0, E^1, r, s)$ consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. We will always assume that the graph is **row-finite** which means that each vertex emits at most finitely many edges. Later we will also assume that the graph is *locally finite* which means it is row-finite and each vertex receives at most finitely many edges. We write E^n for the set of paths $\mu = \mu_1\mu_2\cdots\mu_n$ of length $|\mu| := n$; that is, sequences of edges μ_i such that $r(\mu_i) = s(\mu_{i+1})$ for $1 \le i < n$. The maps r, s extend to $E^* := \bigcup_{n\ge 0} E^n$ in an obvious way. A *loop* in E is a path $L \in E^*$ with $s(L) = r(L)$; we say that a loop L has an exit if there is $v = s(L_i)$ for some i which emits more than one edge. If $V \subseteq E^0$ then we write $V \ge w$ if there is a path $\mu \in E^*$ with $s(\mu) \in V$ and $r(\mu) = w$ (we also sometimes say that w is downstream from V). A *sink* is a vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$, a *source* is a vertex $w \in E^0$ with $r^{-1}(w) = \emptyset$.

A *Cuntz-Krieger E-family* in a C^*-algebra B consists of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ satisfying the *Cuntz-Krieger relations*

$$S_e^* S_e = p_{r(e)} \text{ for } e \in E^1 \text{ and } p_v = \sum_{\{e : s(e)=v\}} S_e S_e^* \text{ whenever } v \text{ is not a sink.}$$

It is proved in [KPR, Theorem 1.2] that there is a universal C^*-algebra $C^*(E)$ generated by a non-zero Cuntz-Krieger E-family $\{S_e, p_v\}$. A product $S_\mu := S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n}$ is non-zero precisely when $\mu = \mu_1\mu_2\cdots\mu_n$ is a path in E^n. Since the Cuntz-Krieger relations imply that the projections $S_eS_e^*$ are also mutually orthogonal, we have $S_e^*S_f = 0$ unless $e = f$, and words in $\{S_e, S_f^*\}$ collapse to products of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$ satisfying $r(\mu) = r(\nu)$ (cf. [KPR, Lemma 1.1]). Indeed, because the family $\{S_\mu S_\nu^*\}$ is closed under multiplication and involution, we have

$$C^*(E) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}. \tag{7.2}$$

The algebraic relations and the density of $\operatorname{span}\{S_\mu S_\nu^*\}$ in $C^*(E)$ play a critical role. We adopt the conventions that vertices are paths of length 0, that $S_v := p_v$ for $v \in E^0$, and that all paths μ, ν appearing in (7.2) are non-empty; we recover S_μ, for example, by taking $\nu = r(\mu)$, so that $S_\mu S_\nu^* = S_\mu p_{r(\mu)} = S_\mu$.

If $z \in S^1$, then the family $\{zS_e, p_v\}$ is another Cuntz-Krieger E-family which generates $C^*(E)$, and the universal property gives a homomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(S_e) = zS_e$ and $\gamma_z(p_v) = p_v$. The homomorphism $\gamma_{\overline{z}}$ is an inverse for γ_z, so $\gamma_z \in \operatorname{Aut} C^*(E)$, and a routine $\epsilon/3$ argument using (7.2) shows that γ is a strongly continuous action of S^1 on $C^*(E)$. It is called the *gauge action.* Because S^1 is compact, averaging over γ with respect to the normalised Haar measure gives an expectation Φ of $C^*(E)$ onto the fixed-point algebra $C^*(E)^\gamma$:

$$\Phi(a) := \frac{1}{2\pi}\int_{S^1} \gamma_z(a)\, d\theta \quad \text{for} \quad a \in C^*(E), \quad z = e^{i\theta}.$$

The map Φ is positive, has norm 1, and is faithful in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.

From Equation (7.2), it is easy to see that a graph C^*-algebra is unital if and only if the underlying graph is finite. When we consider infinite graphs, we always obtain a quasi-local algebra.

Example. For a graph C^*-algebra $A = C^*(E)$, Equation (7.2) shows that

$$A_c = \text{span}\{S_\mu S_\nu^* : \mu, \nu \in E^* \text{ and } r(\mu) = r(\nu)\}$$

is a dense subalgebra. It has local units because

$$p_v S_\mu S_\nu^* = \begin{cases} S_\mu S_\nu^* & v = s(\mu) \\ 0 & \text{otherwise} \end{cases}.$$

Similar comments apply to right multiplication by $p_{s(\nu)}$. By summing the source and range projections (without repetitions) of all $S_{\mu_i} S_{\nu_i}^*$ appearing in a finite sum

$$a = \sum_i c_{\mu_i,\nu_i} S_{\mu_i} S_{\nu_i}^*$$

we obtain a local unit for $a \in A_c$. By repeating this process for any finite collection of such $a \in A_c$ we see that A_c has local units.

7.3 Graph C^*-algebras with semifinite graph traces. This section considers the existence of (unbounded) traces on graph algebras. We denote by A^+ the positive cone in a C^*-algebra A, and we use extended arithmetic on $[0, \infty]$ so that $0 \times \infty = 0$. From [PhR] we take the definition:

Definition 7.9. *A trace on a C^*-algebra A is a map $\tau : A^+ \to [0, \infty]$ satisfying*

1) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in A^+$

2) $\tau(\lambda a) = \lambda\tau(a)$ for all $a \in A^+$ and $\lambda \geq 0$

3) $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$

We say: that τ is faithful if $\tau(a^*a) = 0 \Rightarrow a = 0$; that τ is semifinite[3] if $\{a \in A^+ : \tau(a) < \infty\}$ is norm dense in A^+ (or that τ is densely defined); that τ is lower semicontinuous if whenever $a = \lim_{n\to\infty} a_n$ in norm in A^+ we have $\tau(a) \leq \liminf_{n\to\infty} \tau(a_n)$.

We may extend a (semifinite) trace τ by linearity to a linear functional on (a dense subspace of) A. Observe that the domain of definition of a densely defined trace is a two-sided ideal $I_\tau \subset A$.

Lemma 7.10. *Let E be a row-finite directed graph and let $\tau : C^*(E) \to \mathbb{C}$ be a semifinite trace. Then the dense subalgebra*

$$A_c := \text{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$$

is contained in the domain I_τ of τ.

[3]The use of semifinite here is different from the von Neumann setting. For a von Neumann algebra, a trace is semifinite if the domain is weakly dense. For a C^*-algebra a trace is semifinite if the domain is norm dense.

It is convenient to denote by $A = C^*(E)$ and $A_c = \text{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$.

Lemma 7.11. *Let E be a row-finite directed graph.*

(i) *If $C^*(E)$ has a faithful semifinite trace then no loop can have an exit.*

(ii) *If $C^*(E)$ has a gauge-invariant, semifinite, lower semicontinuous trace τ then $\tau \circ \Phi = \tau$ and*

$$\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu} \tau(p_{r(\mu)}).$$

In particular, τ is supported on $C^*(\{S_\mu S_\mu^* : \mu \in E^*\})$.

Whilst the condition that no loop has an exit is necessary for the existence of a faithful semifinite trace, it is not sufficient.

One of the advantages of graph C^*-algebras is the ability to use both graphical and analytical techniques. There is an analogue of the above discussion of traces in terms of the graph.

Definition 7.12 (cf. [T]). *If E is a row-finite directed graph, then a graph trace on E is a function $g : E^0 \to \mathbb{R}^+$ such that for any $v \in E^0$ we have*

$$g(v) = \sum_{s(e)=v} g(r(e)). \tag{7.3}$$

If $g(v) \neq 0$ for all $v \in E^0$ we say that g is faithful.

Remark. One can show by induction that if g is a graph trace on a directed graph with no sinks, and $n \geq 1$

$$g(v) = \sum_{s(\mu)=v,\ |\mu|=n} g(r(\mu)). \tag{7.4}$$

For graphs with sinks, we must modify this formula to take into account paths of length less than n which end on sinks. To deal with this more general case we write

$$g(v) = \sum_{s(\mu)=v,\ |\mu|\preceq n} g(r(\mu)) \geq \sum_{s(\mu)=v,\ |\mu|=n} g(r(\mu)), \tag{7.5}$$

where $|\mu| \preceq n$ means that μ is of length n or is of length less than n and terminates on a sink.

As with traces on $C^*(E)$, it is easy to see that a necessary condition for E to have a faithful graph trace is that no loop has an exit.

Proposition 7.13. *Let E be a row-finite directed graph. Then there is a one-to-one correspondence between faithful graph traces on E and faithful, semifinite, lower semicontinuous, gauge invariant traces on $C^*(E)$.*

There are several steps in the construction of a spectral triple. We begin in Subsection 7.3.1 by constructing a C^*-module. We define an unbounded operator $\mathcal{D}$ on this C^*-module as the generator of the gauge action of S^1 on the graph algebra. We show in Subsection 7.3.2 that $\mathcal{D}$ is a regular self-adjoint operator on the C^*-module. We use the phase of $\mathcal{D}$ to construct a Kasparov module.

7.3.1 Building a C^*-module. Readers unfamiliar with C^*-modules should understand that they share some properties of a Hilbert space, except that the inner product takes values in a C^*-algebra F, which acts on (the right of) the module. Consequently they are a special class of Banach spaces. This introduces many subtleties into the theory. Fortunately the examples below are straightforward; if necessary more information can be found in [La, RW].

So, in brief, a right C^*-module X for the C^*-algebra F is a linear space with an action (on the right) of F and an inner product

$$(\cdot|\cdot) : X \times X \to F$$

linear in the second variable, and satisfying $(x|y)^* = (y|x)$ and that $(x|x) \geq 0$ in the sense of positive elements of F, and $(x|x) = 0$ if and only if $x = 0$. We also require that X is complete for the norm $\|x\|^2 = \|(x|x)\|_F$.

The useful things to know concern operators on these modules which commute with the right action of the C^*-algebra. One fact is that NOT all F-linear maps $X \to X$ possess adjoints for the inner product. The collection of adjointable endomorphisms (those with an adjoint) is denoted $\mathrm{End}_F(X)$. The adjointable endomorphisms form a C^*-algebra with respect to the adjoint operation and operator norm. Amongst these endomorphisms are the rank one endomorphisms $\Theta_{x,y}$, x, $y \in X$, defined on $z \in X$ by

$$\Theta_{x,y} z := x(y|z)_R.$$

Exercise. What is the adjoint of $\Theta_{x,y}$?

Finite sums of rank one endomorphisms are called finite rank. The finite rank endomorphisms generate a closed ideal in $\mathrm{End}_F(X)$. This ideal is called the ideal of compact endomorphisms and is denoted $\mathrm{End}^0_F(X)$.

Two important things should be noted:

(i) We have a notion of compact, so we have a notion of Fredholm (invertible modulo compacts), and so we have a notion of index. In this case the index is a difference of two F-modules, and this difference defines an element of $K_0(F)$. See [GVF] for a thorough discussion.

(ii) This notion of compactness need have nothing whatsoever to do with the compactness of operators on Hilbert space or even the notion of compactness in semifinite von Neumann algebras.

The actual C^*-modules we will consider in these notes are not complicated and thus we do not need to delve deeply into the complications of general C^*-module theory. Also, the constructions of this subsection work for any locally finite graph.

Let $A = C^*(E)$, where E is any locally finite directed graph. Let $F = C^*(E)^\gamma$ be the fixed point subalgebra for the gauge action. Finally, let A_c, F_c be the dense subalgebras of A, F given by the (finite) linear span of the generators. We make A a right inner product F-module. The right action of F on A is by right multiplication. The inner product is defined by

$$(x|y)_R := \Phi(x^*y) \in F.$$

Here Φ is the canonical expectation. It is simple to check the requirements that $(\cdot|\cdot)_R$ defines an F-valued inner product on A. The requirement $(x|x)_R = 0 \Rightarrow x = 0$ follows from the faithfulness of Φ.

Definition 7.14. *Define X to be the C^*-F-module completion of A for the C^*-module norm*

$$\|x\|_X^2 := \|(x|x)_R\|_A = \|(x|x)_R\|_F = \|\Phi(x^*x)\|_F.$$

Define X_c to be the pre-C^*-F_c-module with linear space A_c and the inner product $(\cdot|\cdot)_R$.

Remark. Typically, the action of F does not map X_c to itself, so we may only consider X_c as an F_c module. This is a reflection of the fact that F_c and A_c are quasilocal, not local.

The inclusion map $\iota : A \to X$ is continuous since

$$\|a\|_X^2 = \|\Phi(a^*a)\|_F \leq \|a^*a\|_A = \|a\|_A^2.$$

We can also define the gauge action γ on $A \subset X$, and as

$$\begin{aligned}\|\gamma_z(a)\|_X^2 &= \|\Phi((\gamma_z(a))^*(\gamma_z(a)))\|_F = \|\Phi(\gamma_z(a^*)\gamma_z(a))\|_F \\ &= \|\Phi(\gamma_z(a^*a))\|_F = \|\Phi(a^*a)\|_F = \|a\|_X^2,\end{aligned}$$

for each $z \in S^1$, the action of γ_z is isometric on $A \subset X$ and so extends to a unitary U_z on X. This unitary is F linear, adjointable, and we obtain a strongly continuous action of S^1 on X, which we still denote by γ.

For each $k \in \mathbb{Z}$, the projection onto the k-th spectral subspace for the gauge action defines an operator Φ_k on X by

$$\Phi_k(x) = \frac{1}{2\pi}\int_{S^1} z^{-k}\gamma_z(x)d\theta, \quad z = e^{i\theta}, \quad x \in X.$$

Observe that on generators we have $\Phi_k(S_\alpha S_\beta^*) = S_\alpha S_\beta^*$ when $|\alpha| - |\beta| = k$ and is zero when $|\alpha| - |\beta| \neq k$. The range of Φ_k is

$$\text{Range } \Phi_k = \{x \in X : \gamma_z(x) = z^k x \ \text{ for all } z \in S^1\}. \tag{7.6}$$

These ranges give us a natural $\mathbb{Z}$-grading of X.

Remark. If E is a finite graph with no loops, then for k sufficiently large there are no paths of length k and so $\Phi_k = 0$. This will obviously simplify many of the convergence issues below.

Lemma 7.15. *The operators Φ_k are adjointable endomorphisms of the F-module X such that $\Phi_k^* = \Phi_k = \Phi_k^2$ and $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$. If $K \subset \mathbb{Z}$ then the sum $\sum_{k\in K}\Phi_k$ converges strictly to a projection in the endomorphism algebra. The sum $\sum_{k\in\mathbb{Z}}\Phi_k$ converges to the identity operator on X.*

Corollary 7.16. *Let $x \in X$. Then with $x_k = \Phi_k x$ the sum $\sum_{k\in\mathbb{Z}} x_k$ converges in X to x.*

7.3.2 The Kasparov module. In this subsection we assume that E is locally finite and furthermore has no sources. That is, every vertex receives at least one edge.

Since we have the gauge action defined on X, we may use the generator of this action to define an unbounded operator $\mathcal{D}$. We will not define or study $\mathcal{D}$ from the generator point of view, rather taking a more bare-hands approach. It is easy to check that $\mathcal{D}$ as defined below is the generator of the S^1 action.

The theory of unbounded operators on C^*-modules that we require is all contained in Lance's book, [La, Chapters 9,10]. We quote the following definitions (adapted to our situation).

Definition 7.17. *Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D} : \mathrm{dom}\,\mathcal{D} \subset Y \to Y$ is a B-linear operator defined on a dense B-submodule $\mathrm{dom}\,\mathcal{D} \subset Y$. The operator $\mathcal{D}$ is closed if the graph*

$$G(\mathcal{D}) = \{(x|\mathcal{D}x)_R : x \in \mathrm{dom}\,\mathcal{D}\}$$

is a closed submodule of $Y \oplus Y$.

If $\mathcal{D} : \mathrm{dom}\,\mathcal{D} \subset Y \to Y$ is densely defined and unbounded, define a submodule

$$\mathrm{dom}\,\mathcal{D}^* := \{y \in Y : \text{ there exists } z \in Y \text{ such that for all } \ x \in \mathrm{dom}\,\mathcal{D}, (\mathcal{D}x|y)_R = (x|z)_R\}.$$

Then for $y \in \mathrm{dom}\ \mathcal{D}^*$ define $\mathcal{D}^*y = z$. Given $y \in \mathrm{dom}\,\mathcal{D}^*$, the element z is unique, so $\mathcal{D}^* : \mathrm{dom}\,\mathcal{D}^* \to Y$, $\mathcal{D}^*y = z$ is well-defined, and moreover is closed.

Definition 7.18. *Let Y be a right C^*-B-module. A densely defined unbounded operator $\mathcal{D} : \mathrm{dom}\,\mathcal{D} \subset Y \to Y$ is symmetric if for all $x, y \in \mathrm{dom}\,\mathcal{D}$*

$$(\mathcal{D}x|y)_R = (x|\mathcal{D}y)_R.$$

A symmetric operator $\mathcal{D}$ is self-adjoint if $\mathrm{dom}\ \mathcal{D} = \mathrm{dom}\,\mathcal{D}^*$ (and so $\mathcal{D}$ is necessarily closed). A densely defined unbounded operator $\mathcal{D}$ is regular if $\mathcal{D}$ is closed, $\mathcal{D}^*$ is densely defined, and $(1 + \mathcal{D}^*\mathcal{D})$ has dense range.

The extra requirement of regularity is necessary in the C^*-module context for the continuous functional calculus, and is not automatic, [La, Chapter 9]. With these definitions in hand, we return to our C^*-module X.

Proposition 7.19. *Let X be the right C^*-F-module of Definition 7.14. Define $X_{\mathcal{D}} \subset X$ to be the linear space*

$$X_{\mathcal{D}} = \{x = \sum_{k\in\mathbb{Z}} x_k \in X : \|\sum_{k\in\mathbb{Z}} k^2 (x_k|x_k)_R\| < \infty\}.$$

For $x = \sum_{k\in\mathbb{Z}} x_k \in X_{\mathcal{D}}$ define

$$\mathcal{D}x = \sum_{k\in\mathbb{Z}} k x_k.$$

Then $\mathcal{D} : X_{\mathcal{D}} \to X$ is a self-adjoint regular operator on X.

Remark. Any $S_\alpha S_\beta^* \in A_c$ is in $X_{\mathcal{D}}$ and

$$\mathcal{D} S_\alpha S_\beta^* = (|\alpha| - |\beta|) S_\alpha S_\beta^*.$$

There is a continuous functional calculus for self-adjoint regular operators, [La, Theorem 10.9], and we use this to obtain spectral projections for $\mathcal{D}$ at the C^*-module level. Let $f_k \in C_c(\mathbb{R})$ be 1 in a small neighbourhood of $k \in \mathbb{Z}$ and zero on $(-\infty, k-1/2] \cup [k+1/2, \infty)$. Then it is clear that

$$\Phi_k = f_k(\mathcal{D}).$$

That is, the spectral projections of $\mathcal{D}$ are the same as the projections onto the spectral subspaces of the gauge action.

The next Lemma is the first place where we need our graph to be locally finite and have no sources.

Lemma 7.20. *Assume that the directed graph E is locally finite and has no sources. For all $a \in A$ and $k \in \mathbb{Z}$, $a\Phi_k \in \mathrm{End}^0_F(X)$, the compact endomorphisms of the right F-module X. If $a \in A_c$ then $a\Phi_k$ is finite rank.*

Remark. The proof actually shows that for $k \geq 0$

$$\Phi_k = \sum_{|\rho|=k} \Theta^R_{S_\rho, S_\rho}$$

where the sum converges in the strict topology. A similar formula holds for $k < 0$.

Lemma 7.21. *Let E be a locally finite directed graph with no sources. For all $a \in A$, $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism of the F-module X.*

Proof. First let $a = p_v$ for $v \in E^0$. Then the sum

$$R_{v,N} := p_v \sum_{k=-N}^{N} \Phi_k (1+k^2)^{-1/2}$$

is finite rank, by Lemma 7.20. We will show that the sequence $\{R_{v,N}\}_{N\geq 0}$ is convergent with respect to the operator norm $\|\cdot\|_{\mathrm{End}}$ of endomorphisms of X. Indeed, assuming that $M > N$,

$$\begin{aligned}\|R_{v,N} - R_{v,M}\|_{\mathrm{End}} &= \|p_v \sum_{k=-M}^{-N} \Phi_k (1+k^2)^{-1/2} + p_v \sum_{k=N}^{M} \Phi_k (1+k^2)^{-1/2}\|_{\mathrm{End}} \\ &\leq 2(1+N^2)^{-1/2} \to 0, \end{aligned} \tag{7.7}$$

since the ranges of the $p_v\Phi_k$ are orthogonal for different k. Thus, using the argument from Lemma 7.20, $a(1+\mathcal{D}^2)^{-1/2} \in \mathrm{End}^0_F(X)$. Letting $\{a_i\}$ be a Cauchy sequence from A_c, we have

$$\|a_i(1+\mathcal{D}^2)^{-1/2} - a_j(1+\mathcal{D}^2)^{-1/2}\|_{\mathrm{End}} \le \|a_i - a_j\|_{\mathrm{End}} = \|a_i - a_j\|_A \to 0,$$

since $\|(1+\mathcal{D}^2)^{-1/2}\| \le 1$. Thus the sequence $a_i(1+\mathcal{D}^2)^{-1/2}$ is Cauchy in norm and we see that $a(1+\mathcal{D}^2)^{-1/2}$ is compact for all $a \in A$. □

It eventuates that the previous lemmas have proved that we have a Kasparov module. This is an extension of the notion of Fredholm module, but now, instead of a Hilbert space, we have a C^*-module. As for Fredholm modules and spectral triples, they come in two flavours, even and odd.

Definition 7.22. *An* **odd Kasparov A-B-module** *consists of a countably generated ungraded right B-C^*-module E, with $\phi : A \to \mathrm{End}_B(E)$ a $*$-homomorphism, together with $P \in \mathrm{End}_B(E)$ such that $a(P-P^*)$, $a(P^2-P)$, $[P,a]$ are all compact endomorphisms. Alternatively, for $V = 2P-1$, $a(V-V^*)$, $a(V^2-1)$, $[V,a]$ are all compact endomorphisms for all $a \in A$. One can modify P to $\tilde{P}$ so that $\tilde{P}$ is self-adjoint; $\|\tilde{P}\| \le 1$; $a(P-\tilde{P})$ is compact for all $a \in A$ and the other conditions for P hold with $\tilde{P}$ in place of P without changing the module E. If P has a spectral gap about 0 (as happens in the cases of interest here) then we may and do assume that $\tilde{P}$ is in fact a projection without changing the module, E.*

An **even Kasparov A-B-module** *has, in addition to the above data, a grading by a self-adjoint endomorphism Γ with $\Gamma^2 = 1$ and $\phi(a)\Gamma = \Gamma\phi(a)$, $V\Gamma + \Gamma V = 0$.*

Just as suitable equivalence relations turned Fredholm modules into a cohomology theory for C^*-algebras, so too there are relations which turn Kasparov A-B-modules into a *bivariant* theory, $KK^*(A,B)$. This works so that

$$KK^j(A,\mathbb{C}) = K^j(A),\ K\text{-homology}, \qquad KK^j(\mathbb{C},A) = K_j(A),\ K\text{-theory}.$$

By [K, Lemma 2, Section 7], the pair (ϕ, P) determines a $KK^1(A,B)$ class, and every class has such a representative. As for Fredholm modules, Kasparov modules have an unbounded version as well.

Definition 7.23. *An* **odd unbounded Kasparov A-B-module** *consists of a countably generated ungraded right B-C^*-module E, with $\phi : A \to \mathrm{End}_B(E)$ a $*$-homomorphism, together with an unbounded self-adjoint regular operator $\mathcal{D} : \mathrm{dom}\,\mathcal{D} \subset E \to E$ such that $[\mathcal{D},a]$ is bounded for all a in a dense $*$-subalgebra of A and $a(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism of E for all $a \in A$. An* **even unbounded Kasparov A-B-module** *has, in addition to the previous data, a $\mathbb{Z}_2$-grading with A even and $\mathcal{D}$ odd, as in Definition 7.22.*

So, now we can state a theorem about graph algebras.

Proposition 7.24. *Assume that the directed graph E is locally finite and has no sources. Let $V = \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$. Then (X,V) defines an odd Kasparov module, and so a class in $KK^1(A,F)$.*

Proof. We will use the approach of [K, Section 4]. We need to show that various operators belong to $\mathrm{End}^0_F(X)$. Notice that $V - V^* = 0$, so $a(V-V^*)$ is compact for all $a \in A$. Also $a(1-V^2) = a(1+\mathcal{D}^2)^{-1}$ which is compact from Lemma 7.21 and the boundedness of $(1+\mathcal{D}^2)^{-1/2}$. Finally, we need to show that $[V,a]$ is compact for all $a \in A$. First we suppose that $a = a_m$ is homogenous for the $\mathbb{T}^1$ action. Then

$$\begin{aligned}[V,a] &= [\mathcal{D},a](1+\mathcal{D}^2)^{-1/2} - \mathcal{D}(1+\mathcal{D}^2)^{-1/2}[(1+\mathcal{D}^2)^{1/2},a](1+\mathcal{D}^2)^{-1/2} \\ &= b_1(1+\mathcal{D}^2)^{-1/2} + Vb_2(1+\mathcal{D}^2)^{-1/2},\end{aligned}$$

where $b_1 = [\mathcal{D}, a] = ma$ and $b_2 = [(1+\mathcal{D}^2)^{1/2}, a]$. Provided that $b_2(1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism, Lemma 7.21 will show that $[V, a]$ is compact for all homogenous a. So consider $[(1+\mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2}$ acting on $x = \sum_{k\in\mathbb{Z}} x_k$. We find

$$\sum_{k\in\mathbb{Z}}[(1+\mathcal{D}^2)^{1/2}, S_\mu S_\nu^*](1+\mathcal{D}^2)^{-1/2}x_k = \sum_{k\in\mathbb{Z}}\Big((1+(|\mu|-|\nu|+k)^2)^{1/2} - (1+k^2)^{1/2}\Big)(1+k^2)^{-1/2}S_\mu S_\nu^* x_k = \sum_{k\in\mathbb{Z}} f_{\mu,\nu}(k) S_\mu S_\nu^* \Phi_k x. \tag{7.8}$$

The function

$$f_{\mu,\nu}(k) = \Big((1+(|\mu|-|\nu|+k)^2)^{1/2} - (1+k^2)^{1/2}\Big)(1+k^2)^{-1/2}$$

goes to 0 as $k \to \pm\infty$, and as the $S_\mu S_\nu^*\Phi_k$ are finite rank with orthogonal ranges, the sum in (7.8) converges in the endomorphism norm, and so converges to a compact endomorphism. For $a \in A_c$ we write a as a finite linear combination of generators $S_\mu S_\nu^*$, and apply the above reasoning to each term in the sum to find that $[(1+\mathcal{D}^2)^{1/2}, a](1+\mathcal{D}^2)^{-1/2}$ is a compact endomorphism. Now let $a \in A$ be the norm limit of a Cauchy sequence $\{a_i\}_{i\geq 0} \subset A_c$. Then

$$\|[V, a_i - a_j]\|_{\mathrm{End}} \leq 2\|a_i - a_j\|_{\mathrm{End}} \to 0,$$

so the sequence $[V, a_i]$ is also Cauchy in norm, and so the limit is compact. □

7.4 The gauge spectral triple of a graph algebra. In this section we will construct a semifinite spectral triple for those graph C^*-algebras which possess a faithful gauge invariant trace, τ. Recall from Proposition 7.13 that such traces arise from faithful graph traces.

We will begin with the right F_c-module X_c. In order to deal with the spectral projections of $\mathcal{D}$ we will also assume throughout this section that E is locally finite and has no sources. This ensures, by Lemma 7.20, that for all $a \in A$ the endomorphisms $a\Phi_k$ of X are compact endomorphisms.

As in the proof of Proposition 7.13, we define a $\mathbb{C}$-valued inner product on X_c:

$$\langle x, y\rangle := \tau((x|y)_R) = \tau(\Phi(x^*y)) = \tau(x^*y).$$

This inner product is linear in the second variable. We define the Hilbert space $\mathcal{H} = L^2(X,\tau)$ to be the completion of X_c for $\langle\cdot,\cdot\rangle$. We need a few lemmas in order to obtain the ingredients of our spectral triple.

Lemma 7.25. *The C^*-algebra $A = C^*(E)$ acts on $\mathcal{H}$ by an extension of left multiplication. This defines a faithful nondegenerate $*$-representation of A. Moreover, any endomorphism of X leaving X_c invariant extends uniquely to a bounded linear operator on $\mathcal{H}$.*

Lemma 7.26. *Let $\mathcal{H}, \mathcal{D}$ be as above and let $|\mathcal{D}| = \sqrt{\mathcal{D}^*\mathcal{D}} = \sqrt{\mathcal{D}^2}$ be the absolute value of $\mathcal{D}$. Then for $S_\alpha S_\beta^* \in A_c$, the operator $[|\mathcal{D}|, S_\alpha S_\beta^*]$ is well-defined on X_c, and extends to a bounded operator on $\mathcal{H}$ with*

$$\|[|\mathcal{D}|, S_\alpha S_\beta^*]\|_\infty \leq \Big||\alpha| - |\beta|\Big|.$$

Similarly, $\|[\mathcal{D}, S_\alpha S_\beta^*]\|_\infty = \Big||\alpha| - |\beta|\Big|$.

Corollary 7.27. *The algebra A_c is contained in the smooth domain of the derivation δ where for $T \in \mathcal{B}(\mathcal{H})$, $\delta(T) = [|\mathcal{D}|, T]$. That is,*

$$A_c \subseteq \bigcap_{n\geq 0} \mathrm{dom}\ \delta^n.$$

Definition 7.28. *Define the $*$-algebra $\mathcal{A} \subset A$ to be the completion of A_c in the δ-topology. By Lemma 5.5, $\mathcal{A}$ is Fréchet and stable under the holomorphic functional calculus.*

Lemma 7.29. *If $a \in \mathcal{A}$ then $[\mathcal{D}, a] \in \mathcal{A}$ and the operators $\delta^k(a)$, $\delta^k([\mathcal{D}, a])$ are bounded for all $k \geq 0$. If $\phi \in F \subset \mathcal{A}$ and $a \in \mathcal{A}$ satisfy $\phi a = a = a\phi$, then $\phi[\mathcal{D}, a] = [\mathcal{D}, a] = [\mathcal{D}, a]\phi$. The norm closed algebra generated by $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ is A. In particular, $\mathcal{A}$ is quasi-local.*

We leave the straightforward proofs of these statements to the reader.

7.4.1 Traces and compactness criteria. We still assume that E is a locally finite graph with no sources and that τ is a faithful semifinite lower semicontinuous gauge invariant trace on $C^*(E)$. We will define a von Neumann algebra $\mathcal{N}$ with a faithful semifinite normal trace $\tilde{\tau}$ so that $\mathcal{A} \subset \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ and $\mathcal{H}$ are as defined in the last subsection. Moreover the operator $\mathcal{D}$ will be affiliated to $\mathcal{N}$. The aim of this subsection will then be to prove the following result.

Theorem 7.30. *Let E be a locally finite graph with no sources, and let τ be a faithful, semifinite, gauge invariant, lower semicontinuous trace on $C^*(E)$. Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a QC^∞, $(1, \infty)$-summable, odd, local, semifinite spectral triple (relative to $(\mathcal{N}, \tilde{\tau})$). For all $a \in \mathcal{A}$, the operator $a(1 + \mathcal{D}^2)^{-1/2}$ is not trace class. If $v \in E^0$ has no sinks downstream then*

$$\tilde{\tau}_\omega(p_v(1 + \mathcal{D}^2)^{-1/2}) = 2\tau(p_v),$$

where $\tilde{\tau}_\omega$ is any Dixmier trace associated to $\tilde{\tau}$.

We require the definitions of $\mathcal{N}$ and $\tilde{\tau}$, along with some preliminary results.

Definition 7.31. *Let $\mathrm{End}^{00}_F(X_c)$ denote the algebra of finite rank operators on X_c acting on $\mathcal{H}$. Define $\mathcal{N} = (\mathrm{End}^{00}_F(X_c))''$, and let $\mathcal{N}_+$ denote the positive cone in $\mathcal{N}$.*

Definition 7.32. *Let $T \in \mathcal{N}$ and $\mu \in E^*$. Let $|v|_k =$ the number of paths of length k with range v, and define for $|\mu| \neq 0$*

$$\omega_\mu(T) = \langle S_\mu, TS_\mu\rangle + \frac{1}{|r(\mu)|_{|\mu|}}\langle S_\mu^*, TS_\mu^*\rangle.$$

For $|\mu| = 0$, $S_\mu = p_v$, for some $v \in E^0$, set $\omega_\mu(T) = \langle S_\mu, TS_\mu\rangle$. Define

$$\tilde{\tau} : \mathcal{N}_+ \to [0, \infty], \quad by \quad \tilde{\tau}(T) = \lim_{L \nearrow} \sum_{\mu \in L \subset E^*} \omega_\mu(T)$$

where L is in the net of finite subsets of E^*.

Remark. For $T, S \in \mathcal{N}_+$ and $\lambda \geq 0$ we have

$$\tilde{\tau}(T + S) = \tilde{\tau}(T) + \tilde{\tau}(S) \quad \text{and} \quad \tilde{\tau}(\lambda T) = \lambda\tilde{\tau}(T) \;\; \text{where} \;\; 0 \times \infty = 0.$$

Proposition 7.33. *The function $\tilde{\tau} : \mathcal{N}_+ \to [0, \infty]$ defines a faithful normal semifinite trace on $\mathcal{N}$. Moreover,*

$$\mathrm{End}^{00}_F(X_c) \subset \mathcal{N}_{\tilde{\tau}} := \mathrm{span}\{T \in \mathcal{N}_+ : \tilde{\tau}(T) < \infty\},$$

the domain of definition of $\tilde{\tau}$, and

$$\tilde{\tau}(\Theta^R_{x,y}) = \langle y, x\rangle = \tau(y^*x), \quad x, y \in X_c.$$

Notation If $g : E^0 \to \mathbb{R}_+$ is a faithful graph trace, we shall write τ_g for the associated semifinite trace on $C^*(E)$, and $\tilde{\tau}_g$ for the associated faithful, semifinite, normal trace on $\mathcal{N}$ constructed above.

Lemma 7.34. *Let E be a locally finite graph with no sources and a faithful graph trace g. Let $v \in E^0$ and $k \in \mathbb{Z}$. Then*

$$\tilde{\tau}_g(p_v\Phi_k) \leq \tau_g(p_v)$$

with equality when $k \leq 0$ or when $k > 0$ and there are no sinks within k vertices of v.

Proposition 7.35. *Assume that the directed graph E is locally finite, has no sources and has a faithful graph trace g. For all $a \in A_c$ the operator $a(1+\mathcal{D}^2)^{-1/2}$ is in the ideal $\mathcal{L}^{(1,\infty)}(\mathcal{N}, \tilde{\tau}_g)$.*

Remark. Using Proposition 7.8, one can check that

$$\mathrm{res}_{s=0}\tilde{\tau}_g(p_v(1+\mathcal{D}^2)^{-1/2-s}) = \frac{1}{2}\tilde{\tau}_{g\omega}(p_v(1+\mathcal{D}^2)^{-1/2}). \tag{7.9}$$

We will require this formula when we apply the local index formula.

Corollary 7.36. *Assume E is locally finite, has no sources and has a faithful graph trace g. Then for all $a \in A$, $a(1+\mathcal{D}^2)^{-1/2} \in \mathcal{K}_{\mathcal{N}}$.*

7.5 The index pairing. Having constructed semifinite spectral triples for graph C^*-algebras arising from locally finite graphs with no sources and a faithful graph trace, we can apply the semifinite local index formula described in [CPRS2]. See also [CPRS3, CM, H].

There is a C^*-module index, which takes its values in the K-theory of the core. The numerical index is obtained by applying the trace $\tilde{\tau}$ to the difference of projections representing the K-theory class coming from the C^*-module index.

Thus, for any unitary u in a matrix algebra over the graph algebra A,

$$\langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle \in \tilde{\tau}_*(K_0(F)).$$

We compute this pairing for unitaries arising from loops (with no exit), which provide a set of generators of $K_1(\mathcal{A})$. To describe the K-theory of the graphs we are considering, we employ the notion of ends.

Definition 7.37. *Let E be a row-finite directed graph. An* end *will mean a sink, a loop without exit or an infinite path with no exits.*

Remark. We shall identify an end with the vertices which comprise it. Once on an end (of any sort) the graph trace remains constant.

Lemma 7.38. *Let $C^*(E)$ be a graph C^*-algebra such that no loop in the locally finite graph E has an exit. Then,*

$$K_0(C^*(E)) = \mathbb{Z}^{\#\text{ends}}, \qquad K_1(C^*(E)) = \mathbb{Z}^{\#\text{loops}}.$$

If $A = C^*(E)$ is non-unital, we will denote by A^+ the algebra obtained by adjoining a unit to A; otherwise we let A^+ denote A.

Definition 7.39. *Let E be a locally finite graph such that $C^*(E)$ has a faithful graph trace g. Let L be a loop in E, and denote by $p_1, \ldots, p_n$ the projections associated to the vertices of L and by $S_1, \ldots, S_n$ the partial isometries associated to the edges of L, labelled so that $S_n^*S_n = p_1$ and*

$$S_i^*S_i = p_{i+1},\ i = 1, \ldots, n-1, \quad S_iS_i^* = p_i,\ i = 1, \ldots, n.$$

Lemma 7.40. *Let $A = C^*(E)$ be a graph C^*-algebra with faithful graph trace g. For each loop L in E we obtain a unitary in A^+,*

$$u = 1 + S_1 + S_2 + \cdots + S_n - (p_1 + p_2 + \cdots + p_n),$$

whose $K_1(A)$ class does not vanish. Moreover, distinct loops give rise to distinct $K_1(A)$ classes, and we obtain a complete set of generators of $K_1(A)$ in this way.

Proposition 7.41. *Let E be a locally finite graph with no sources and a faithful graph trace g and $A = C^*(E)$. The pairing between the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Theorem 7.30 with $K_1(A)$ is given on the generators of Lemma 7.40 by*

$$\langle [u], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = -\sum_{i=1}^{n} \tau_g(p_i) = -n\tau_g(p_1).$$

Proof. The semifinite local index formula, [CPRS2] provides a general formula for the Chern character of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. In our setting it is given by a one-cochain

$$\phi_1(a_0, a_1) = \mathrm{res}_{s=0}\sqrt{2\pi i}\, \tilde{\tau}_g(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-1/2-s}),$$

and the pairing (with $P = \chi_{[0,\infty)}(\mathcal{D})$) is given by

$$\mathrm{Index}(PuP) = \langle [u], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = \frac{1}{\sqrt{2\pi i}}\phi_1(u, u^*).$$

Now $[\mathcal{D}, u^*] = -\sum S_i^*$ and $u[\mathcal{D}, u^*] = -\sum_{i=1}^n p_i$. Using Equation (7.9) and Proposition 7.35,

$$\mathrm{Index}(PuP) = -\mathrm{res}_{s=0}\tilde{\tau}_g(\sum_{i=1}^{n} p_i(1 + \mathcal{D}^2)^{-1/2-s}) = -\sum_{i=1}^{n} \tau_g(p_i) = -n\tau_g(p_1),$$

the last equalities following since all the p_i have equal trace and there are no sinks 'downstream' from any p_i, since no loop has an exit. □

Remark. The C^*-algebra of the graph consisting of a single edge and single vertex is $C(S^1)$ (we choose Lebesgue measure as our trace, normalised so that $\tau(1) = 1$). For this example, the spectral triple we have constructed is the Dirac triple of the circle, $(C^\infty(S^1), L^2(S^1), \frac{1}{i}\frac{d}{d\theta})$, (as can be seen from Corollary 7.43 below.) The index theorem above gives the correct normalisation for the index pairing on the circle. That is, if we denote by z the unitary coming from the construction of Lemma 7.40 applied to this graph, then $\langle [\bar{z}], (\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = 1$.

Proposition 7.42. *Let E be a locally finite graph with no sources and a faithful graph trace g, and $A = C^*(E)$. The pairing between the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of Theorem 7.30 with $K_1(A)$ can be computed as follows. Let P be the positive spectral projection for $\mathcal{D}$, and perform the C^* index pairing [KNR]*

$$K_1(A) \times KK^1(A, F) \to K_0(F), \qquad [u] \times [(X, P)] \to [\ker PuP] - [\mathrm{coker}\, PuP].$$

Then we have

$$\mathrm{Index}\, PuP = \tilde{\tau}_g(\ker PuP) - \tilde{\tau}_g(\mathrm{coker}\, PuP) = \tilde{\tau}_{g*}([\ker PuP] - [\mathrm{coker}\, PuP]).$$

Proof. It suffices to prove this on the generators of $K_1(A)$ arising from loops L in E. Let $u = 1 + \sum_i S_i - \sum_i p_i$ be the corresponding unitary in A^+ defined in Lemma 7.40. We will show that $\ker PuP = \{0\}$ and that $\mathrm{coker} PuP =$

$\sum_{i=1}^n p_i\Phi_1$. For $a \in PX$ write $a = \sum_{m\geq 1} a_m$. For each $m \geq 1$ write $a_m = \sum_{i=1}^n p_i a_m + (1 - \sum_{i=1}^n p_i)a_m$. Then

$$\begin{aligned}
PuPa_m &= P(1 - \sum_{i=1}^n p_i + \sum_{i=1}^n S_i)a_m \\
&= P(1 - \sum^n p_i + \sum^n S_i)(\sum^n p_i a_m) + P(1 - \sum^n p_i + \sum^n S_i)(1 - \sum^n p_i)a_m \\
&= P\sum^n S_i a_m + P(1 - \sum^n p_i)a_m \\
&= \sum^n S_i a_m + (1 - \sum^n p_i)a_m.
\end{aligned}$$

It is clear from this computation that $PuPa_m \neq 0$ for $a_m \neq 0$.

Now suppose $m \geq 2$. If $\sum_{i=1}^n p_i a_m = a_m$ then $a_m = \lim_N \sum_{k=1}^N S_{\mu_k} S_{\nu_k}^*$ with $|\mu_k| - |\nu_k| = m \geq 2$ and $S_{\mu_{k1}} = S_i$ for some i. So we can construct b_{m-1} from a_m by removing the initial S_i's. Then $a_m = \sum_{i=1}^n S_i b_{m-1}$, and $\sum_{i=1}^n p_i b_{m-1} = b_{m-1}$. For arbitrary a_m, $m \geq 2$, we can write $a_m = \sum_i p_i a_m + (1 - \sum_i p_i)a_m$, and so

$$\begin{aligned}
a_m &= \sum^n p_i a_m + (1 - \sum^n p_i)a_m \\
&= \sum^n S_i b_{m-1} + (1 - \sum^n p_i)a_m \quad \text{and by adding zero} \\
&= \sum^n S_i b_{m-1} + (1 - \sum^n p_i)b_{m-1} + \big(\sum^n S_i + (1 - \sum^n p_i)\big)(1 - \sum^n p_i)a_m \\
&= ub_{m-1} + u(1 - \sum^n p_i)a_m \\
&= PuPb_{m-1} + PuP(1 - \sum^n p_i)a_m.
\end{aligned}$$

Thus PuP maps onto $\sum_{m\geq 2} \Phi_m X$.

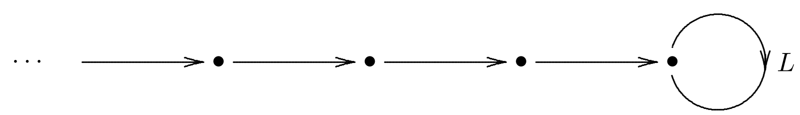
For $m = 1$, if we try to construct b_0 from $\sum_{i=1}^n p_i a_1$ as above, we find $PuPb_0 = 0$ since $Pb_0 = 0$. Thus $\mathrm{coker} PuP = \sum^n p_i \Phi_1 X$. By Proposition 7.41, the pairing is then

$$\begin{aligned}
\mathrm{Index}\, PuP &= -\sum^n \tau_g(p_i) = -\tilde{\tau}_g(\sum^n p_i \Phi_1) \\
&= -\tilde{\tau}_{g*}([\mathrm{coker}\, PuP]) = -\tilde{\tau}_g(\mathrm{coker}\, PuP). \qquad (7.10)
\end{aligned}$$

Thus we can recover the numerical index using $\tilde{\tau}_g$ and the C^*-index. □

The following example shows that the semifinite index provides finer invariants of directed graphs than those obtained from the ordinary index. The ordinary index computes the pairing between the K-theory and K-homology of $C^*(E)$, while the semifinite index also depends on the core and the gauge action.

Corollary 7.43 (Example). *Let $C^*(E_n)$ be the algebra determined by the graph*



where the loop L has n edges. Then $C^*(E_n) \cong C(S^1) \otimes \mathcal{K}$ for all n, but n is an invariant of the pair of algebras $(C^*(E_n), F_n)$ where F_n is the core of $C^*(E_n)$.

Proof. Observe that the graph E_n has a one-parameter family of faithful graph traces, specified by $g(v) = r \in \mathbb{R}_+$ for all $v \in E^0$. First consider the case where the graph consists only of the loop L. The C^*-algebra A of this graph is isomorphic to $M_n(C(S^1))$, via

$$S_i \to e_{i,i+1},\ i = 1, \dots, n-1, \quad S_n \to \mathrm{Id}_{S^1}\, e_{n,1},$$

where the $e_{i,j}$ are the standard matrix units for $M_n(\mathbb{C})$. The unitary

$$S_1 S_2 \cdots S_n + S_2 S_3 \cdots S_1 + \cdots + S_n S_1 \cdots S_{n-1}$$

is mapped to the orthogonal sum $\mathrm{Id}_{S^1} e_{1,1} \oplus \mathrm{Id}_{S^1} e_{2,2} \oplus \cdots \oplus \mathrm{Id}_{S^1} e_{n,n}$. The core F of A is $\mathbb{C}^n = \mathbb{C}[p_1, \dots, p_n]$. Since $KK^1(A, F)$ is equal to

$$\oplus^n KK^1(A, \mathbb{C}) = \oplus^n KK^1(M_n(C(S^1)), \mathbb{C}) = \oplus^n K^1(C(S^1)) = \mathbb{Z}^n$$

we see that n is the rank of $KK^1(A, F)$ and so an invariant, but let us link this to the index computed in Propositions 7.41 and 7.42 more explicitly. Let $\phi : C(S^1) \to A$ be given by $\phi(id_{S^1}) = S_1 S_2 \cdots S_n \oplus \sum_{i=2}^n e_{i,i}$. We observe that $\mathcal{D} = \sum_{i=1}^n p_i \mathcal{D}$ because the 'off-diagonal' terms are $p_i \mathcal{D} p_j = \mathcal{D} p_i p_j = 0$. Since $S_1 S_1^* = S_n^* S_n = p_1$, we find (with P the positive spectral projection of $\mathcal{D}$)

$$\phi^*(X, P) = (p_1 X, p_1 P p_1) \oplus \text{degenerate module} \in KK^1(C(S^1), F).$$

Now let $\psi : F \to \mathbb{C}^n$ be given by $\psi(\sum_j z_j p_j) = (z_1, z_2, ..., z_n)$. Then

$$\psi_* \phi^*(X, P) = \oplus_{j=1}^n (p_1 X p_j, p_1 P p_1) \in \oplus^n K^1(C(S^1)).$$

Now $X \cong M_n(C(S^1))$, so $p_1 X p_j \cong C(S^1)$ for each $j = 1, \dots, n$. It is easy to check that $p_1 \mathcal{D} p_1$ acts by $\frac{1}{i}\frac{d}{d\theta}$ on each $p_1 X p_j$, and so our Kasparov module maps to

$$\psi_* \phi^*(X, P) = \oplus^n (C(S^1), P_{\frac{1}{i}\frac{d}{d\theta}}) \in \oplus^n K^1(C(S^1)),$$

where $P_{\frac{1}{i}\frac{d}{d\theta}}$ is the positive spectral projection of $\frac{1}{i}\frac{d}{d\theta}$. The pairing with Id_{S^1} is nontrivial on each summand, since $\phi(\mathrm{Id}_{S^1}) = S_1 \cdots S_n \oplus \sum_{i=2}^n e_{i,i}$ is a unitary mapping $p_1 X p_j$ to itself for each j. So we have, [HR],

$$\begin{aligned} \mathrm{Id}_{S^1} \times \psi_* \phi^*(X, P) &= \sum_{j=1}^n \mathrm{Index}(P \, \mathrm{Id}_{S^1} P : p_1 P X p_j \to p_1 P X p_j) \\ &= -\sum_{j=1}^n [p_j] \in K_0(\mathbb{C}^n). \end{aligned} \tag{7.11}$$

By Proposition 7.42, applying the trace to this index gives $-n\tau_g(p_1)$. Of course in Proposition 7.42 we used the unitary $S_1 + S_2 + \cdots + S_n$; however in $K_1(A)$

$$[S_1 S_2 \cdots S_n] = [S_1 + S_2 + \cdots + S_n] = [\mathrm{Id}_{S^1}].$$

To see this, observe that

$$(S_1 + \cdots + S_n)^n = S_1 S_2 \cdots S_n + S_2 S_3 \cdots S_1 + \cdots + S_n S_1 \cdots S_{n-1}.$$

This is the orthogonal sum of n copies of Id_{S^1}, which is equivalent in K_1 to $n[\mathrm{Id}_{S^1}]$. Finally, $[S_1 + \cdots + S_n] = [\mathrm{Id}_{S^1}]$ and so

$$[(S_1 + \cdots + S_n)^n] = n[S_1 + \cdots + S_n] = n[\mathrm{Id}_{S^1}].$$

Since we have cancellation in K_1, this implies that the class of $S_1 + \cdots + S_n$ coincides with the class of $S_1 S_2 \cdots S_n$.

Having seen what is involved, we now add the infinite path on the left. The core becomes $\mathcal{K} \oplus \mathcal{K} \oplus \cdots \oplus \mathcal{K}$ (n copies). Since $A = C(S^1) \otimes \mathcal{K} = M_n(C(S^1)) \otimes \mathcal{K}$, the intrepid reader can go through the details of an argument similar to the one above, with entirely analogous results. □

The invariants obtained from the semifinite index are finer than the isomorphism class of $C^*(E)$, depending as they do on $C^*(E)$ and the gauge action, or equivalently $C^*(E)$ and F.

7.6 The relationship between semifinite triples and KK-theory. In order to construct a semifinite spectral triple for a graph algebra with gauge invariant trace, we first constructed a Kasparov module. The numerical index we computed was then compatible with the Kasparov product (K-theory-valued index). The question is whether this is always the case. The following proposition from [KNR] gives an affirmative answer. While stated for unital algebras, it can be generalised to the non-unital setting.

Proposition 7.44. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a unital semifinite spectral triple relative to $(\mathcal{N}, \tau)$. Suppose that the norm closure $A = \overline{\mathcal{A}}$ of $\mathcal{A}$ is a separable C^*-algebra. Let $P = \chi_{[0,\infty)}(\mathcal{D})$ and let $u \in \mathcal{A}$ be unitary. For V a closed subspace of $\mathcal{H}$ we write Q_V for the orthogonal projection onto V. Set $\mathcal{D}_t = (1-t)\mathcal{D} + tu^*\mathcal{D}u = \mathcal{D} + tu^*[\mathcal{D}, u]$; then the unbounded semifinite spectral flow of the path $t \mapsto \mathcal{D}_t$ is given by*

$$\mathrm{sf}\{\mathcal{D}_t\} = \mathrm{Index}(PuP) = \tau_*\big(Q_{\ker(pup+1-p)}\big) - \tau_*\big(Q_{\ker(pu^*p+1-p)}\big)$$

where $\tau_* : K_0(\mathcal{K}_\mathcal{N}) \to \mathbb{R}$ is the homomorphism induced by the trace τ and $p = \chi_{[0,\infty)}(F_\mathcal{D})$. In addition there exists a separable C^*-algebra $B \subseteq \mathcal{K}_\mathcal{N}$ and a class $[\mathcal{D}_B] \in KK^1(A, B)$ such that

$$\mathrm{sf}\{\mathcal{D}_t\} = \tau\big(i_*([u] \otimes_A [\mathcal{D}_B])\big)$$

where $i : B \to \mathcal{K}_N$ is the inclusion and $[u] \in K_1(A)$ is the class of the unitary.

Thus semifinite index theory is a special, computable, case of Kasparov theory. The greater the constraint we can place on the 'right-hand' algebra B, the more constraint we place on the possible values of the index. Since the index is *a priori* any real number, this can be very important.

For the graph of the previous Section, the index actually tells us the value of the graph trace on a projection (analytic input), and the number of vertices on the loop (topological data).

7.7 Modular spectral triples, type III von Neumann algebras and KMS states. The examples of spectral triples associated to graph algebras that we have discussed to this point are not sufficient for all applications. To illustrate this point: the Cuntz algebras $O_n : n = 2, 3, \ldots$ are graph algebras. Following the methods described previously enables us to construct a Kasparov module. However, O_n has no non-trivial traces, so we cannot construct a semifinite spectral triple using the methods we employed previously. The Cuntz algebras do however admit twisted traces. By this we mean there are densely defined non-$*$-automorphisms σ and states ϕ on O_n such that $\phi(ab) = \phi(\sigma(b)a)$ for all a, b in a dense subalgebra. These twisted traces are in fact KMS states for certain actions of the reals (the theory of which is developed in the lectures of Marcelo Laca). It is a subject of ongoing research at the moment to understand how to construct an index theory for such KMS states. The idea which has been expounded in [CPR2] is to modify the definition of spectral triple so as to accommodate the twist. Because the von Neumann algebra generated by the algebra in the GNS representation associated to a twisted trace may be type III there is no elementary way to do this. However, there is a proposal that gives interesting results for a variety of examples, including the Cuntz algebras, [CPR2], the algebra $SU_q(2)$, [CRT] and the graph algebras that arise in number theory [ConM]. Also, a class of examples extending the Cuntz algebra examples, constructed using a topological version of the group measure space construction, is contained in [CPPR].

Recently there has been substantial progress for the general situation where one studies KMS states for periodic actions of the reals on general C^*-algebras, [CNNR]. In this framework a general index theorem has been proved. There is an application of this KMS point of view to number theory in the form of constructing invariants of Mumford curves [CMR]. A discussion would however take us too far afield. Nevertheless [CNNR, CMR] indicate that there are connections between ideas from quantum statistical mechanics in the form of KMS states, equivariant KK-theory and the geometry of singular spaces.

A Unbounded operators on Hilbert space

This appendix is based primarily on [HR], but also see [RS].

Definition A.1. *An unbounded operator D on a Hilbert space $\mathcal{H}$ is a linear map from a subspace* $\mathrm{Dom}\, D \subset \mathcal{H}$ *(called the domain of D) to $\mathcal{H}$. The unbounded operator D is said to be densely defined if* $\mathrm{Dom}\, D$ *is dense in $\mathcal{H}$.*

We are really only interested in densely defined operators.

Definition A.2. *If D, D' are unbounded operators on $\mathcal{H}$ and* $\mathrm{Dom}\, D \subset \mathrm{Dom}\, D'$ *and $D\xi = D'\xi$ for all $\xi \in$* $\mathrm{Dom}\, D$*, then we write $D \subseteq D'$ and say that D' is an extension of D.*

Definition A.3. *If D is an (unbounded) operator on $\mathcal{H}$, the graph of D is the subspace* $\{(\xi, D\xi) : \xi \in \mathrm{Dom}\, D\} \subset \mathcal{H} \times \mathcal{H}$*. The operator D is said to be closed if the graph is a closed subspace of $\mathcal{H} \times \mathcal{H}$. The operator D is said to be closable if D has a closed extension D'.*

If $\mathrm{Dom}\, D$ is all of $\mathcal{H}$ and D is closed, then the closed graph theorem shows that D is bounded. For an unbounded operator D to be closed we must have: whenever $\{\xi_k\}_{k\geq 1} \subset \mathrm{Dom}\, D$ is a convergent sequence such that $\{D\xi_k\}_{k\geq 1}$ is also a convergent sequence we have $\lim_{k\to\infty} D\xi_k = D \lim_{k\to\infty} \xi_k$.

Any closable operator has a closure $\bar{D} \supseteq D$ which is the operator whose graph is the closure of the graph of D.

Definition A.4. *Let D be an unbounded densely defined operator on $\mathcal{H}$. Define*

$$\mathrm{Dom}\, D^* = \{\eta \in \mathcal{H} : \textit{for all } \ \xi \in \mathrm{Dom}\, D\ \exists \rho \in \mathcal{H} \text{ such that } \langle D\xi, \eta\rangle = \langle \xi, \rho\rangle\}.$$

Then we define $D^* : \mathrm{Dom}\, D^* \to \mathcal{H}$ by $D^*\eta = \rho$. This is well-defined, and the operator D^* is closed.

Exercise. Prove the two assertions of the definition.

Definition A.5. *An operator D is symmetric if $D \subseteq D^*$, so*

$$\langle D\xi, \eta\rangle = \langle \xi, D\eta\rangle \qquad \text{for all } \xi, \eta \in \mathrm{Dom}\, D.$$

The operator D is self-adjoint if $D = D^*$, so D is symmetric and $\mathrm{Dom}\, D = \mathrm{Dom}\, D^*$.

Despite appearances, there is a world of difference between symmetric and self-adjoint operators. If D is symmetric then it is closable and $D \subseteq \bar{D} \subseteq D^*$. If $\mathrm{Dom}\, \bar{D} = \mathrm{Dom}\, D^*$ then we say that D is essentially self-adjoint, meaning it has a unique self-adjoint extension.

Let D be a closed operator, and give $\mathrm{Dom}\, D$ the graph norm

$$\|\xi\|_D^2 = \|\xi\|^2 + \|D\xi\|^2.$$

Then $\mathrm{Dom}\, D$ is closed in the topology coming from the graph norm. The resolvent set of D is the set of all $\lambda \in \mathbb{C}$ such that the operator

$$(D - \lambda\, \mathrm{Id}_{\mathcal{H}}) : \mathrm{Dom}\, D \to \mathcal{H}$$

has a two-sided inverse. Any such inverse is a bounded operator from $\mathcal{H}$ to $\mathrm{Dom}\, D$ and so is a bounded operator.

The spectrum of D is the complement of the resolvent set, i.e. those $\lambda \in \mathbb{C}$ such that $(D - \lambda\, \mathrm{Id}_{\mathcal{H}})$ is not invertible.

Lemma A.6. *The spectrum of a self-adjoint operator is real.*

This allows us, after some effort, to come up with a functional calculus for self-adjoint operators. This functional calculus allows us to define $f(D)$ for any bounded Borel function on the spectrum of D. If $f_n \to f$ pointwise, then $f_n(D) \to f(D)$ in the strong operator topology. With suitable care with domains, it is also possible to define unbounded Borel functions of D. For a thorough discussion of this, see [RS].

Two important results that we exploit in the text are:

(i) Any differential operator on a manifold-without-boundary is closable.
(ii) Every symmetric differential operator on a compact manifold-without-boundary is essentially self-adjoint.

Proofs of these two results can be found in [HR].

Finally, an unbounded operator D on a Hilbert space $\mathcal{H}$ is said to be affiliated to a von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ if for all projections p in the commutant of $\mathcal{N}$ we have $p : \mathrm{dom}\, D \to \mathrm{dom}\, D$ and $Dp = pD$.

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